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Some remarks on invariant eigendistributions on semisimple Lie groups

By

Takeshi HIRAI

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Introduction

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Denote by $C_0^\infty(G)$ the set of all indefinitely differentiable functions on G which vanish outside some compact sets. For a differential operator D on G, we define its adjoint D^* as

$$\int_G Df_1(g)f_2(g)dg = \int_G f_1(g)\ D^*f_2(g)dg \qquad (f_1, f_2 \in C_0^\infty(G)),$$

where dg is a Haar measure on G. For any distribution π on G, we put $(D\pi)(f)=\pi(D^*f)\ (f\in C_0^\infty(G))$. A differential operator on G is called Laplace operator if it is invariant under both left and right translations. As usual, let us identify every $X\in\mathfrak{g}$ with a left-invariant differential operator on G. Then the center $\mathfrak{Z}$ of the universal envelopping algebra $U(\mathfrak{g}_c)$ of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ is the algebra of all Laplace operators on G. The correspondence $D\to D^*$ on $U(\mathfrak{g}_c)$ is its anti-automorphism generated by $X\to -X\ (X\in\mathfrak{g})$.

A distribution π on G is called invariant if it is invariant under any inner automorphism of G. It is called eigendistribution if there exists a homomorphism λ of $\mathfrak{Z}$ into $\boldsymbol{C}$ such that $Z\pi=\lambda(Z)\pi\ (Z\in\mathfrak{Z})$. Here λ is called the infinitesimal character of π. Let Z_G be the center of G.

If there exists a homomorphism χ of Z_G into $\boldsymbol{C}^*$ such that $\pi(zg)=\chi(z)\pi(g)(z\in Z_G)$, π is called Z_G-simple.

Now let $g\rightarrow T(g)(g\in G)$ be a representation of G by bounded operators on a Hilbelt space $\mathcal{H}$. Put for any $f\in C_0^\infty(G)$,

$$T(f)=\int_G T(g)f(g)dg.$$

A representation $(T, \mathcal{H})$ is called (topologically) irreducible if $\mathcal{H}$ has no closed invariant subspace except $\{0\}$ and $\mathcal{H}$ itself. An irreducible representation $(T, \mathcal{H})$ is called quasi-simple [2(a), I] if there exist homomorphisms χ of Z_G into $\boldsymbol{C}^*$ and λ of $\mathfrak{Z}$ into $\boldsymbol{C}$ such that

$$T(z)=X(z)1_{\mathcal{H}} \ (z\in Z_G), \quad T(Z)=\lambda(Z)1_{\mathcal{H}^0} \ (Z\in\mathfrak{Z}),$$

where $\mathcal{H}^0$ is the Gårding subspace of $\mathcal{H}$ spanned by all $T(f)v$ ($f\in C_0^\infty(G)$, $v\in\mathcal{H}$) and $1_{\mathcal{H}^0}$ denotes the identity operator on $\mathcal{H}^0$. The character π of such representation can be defined as to be the distribution $\pi(f)=\mathrm{tr}(T(f))(f\in C_0^\infty(G))$[2(a), II]. Then π is a Z_G-simple invariant eigendistribution corresponding to χ and λ. Call it simply *irreducible character*.

Denote by $\mathfrak{A}(\lambda)$ (or $\mathfrak{C}(\lambda)$) the set of all invariant eigendistributions on G (or linear combinations of irreducible characters) with infinitesimal character λ. Then $\mathfrak{A}(\lambda)\supset\mathfrak{C}(\lambda)$. One of the purposes of this paper is to study the problem whether $\mathfrak{A}(\lambda)=\mathfrak{C}(\lambda)$ for all λ or not. Here we give an elementary proof of existence on $SL(n, \boldsymbol{R})$ ($n\geqslant 3$) of tempered invariant eigendistributions which can not be expressed as linear combinations of irreducible characters. Moreover for $SL(n, \boldsymbol{R})$, all irreducible characters and all invariant eigendistributions with certain infinitesimal characters λ are obtained. Therefore we know exactly the difference of $\mathfrak{A}(\lambda)$ and $\mathfrak{C}(\lambda)$ for such λ. For complex classical groups $SL(n, \boldsymbol{C})$, $SO(2n+1, \boldsymbol{C})$, $Sp(n, \boldsymbol{C})$ and $SO(2n, \boldsymbol{C})$, we see that if $n\leqslant 3$, $\mathfrak{A}(\lambda)=\mathfrak{C}(\lambda)$ for any λ and that if $n\geqslant 4$, $\mathfrak{A}(\lambda)\neq\mathfrak{C}(\lambda)$ for some λ.

§ 1. Preliminary results

Let us introduce some notations and make some general statements. Let G, $\mathfrak{g}$ be as before and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Denote by P the set of all positive roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$ with respect to a lexicographic order. A root α is called real (or imaginary) if it takes only real (or imaginary) values on $\mathfrak{h}$. Denote by P_R (or P_I) the set of all real (or imaginary) positive roots. Let W_c be the Weyl group of $(\mathfrak{g}_c, \mathfrak{h}_c)$. Let H be the Cartan subgroup of G corresponding to $\mathfrak{h}$ and $W_G(H)$ the factor group of the normalizer of $\mathfrak{h}$ in G by the center H_0 of H. For any root α, let $X_\alpha \in \mathfrak{g}_c$ be its non-zero root vector and put $\mathrm{Ad}(h)X_\alpha = \xi_\alpha(h)X_\alpha$ $(h \in H)$. Define for $h \in H$,

$$(1.\ 1) \qquad \Delta'(h) = \prod_{\alpha \in P}(1-\xi_\alpha(h)^{-1}), \quad \Delta'_R(h) = \prod_{\alpha \in P_R}(1-\xi_\alpha(h)^{-1}).$$

Replacing G, if necessary, by a certain covering group which covers G finitely many times, we may assume that there exists a connected complex semisimple Lie group G_c with the following two properties. (a) Let ρ be the half-sum of all $\alpha \in P$ and H_c the Cartan subgroup of G_c corresponding to $\mathfrak{h}_c$. Then $\xi_\rho(\exp X) = e^{\rho(X)}$ $(X \in \mathfrak{h}_c)$ defines a one-valued function on H_c. (b) The injection j of $\mathfrak{g}$ into $\mathfrak{g}_c$ can be lifted up a homomorphism j' of G into G_c. The function $\xi_\rho \circ j'$ on H is denoted again by ξ_ρ. Now put

$$(1.\ 2) \qquad \nabla(h) = \xi_\rho(h)\mathrm{sign}(\Delta'_R(h))\Delta'(h) \qquad (h \in H).$$

Then for any $w \in W_G(H)$, there exists $\epsilon(w) = \pm 1$ such that $\nabla(wh) = \epsilon(w)\nabla(h)$.

