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Deformations of G-structures and infinitesimal automorphisms

By

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When we consider a transitive G-structure g_0 on a compact differentiable manifold M, another G-structure g on M is said to be locally equivalent to g_0, if there exists a local transformation f of a neighborhood U of each point of M such that the G-structure induced by f from g_0 is equal to g on U, and g is said to be globally equivalent to g_0, if there exists a global transformation f of M such that the G-structure induced by f from g_0 is equal to g on M. The theory of deformations of G-structures is considered to represent a difference between the local equivalence and the global equivalence of G-structures. In our paper, we take note of a certain global property for G-structures and we consider the extent of G-structures which are locally equivalent to g_0 and have the global property. We represent the extent in the space of G-structures, using the theory of deformations, and describe a relation between the global property and the equivalence of G-structures.

We suppose throughout our paper that G is closed and of finite type and the transitive G-structure g_0 satisfies the following condition. When $\tilde{g}_0$ denotes the lift of g_0 by p on the universal covering manifold $\tilde{M}$ of M, where p is the covering projection, and $\mathfrak{A}(\tilde{g}_0)$ denotes the sheaf of germs of infinitesimal automorphisms of $\tilde{g}_0$, the Lie algebra of the Lie group of automorphisms of $\tilde{g}_0$ is equal to $H^0(M, \mathfrak{A}(\tilde{g}_0))$.

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Then the G-structures on M correspond one-to-one to the cross-sections of the associated bundle $F(M)/G$ of the frame bundle $F(M)$ over M. The set of all G-structures forms a Banach manifold $\mathcal{G}$ as the space of cross-sections with respect to a riemannian metric on the bundle of jets of cross-sections. We regard the whole set $\mathcal{D}$ of G-structures locally equivalent to g_0 as a subspace of $\mathcal{G}$. Then deformations of g_0 are given by curves in $\mathcal{D}$ through g_0. Let us take note of the equivalence of the infinitesimal automorphisms as a global property. We also regard the whole set $\mathcal{J}$ of G-structures having the infinitesimal automorphisms equivalent to those of g_0 as a subspace of $\mathcal{G}$. *A* deformation g_t of g_0 is said to *have the equivalent infinitesimal automorphisms,* if each G-structure of g_t has the infinitesimal automorphisms equivalent to those of g_0, that is, if g_t is a curve in $\mathcal{J}\cap\mathcal{D}$ through g_0. Let $\mathcal{E}$ be the subspace of $\mathcal{G}$ consisting of G-structures globally equivalent to g_0 and $\mathcal{S}$ be one consisting of G-structures having the same infinitesimal automorphisms as g_0. The group $\mathrm{Diff}(M)$ of diffeomorphisms of M is a transformation group of $\mathcal{G}$, under which $\mathcal{E}$ and $\mathcal{J}$ are the obrits of g_0 and $\mathcal{S}$ respectively. Then there exists a differentiable submanifold $\mathcal{V}$ of $\mathcal{G}$ such that $\mathcal{J}\cap\mathcal{D}$ in a neighborhood U_{g_0} of g_0 is the image of $\mathcal{V}$ transformed by the elements of a neighborhood U_e of e in $\mathrm{Diff}(M)$. The tangent space of $\mathcal{V}$ at g_0 is isomorphic to some subspace $\mathcal{K}$ of the kernel of the homomorphism ω: $H^1(M, \mathfrak{A}(g_0))\rightarrow H^1(M, \mathfrak{N})$ induced by the injection $\mathfrak{A}(g_0)\rightarrow\mathfrak{N}$, where $\mathfrak{A}(g_0)$ is the sheaf of germs of infinitesimal automorphisms of g_0 and $\mathfrak{N}$ is the sheaf of normalizer of $\mathfrak{A}(g_0)$ in the sheaf of germs of vector fields on M. Thus G-structures not globally equivalent to g_0 with respect to the elements of U_e, but locally equivalent to g_0 and having the infinitesimal automorphisms equivalent to those of g_0, exist in U_{g_0} to the extent of $\mathcal{K}$. As for deformations of g_0, infinitesimal deformations corresponding to elements of $\mathcal{K}$ can be extended to deformations having the equivalent infinitesimal automorphisms, and classes of germs of deformations having the equivalent infinitesimal automorphisms are represented uniquely by curves in $\mathcal{V}$ through g_0. Then we have the following proposition as a special case. If the homomorphism ω is injective, deformations having the equivalent infinitesimal automorphisms are trivial.

§1. The space of G-structures and the group of diffeomorphisms.

Let M be a compact differentiable manifold of class C^∞ with dimension n. G-structures on M are reductions of the structure group of the frame bundle $F(M)$ over M to a subgroup G of $GL(n)$, where we suppose G to be closed and of finite type. They are represented as submanifolds $B_G(M)$ of $F(M)$. Let $F(M)/G$ be the quotient space of $F(M)$ by G. Then G-structures are represented as cross-sections of $F(M)/G$, where the image of $B_G(M)$ by the quotient projection π' of $F(M)$ onto $F(M)/G$ is the corresponding cross-section of $F(M)/G$. In our paper, we represent G-structures by not only submanifolds but cross-sections.

Remark about the class of differentiable G-structures. If the class of differentiable G-structures is C^r, $B_G(M)$ is of class C^r and the s-th prolongation of $B_G(M)$ is of class C^{r-s}. Then we take $r>k$, in order that the k-th prolongation of $B_G(M)$ with $\{e\}$-structure may be of class C^1, where k is the order of G of finite type. Moreover, we suppose r to be finite.

Let B_V be a finite dimensional vector bundle over M of class C^r. The whole of r-jets of cross-sections of B_V is a vector bundle over M which is denoted by B_V^r. We define a norm on each fibre of B_V^r which is continuously dependent to $x\in M$, that is, $\|\phi^r(x)\|$ is continuous on x for any continuous local cross-section ϕ^r of B^r. Let us define a norm $\|\phi\|^{(r)}$ of C^r-cross-section ϕ of B_V by $\operatorname*{Max}_{x\in M}\|j_x^r\phi\|$, where $j_x^r\phi$ is the r-jet of ϕ at x and $\|j_x^r\phi\|$ is the norm of $j_x^r\phi$ in the fibre $B_V^r(x)$ of B_V^r over x. Then the whole of C^r-cross-sections of B_V is a Banach space with respect to the above norm. Let us denote this space by $\Gamma^{(r)}(B_V)$.

Lemma 1. *Let B_V and B_W be vector bundles of class C^r over M and $\eta: B_V \to B_W$ be a fibre mapping of class C^r such that η is infinitely partial differentiable with respect to the fibre of B_V, every partial derivative of η of any order with respect to the fibre is also of class C^r and the diffeomorphism of M induced by η is identity. Then, the mapping $\bar{\eta}: \Gamma^{(r)}(B_V) \to \Gamma^{(r)}(B_W)$ defined by $(\bar{\eta}\phi)(x) = \eta(\phi(x))$ is of class C^∞.*

Proof. η induces the continuous mapping η^r of B_V^r in B_W^r well defined by $\eta^r(j_x^r\phi) = j_x^r(\bar{\eta}\phi)$ for any $j_x^r\phi \in B_V^r$. Then η^r is infinitely partial differentiable with respect to the fibre and every partial derivative of η^r of any order with respect to the fibre is continuous on B_V^r. Let ϕ_0 be a fixed element of $\Gamma^{(r)}(B_V)$. We have

$$\|\eta^r(j_x^r\phi + j_x^r\phi_0) - \eta^r(j_x^r\phi_0) - d\eta^r_{j_x^r\phi_0}(j_x^r\phi)\| < \|j_x^r\phi\|^2 \cdot K$$

for any element ϕ of $\Gamma^{(r)}(B_V)$ such that $\|\phi\|^{(r)} < \epsilon$ for a fixed ϵ, where K is a constant independent to x and $d\eta^r_{j_x^r\phi_0}$ is the partial differential of η^r at $j_x^r\phi_0$ with respect to the fibre of B_V^r, because every 2nd partial derivative of η^r with respect to the fibre of B_V^r is bounded on an open set $\bigcup_{x\in M}\{j_x^r\phi \in B_V^r(x);\, \|j_x^r\phi - j_x^r\phi_0\| < \epsilon\}$ of B_V^r. Let $\overline{d\eta}_{\phi_0}$ be a continuous linear mapping of $\Gamma^{(r)}(B_V)$ into $\Gamma^{(r)}(B_W)$ defined by $\overline{d\eta}_{\phi_0}(\phi)(x) = d\eta_{\phi_0}\cdot(\phi(x))$, where $d\eta_{\phi_0}: B_V \to B_W$ is the partial differential of η at ϕ_0 with respect to the fibre. Since the mapping of B_V^r into B_W^r induced by $d\eta_{\phi_0}$ is $d\eta^r_{j_x^r\phi_0}$ over x, we have

$$\begin{aligned}
&\|\bar{\eta}(\phi_0+\phi) - \bar{\eta}(\phi_0) - \overline{d\eta}_{\phi}(\phi)\|^{(r)} \\
&= \operatorname*{Max}_{x\in M} \|j_x^r(\bar{\eta}(\phi_0+\phi) - \bar{\eta}(\phi_0) - d\bar{\eta}_{\phi}(\phi))\| \\
&= \operatorname*{Max}_{x\in M} \|\eta^r(j_x^r(\phi_0+\phi)) - \eta^r(j_x^r(\phi_0)) - d\eta^r_{j_x^r\phi_0}(j_x^r\phi)\| \\
&< \operatorname*{Max}_{x\in M} \|j_x^r\phi\|^2 \cdot K \\
&= (\|\phi\|^{(r)})^2 \cdot K, \quad \text{for } \|\phi\|^{(r)} < \epsilon
\end{aligned}$$