Let G' be the set of all regular elements of G and put $H' = H \cap G'$, $G_H = \bigcup_{g \in G} gH'g^{-1}$. Define for any $f \in C_0^\infty(G)$, a function F_f on H' as

$$(1.\ 3) \qquad F_f(h) = \overline{\nabla(h)} \int_{G/H_0} f(ghg^{-1})d\tilde{g},$$

where $\tilde{g} = gH_0$, $d\tilde{g}$ is an invariant measure on G/H_0 and $\bar{a}$ denotes the complex conjugate of $a \in \boldsymbol{C}$.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$. Moreover let $\mathfrak{h}^0$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}^0=\mathfrak{a}+\mathfrak{h}^0\cap\mathfrak{k}$. Assume that the order in the set of roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$ is compatible with one in the set of roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$. Put

$$\mathfrak{n}_c=\sum_{\alpha\in P,\alpha|_{\mathfrak{a}}\neq 0} CX_\alpha, \quad \mathfrak{n}=\mathfrak{n}_c\cap\mathfrak{g}.$$

Let K, A and N be the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$. Then $G=KAN$ is Iwasawa decomposition of G. The Cartan subgroup corresponding to $\mathfrak{h}^0$ is denoted by H^0. We see easily that for $H=H^0$,

$$F_f(h)=\overline{\xi_\rho(h)}\,\overline{\prod_{\alpha\in P_I}(1-\xi_\alpha(h)^{-1})}\int_N\int_{K/Z_G} f(khnk^{-1})dk\, dn,$$

where $k=kZ_G$, dk and dn denote appropriate invariant measures on K/Z_G and N respectively. Hence,

Lemma 1.1. *For $H=H^0$, the function F_f on H' can be extended to an indefinitely differentiable function on the whole H with compact support.*

Now let $I(\mathfrak{h}_c)$ be the subset of $U(\mathfrak{h}_c)$ consisting of all W_c-invariant elements.

Lemma 1.2. (See [2(b), p. 118] and [2(c), Th. 3].) *There exists unique isomorphism $\gamma=\gamma^{\mathfrak{h}}$ of $\mathfrak{Z}$ onto $I(\mathfrak{h}_c)$ such that $F_{Zf}=\gamma(Z)F_f$ $(Z\in\mathfrak{Z})$. This γ satisfies that $\gamma(Z^*)=(\gamma(Z))^*$.*

A homomorphism of $I(\mathfrak{h}_c)$ into $\boldsymbol{C}$ is always induced by some $\mu\in\mathfrak{h}_c^*$, where $\mathfrak{h}_c^*$ is the dual of $\mathfrak{h}_c$. Denote this one by λ_μ. Then $\lambda_\mu=\lambda_{\mu'}$ if and only if $\mu'=\sigma\mu$ for some $\sigma\in W_c$. We say λ_μ is regular if $\mu\neq\sigma\mu$ for all $\sigma\in W_c$ not equal to the identity. We sometimes identify the homomorphism λ of $\mathfrak{Z}$ into $\boldsymbol{C}$ and the one $\lambda\circ\gamma^{-1}$ of $I(h_c)$. For a fixed λ, let us consider an analytic function κ on H^0 satisfying for some $\mu \in (\mathfrak{h}_c^0)^*$ such that $\lambda = \lambda_\mu \circ \gamma$ the following equations:

$$\kappa(wh) = \epsilon(w)\kappa(h) \qquad (w \in W_G(H^0),\ h \in H^0), \tag{1. 4}$$

$$D\kappa = \lambda_\mu(D)\kappa \qquad (D \in I(\mathfrak{h}_c^0(G)). \tag{1. 5}$$

Define a function π on G from κ as follows: for any $g \notin G_{H^0}$, $\pi(g)=0$; for $g \in G_{H^0}$, $\pi(g) = (\nabla(h_g))^{-1}\ \kappa(h_g)$, where $h_g \in H^0$ is an element such that $g = g_0 h_g g_0^{-1}$ for some $g_0 \in G$. Consider the distribution defined as

$$\pi(f) = <\pi, f> = \int_G f(g)\pi(g)dg \qquad (f \in C_0^\infty(G)).$$

Then using the above two lemmas on F_f, we obtain

Proposition 1. *The distribution π defined above is an invariant eigendistribution on G with the infinitesimal character $\lambda = \lambda_\mu \circ \gamma$ which vanishes identically outside the closure of G_{H^0}.*

Proof. Chose a Haar measure dh on a Cartan subgroup H appropriately, then for any integrable function φ on G_H,

$$\int_{G_H} \varphi(g)dg = \int_H \int_{G/H_0} \varphi(ghg^{-1})dg \cdot |\nabla(h)|^2 dh.$$

Therefore applying this formula for $H = H^0$,

$$<\pi, f> = \int_{H^0} F_f \cdot \kappa dh.$$

$$\begin{aligned} <Z\pi, f> = <\pi, Z^* f> &= \int_{H^0} F_{Z^* f} \cdot \kappa dh = \int_{H^0} \gamma(Z^*) F_f \cdot \kappa dh \\ &= \int_{H^0} \gamma(Z)^* F_f \cdot \kappa dh = \int_{H^0} F_f \cdot \gamma(Z) \kappa dh \\ &= \int_{H^0} F_f \cdot \lambda_\mu(\gamma(Z)) \kappa dh = \lambda(Z) <\pi, f>. \end{aligned}$$

Q.E.D.

Denote by $\mathfrak{A}_{H^0}(\lambda_\mu)$ the set of all invariant eigendistributions π obtained from the analytic functions κ on H^0 as above. Using Lemma 2.4 in §2, we can prove as in [3(b)] the following proposition (see [3(c)]). But this one is not used to prove that for $SL(n, \boldsymbol{R})(n \geqslant 3)$, $\mathfrak{A}(\lambda) \neq \mathfrak{C}(\lambda)$ for some λ.

Proposition 2. *The set $\mathfrak{A}_{H^0}(\lambda_\mu)$ is equal to the set of all invariant eigendistributions on G with infinitesimal character $\lambda = \lambda_\mu \circ \gamma$ which vanish identically outside the closure of G_{H^0}.*

§ 2. Review on known results

Here we summalize some known results in the form of a certain number of lemmas. Two quasi-simple irreducible representations T_i on $\mathcal{H}_i (i=1, 2)$ are said to be infinitesimally equivalent [2(a), I, p. 230] if the corresponding representations of $U(\mathfrak{g}_c)$ on $\mathcal{H}_i^\infty = \sum_\delta \mathcal{H}_i(\delta)$ (algebraic sum) are algebraically equivalent, where δ denotes an equivalent class of irreducible representations of K and $\mathcal{H}_i(\delta)$ denotes the subspace consisting of all vectors transformed under $T_i(k)$ $(k \in K)$ according to δ. Then,

Lemma 2.1 [2(a), III]. *Two quasi-simple irreducible representations of G have the same character if and only if they are infinitesimally equivalent. Two unitary irreducible representations have the same character if and only if they are unitary equivalent.*

Let M be the centralizer of A in K. Take $\mu^\mathfrak{a} \in \mathfrak{a}_c^*$ and a finite-dimensional irreducible representation ν of M. Then $L = (\mu^\mathfrak{a}, \nu)$ defines canonically a representation of MAN. Inducing this one from MAN to G, we obtain a representation T^L on a Hilbert space $\mathcal{H}^L$ consisting certain vector-valued functions on K(see e.g., [3(a)]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two closed invariant subspaces of $\mathcal{H}^L$ such that $\mathcal{H}_1 \supset \mathcal{H}_2$. If the representation induced on $\mathcal{H}_1 / \mathcal{H}_2$ is irreducible, it is called irreducible constituent of T^L. Then we know from Th. 4 of [2(a), II] the following

Lemma 2.2. *For $G=SL(n, \boldsymbol{R})$ or a connected complex semisimple Lie group, any quasi-simple irreducible representation of G is infinitesimally equivalent to an irreducible constituent of some T^L.*

We use the following lemmas in § 5.

Lemma 2.3. *Let $T_1, T_2, \ldots, T_d$ be the set of quasi-simple irreducible representations of G any two of which are not infinitesimally equivalent. Then their characters are linearly independent.*

Lemma 2.4 [2(d)]. *Any invariant eigendistribution π on G coincides with a locally summable function on G which is analytic on G'. Moreover for every Cartan subgroup H, the function $\kappa^{\mathfrak{h}}=\nabla\cdot(\pi|_{H'})$ on $H'=H\cap G'$ can be extended to an analytic function on $H'(R)=\{h\in H, \Delta_R'(h)\neq 0\}$.*

Let λ be the infinitesimal character of π and chose $\mu^{\mathfrak{h}}\in\mathfrak{h}_c^*$ such that $\lambda=\lambda_{\mu^{\mathfrak{h}}}\circ\gamma^{\mathfrak{h}}$. Then $\kappa^{\mathfrak{h}}$ on $H'(R)$ satisfies the analogous equations as (1. 4) and (1. 5):

$$\kappa^{\mathfrak{h}}(wh)=\epsilon(w)\kappa^{\mathfrak{h}}(h) \qquad (w\in W_G(H),\ h\in H'(R)), \tag{2. 1}$$

$$D\kappa^{\mathfrak{h}}=\lambda_{\mu^{\mathfrak{h}}}(D)\kappa^{\mathfrak{h}} \qquad (D\in I(\mathfrak{h}_c)). \tag{2. 2}$$

Suppose that $\mathfrak{h}=\mathfrak{h}_-+\mathfrak{h}_+$, where $\mathfrak{h}_-=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{h}_+=\mathfrak{h}\cap\mathfrak{p}$. Then putting $H_-=H\cap K$, $H=H_-\exp\mathfrak{h}_+$. For any connected component F of $H'(R)$, take $h_0\in H_-$ on the boundary of F. As a solution of (2. 2), $\kappa^{\mathfrak{h}}$ is expressed as

$$\kappa^{\mathfrak{h}}(h_0\exp X)=\sum_{\sigma\in W_c} p_\sigma(X)\exp\{\mu^{\mathfrak{h}}(\sigma X)\} \tag{2. 3}$$

if $X\in\mathfrak{h}$ is sufficiently small and $h_0\exp X\in F$, where p_σ 's are some polynomial functions on $\mathfrak{h}$. If K is compact or π is Z_G-simple, all p_σ can be taken as not to depend on $\mathfrak{h}_-$, because $F=F\exp\mathfrak{h}_-$. Define $\mathfrak{A}'(\lambda)$ (or $\mathfrak{A}''(\lambda)$, in case when K is compact) as the subset of $\mathfrak{A}(\lambda)$ consisting of such π that for any F and H, all p_σ in the expression (2. 3) can be taken as to be constants (or polynomials with constant terms zero).

§ 3. Invariant eigendistributions on $SL(n, \boldsymbol{R})$

In this section, let $G=SL(n, \boldsymbol{R})$ and H^0 its Cartan subgroup consisting of all diagonal matrices in G. Let us calculate all analytic functions κ on H^0 satisfying (1. 4) and (1. 5). Denote by $d(a_1, a_2, \dots, a_n)$ the diagonal matrix with diagonal elements $a_1, a_2, \dots, a_n$. For $h=d(a_1, a_2, \dots, a_n)\in H^0$,

$$\nabla(h)=|\prod_{i<j}(a_i-a_j)|. \tag{3. 1}$$

The Weyl group $W_G(H^0)$, simply denoted by W, is isomorphic to W_c and to the symmetric group $\mathfrak{S}_n$ of order n as permutation group of $a_1, a_2, \dots, a_n$. Let $\epsilon_j=\pm 1 (1\leqslant j\leqslant n)$ such that $\epsilon_1\epsilon_2\dots\epsilon_n=1$ and put $\epsilon=(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$. Denote by $H^0(\epsilon)$ the connected component of H^0 containing $d(\epsilon_1 e^{t_1}, \epsilon_2 e^{t_2}, \dots, \epsilon_n e^{t_n})$, where $t_j\in\boldsymbol{R}$. Put $I_k=\{1, 2, \dots, 2k\}$, $J_k=\{2k+1, 2k+2, \dots, n\}$ and let $\epsilon^{(k)}$ be such row ϵ that $\epsilon_j=-1$ for $l\in I_k$ and $\epsilon_j=1$ for $j\in J_k$. Put $H_k^0=H^0(\epsilon^{(k)})$. Any $H^0(\epsilon)$ is conjugate to some H_k^0 under W. It is sufficient to determine the restrictions κ_k of κ on H_k for $0\leqslant k\leqslant[n/2]$ because for $h\in H^0(\epsilon)=wH_k^0$ $(w\in W)$, $\kappa(h)=\kappa_k(w^{-1}h)$. The subgroup $W_k=\{w\in W;\ wH_k^0=H_k^0\}$ is isomorphic to $\mathfrak{S}_{2k}\times\mathfrak{S}_{n-2k}$ and (1. 4) is rewritten as