Therefore, we have $\lim_{\|\phi\|\to 0} \|\bar{\eta}(\phi_0+\phi)-\bar{\eta}(\phi_0)-\overline{d\eta}_{\phi_0}(\phi)\|^{(r)} / \|\phi\|^{(r)}=0$, that is, $\bar{\eta}$ is differentiable at ϕ_0. Next, we consider the bundle Hom $(B_V; B_W)$ of which the fibre over each x is a linear space of homomorphisms of $B_V(x)$ into $B_W(x)$. The bundle $\mathrm{Hom}^r(B_V; B_W)$ of r-jets of cross-sections of Hom $(B_V; B_W)$ can be identified with a subbundle of the bundle $\mathrm{Hom}(B_V^r : B_W^r)$, of which each fibre has a norm continuously dependent to $x\in M$ defined by the norm of $B_V^r(x)$ and that of $B_W^r(x)$. The space $\Gamma^{(r)}(\mathrm{Hom}(B_V; B_W))$ with respect to the above norm can be identified with a subspace of the Banach space $L(\Gamma^{(r)}(B_V); \Gamma^{(r)}(B_W))$ of continuous linear mapping of $\Gamma^{(r)}(B_V)$ into $\Gamma^{(r)}(B_W)$. Let $d\eta$ be the partial derivative of η with respect to the fibre of B_V and then it is a fibre mapping of B_V into Hom $(B_V; B_W)$. If we take the bundle Hom $(B_V; B_W)$ instead of B_W, the mapping $d\eta$ satisfies the condition of η in Lemma 1 and we have a differentiable mapping

$$\overline{d\eta} : \Gamma^{(r)}(B_V) \longrightarrow \Gamma^{(r)}(\mathrm{Hom}(B_V; B_W)) \subset L(\Gamma^{(r)}(B_V); \Gamma^{(r)}(B_W))$$

induced from $d\eta$, such that $\overline{d\eta}(\phi_0)$ for any $\phi_0 \in \Gamma^{(r)}(B_V)$ is the differential of $\bar{\eta}$ at ϕ_0. Following the above argument for any order of the differential of $\bar{\eta}$ in succession, we conclude the mapping $\bar{\eta}$ is of class C^∞.

Remark. Even if η is not a mapping of the whole space of B_V into B_W but a mapping of a fibre subspace B' of B_V into B_W, Lemma 1 is right for $\Gamma^{(r)}(B')$ instead of $\Gamma^{(r)}(B_V)$.

Let B be a fibre bundle over M of class C^∞ and let us define a riemannian metric on B of class C^∞. Let B^r denote a bundle of r-jets of cross-sections of B. Since B^r is a C^∞-bundle over B, we can define a riemannian metric on B^r of class C^∞ based on the metric on B such that $\rho_x(\pi^r b, \pi^r b') \leqq \rho_x^r(b, b')$ for each x, where $b, b' \in B^r(x)$, $\pi^r : B^r \to B$ is the canonical projection and ρ_x(resp. ρ_x^r) is the distance along each fibre $B(x)$(resp. $B^r(x)$). Let $\Gamma^{(r)}(B)$ be the whole of C^r-cross-sections of B with the metric defined by

$$\rho^{(r)}(\phi, \psi)=\operatorname*{Max}_{x\in M} \rho_x^r(j_x^r\phi, j_x^r\psi) \text{ for } \phi, \psi\in\Gamma^{(r)}(B).$$

Applying the notion of the Banach manifold (see [2]) to $\Gamma^{(r)}(B)$, under the fact of Lemma 1 which gives the smoothness of the coordinate transformation, we have

Proposition 1. *The metric space $\Gamma^{(r)}(B)$ is a Banach manifold of class C^∞. The tangent space of $\Gamma^{(r)}(B)$ at ϕ is the Banach space $\Gamma^{(r)}(V_\phi(B))$, where $V_\phi(B)$ is the bundle of vertical vectors of B at ϕ.*

Definition. *The space $\mathcal{G}$ of G-structures* of class C^r on M is the Banach manifold $\Gamma^{(r)}(F(M)/G)$ of class C^∞, with respect to a riemannian metric of the bundle space of r-jets of cross-sections of $F(M)/G$.

The tangent space $T_g(\mathcal{G})$ at $g\in\mathcal{G}$ is the Banach space $\Gamma^{(r)}(V_g(F(M)/G))$, where $V_g(F(M)/G)$ is the vertical vector bundle of $F(M)/G$ at g.

Let B and B' be fibre bundles of class C^∞ over M and $\xi : B\rightarrow B'$ be a fibre mapping of class C^r such that ξ is infinitely partial differentiable with respect to the fibre of B, every partial derivative of any order of ξ with respect to the fibre is of class C^r and the diffeomorphism of M induced by ξ is identity. Let us define the mapping $\bar{\xi} : \Gamma^{(r)}(B)\rightarrow\Gamma^{(r)}(B')$ by $(\bar{\xi}\phi)(x)=\xi(\phi(x))$. Since ξ induces a mapping $\dot{\bar{\xi}}$ of the tangent space $\Gamma^{(r)}(V_\phi(B))$ to $\Gamma^{(r)}(V_\psi(B'))$ for any $\xi\in\Gamma^{(r)}(B)$, $\psi\in\Gamma^{(r)}(B')$ and $\dot{\bar{\xi}}$ is of class C^∞ by Lemma 1, we have

Proposition 2. *The mapping $\bar{\xi}$ is of class C^∞.*

Let us define a riemannian metric of class C^∞ on M. The product mainfold $M\times M$ is a trivial bundle over M and the space $C^{(r')}(M)$ of $C^{r'}$-transformations of M is the Banach manifold $\Gamma^{(r')}(M\times M)$ of $C^{r'}$-

cross-sections of the above bundle with respect to a riemannian metric of the bundle $J^{r'}(M\times M)$ of r'-jets based on the product riemannian metric of $M\times M$. Any element of the ϵ-neighborhood of identity of $C^{(r')}(M)$ is a $C^{r'}$-diffeomorphism of M by the definition of the metric of $C^{(r')}(M)$ and then the set of $C^{r'}$-diffeomorphisms of M is an open subspace of $C^{(r')}(M)$. Let $\rho^{(r')}$ be the metric of $C^{(r')}(M)$. We define the metric $\rho(f_1, f_2)=\rho^{(r')}(f_1, f_2)$ on the set $\mathrm{Diff}^{(r')}(M)$ of $C^{r'}$-diffeomorphisms of M. Then $\mathrm{Diff}^{(r')}(M)$ is a Banach manifold of class C^{∞}. The tangent space of $\mathrm{Diff}^{(r')}(M)$ at any f is the Banach space $\Gamma^{(r')}(T(M))$ with respect to the norm of each fibre of $J^{r'}(T(M))$ based on the metric of $J^{r'}(M\times M)$, where $T(M)$ is the tangent bundle which is identified with the vertical vector bundle of the trivial bundle $M\times M$ at f.

§2. Infinitesimal automorphisms.

Let θ be a vector field of class C^{r+1} on an open set U of M. For $g\in\mathcal{G}$, let g' be a cross-section of $F(M)$ on U such that $\pi' g'=g$ and $\mathcal{L}_\theta g'$ be the Lie derivative of a tensor field g' with respect to θ. If we set $\mathcal{L}_\theta g'=g'\times\mathfrak{a}$, then $\mathfrak{a}$ is a $\mathfrak{gl}$-valued function on U. Since $F(M)\times\mathfrak{gl}$ is the bundle of vertical vectors of $F(M)$, $g'\times\mathfrak{a}$ is a vertical vector field of $F(M)$ at g'. The bundle of vertical vectors of $F(M)/G$ is an associated bundle $F(M)\underset{G}{\times}\mathfrak{f}$ of $F(M)$ by the linear isotropy representation ι_s: $G\to GL(\mathfrak{f})$, where $\mathfrak{f}=\mathfrak{gl}/\mathfrak{g}$. Then $g'\underset{G}{\times}q\cdot\mathfrak{a}$ is a vertical vector field of $F(M)/G$ at g, where q is the projection $\mathfrak{gl}\to\mathfrak{f}$. This field is determined by θ and g, that is, $g'\underset{G}{\times}q\cdot\mathfrak{a}$ is independent to a choice of g' such that $\pi' g'=g$. We denote $g'\underset{G}{\times}q\cdot\mathfrak{a}$ by $\mathcal{L}_\theta g$. Then θ is an infinitesimal automorphism of g, if and only if $\mathcal{L}_\theta g=0$.

By the condition of g_0 in Introduction, g_0 is of class C^{∞}. When θ is a global vector field of class C^{r+1} on M, $\mathcal{L}_\theta g_0$ is a global C^r-cross-section of the vertical vector bundle $V_{g_0}(F(M)/G)$ of $F(M)/G$ at g_0. Then we have a linear mapping $\bar{\delta}_{g_0}$ of the Banach space $\Gamma^{(r+1)}(T(M))$ of all vector fields of class C^{r+1} on M into the Banach space $\Gamma^{(r)}(V_{g_0}(F(M)/G))$ of all C^r-cross-sections of $V_{g_0}(F(M)/G)$, such that $\bar{\delta}_{g_0}\theta=\mathcal{L}_\theta g_0$.