$$\kappa_k(wh)=\kappa_k(h) \qquad (w\in W_k,\ h\in H_k^0). \tag{3. 2}$$

Any element $\mu\in(\mathfrak{h}_c^0)^*$ is expressed uniquely as $\mu=(\mu_1, \mu_2, \dots, \mu_n)$, where $\mu_j\in\boldsymbol{C}$ and $\mu_1+\mu_2+\dots+\mu_n=0$, in such a way that

$$\mu(d(t_1, t_2, \dots, t_n))=\sum_{1\leqslant j\leqslant n}\mu_j t_j=(\mu, t) \quad \text{(put)}. \tag{3. 3}$$

To study the equations (1. 4) and (1. 5), it is convenient to replace G by the reductive group $^{+}G=\{g\in GL(n, \boldsymbol{R});\det g>0\}$. The results in §1 can be translated for ^{+}G word for word. Denote by $^{+}\mathfrak{h}^{0}$, $^{+}H^{0}$, $^{+}H^{0}(\epsilon)$, $^{+}H_{k}^{0}$, $^{+}\boldsymbol{V}$, $^{+}\kappa$ and $^{+}\kappa_{k}$ the analogous objects as $\mathfrak{h}^{0}$, H^{0}, $H^{0}(\epsilon)$, H_{k}^{0}, $\boldsymbol{V}$, κ and κ_{k} respectively. Then for $h=d(a_{1}, a_{2}, \ldots, a_{n})\in{}^{+}H^{0}$,

$$(3.\ 4)\qquad {}^{+}\boldsymbol{V}(h)=(a_{1}a_{2}\ldots a_{n})^{-\frac{n-1}{2}}\Big|\prod_{i>j}(a_{i}-a_{j})\Big|.$$

The Weyl groups are the same for G and ^{+}G. Denote by $\boldsymbol{t}_{j}$ the differential operator $\partial/\partial t_{j}$ on $^{+}H^{0}$. Then $I(^{+}\mathfrak{h}_{c}^{0})$, considered as the algebra of differential operators on $^{+}H^{0}$, is nothing but the symmetric polynomials of $\boldsymbol{t}_{1}, \boldsymbol{t}_{2}, \ldots, \boldsymbol{t}_{n}$. For any $\mu=(\mu_{1}, \mu_{2}, \ldots, \mu_{n})\in(^{+}\mathfrak{h}_{c}^{0})^{*}$ and $D(\boldsymbol{t})\in I(^{+}\mathfrak{h}_{c}^{0})$, $\lambda_{\mu}(D(\boldsymbol{t}))=D(\mu)$. We restrict ourselves to treat μ such that $\mu_{1}+\mu_{2}+\ldots+\mu_{n}=0$. Then for any such μ, there exists a one-one correspondence between the set of all solutions κ of the equations (1. 4), (1. 5) on H^{0} and that of $^{+}\kappa$ of the corresponding equations on $^{+}H^{0}$, by restricting $^{+}\kappa$ on H^{0}. Therefore it is sufficient for us to study the following equations: for $0\leqslant k\leqslant[n/2]$,

$$(3.\ 5)\qquad {}^{+}\kappa_{k}(wh)={}^{+}\kappa_{k}(h)\qquad(h\in{}^{+}H_{k}^{0},\ w\in W_{k}),$$

$$(3.\ 6)\qquad D(\boldsymbol{t})^{+}\kappa_{k}=D(\mu)^{+}\kappa_{k}\qquad(D\in I(^{+}\mathfrak{h}_{c}^{0})).$$

Put $W(\mu)=\{\tau\in W;\ \tau\mu=\mu\}$ and $\bar{\sigma}=\sigma W(\mu)$ for $\sigma\in W$. Any solution of (3. 6) is expressed uniquely as follows: for $h=d(\epsilon_{1}e^{t_{1}}, \epsilon_{2}e^{t_{2}}, \ldots, \epsilon_{n}e^{t_{n}})\in{}^{+}H_{k}^{0}$,

$$(3.\ 7)\qquad {}^{+}\kappa_{k}(h)=\sum_{\bar{\sigma}\in W/W(\mu)}p_{\bar{\sigma}}(t)\exp((\sigma\mu, t)),$$

where $p_{\bar{\sigma}}$'s are some polynomials of $t=(t_{1}, t_{2}, \ldots, t_{n})$. Let us rewrite (3. 5) and (3. 6) in terms of $p_{\bar{\sigma}}$'s. The equation (3. 5) is written as

$$(3.\ 8)\qquad wp_{\bar{\sigma}}=p_{\overline{w\sigma}}\qquad(w\in W_{k},\ \sigma\in W),$$

where $wp_{\bar{\sigma}}(t)=p_{\bar{\sigma}}(w^{-1}t)$. Take a complete system Σ of representatives of the double coset space $W_k\backslash W/W(\mu)$. Then (3. 8) means that it is sufficient to determin $p_{\bar\sigma}$ for $\sigma\in\Sigma$ and that for any $\sigma\in\Sigma$,

$$wp_{\bar\sigma}=p_{\bar\sigma} \qquad (w\in W_k\cap W(\sigma\mu)). \tag{3. 9}$$

Now let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the set of different numbers in $\mu_1, \mu_2, \ldots, \mu_n$ and put $A_r=\{j;\ \mu_j=\alpha_r\}$. Define $\sigma(i)$ as $(\sigma^{-1}\mu)_i=\mu_{\sigma(i)}$. Then $\sigma A_r=\{j;\ (\sigma\mu)_j=\alpha_r\}$. For any subset A of $\{1, 2, \ldots, n\}$, put

$$D_m(A)=\sum_{i\in A} t_i^m, \quad W(A)=\{w\in W;\ wA=A,\ w(i)=i \text{ for any } i\notin A\}$$