Proposition 3. *The linear mapping $\bar{\delta}_{g_0}$ is continuous.*

Proof. Since a vertical vector $\mathcal{L}_\theta g_0(x)$ is determined by $j_x^1\theta$ and $j_x^1 g_0$, we have a mapping L of the bundle $J^1(T(M))$ of 1-jets of vector fields on M into $V_{g_0}(F(M)/G)$ such that $L(j_x^1\theta)=\mathcal{L}_\theta g_0$ for any vector field θ on a neighborhood of x, and L is a bundle mapping of vector bundles. Then L induces a continuous linear mapping $\bar{L}$ of $\Gamma^{(r)}(J^1(T(M)))$ into $\Gamma^{(r)}(V_{g_0}(F(M)/G))$ and a correspondence defined by $\theta\rightarrow j^1\theta$ is an imbedding im of $\Gamma^{(r+1)}(T(M))$ into $\Gamma^{(r)}(J^1(T(M)))$. Then $\bar{L}\cdot im$ is continuous linear and $\bar{\delta}_{g_0}=\bar{L}\cdot im$.

Taking the germs of each cross-section, the correspondence $\theta\rightarrow\mathcal{L}_\theta g_0$ induces a sheaf homomorphism $\delta_{g_0}:\mathfrak{T}\rightarrow\mathfrak{V}$, where $\mathfrak{T}$ is a sheaf of germs of vector fields of class C^{r+1} on M and $\mathfrak{V}$ is a sheaf of germs of C^r-cross-sections of $V_{g_0}(F(M)/G)$. The kernel of δ_{g_0} is the sheaf $\mathfrak{A}(g_0)$ of germs of infinitesimal automorphisms of g_0. Since G is of finite type, the sheaf $\mathfrak{A}(g_0)$ is locally constant and its stalks are finite dimensional vector spaces. Then the set $\Gamma(\mathfrak{A}(g_0))$ $(=H^0(M, \mathfrak{A}(g_0)))$ of global infinitesimal automorphisms is a finite dimensional vector space which is a subspace of $\Gamma^{(r+1)}(T(M))$. Thus we have a closed complement D of $\Gamma(\mathfrak{A}(g_0))$ in $\Gamma^{(r+1)}(T(M))$ and $\bar{\delta}_{g_0}$ is isomorphic on D.

A C^{r+1}-diffeomorphism f of M induces a C^r-diffeomorphism f' of $F(M)$ such that $f'(b\cdot a)=(f'(b))\cdot a$ for any $a\in G$ and $b\in F(M)$, and then it induces a C^r-diffeomorphism f^* of $F(M)/G$ such that $\pi'(f'(b))=f^*\pi'(b)$ and $f(\pi(b'))=\pi(f^*(b'))$ where $b'\in F(M)/G$. Let us define $\bar{f}g$ by $(\bar{f}g)(x)=f^{*-1}(g(f(x)))$ for any $g\in\mathcal{G}$. Then $\bar{f}g$ is a new C^r-cross-section of $F(M)/G$ and $\bar{f}$ is a transformation of the space $\mathcal{G}$. The partial differential of f^* with respect to the fibre of $F(M)/G$ is a diffeomorphism f^{**} of the vertical vector bundle defined by $f^{**}v=f'(b)\underset{G}{\times}\mathfrak{a}$, where a vertical vector v is an element $b\underset{G}{\times}\mathfrak{a}$ of $F(M)\underset{G}{\times}\mathfrak{f}$, and f^{**} induces a transformation $\bar{f}^{**}$ of the vertical vector fields $\bar{v}$ by $(\bar{f}^{**}\bar{v})(x)=f^{**-1}(\bar{v}(f(x)))$.

Proposition 4. $\bar{f}^{**}(\mathcal{L}_\theta g)=\mathcal{L}_{f\theta}(\bar{f}g)$.

Proof. If $\pi' g'=g$ and $\mathcal{L}_\theta g'=g'\times \mathfrak{a}$, then

$$(\mathcal{L}_{f^{-1}\theta}(f'^{-1}g'))(x)=f''^{-1}(\mathcal{L}_\theta g'(f(x)))$$
$$=f''^{-1}[g'(f(x))\times \mathfrak{a}(f(x))],$$

where f'' is a diffeomorphism of vertical vector bundle of $F(M)$ induced by f'. Therefore, we have

$$[\mathcal{L}_{f\theta}(\bar{f}g)](x)=f''^{-1}[g'(f(x))]\underset{G}{\times} q[\mathfrak{a}(f(x))]$$
$$=f^{**-1}[g'(f(x))\underset{G}{\times} q(\mathfrak{a}(f(x))]$$
$$=f^{**-1}[\mathcal{L}_\theta g(f(x))]=(\bar{f}^{**}(\mathcal{L}_\theta g))(x).$$

§ 3. Transitive *G*-structures and associated *G*-structures.

In our paper, we suppose that the G-structure g_0 is transitive, that is, the local automorphisms of g_0 act locally transitive on M and moreover those of every prolongation $B_{G^{(\mu)}}^{\mu}$ of $B_G(=g_0)$ act locally transitive on $B_{G^{(\mu)}}^{\mu}$, (see [4], Appendix I).

If and only if n is an element of the normalizer N of G in $GL(n)$, the right translation of g_0 by n is also a G-structure, which is called to be associated to g_0. By the theory of G-structures (see [1]), we have

Proposition 5. *A G-structure g is associated to* g_0, *if and only if g has the same local infinitesimal automorphisms as* g_0.

The product space $N\times M$ is a trivial C^∞-bundle over M and then a mapping of $N\times M$ into $F(M)/G$ defined by $n\times x\rightarrow g_0(x)\cdot n$ satisfies the condition of ξ of Proposition 2, because g_0 is of class C^∞. Since N is a closed submanifold of $\Gamma^{(r)}(N\times M)$ as the constant cross-sections, the mapping $\rho_{g_0}: N\rightarrow \mathcal{G}$ defined by $\rho_{g_0}(n)=g_0\cdot n$ is of class C^∞ by Proposition 2. For each x, the set $\{g_0(x)\cdot n;\ n\in N\}$ is a closed submanifold of the fibre of $F(M)/G$ over x. Thus we have

Proposition 6. *Let $\mathcal{A}$ be the set of G-structures of class C^r associated to g_0. Then $\mathcal{A}$ is a closed submanifold of $\mathcal{G}$.*

If and only if $n\cdot n'^{-1}\in G$, we have $\rho_{g_0}(n)=\rho_{g_0}(n')$. Then ρ_g induces a C^∞-imbedding $\bar{\rho}_g : N/G\to\mathcal{G}$ such that $\bar{\rho}_g q'=\rho_g$, where $q' : N\to N/G$.

We consider a mapping of $\mathrm{Diff}^{(r+1)}(M)\times N/G$ into $\mathcal{G}$ defined by $f\times\bar{n}\to\bar{f}\cdot\bar{\rho}_{g_0}(\bar{n})$. Let $J^1(M,\alpha)$ (resp. $J^1(M,\beta)$) be the fibre bundle of invertible 1-jets of diffeomorphisms of M with the source projection α(resp. the target projection β) as the bundle projection. Each jet j^1f^{-1} with source y and target x operates on the fibre of $F(M)/G$ over y such that $(j^1f^{-1})\cdot g(y)=(\bar{f}g)(x)$ for each $g\in\mathcal{G}$. The product space $J^1(M,\beta)\times N/G$ is also a C^∞-bundle over M with the projection $\bar{\beta}$: $(j^1f,\bar{n})\to\beta(j^1f)$. Since g_0 is of class C^∞, the mapping of $J^1(M,\beta)\times N/G$ into $F(M)/G$ defined by $(j^1f^{-1},\bar{n})\to(j^1f^{-1})\cdot(\bar{\rho}_{g_0}(\bar{n})(y))$ is a fibre mapping of class C^∞. Then, by Proposition 2 we have a C^∞-mapping $\tau' : \Gamma^{(r)}(J^1(M,\beta)\times N/G)\to\Gamma^{(r)}(F(M)/G)$. The space $\Gamma^{(r)}(J^1(M,\beta)\times N/G)$ is C^∞-diffeomorphic to $\Gamma^{(r)}(J^1(M,\beta))\times\Gamma^{(r)}(M,N/G)$. On the other hand, the correspondence $j^1_xf\to j^1_yf^{-1}$, where $y=f(x)$, gives a C^∞-isomorphism of the bundle $J^1(M,\alpha)$ onto $J^1(M,\beta)$, which induces a C^∞-diffeomorphism $\iota : \Gamma^{(r)}(J^1(M,\alpha))\to\Gamma^{(r)}(J^1(M,\beta))$ such that $\iota(j^1f)=j^1f^{-1}$. From the definition of the Banach manifold $\mathrm{Diff}^{(r+1)}(M)$ in §1, we have a C^∞-injection ι' of $\mathrm{Diff}^{(r+1)}(M)$ into $\Gamma^{(r)}(J^1(M,\alpha))$ such that $\iota'(f)=j^1f$, and we have a C^∞-injection ι'' of $\mathrm{Diff}^{(r+1)}(M)$ into $\Gamma^{(r)}(J^1(M,\beta))$ such that $\iota''(f)=\iota\cdot\iota'(f)=j^1f^{-1}$. The C^∞-manifold N/G, of which each element can be considered as a constant mapping of M into N/G, is a C^∞-submanifold of $\Gamma^{(r)}(M,N/G)$ and then we have a C^∞-injection $\kappa: N/G\to\Gamma^{(r)}(M,N/G)$. Let τ be the composed mapping of

$$\tau''\times\kappa : \mathrm{Diff}^{(r+1)}(M)\times N/G\to\Gamma^{(r)}(J^1(M,\beta))\times\Gamma^{(r)}(M,N/G),$$

the isomorphism: $\Gamma^{(r)}(J^1(M,\beta))\times\Gamma^{(r)}(M,N/G)$

$$\to\Gamma^{(r)}(J^1(M,\beta)\times N/G)$$

and τ': $\Gamma^{(r)}(J^1(M,\beta)\times N/G)\rightarrow\Gamma^{(r)}(F(M)/G)$.