Using the same method as in [3(b), §9], we can prove the following

Lemma 3.1. *The system of equations (3. 5) and (3. 6) is expressed in terms of $p_{\bar\sigma}$ $(\sigma\in\Sigma)$ as*

$$\begin{cases} wp_{\bar\sigma}=p_{\bar\sigma} & (w\in W_k\cap W(\sigma A_r),\ 1\leqslant r\leqslant N), \\ D_m(\sigma A_r)p_{\bar\sigma}=0 & (m\geqslant 1,\ 1\leqslant r\leqslant N). \end{cases} \tag{3. 10}$$

Fix $\sigma\in\Sigma$ and r and put $A=\sigma A_r\cap I_k$, $B=\sigma A_r\cap J_k$, $p=p_{\bar\sigma}$, then $A\cap B=\phi$ and the above equations for σ and r are

$$\begin{cases} wp=p & (w\in W(A)\cap W(B)), \\ D_m(A\cap B)p=0 & (m\geqslant 1). \end{cases} \tag{3. 11}$$

if A or $B=\phi$, the polynomial p does not contain the variables $t_j (j\in A\cap B)$ explicitely (see [3(b), §9]). If $A\neq\phi$ and $B\neq\phi$, the equation (3. 11) has the following solution:

$$p(t)=(\sharp A)^{-1}\sum_{j\in A} t_j-(\sharp B)^{-1}\sum_{j\in B} t_j, \tag{3. 12}$$

where $\sharp A$ denotes the number of elements in A. Restricting this solution $p(t)$ from ${}^+H_k^0$ to H_k^0, we always obtain non-zero function.

Now denote by $\mathfrak{A}'_{H^0}(\lambda)$ and $\mathfrak{A}''_{H^0}(\lambda)$ the sets $\mathfrak{A}_{H^0}(\lambda)\cap\mathfrak{A}'(\lambda)$ and

$\mathfrak{A}_{H^0}(\lambda)\cap\mathfrak{A}''(\lambda)$. Then we obtain from the above arguments the following

Proposition 3. *For* $SL(n, \boldsymbol{R})$, $\mathfrak{A}_{H_0}(\lambda)=\mathfrak{A}'_{H^0}(\lambda)+\mathfrak{A}''_{H^0}(\lambda)$ *(direct sum). When $n=2$, always* $\mathfrak{A}''_{H^0}(\lambda)=\{0\}$ *and* $\mathfrak{A}_{H^0}(\lambda)=\mathfrak{A}'_{H^0}(\lambda)$. *When* $n \geqslant 3$, $\mathfrak{A}''_{H^0}(\lambda)=\{0\}$ *or* $\neq\{0\}$ *according as λ is regular or not.*

§ 4. Irreducible characters of $SL(n, \boldsymbol{R})$

In this and the next sections, we calculate all irreducible characters of $G=SL(n, \boldsymbol{R})$ with certain infinitesimal characters λ. Let us apply Lem's 2.1 and 2.2. Put $\mathfrak{a}=\mathfrak{h}^0$ and $K=SO(n)$, then $M=\{d(\epsilon_1, \epsilon_2, \dots, \epsilon_n)\}$ and $MA=H^0$. Take $\mu=(\mu_1, \mu_2, \dots, \mu_n)\in(\mathfrak{h}_c^0)^*$ and let $\nu=(\nu_1, \nu_2, \dots, \nu_n)$ be a row of $\nu_j=0$ or 1. Then ν determines a character of M and the pair (μ, ν) determines a character $\chi^{\mu,\nu}$ of $H^0=MA$ as

$$\chi^{\mu,\nu}(h)=\prod_{1\leqslant j\leqslant n}|a_j|^{\mu_j}(a_j/|a_j|)^{\nu_j}, \tag{4. 1}$$

where $h=d(a_1, a_2, \dots, a_n)$. Consider the induced representation of $\chi^{\mu,\nu}$ defined in §2 and denote it by $T^{\mu,\nu}$ (see also [1]). Then we see that $T^{\mu,\nu}$ and its character $\pi^{\mu,\nu}$ satisfy

$$T^{\mu,\nu}(z)=\nu(z)1 \ \ (z\in Z_G), \quad T^{\mu,\nu}(Z)=\lambda_\mu(Z)1 \ \ (Z\in\mathfrak{Z});$$
$$\pi^{\mu,\nu}(zg)=\nu(z)\pi^{\mu,\nu}(g) \ \ (z\in Z_G), \quad Z\pi^{\mu,\nu}=\lambda_\mu(Z)\pi^{\mu,\nu} \ \ (Z\in\mathfrak{Z}).$$

This character is a function on G which vanishes identically outside G_{H^0} and is given on G_{H^0} as follows:

$$\pi^{\mu,\nu}(h)=\nabla(h)^{-1}\kappa^0_{\mu,\nu}(h) \qquad (h\in H^0\cap G'),$$

where putting $W=W_G(H^0)$,

$$\kappa^0_{\mu,\nu}(h)=\sum_{w\in W}\chi^{\mu,\nu}(wh). \tag{4. 2}$$

Therefore it follows from the results in §3 that for any $\mu\in(\mathfrak{h}_c^0)^*$, the space $\mathfrak{A}'_{H^0}(\lambda_\mu)$ is spanned by $\pi^{\mu,\nu}$, when ν runs over all possible rows, whence $\mathfrak{C}(\lambda_\mu)\supset\mathfrak{A}'_{H^0}(\lambda_\mu)$. Note that $\pi^{\mu,\nu}=\pi^{\mu',\nu'}$ if and only if there exists some $w\in W_G(H^0)$ such that $\chi^{\mu',\nu'}(h)=\chi^{\mu,\nu}(wh)$ $(h\in H^0)$.

To apply Lem. 2.2, we must decompose $T^{\mu,\nu}$ into irreducible constituents. We call $\mu\in(\mathfrak{h}_c^0)^*$ imaginary if it takes on $\mathfrak{h}^0$ only pure-imaginary values. If μ is imaginary, $T^{\mu,\nu}$ is unitary and its irreducibility is studied in [1]. Put $\tilde{G}=\{g\in GL(n, \boldsymbol{R});\ \det g=\pm 1\}$ and let $\tilde{H}^0$ be its subgroup consisting of all diagonal matrices in $\tilde{G}$. Extend $\chi^{\mu,\nu}$ from H^0 to $\tilde{H}^0$ by (4. 1) and construct its induced representation $\tilde{T}^{\mu,\nu}$ of $\tilde{G}$ analogously as $T^{\mu,\nu}$. Then the restriction of $\tilde{T}^{\mu,\nu}$ on G is exactly $T^{\mu,\nu}$.