Then τ is of class C^∞ from $\mathrm{Diff}^{(r+1)}(M)\times N/G$ into $\mathcal{G}$ such that $\tau(f\times\bar{n})=\bar{f}\bar{\rho}_{g_0}(\bar{n})$. By the definition of $\mathcal{L}_\theta g_0$, the partial differential of τ at (identity$\times\bar{e}$) with respect to $\mathrm{Diff}^{(r+1)}(M)$ is a continuous linear mapping $\bar{\delta}_{g_0}$ in Proposition 3. The partial differential of τ at (identity$\times\bar{e}$) with respect to N/G is a continuous linear mapping of $\mathfrak{n}/\mathfrak{g}$ into $\Gamma^{(r)}(V_{g_0}(F(M)/G))$ defined by $\dot{n}\rightarrow g'_0\underset{G}{\times}\dot{n}$, where $g_0=\pi' g'_0$. Thus we have

Proposition 7. *The mapping τ is of class C^∞. The differential of τ at (identity$\times\bar{e}$) is a continuous linear mapping of $\Gamma^{(r+1)}(T(M))\times\mathfrak{n}/\mathfrak{g}$ into $\Gamma^{(r)}(V_{g_0}(F(M)/G))$ defined by $\theta\times\dot{n}\rightarrow\bar{\delta}_{g_0}\theta+(g'_0\underset{G}{\times}\dot{n})$ where $\pi' g'_0=g_0$.*

§ 4. Transformation of the infinitesimal automorphisms.

Proposition 8. *Let f be a local diffeomorphism of class C^{r+1} with domain U. A local isomorphism of the sheaf $\mathfrak{T}$ induced by f maps a potion $\mathfrak{A}(g_0)|U$ of $\mathfrak{A}(g_0)$ over U onto $\mathfrak{A}(g_0)|f(U)$, if and only if a G-structure $\bar{f}g_0$ induced from g_0 by f has the same infinitesimal automorphisms as g_0 on $f(U)$.*

Proof. By Proposition 4, we have $f(\mathfrak{A}(g_0)|U)=\mathfrak{A}(\bar{f}g_0)|f(U)$. Then $\mathfrak{A}(g_0)|f(U)=f(\mathfrak{A}(g_0)|U)$, if and only if $\mathfrak{A}(g_0)|f(U)=\mathfrak{A}(\bar{f}g_0)|f(U)$.

Proposition 9. *Let $f(t)$ be the 1-parameter diffeomorphisms* ($\exp t\theta$) *generated by a local vector field θ. If and only if each $f(t)$ satisfies the condition of Proposition 8, the germs of θ belong to the sheaf $\mathfrak{N}$ of normalizer of $\mathfrak{A}(g_0)$ in $\mathfrak{T}$.*

Proof. Let U be an open set of M and θ be a vector field on U. For an open set $V\subset U$, each $f(t)$ is diffeomorphism with domain V for a suitable small interval of $|t|$ such that $f(t)\cdot V\subset U$. If $f(t)(\mathfrak{A}(g_0)|V)$

$=\mathfrak{A}(g_0)|f(t)V$, then $[\theta, \lambda]$ is an infinitesimal automorphism for any infinitesimal automorphisms λ on V. Since V is any open set of U, the germ of θ at any $x\in U$ is in $\mathfrak{N}$. Conversely, let n be a vector field on U such that its germs belong to $\mathfrak{N}$. Then $e^{\mathrm{ad}(tn)}$ is an infinitesimal automorphism on U for any λ. Local diffeomorphisms $(\exp tn)\cdot(\exp s\lambda)\cdot(\exp tn)^{-1}$ for a small fixed $|t|$ are local automorphisms $a(s)$ of g_0 for a suitable small $|s|$ and on a suitable domain such that the above compositions are considerable, because $(\exp tn)\cdot(\exp s\lambda)\cdot(\exp tn)^{-1}=\exp(e^{\mathrm{ad}(tn)}s\lambda)$. By Proposition 4,

$$\mathcal{L}_{(\exp tn)\lambda}g_0=(\exp tn)^{**}\mathcal{L}_\lambda((\exp tn)^{-1}g_0).$$

On the notation $(\ )'$ in §2, we have

$$\begin{aligned}\mathcal{L}_\lambda((\overline{\exp tn})^{-1}g_0)'(x)&=\Big[\frac{d}{ds}(\exp s\lambda)'^{-1}(\exp tn)'^{-1}\\&\qquad g'_0((\exp tn)(\exp s\lambda)x)\Big]_{s=0}\\&=\Big[\frac{d}{ds}(\exp tn)'^{-1}a(s)'^{-1}(g'_0(a(s)(\exp tn)x))\Big]_{s=0}\\&=\Big[(\exp tn)'^{-1}\frac{d}{ds}g'(s)((\exp tn)x)\Big]_{s=0},\end{aligned}$$

where $g'(s)=\overline{a(s)}'^{-1}g_0$. Here $a'(s)(y)\in G$, if we set $g'(s)(y)=g'_0(y)(a'(s)(y))$ and then $\mathfrak{a}(y)\in\mathfrak{g}$, if we set $\Big[\frac{d}{ds}g'(s)(y)\Big]_{s=0}=g'_0(y)\underset{G}{\times}\mathfrak{a}(y)$. Therefore,

$$\begin{aligned}&\mathcal{L}_\lambda((\overline{\exp tn})^{-1}g_0)(x)\\&\qquad=(\exp tn)^{**-1}[g'_0((\exp tn)x)\underset{G}{\times}\mathfrak{a}((\exp tn)x)]=0\end{aligned}$$

and then $(\exp tn)\lambda$ is a local infinitesimal automorphism.

Proposition 10. *The dimension of the stalk of* $\mathfrak{N}$ *is finite and constant for every* $x\in M$.

Proof. For a point $x_0\in M$, the adjoint representation of $\mathfrak{N}(x_0)$

on $\mathfrak{A}(g_0)(x_0)$ defines a homomorphism K from an additive group $\mathfrak{N}(x_0)$ into an additive group $\mathrm{Hom}(\mathfrak{A}(g_0)(x_0))$, where $\mathfrak{N}(x_0)$ (resp. $\mathfrak{A}(g_0)(x_0)$) is a stalk of $\mathfrak{N}$ (resp. $\mathfrak{A}(g_0)$) at x_0. Each element of kernel of K is the germ of vector field n on a neighborhood of x_0 at x_0 such that $[n, \lambda]=0$ for any infinitesimal automorphisms λ on U. Since g_0 is transitive, there exist n independent infinitesimal automorphisms λ_i $(i=1, \dots, n)$ on U. The condition $[n, \lambda_i]=0$ $(i=1, \dots, n)$ is a system $\lambda_j(n^i)=\sum_k n^k c^i_{kj}$ $(i, j=1, \dots, n) \dots (*)$ of linear differential equations, where $n=\sum_i n^i\lambda_i$ and $[\lambda_i, \lambda_j]=\sum_k c^k_{ij}\lambda_k$. By the uniqueness of solution for the initial condition $n(x_0)$, the dimension of the solutions is finite. Therefore the dimension of kernel of K is finite. Since dim. (Hom $(\mathfrak{A}(g_0)(x_0)))$ is finite, dim. $\mathfrak{N}(x_0)$ is finite. Since g_0 is transitive, there exists a local automorphism f of a neighborhood of x onto that of x' for any x, x' of M and f induces an isomorphism of $\mathfrak{N}(x)$ onto $\mathfrak{N}(x')$. Then, dim. $\mathfrak{N}(x)=$dim. $\mathfrak{N}(x')$.

Proposition 11. *The sheaf $\mathfrak{N}$ is locally constant.*

Proof. Since dim. $\mathfrak{N}(x_0)$ is finite, $\mathfrak{N}(x_0)$ is the germs of vector fields n on some common neighborhood U of x_0 such that $[n, \lambda]$ are infinitesimal automorphisms on U for any infinitesimal automorphisms λ on U. Let $(\mathfrak{N}, U)$ denote the whole of such vector fields n on U and let $n_1, n_2 \in (\mathfrak{N}, U)$. If $n_1=n_2$ on an open set V of U, then $[n_1-n_2, \lambda]=0$ on U for any λ. Then n_1-n_2 is a solution of the system (*) in Proof of Proposition 10 and then $n_1=n_2$ on U. Therefore each vector field of $(\mathfrak{N}, U)$ has a respectively different germ at any $x \in U$. Since dim. $\mathfrak{N}(x)$ is constant, the whole of germs of vector fields of $(\mathfrak{N}, U)$ at every point of U is the portion $\mathfrak{N}|U$. Therefore $\mathfrak{N}$ is locally constant.