Lemma 4.1 [1]. *The representation $\tilde{T}^{\mu,\nu}$ of $\tilde{G}$ is always irreducible if μ is imaginary.*

Put $u_0=d(-1, -1, \ldots, -1)$ if n is odd and $u_0=d(1, 1, \ldots, 1, -1)$ if n is even. Then $\tilde{G}=G\cap Gu_0$. Using the general theory of group representations, we obtain from the above lemma the following

Lemma 4.2. *When n is odd, $T^{\mu,\nu}$ is always irreducible. When n is even, if it is reducible, it is a direct sum of two inequivalent irreducible representations T and T' such that T' is unitary equivalent to the representation $g\rightarrow T(u_0gu_0^{-1})(g\in G)$.*

Note that $\lambda_\mu=\lambda_{\mu'}$ if and only if $\mu'=\sigma\mu$ for some $\sigma\in W_c$. Then,

Proposition 4. *Suppose n is odd. If $\mu\in(\mathfrak{h}_c^0)^*$ is imaginary, the characters $\pi^{\mu,\nu}$ give all irreducible characters of G with infinitesimal character λ_μ and $\mathfrak{C}(\lambda_\mu)=\mathfrak{A}'_{H^0}(\lambda_\mu)$.*

Thus, Prop's 3 and 4 give us an elementary proof of the following theorem in the case when n is odd, because $\mathfrak{A}''_{H^0}(\lambda_\mu)\neq\{0\}$ for some λ_μ.

Theorem 1. *For $SL(n, \boldsymbol{R})(n\geqslant 3)$, there exist tempered invariant eigendistributions on it which can not be expressed as linear combinations of irreducible characters (for the definition of temperedness, see* [2(e)]).

Note that for $SL(n, \boldsymbol{R})$, every element in $\mathfrak{A}(\lambda_\mu)$ is tempered if μ is imaginary.

Apply Lem. 2.4 and consider the equations (2. 1) and (2. 2) for every $H=H^r(0\leqslant r\leqslant[n/2])$. Then, using Prop. 2, we obtain

Proposition 5. *When n is odd, $\mathfrak{A}(\lambda_\mu)=\mathfrak{A}_{H^0}(\lambda_\mu)$ if $\mu\in(\mathfrak{h}_c^0)^*$ is imaginary.*

§ 5. Irreducible characters of $SL(n, \boldsymbol{R})$ for even n

Now suppose $n=2s$ is even. To calculate all irreducible characters, we apply Lem's 2.3 and 2.4. Put

$$u(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

and let

$$d(e^{\tau_1}u(\theta_1), e^{\tau_2}u(\theta_2), \dots, e^{\tau_r}u(\theta_r), \epsilon_1 e^{t_1}, \dots, \epsilon_2 e^{t_2}, \dots, \epsilon_{n-2r}e^{t_{n-2r}})$$

be the blockwise diagonal matrix with r blocks of 2×2. Denote by H^r the set all such matrices in G. Then $H^0, H^1, \dots, H^r$ form a complete system of Cartan subgroups of G which are not conjugate to each other under inner automorphisms.

Suppose that $\mu\in(\mathfrak{h}_c^0)^*$ is imaginary as before and $T^{\mu,\nu}$ is reducible. Let T, T' be as in Lem. 4.2 and let π, π' be their characters. Then,

$$(5.\ 1)\qquad \pi+\pi'=\pi^{\mu,\nu},\quad \pi'(g)=\pi(u_0 g u_0^{-1})\qquad (g\in G').$$

When $0\leqslant r<s$, $u_0 h u_0^{-1}=h$ for any $h\in H^r$, whence $\pi'(h)=\pi(h)$ on $H^{r\prime}=H^r\cap G'$. Therefore,

$$(5.\ 2)\qquad \begin{cases} \pi=\pi'=2^{-1}\pi^{\mu,\nu} \text{ on } H^{0'}; \\ \pi=\pi'=2^{-1}\pi^{\mu,\nu}=0 \text{ on } H^{r'} \text{ for } 0<r<s. \end{cases}$$

Moreover, since T is not equivalent to T', $\pi\neq\pi'$. Hence $\pi=-\pi'\neq 0$ on H^s.

We note here that it follows from (5. 2) that any non-zero element in $\mathfrak{A}'_{H^0}(\lambda_\mu)$ can not be expressed as a linear combination of irreducible characters, which proves Th. 1 in the case when n is even.

On the other hand, a study of the equations (2. 1), (2. 2) for $\lambda=\lambda_\mu\circ\gamma$ on $H=H^r(0\leqslant r\leqslant s)$ gives us more exact results. Denote in general a solution $\kappa^{\mathfrak{h}}$ on $H=H^r$ by κ^r. For $0<r<s$, always $\kappa^r=0$ on $H^{r'}(R)$. Let M^s be the set of all $\mu'=(\mu'_1, \mu'_2, \dots, \mu'_n)\in(\mathfrak{h}_c^0)^*$ such that $\mu'_1=\mu'_2$, $\mu'_3=\mu'_4$, $\dots$, $\mu'_{2s-1}=\mu'_{2s}$. When $\sigma\mu\notin M_s$ for any $\sigma\in W_c$, always $\kappa^s=0$ on $H^{s'}(R)=H^s$. When $\mu'=\sigma_0\mu\in M_s$ for some $\sigma_0\in W_c$, κ^s is a constant multiple of

$$(5.\ 3)\qquad \eta(h)=2^{s-1}\prod_{\sigma\in\mathfrak{S}_s}\exp\{(\mu'_{2i-1}+\mu'_{2i})\tau_{\sigma(j)}\},$$

where $h=d(e^{\tau_1}u(\theta_1), e^{\tau_2}u(\theta_2), \dots, e^{\tau_s}u(\theta_s))$.

Let $\pi^{\mu,\nu^i}(1\leqslant i\leqslant N_0)$ be the set of all different $\pi^{\mu,\nu}$. Then it follows from the above arguments and Lem. 2.3 that $\dim\mathfrak{C}(\lambda_\mu)\leqslant N_0+1$ and that at most one T^{μ,ν^i} is reducible.