Since the dimension of the space $\Gamma(\mathfrak{N}, M)$ is finite, we have by Palais' theorem ([5])

Proposition 12. *Let $N(g_0)$ be the group of C^{r+1}-diffeomorphisms of M which map all the local infinitesimal automorphisms of g_0 onto* *themselves. Then $N(g_0)$ is a Lie group.*

Let $\tilde{g}_0$ be the lift of g_0 on the universal covering manifold $\tilde{M}$ of M. We have a Lie group $N(\tilde{g}_0)$ in the similar way to $N(g_0)$. We denote by $\tilde{\mathfrak{X}}$ the sheaf of germs of vector C^{r+1}-fields on $\tilde{M}$ and by $\mathfrak{A}(\tilde{g}_0)$ that of infinitesimal automorphisms of $\tilde{g}_0$. The Lie algebra of $N(\tilde{g}_0)$ is a subalgebra of $\Gamma(\tilde{\mathfrak{N}}, \tilde{M})$, where $\tilde{\mathfrak{N}}$ is the sheaf of normalizer of $\mathfrak{A}(\tilde{g}_0)$ in $\tilde{\mathfrak{X}}$.

§ 5. Deformations of a transitive G-structure.

Definition. Let g_t be a 1-parameter family of G-structures of class C^r parametrized by t of a neighborhood I of 0 in R. A family g_t is a *deformation* of g_0, if there exist an open covering $\{U_i;\ i\in J\}$ of M and a family $\{f_i(x, t);\ i\in J\}$ of local continuous transformations of $M\times I$ such that (i) the domain of f_i is $U_i\times I$, (ii) $f_i(x, t)$ for each fixed t is a local C^{r+1}-transformation of $M\times t$, (iii) partial derivatives of $f_i(x, t)$ of any order $(\leqq r+1)$ with respect to x are continuous on $U_i\times I$, (iv) $f_i^*(x, t)^{-1}g_0(f(x, t))=g_t(x)$ for $x\in U_i$, (v) $f_i(0)=$ identity for each i and (vi) $\{f_i(U_i, t);\ i\in J\}$ for each t is an open covering of M. Each G-structure of a deformation of g_0 is called to be *deformable* to g_0.

Two transitive $\{e\}$-structures are locally equivalent, if they have the same constant structure function (see [6]). When we follow the proof of the above fact, while parametrizing by t, we have

Lemma 2. *Let $\{\theta^\alpha(x, t);\ \alpha\in N\}$ be a system of independent continuous 1-forms on $R^N\times I$ such that each 1-form is of class $C^{r'}$ on $R^N\times t$ for each t, partial derivatives of $\theta^\alpha(x, t)$ of any order $(\leqq r')$ with respect to x are continuous on $R^N\times I$, $<\theta^\alpha, \frac{\partial}{\partial t}>=0$ and $c^\alpha_{\beta\gamma}$ are constant, where $d_x\theta^\alpha=\sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma$ and d_x is the exterior differentiation with respect to x. Then there exist a neighborhood U of each point of R^N and a homeomorphism $\phi(x, t)$ of $U\times I$ into $R^N\times I$ such that $\phi(x, t)$*

for any fixed t is a $C^{r'+1}$-diffeomorphism of $U\times t$, partial derivatives of $\phi(x, t)$ of any order ($\leqq r'+1$) with respect to x are continuous on $R^N\times I$, $\phi^*(x, t)^{-1}\theta^\alpha(\theta(x, t), 0)=\theta^\alpha(x, t)$ and $\phi(x, 0)=identity$.

Since g_0 is transitive, the k-th prolongation of g_0 is an $\{e\}$-structure with a constant structure function, where k is the order of G. Let $\mathcal{D}$ denote the subspace of $\mathcal{G}$ consisting of G-structures of which k-th prolongations have the same constant functions as that of g_0.

Proposition 13. *A deformation of g_0 is a continuous mapping $g(t)$ of a neighborhood I of 0 in R into $\mathcal{D}$ with $g(0)=g_0$ and conversely.*

Proof. From (ii) and (iii), a correspondence $x\rightarrow j_x^r(\bar{f}_i(t)g_0)$ for $x\in U_i$ defines a continuous cross-section of $J^r(F(M)/G)\times I$ over $U_i\times I$, by the application of arguments in Proof of Proposition 7 on $f_i(t)U_i$, where $\{f_i, U_i;\ i\in J\}$ defines the deformation g_t and

$$(\bar{f}_i(t)g_0)(x)=f_i^*(x, t)^{-1}g_0(f_i(x, t)).$$

Since $\bar{f}_i(t)g_0=g_t, j^rg_t$ is a continuous section of $J^r(F(M)/G)\times I$. Therefore g_t is a curve in $\mathcal{G}$. Since g_t is locally equivalent to g_0, each g_t has the same structure function as g_0. Therefore g_t is a curve in $\mathcal{D}$ through g_0. Conversely, let $\{V_i;\ i\in J\}$ be an open covering of M such that the restriction of the bundle $F(M)$ on each V_i are the product $V_i\times GL(n)$. Since a curve $g(t)$ in $\mathcal{D}$ through g_0 is an 1-parameter family of G-structures continuously dependent to t and $k<r$ by remark of §1, a set of the portions of the manifolds of k-th prolongations of $g(t)$ on V_i for all $t\in I$ constructs a domain $V_i\times V\times I$ on $R^N\times I$, where N is the dimension of the manifold of k-th prolongation, and the $\{e\}$-structures of k-th prolongations of $g(t)$ construct a system $\{\theta^\alpha(x, t);\ \alpha\in N\}$ of 1-forms which satisfies the condition of Lemma 2 with $r'=r-k$. By Lemma 2, we have a neighborhood $U_x(\subset V_i)$ of each point x of V_i and a homeomorphism $\phi'_x(x', t)$ of $U_x\times V\times I$ into $R^N\times I$ such that they satisfy the condition of the

conclusion of Lemma 2. Since local automorphisms of the prolongation of the G-structure induce those of the G-structure, $\phi'_x(x', t)$ induces a diffeomorphism $\phi_x(x'', t)$ of $U_x\times I$ into $V_i\times I$ such that $\phi_x(x'', t)$ is a C^{r+1}-diffeomorphism of $U_x\times t$ into $V_i\times t$ for any fixed t, partial derivatives of $\phi_x(x'', t)$ of any order ($\leqq r+1$) with respect to x'' are continuous on $U_x\times I$ and $\phi_x^*(x'', t)^{-1}g_0(\phi_x(x'', t))=g(t)(x'')$. Then for a suitable index J, $\{U_\lambda;\ \lambda\in J\}$ and $\{\phi_{x_\lambda}(x, t);\ \lambda\in J\}$ define a deformation g_t of g_0 such that $g(t)=g_t$. This fact holds good, even if we use any one of $g(t)$ in place of g_0. Then we have Propostion, extending the above proof on I successively.

Two deformations g_t^1 and g_t^2 of g_0 is said to have the same *germ of deformation* at 0 if there exists a positive number t_0 such that $g_t^1=g_t^2$ on $(-t_0, t_0)$. If there is a positive number t'_0 and a continuous family $\{f_t;\ t\in(-t'_0, t'_0)\}$ in $\mathrm{Diff}^{(r+1)}(M)$ through $e=f_0$ such that $\bar{f}_t g_t^1=g_t^2$ for any $t\in(-t'_0, t'_0)$, the germ of g_t^1 at 0 is said to be equivalent to that of g_t^2. Thus we have the equivalence class of germs of deformations. Let $\phi(x, t)$ be a local transformation of $M\times I$ such that $\phi(x, 0)$ is identity and $\phi(x, t)$ for any fixed t is a local automorphism of g_0. Let $[A(g_0)\times t]$ denote the whole of germs of such $\phi(x, t)$ at every point of $M\times 0$. Then $[A(g_0)\times t]$ is a sheaf of group on M and we have the 1-chomology set $H^1(M, [A(g_0)\times t])$. It is well known that $H^1(M, [A(g_0)\times t])$ is one-to-one correspondent to the whole of equivalence classes of germs of deformations of g_0(see [3] or [7]).

§ 6. G-structures having the same infinitesimal automorphisms

Let $N_e(\tilde{g}_0)$ (resp. $A_e(\tilde{g}_0)$) be the e-component of $N(\tilde{g}_0)$ (resp. $A(\tilde{g}_0)$).

On the notation and the argument of §4, a G-structure $\bar{\bar{f}}\tilde{g}_0$ on $\tilde{M}$ for $\tilde{f}\in N_e(\tilde{g}_0)$ has the same infinitesimal automorphisms as $\tilde{g}_0$. By Proposition 6, there exists an element a of N such that $\bar{\bar{f}}\tilde{g}_0=\tilde{g}_0\cdot a$. If $\bar{\bar{f}}\tilde{g}_0=\tilde{g}_0\cdot a=\tilde{g}_0\cdot a'$, then $a\cdot a'^{-1}\in G$. Thus we have a mapping σ: $N_e(\tilde{g}_0)\to N/G$ defined by $\sigma(\tilde{f})=q'(a)$ where $\bar{\bar{f}}\tilde{g}_0=\tilde{g}_0\cdot a$ and $q' : N\to N/G$.