Suppose $\mu\in M_s$ and $\nu^1=(1, 0, 1, 0, \dots, 1, 0)$. Let us prove that T^{μ,ν^1} is reducible. Put

$$^+G_2=\{\delta\in GL(2, \boldsymbol{R});\ \det\delta>0\},\quad D_s=\{d(\delta_1, \delta_2, \dots, \delta_s)\in G;\ \delta_j\in{}^+G_2\},$$

and $P_s=D_sN$. Denote by $D^{\pm}_{1/2,c}$ the irreducible unitary representations of $^+G_2$ with the following characters respectively: for $d(\epsilon e^{t_1}, \epsilon e^{t_2})$ and $e^\tau u(\theta)\in{}^+G_2$,

$$\frac{\epsilon e^{c(t_1+t_2)}}{\left|e^{\frac{t_1-t_2}{2}}-e^{-\frac{t_1-t_2}{2}}\right|}\quad\text{and}\quad\frac{\mp e^{2c\tau}}{e^{i\theta}-e^{-i\theta}}.$$

Consider the representation L of D_s obtained from the Kronecker product $D^{\beta_1}_{1/2,\mu_1}\otimes D^{\beta_2}_{1/2,\mu_3}\otimes\cdots\otimes D^{\beta_s}_{1/2,\mu_{2s-1}}$ with $\beta_j=\pm$. Extend L to the parabolic subgroup P_s and induce it from P_s to G, then we obtain a unitary representation T^L of G(see [3(a)]). Its character π^L is given by Th. 2 in [3(a)] as follows. Let q be the number of β_j such that $\beta_j=+$. Then,

$$\pi^L=\begin{cases}(-1)^q\nabla^{-1}\cdot\eta & \text{on } H^s;\\ 0 & \text{on } H^r(0<r<s);\\ 2^{-1}\pi^{\mu,\nu^1} & \text{on } H^0.\end{cases} \tag{5. 4}$$

Let $T^{\mu}_{\pm}$ be the induced representations for which $\beta_1=\pm$, $\beta_2=\beta_3=\dots=\beta_s=+$ and let $\pi^{\mu}_{\pm}$ be their characters. Then $\pi^{\mu}_{+}+\pi^{\mu}_{-}=\pi^{\mu,\nu^1}$. Therefore T^{μ,ν^1} is equivalent to the direct sum of T^{μ}_{+} and T^{μ}_{-} and the latters are irreducible. Thus,

Proposition 6. *Suppose $n=2s$ is even and $\mu\in(\mathfrak{h}^0_c)^*$ is imaginary.* (a) *When $\sigma\mu\notin M_s$ for any $\sigma\in W_c$, all $T^{\mu,\nu}$ are irreducible and $\pi^{\mu,\nu}$'s give all irreducible characters of G with infinitesimal character λ_μ and $\mathfrak{C}(\lambda_\mu)=\mathfrak{A}'_{H^0}(\lambda_\mu)$.* (b) *When $\mu\in M_s$, let $\pi^{\mu,\nu^i}(1\leqslant i\leqslant N_0)$ be all different $\pi^{\mu,\nu}$ and $\gamma^1=(1, 0, 1, 0, \dots, 1, 0)$. Then T^{μ,ν^1} is equivalent to the direct sum of T^{μ}_{+} and T^{μ}_{-} and all other T^{μ,ν^i} are irreducible. All irreducible characters on G with infinitesimal character λ_μ are π^{μ}_{+}, π^{μ}_{-} and $\pi^{\mu,\nu^i}(i\neq 1)$, and $\mathfrak{C}(\lambda_\mu)=\mathfrak{A}'_{H^0}(\lambda_\mu)+\boldsymbol{C}(\pi^{\mu}_{+}-\pi^{\mu}_{-})$.*

Analogously as Prop. 5, we obtain also

Proposition 7. *Suppose that $\mu\in(\mathfrak{h}^0_c)^*$ is imaginary. In the case* (a), *$\mathfrak{A}(\lambda_\mu)=\mathfrak{A}_{H^0}(\lambda_\mu)$. In the case* (b), *$\mathfrak{A}(\lambda_\mu)=\mathfrak{A}_{H^0}(\lambda_\mu)+\boldsymbol{C}(\pi^{\mu}_{+}-\pi^{\mu}_{-})$.*

Moreover we can prove for $SL(n, \boldsymbol{R})$ the following generalization of Prop. 3(cf. [3(c)]).

Proposition 8. *For any homomorphism* λ *of* $\mathfrak{Z}$ *into* $\boldsymbol{C}$, $\mathfrak{A}(\lambda)=\mathfrak{A}'(\lambda)+\mathfrak{A}''(\lambda)$ (*direct sum*) *and* $\mathfrak{C}(\lambda)\supset\mathfrak{A}'(\lambda)$. *Especially when* $n=2$, *always* $\mathfrak{A}''(\lambda)=\{0\}$ *and* $\mathfrak{A}(\lambda)=\mathfrak{A}'(\lambda)=\mathfrak{C}(\lambda)$. *When* $n\geqslant 3$, $\mathfrak{A}''(\lambda)=\{0\}$ *or* $\neq\{0\}$ *according as* λ *is regular or not.*

We proved in §4 and §5 that on $G=SL(n, \boldsymbol{R})$ $\mathfrak{C}(\lambda_\mu)=\mathfrak{A}'(\lambda_\mu)$ for any imaginary $\mu\in(\mathfrak{h}_c^0)^*$.

§6. The case of complex semi-simple Lie groups

In this section let G be a connected complex semisimple Lie group and $H=H^0$ its Cartan subgroup. Then we can apply Prop's 1 and 2. For any root α of $(\mathfrak{g}, \mathfrak{h})$, define $H_\alpha\in\mathfrak{h}$ as $\alpha(X)=\langle H_\alpha, X\rangle\ (X\in\mathfrak{h})$, where $\langle , \rangle$ denotes the Killing form of $\mathfrak{g}$. Let $X\to\bar{X}(X\in\mathfrak{h})$ be the conjugation of $\mathfrak{h}$ with respect to the real subalgebra spanned by $H_\alpha(\alpha\in P)$. Denote by $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$ over $\boldsymbol{C}$. Then any character χ of H can be expressed uniquely as

$$\chi(\exp X)=\exp\ \{p(X)+q(\bar{X})\} \qquad (X\in\mathfrak{h}),$$

where $p, q\in\mathfrak{h}^*$. (Note that H is connected.) Denote χ by (p, q) and consider it also as an element of $\mathfrak{h}^*$. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. It operates on $\chi=(p, q)$ as $w\chi=(wp, wq)\ (w\in W)$. The Weyl group W_c of $(\mathfrak{g}_c, \mathfrak{h}_c)$ is isomorphic to $W\times W$ in such a way that $\sigma=(w, w')\ (w, w'\in W)$ operates on $\chi=(p, q)$ as $\sigma\chi=(wp, w'q)$. Let T^χ be the induced representation of χ on a Hilbert space $\mathcal{H}^\chi$ defined in §2 and π^χ its character. Let $J(\mathfrak{h}_c)$ be the set of W-invariant analytic differential operators on H. Then $I(\mathfrak{h}_c)$ is generated by $J(\mathfrak{h}_c)$ and $\overline{J(\mathfrak{h}_c)}$. We see from these facts that $Z\pi^\chi=\lambda_\chi(Z)\pi^\chi\ (Z\in\mathfrak{Z})$, and that $\pi^\chi=\pi^{\chi'}$(or $\lambda_\chi=\lambda_{\chi'}$) if and only if $\chi'=w\chi$ (or $=\sigma\chi$) for some $w\in W$(or $\sigma\in W_c$). A study of the equations (1. 4) and (1. 5) gives us the following