Proposition 14. *The mapping σ is an anti-homomorphism and of class C^∞ from the Lie group $N_e(\tilde{g}_0)$ into N/G.*

Proof. Since $(\overline{\tilde{f}_1\cdot\tilde{f}_2})\tilde{g}_0=\bar{\tilde{f}}_1(\bar{\tilde{f}}_2\tilde{g}_0)=(\tilde{g}_0\cdot a_2)\cdot a_1=\tilde{g}_0\cdot(a_2a_1)$, the mapping σ is an anti-homomorphism. Since $N_e(\tilde{g}_0)$ is a Lie transformation group of $\tilde{M}$, the correspondence $\tilde{f}\to\tilde{f}(\tilde{x})$ for a fixed $\tilde{x}\in\tilde{M}$ defines a C^∞-mapping γ: $N_e(\tilde{g}_0)\to\tilde{M}$. Moreover, $N_e(\tilde{g}_0)$ is a Lie transformation group of $F(\tilde{M})/G$, that is, the correspondence $y\times\tilde{f}\to\tilde{f}^*(y)$ defines a C^∞-mapping of $F(\tilde{M})/G\times N_e(\tilde{g}_0)$ into $F(\tilde{M})/G$, where $\tilde{f}^*$ is a transformation of $F(\tilde{M})/G$ induced by $\tilde{f}$. Since $\tilde{f}(\gamma(\tilde{f}))=\tilde{x}$, the composed mapping

$$\sigma' : \tilde{f} \longrightarrow \gamma(\tilde{f}) \longrightarrow \tilde{g}_0(\gamma(\tilde{f})) \longrightarrow \tilde{f}^{*-1}(\tilde{g}_0(\gamma(\tilde{f})))$$

is of class C^∞ from $N_e(\tilde{g}_0)$ into the fibre $F(\tilde{M})/G|\tilde{x}$ of $F(\tilde{M})/G$ over $\tilde{x}$. By the right translation of $F(\tilde{M})/G|\tilde{x}$ by N, we have an imbedding $\nu_{\tilde{x}}$ of N/G into $F(\tilde{M})/G|\tilde{x}$ such that $\nu_{\tilde{x}}q'(a)=\tilde{g}_0(\tilde{x})\cdot a$ for $a\in N$. If $\bar{\tilde{f}}\tilde{g}_0=\tilde{g}_0\cdot a$. we have

$$\sigma'(\tilde{f})=\tilde{f}^{*-1}(\tilde{g}_0(\tilde{f}(\tilde{x})))=\bar{\tilde{f}}\tilde{g}_0(\tilde{x})=\tilde{g}_0(\tilde{x})\cdot a=\nu_{\tilde{x}}q'(a).$$

Then we have a C^∞-mapping $\nu_{\tilde{x}}^{-1}\sigma'$ of $N_e(\tilde{g}_0)$ into N/G, which is σ.

Proposition 15. *For each $\tilde{f}\in N_e(\tilde{g}_0)$, the G-structure $\bar{\rho}_{g_0}\cdot\sigma(\tilde{f})$ is deformable to g_0 and has the same infinitesimal automorphisms as g_0, where $\bar{\rho}_{g_0}$ is the C^∞-imbedding of N/G into $\mathfrak{G}$ in* §3.

Proof. Since $N_e(\tilde{g}_0)$ is arcwise connected, we have a curve $\tilde{f}(t)$ of $N_e(\tilde{g}_0)$ for t of an interval I such that $\tilde{f}(0)=$ identity and $\tilde{f}(t_0)=\tilde{f}$ for some t_0. There exists an open neighborhood $\tilde{U}_{\tilde{x}}$ of each $\tilde{x}\in\tilde{M}$ such that the covering mapping p: $\tilde{M}\to M$ is diffeomorphic on $\tilde{f}(t)(\tilde{U}_{\tilde{x}})$ for every $t\in I$. Then the correspondence $(p(y), t)\to(p(\tilde{f}(t)y), t)$ is a continuous transformation $f_{\tilde{x}}(x, t)$ of an open neighborhood $p(\tilde{U}_{\tilde{x}})\times I$ into $M\times I$ such that $f_{\tilde{x}}(x, t)$ for any fixed t is a local C^{r+1}-transformation of $M\times t$ and a system $\{f_{\tilde{x}}(p(\tilde{U}_{\tilde{x}})\times I;\ \tilde{x}\in\tilde{M}\}$ is an open covering of $M\times I$. If $f_{\tilde{x}}(p(\tilde{U}_{\tilde{x}})\times t)\cap f_{\tilde{x}'}(p(\tilde{U}_{\tilde{x}'})\times t)=V\neq\phi$ a diffeomorphism $f_{\tilde{x}}(x, t)^{-1}f_{\tilde{x}'}(x, t)$ is a local automorphism of g_0 on $(f_{\tilde{x}})^{-1}V$, because $\bar{\bar{f}}(t)\tilde{g}_0=\tilde{g}_0\cdot a(t)=\bar{p}(g_0\cdot a(t))$ and then $(\bar{f}_{\tilde{x}}g_0)(V)$ $=(\bar{f}_{\tilde{x}'}g_0)(V)=(g_0a(t))(V)$ where $\sigma(f(t))=q'(a(t))$. Since M is compact, there exists a finite index J such that $\{p(\tilde{U}_{\tilde{x}_j}); j\in J\}$ and $\{f_{\tilde{x}_j}; j\in J\}$ satisfy the conditions of definition of deformations. Since $\bar{f}_{\tilde{x}_j}g_0=(g_0\cdot a(t))(p(\tilde{U}_{\tilde{x}_j})\times t)=(\bar{\rho}_{g_0}\sigma(\tilde{f}(t)))|(p(\tilde{U}_{\tilde{x}_j})\times t)$, the family $\bar{\rho}_{g_0}\cdot\sigma(\tilde{f}(t))$ is a deformation of g_0. Since each G-structure of $\bar{\rho}_{g_0}\cdot\sigma(\tilde{f}(t))$ is associated to g_0, it has the same infinitesimal automorphisms as g_0.

By the condition of g_0 in Introduction, $H^0(\tilde{M}, \mathfrak{A}(\tilde{g}_0))$ is the Lie algebra of the Lie group $A(\tilde{g}_0)$ of automorphisms of $\tilde{g}_0$. Then, if $\tilde{f}_U(t)$ is a 1-parameter family of local automorphisms such that $\tilde{f}_U(t)(\tilde{x})$ is continuous on $\tilde{U}\times I$ and $\tilde{f}_{\tilde{U}}(0)$ is identity, each of $\tilde{f}_{\tilde{U}}(t)$ can be extended to a unique element of $A_e(\tilde{g}_0)$. Let $\tilde{g}$ be a G-structure locally equivalent to $\tilde{g}_0$. Let $\tilde{\psi}_{\tilde{U}}$ be a local bi-G-mapping of $\tilde{g}$ into $\tilde{g}_0$ on an open neighborhood $\tilde{U}$, satisfying the condition that there exists an 1-parameter family $\tilde{\psi}_{\tilde{U}}(t)$ of local bi-G-mappings such that $\tilde{\psi}_{\tilde{U}}(t)(\tilde{x})$ is continuous on $\tilde{U}\times I$, $\tilde{\psi}_{\tilde{U}}(1)=\tilde{\psi}_{\tilde{U}}$ and $\tilde{\psi}_{\tilde{U}}(0)$ is identity. Then the germ of a local bi-G-mapping at any y of $\tilde{U}$, satisfying the similar condition to $\psi_{\tilde{U}}$, is the germ of $\tilde{f}\tilde{\psi}_{\tilde{U}}$ at y for some $\tilde{f}$ of $A_e(\tilde{g}_0)$. Therefore the portion of the sheaf of germs of local bi-G-mapping which satisfies the above condition on $\tilde{U}$, is isomorphic to $\tilde{U}\times A_e(\tilde{g}_0)$. Since $\tilde{M}$ is simly connected, the $\tilde{\psi}_{\tilde{U}}$ can be extended to a global G-mapping of $\tilde{g}$ into $\tilde{g}_0$.

Proposition 16. *If $\tilde{g}_t$ is a deformation of $\tilde{g}_0$ such that $A(\tilde{g}_t)$ $=A(\tilde{g}_0)$ for each t, then $\tilde{g}_t$ is trivial.*

Proof. There exists a continuous mapping $\tilde{\psi}_{\tilde{U}}(t)$ of $\tilde{U}\times I$ into $\tilde{M}$ for some $\tilde{U}$ such that for each fixed t, $\tilde{\psi}_{\tilde{U}}(t)$ is a local diffeomorphism of $\tilde{U}$ into $\tilde{M}$ and a bi-G-mapping of $\tilde{g}_t$ into $\tilde{g}_0$ on $\tilde{U}$. Then $\tilde{\psi}_{\tilde{U}}(t)$ can be extended to a continuous mapping $\tilde{\psi}(t)$ of $\tilde{M}\times I$ in $\tilde{M}$ such that for each t, $\tilde{\psi}(t)$ is a G-mapping of $\tilde{g}_t$ into $\tilde{g}_0$. Because $A(\tilde{g}_0)=A(\tilde{g}_t)$ and $\tilde{g}_t$ satisfies the condition of $\tilde{g}_0$, we have a G-mapping $\tilde{\psi}'(t)$ such that

$\tilde{\psi}'(t)|U=\tilde{\psi}_{\tilde{U}}(t)^{-1}$. Since $\tilde{\psi}(t)\cdot\tilde{\psi}'(t):\tilde{M}\to\tilde{M}$ is a G-mapping and $\tilde{\psi}(t)\tilde{\psi}'(t)|U=$identity, $\tilde{\psi}(t)\tilde{\psi}'(t)=$identity, that is, $\tilde{\psi}(t)$ is a diffeomorphism of $\tilde{M}$ such that $\tilde{\psi}(t)\tilde{g}_0=\tilde{g}_t$. Therefore $\tilde{g}_t$ is trivial.