Lemma 6.1. *For any character* $\chi=(p, q)$, $\mathfrak{A}(\lambda_\chi)=\mathfrak{A}'(\lambda_\chi)+$

$\mathfrak{A}''(\lambda_\chi)$ *(direct sum) and* $\mathfrak{A}'(\lambda_\chi)$ *is spnned by* $\{\pi^{\chi'};\ \chi'=(p, wq),\ w\in W\}$, *whence* $\mathfrak{C}(\lambda_\chi)\supset\mathfrak{A}'(\lambda_\chi)$.

We want to prove $\mathfrak{C}(\lambda_\chi)=\mathfrak{A}'(\lambda_\chi)$. Meanwhile we obtain from [3(b), App. II] (*) the following

Proposition 9. *Let* G *be any of* $SL(n, \boldsymbol{C})$, $SO(2n+1, \boldsymbol{C})$, $Sp(n, \boldsymbol{C})$ *and* $SO(2n, \boldsymbol{C})$. *When* $n=2$ *or* 3, $\mathfrak{A}''(\lambda)=\{0\}$ *for any* λ. *When* $n\geqslant 4$, *there always exist some* λ *for which* $\mathfrak{A}''(\lambda)\neq\{0\}$. *Moreover* $\mathfrak{A}''(\lambda)=\{0\}$ *for any* $\lambda=\lambda_\chi$ *with imaginary* $\chi=(p, q)\in\mathfrak{h}_c^*$.

As a corollary of the last assertion of this proposition, we obtain

Theorem 2. *For any complex classical group* G, *a tempered invariant eigendistribution of* G *is always a linear combination of the characters of its irreducible unitary representations.*

Now, to determine $\mathfrak{C}(\lambda)$, we apply Lem's 2.1 and 2.2 and some results of D. P. Zhelobenko in [4(a), (b)]. Suppose, for simplicity, that G is simply connected. Then a pair of $p, q\in\mathfrak{h}^*$ defines a character of H if and only if $p_\alpha-q_\alpha$ is integer for any $\alpha\in P$, where $p_\alpha=2\langle p, \alpha\rangle/\langle\alpha, \alpha\rangle$. A character $\chi=(p, q)$ is called discretely positive if for any $\alpha\in P$, p_α and q_α are not negative integers at the same time. D. P. Zhelobenko [4(a), §11] defined for any discretely positive character χ, "*the minimal representation* $\mu(\chi)$" as the restriction of T^χ on an invariant subspace $\mathcal{N}^\chi$ of $\mathcal{H}^\chi$ with a stronger topology than the one induced from $\mathcal{H}^\chi$ and proved the following facts.

Lemma 6.2. *The representation* $\mu(\chi)$ *is completely irreducible in the sence of R. Godement and the two* $\mu(\chi)$ *and* $\mu(\chi')$ *are equivalent if and only if there exists some* $w\in W$ *such that* $\chi'=w\chi$[4(a), §11]. *Any quasi-simple irreducible representation of* G *is infinitesimally equivalent to some* $\mu(\chi)$ [4(b), Th. 7].

Define the character of $\mu(\chi)$ as that of the restriction of T^{χ} on the closure of $\mathcal{N}^{\chi}$ in $\mathcal{H}^{\chi}$ and denote it by $\bar{\mu}(\chi)$. Then it follows from Lem. 6.2 that for any λ, the set of all irreducible characters with infinitesimal character λ does consist of all different $\bar{\mu}(\chi)$ with discretely positive χ such that $\lambda_{\chi}=\lambda$. This gives us the dimension of $\mathfrak{C}(\lambda)$. On the other hand, by Lem. 6.1, the dimension of $\mathfrak{A}'(\lambda)$ is equal to the number of different π^{χ} such that $\lambda_{\chi}=\lambda$. Thus we see that dim $\mathfrak{C}(\lambda)=$dim $\mathfrak{A}'(\lambda)$, whence $\mathfrak{C}(\lambda)=\mathfrak{A}'(\lambda)$.

Theorem 3. *Let G be a connected complex semisimple Lie group. For any* λ, $\mathfrak{A}(\lambda)=\mathfrak{A}'(\lambda)+\mathfrak{A}''(\lambda)$ *(direct sum) and* $\mathfrak{C}(\lambda)=\mathfrak{A}'(\lambda)$.

This theorem and Prop. 9 give us the following

Theorem 4. *For* $SL(n, \boldsymbol{C})$, $SO(2n+1, \boldsymbol{C})$, $Sp(n, \boldsymbol{C})$ *and* $SO(2n, \boldsymbol{C})$, *if* $n\geqslant 4$, *there always exist invariant eigendistributions on it which can not be expressed as linear combinations of irreducible characters. No such distribution is tempered.*

(*) Errata. In [3(b), App. II]; p. 60, the 2nd line from below should be "$p(t\,;\rho)=p(\tau' t\,;\tau'\rho\tau)$ $(\rho\in W,\ \tau\in\mathfrak{T}(c),\ \tau'\in\mathfrak{T}(d))$"; p. 63, the right hand side of (17′) should be multiplied by $\prod_{j=1}^{n}(e^{z_j}-e^{-z_j})$; p. 66, the 4th and 5th lines from below should be "in another cases, $p(t)$ is symmetric with respect to the union of $t_j(j\in A_k^+\cap B_l^+)$ and $-t_j(j\in A_k^-\cap B_l^-)$ and with respect to the union of $t_i(i\in A_k^+\cap B_l^-)$ and $-t_i(i\in A_k^-\cap B_l^+)$".

Kyoto University

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