We denote by $\mathcal{S}$ the g_0-component of the space of G-structures which are deformable to g_0 and have the same infinitesimal automorphisms as g_0, that is, the g_0-component of $\mathcal{A}\cap\mathcal{D}$.

Proposition 17. *The C^∞-mapping $\bar{p}_{g_0}\sigma$ $(=\mu)$ maps $N_e(\tilde{g}_0)$ on $\mathcal{S}$ and the differential $d\mu$ of μ at e satisfies a formula $\dot{p}(d\mu)\tilde{n}=\mathcal{L}_{\tilde{n}}\tilde{g}_0$ for $\tilde{n}\in\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$, where p is the mapping from $\Gamma(V_{g_0}(F(M)/G))$ onto $\Gamma(V_{\tilde{g}_0}(F(\tilde{M})/G))$ induced by p and $\mathfrak{N}(\tilde{g}_0)$ is the sheaf of vector fields of the Lie algebra of $N(\tilde{g}_0)$.*

Proof. By Proposition 15, $\mu(N_e(\tilde{g}_0))\subset\mathcal{S}$. For any $g\in\mathcal{S}$, let $g(t)$ be an 1-parameter continuous family in $\mathcal{S}$ for $t\in[0, 1]$ such that $g(0)=g_0$ and $g(1)=g$. Then the lift $\tilde{g}(t)=\bar{p}g(t)$ of $g(t)$ is a deformation of $\tilde{g}_0$ on $\tilde{M}$. By Proposition 16, we have an 1-parameter $\tilde{f}(t)$ of C^{r+1}-diffeomorphisms of $\tilde{M}$ such that $\bar{\tilde{f}}(t)\tilde{g}_0=\tilde{g}(t)$ and $\tilde{f}(t)(\tilde{x})$ is continuous on $\tilde{M}\times I$. The G-structure $\tilde{g}(t)$ for each t has the same infinitesimal automorphisms as $\tilde{g}_0$. Since $A(\tilde{g}(t))=A(\bar{\tilde{f}}(t)\tilde{g}_0)=\tilde{f}(t)A(\tilde{g}_0)$, each $\tilde{f}(t)$ transforms $A(\tilde{g}_0)$ onto itself and then $\tilde{f}(t)\in N_e(\tilde{g}_0)$. Therefore, $\mu(\tilde{f}(t))=g(t)$ and the image of μ is $\mathcal{S}$. Moreover, for $\tilde{n}\in\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$ we have

$$p^{**}d\mu(\tilde{n})(x)=\left\{\frac{\mathrm{d}}{\mathrm{d}t}p^*(g_0(x)\cdot a(t))\right\}_{t=0}=\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{g}_0(\tilde{x})\cdot a(t))\right\}_{t=0}$$
$$=\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{f}(t)\tilde{g}_0)(\tilde{x})\right\}_{t=0}=\mathcal{L}_{\tilde{n}}\tilde{g}_0(\tilde{x}),$$

where $p(\tilde{x})=x$, $\tilde{f}(t)=\exp t\tilde{n}$, $\sigma(\tilde{f}(t))=q'a(t)$ and p^* (resp. p^{**}) is the mapping of $F(M)/G$ (resp. $V_{g_0}(F(M)/G)$) onto $F(\tilde{M})/G$ (resp. $V_{\tilde{g}_0}(F(\tilde{M})/G)$) induced by p. Then $\dot{p}(d\mu(\tilde{n}))=L_{\tilde{n}}\tilde{g}_0$.

Theorem 1. *The subspace $\mathcal{S}$ is an immersed submanifold of $\mathcal{G}$.*

Proof. If and only if $\mu(\tilde{f}_i)=\mu(\tilde{f}_2)$ for $\tilde{f}_1, \tilde{f}_2 \in N_e(\tilde{g}_0)$, then $\bar{\tilde{f}}_1\tilde{g}_0= \bar{\tilde{f}}_2\tilde{g}_0$, that is, $\tilde{f}_1\tilde{f}_2^{-1} \in A(\tilde{g}_0)$. Now, $N_e(\tilde{g}_0)\cap A(\tilde{g}_0)$ is closed in $N_e(\tilde{g}_0)$. The differentiable mapping μ induces a differentiable injection $\bar{\mu}$ from a factor space $N_e(\tilde{g}_0)/[N_e(\tilde{g}_0)\cap A(\tilde{g}_0)]$ into the space $\mathcal{G}$. Here, the image of $\bar{\mu}$ is $\mathcal{S}$ and the image of its differential $d\bar{\mu}$ at e is that of $d\mu$, of which the rank is equal to the dimension of $N_e(\tilde{g}_0)/[N_e(\tilde{g}_0)\cap A(\tilde{g}_0)]$. Then $d\bar{\mu}$ is injective and split. Therefore $\mathcal{S}$ is an immersed submanifold in $\mathcal{G}$.

Corollary. *The tangent space of $\mathcal{S}$ at g_0 is the vector space $\Gamma(\delta_{g_0}p'\mathfrak{N}(\tilde{g}_0), M)$ of all the sections of the subsheaf $\delta_{g_0}p'\mathfrak{N}(\tilde{g}_0)$ of V, where p' is the sheaf mapping induced by p.*

Proof. A diagram of sheaves

$$\begin{array}{ccccccccc}
\mathfrak{A}(\tilde{g}_0) & \xrightarrow[i]{} & \mathfrak{N}(\tilde{g}_0) & \xrightarrow[i]{} & \tilde{\mathfrak{N}} & \xrightarrow[i]{} & \tilde{\mathfrak{T}} & \xrightarrow[\delta_{\tilde{g}_0}]{} & \tilde{\mathfrak{V}} \\
\downarrow p' & & \downarrow p' & & \downarrow p' & & \downarrow p' & & \downarrow p' \\
\mathfrak{A}(g_0) & \xrightarrow[i]{} & p'\mathfrak{N}(g_0) & \xrightarrow[i]{} & \mathfrak{N} & \xrightarrow[i]{} & \mathfrak{T} & \xrightarrow[\delta_{g_0}]{} & \mathfrak{V}
\end{array}$$

is commutative, where i is the injection. Since $\mathfrak{N}(\tilde{g}_0)$ is a constant sheaf, we have $\Gamma(\delta_{\tilde{g}_0}\mathfrak{N}(\tilde{g}_0), \tilde{M}) = \delta_{\tilde{g}_0}\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$. Since $\mathcal{L}_{\tilde{n}}\tilde{g}_0 \in \delta_{\tilde{g}_0}\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$ for an $\tilde{n}\in\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$ and $p'\delta_{\tilde{g}_0}\mathfrak{N}(\tilde{g}_0)=\delta_{g_0}p'\mathfrak{N}(\tilde{g}_0)$, we have $d\mu(\tilde{n})\in\Gamma(\delta_{g_0}p'\mathfrak{N}(\tilde{g}_0), M)$ by Proposition 17. Conversely, we have $\dot{p}\dot{g}\in\Gamma(\delta_{\tilde{g}_0}(\mathfrak{N}(\tilde{g}_0), \tilde{M})=\delta_{g_0}\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$ for an $\tilde{n}\in\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$, where $\dot{p}\dot{g}$ is the lift of $\dot{g}$ by p. Then $\dot{g}=d\mu(\tilde{n})$.

§7. Equivalence of G-structures having the same infinitesimal automorphisms.

Since M is compact, there is a positive number ϵ such that the covering mapping p is diffeomorphic on each connected component of $p^{-1}(U_\epsilon(x))$ for the ϵ-neighborhood $U_\epsilon(x)$ of any point x of M. For this ϵ, each diffeomorphism f belonging to the ϵ-neighborhood $D_\epsilon(e)$

of e in $\mathrm{Diff}^{(r+1)}(M)$ induces a diffeomorphism $\tilde{f}$ of $\tilde{M}$ such that $p(\tilde{f}(\tilde{x}))=f(p(\tilde{x}))$ for any $\tilde{x}\in\tilde{M}$ and $\tilde{f}(\tilde{x})$ belongs to the same connected component of $p^{-1}(U_\epsilon(p(\tilde{x})))$ as $\tilde{x}$. The correspondence $f\to\tilde{f}$ defines a continuous injection $\tilde{p}$ of $D_\epsilon(e)$ into the topological group $\mathrm{Diff}(\tilde{M})$ with the compact-open topology, because the topology of $\mathrm{Diff}^{(r+1)}(M)$ is stronger than the compact-open topology. Since the topology of the Lie group $A_e(g_0)$ (resp. $N_e(g_0)$) is the modified compact-open topology (see [5]), the identity component of $A_e(g_0)\cap D_\epsilon(e)$ (resp. $N_e(g_0)\cap D_\epsilon(e)$) is an open neighborhood of e in $A_e(g_0)$ (resp. $N_e(g_0)$).

The Lie algebra $\bar{A}$ (resp. $\bar{N}$) of $A(\tilde{g}_0)$ (resp. $N(\tilde{g}_0)$) is $\Gamma(\mathfrak{A}(\tilde{g}_0), \tilde{M})$ (resp. $\Gamma(\mathfrak{N}(\tilde{g}_0), \tilde{M})$). Let $\dot{N}$ be the lift ${}'p(\Gamma(\mathfrak{N}, M))$ of the Lie algebra $\Gamma(\mathfrak{N}, M)$ of $N(\tilde{g}_0)$. Then $\dot{N}$ and $\bar{A}$ are respectively subalgebra of the Lie algebra $\bar{N}$ of $N(\tilde{g}_0)$. Take a complement $\dot{V}$ of the sum $\dot{N}+\bar{A}$ in $\bar{N}$ and a complement $\dot{N}'$ of $\bar{A}$ in $\dot{N}+\bar{A}$. Then, $\bar{N}=\dot{V}\oplus\dot{N}'\oplus\bar{A}$. Since $A_e(\tilde{g}_0)$ is closed in $N_e(\tilde{g}_0)$ and $\tilde{p}(A_e(g_0)\cap D_\epsilon(e))$ is locally closed in $N_e(\tilde{g}_0)$, we have

Lemma 3. *There exist open neighborhoods $\bar{A}_0$, N'_0 and V_0 of* 0 *in $\bar{A}$, $\dot{N}'$ and $\dot{V}$ respectively, such that the mapping*

$$\Phi:\ (a, b, c)\longrightarrow(\exp a)\cdot(\exp b)\cdot(\exp c)\qquad \text{for } a\in\bar{A}_0,\ b\in N'_0,\ c\in V_0$$

is a diffeomorphism of $\bar{A}_0\oplus N'_0\oplus V_0$ onto an open neighborhood $\tilde{U}$ of e in $N_\epsilon(\tilde{g}_0)$ and $\Phi(N'_0+V_0)\cap A_e(\tilde{g}_0)=e$.

Let $\tilde{V}$ denote the submanifold $\{\exp\tilde{v};\ \tilde{v}\in\tilde{V}_0\}$ of $N_e(\tilde{g}_0)$. The restriction of μ on $\tilde{V}$ is an imbedding and then its image $\mu(\tilde{V})$ is a differentiable submanifold of $\mathcal{S}$ which we denote by $\mathcal{V}$.

Proposition 18. *If $\bar{f}(t)g(t)$ is a curve in $\mathcal{V}$ for a curve $f(t)$ in* $\mathrm{Diff}^{(r+1)}(M)$ *through $e=f(0)$ and for a curve $g(t)$ in $\mathcal{V}$ through $g_0=g(0)$, then there exists $t_0>0$ such that $f(t)$ for $t\in[-t_0, t_0]$ is in $A_e(g_0)$.*

Proof. We have curves $\tilde{n}(t)$ and $\tilde{n}'(t)$ in $\tilde{V}$ with $\tilde{n}(0)=\tilde{n}'(0)=e$ such that $\mu(\tilde{n}(t))=g(t)$ and $\mu(\tilde{n}'(t))=\bar{f}(t)g(t)$. Since $f(t)\subset D_\epsilon(e)$, where $|t|<t_0$ for some $t_0>0$, we have $\bar{\tilde{n}}'(t)g_0=(\overline{p\bar{f}(t)})\bar{\tilde{n}}(t)\tilde{g}_0$, that is, $(\tilde{n}'(t))^{-1}(\tilde{p}f(t))\tilde{n}(t)=\tilde{b}(t)\subset A_e(\tilde{g}_0)$. Then $\tilde{p}f(t)=\tilde{n}'(t)\tilde{b}(t)\tilde{n}(t)^{-1}\subset N_e(\tilde{g}_0)$. Taking a smaller t_0 if necessary, we see that the curves $\tilde{p}f(t)$, $\tilde{b}(t)$, $\tilde{n}'(t)\tilde{b}(t)$ and $\tilde{n}'(t)\tilde{b}(t)\tilde{n}'(t)^{-1}=\tilde{b}'(t)$ are in $\tilde{U}$. Since $\tilde{p}(f(t))\tilde{n}(t)=\tilde{n}'(t)\tilde{b}(t)=\tilde{b}'(t)\tilde{n}'(t)$, $\tilde{p}(f(t))\subset\tilde{U}\cap P(D_\epsilon(e)), \tilde{n}(t)\subset\tilde{V}, \tilde{n}'(t)\subset\tilde{V}$ and $\tilde{b}'(t)\subset\tilde{U}\cap A_e(\tilde{g}_0)$, we have $\tilde{p}(f(t))=\tilde{b}'(t)$ and $\tilde{n}(t)=\tilde{n}'(t)$ by Lemma 3. Therefore, $\tilde{p}(f(t))\subset\tilde{p}(A_e(g_0)\cap D_\epsilon(e))$, that is, $f(t)\subset A_e(g_0)$.

Proposition 19. *If we take a suitable connected neighborhood U_0 of g_0 on $\mathcal{S}$, then for each g of U_0 there exist a unique $g'\in\mathcal{V}$ and an $f\in N_e(g_0)\cap D_\epsilon(e)$ such that $g=\bar{f}g'$, and the correspondence $g\to g'$ is a differentiable mapping of U_0 onto $\mathcal{V}$.*

Proof. If we set $U_0=\mu(\tilde{U})$, then we have Proposition from the definition of $\tilde{U}$ and by Lemma 3.

§8. Deformations having the equivalent infinitesimal automorphisms.

Definition. A deformation $g(t)$ of g_0 is called to *have the equivalent infinitesimal automorphisms,* if each $g(t)$ have the infinitesimal automorphisms equivalent to those of g_0, that is, if there exists a continuous curve $\phi(t)$ in $\mathrm{Diff}^{(r+1)}(M)$ through e such that $\phi(0)=e$ and $\phi(t)A(g_0)=A(g(t))$ for each t.

The composed mapping τ' of $\sigma: N_e(\tilde{g}_0)\to N/G$ and $\tau: \mathrm{Diff}^{(r+1)}(M)\times N/G\to\mathcal{G}$ is a C^∞-mapping of $\mathrm{Diff}^{(r+1)}(M)\times N_e(\tilde{g}_0)$ into $\mathcal{G}$. Moreover, τ' defines a C^∞-mapping τ'' of $\mathrm{Diff}^{(r+1)}(M)\times\mathcal{S}$ into $\mathcal{G}$ by formula $\tau''(id.\times\mu)=\tau'$, such that $\tau''(f,g)=\bar{f}g$ for $f\in\mathrm{Diff}^{(r+1)}(M)$ and $g\in\mathcal{S}$.

Proposition 20. *There exists an open neighborhood U_e of e in* $\mathrm{Diff}^{(r+1)}(M)$ *such that, if and only if a deformation $g(t)$ of g_0 have* *the equivalent infinitesimal automorphisms, $g(t)$ is a curve through g_0 in the image of $U_e \times \mathcal{V}$ by the C^∞-mapping τ'' for t of some neighborhood of 0 in R.*

Proof. The differential of τ'' at (e, g_0) is a continuous linear mapping $\theta + \dot{g} \rightarrow \bar{\delta}_{g_0}\theta + \dot{g}$ for $\theta \in T_e(\mathrm{Diff}^{(r+1)}(M))$ and $\dot{g} \in T_{g_0}(\mathcal{S})$. If $\dot{g}$ is tangent to $\mathcal{V}$, we have $\tilde{v} \in \dot{V}_0$ such that $d\mu(\tilde{v}) = \dot{g}$ and $\dot{p}\dot{g} = \mathcal{L}_{\tilde{v}}\tilde{g}_0$. If $\bar{\delta}_{g_0}\theta + \dot{g} = 0$, we have

$$\dot{p}\bar{\delta}_{g_0}\theta + \dot{p}\dot{g} = \mathcal{L}_{'p\theta}\tilde{g}_0 + \mathcal{L}_{\tilde{v}}\tilde{g}_0 = 0$$

and then $'p\theta \in \bar{A} + \dot{V}$, where $'p$ denote the lift of vector fields on M. Since $'p\theta \in \dot{N}$, we have $'p\theta \in \bar{A} \cap N$ and then $\theta \in \Gamma(\mathfrak{A}(g_0), M)$. Since there exists a closed complement D of $\Gamma(\mathfrak{A}(g_0), M)$ in $T_e(\mathrm{Diff}^{(r+1)}(M))$, we have an open neighborhood U_e of e on $\mathrm{Diff}^{(r+1)}(M)$ and a submanifold C tangent to D at e in U_e such that $\tau''(U_e \times \mathcal{V}) = \tau''(C \times \mathcal{V})$ and τ'' is diffeomorphic on $C \times \mathcal{V}$. If $g(t)$ is a curve in $\tau''(C \times \mathcal{V})$ through g_0, then we have a curve $f(t)$ in C and a curve $v(t)$ in $\mathcal{V}$ such that $g(t) = \bar{f}(t)v(t)$. Therefore,

$$A(g(t)) = A(\bar{f}(t)v(t)) = f(t)A(v(t)) = f(t)A(g_0),$$

that is, $g(t)$ is a deformation having the equivalent infinitesimal automorphisms. Conversely, if for a deformation $g(t)$ of g_0 there exists $f(t)$ such that $A(g(t)) = f(t)A(g_0)$ and $f(0) = \text{identity}$, then $A(g_0) = A(\bar{f}(t)^{-1}g(t))$. By Theorem 1 and Proposition 19, $\bar{f}(t)^{-1}g(t)$ is a curve $v(t)$ in $\mathcal{V}$ for a sufficiently small $|t|$. Then $f(t)$ is in U_e for t of some neighborhood of 0 in R and $g(t) = \bar{f}(t)v(t)$ is in $\tau''(U_e \times \mathcal{V})$.

Taking germs at $t=0$, the above facts are represented in the cohomology with coefficient sheaf as follows. Let $\{f_i(t), U_i; i \in J\}$ be a system of an open covering $\{U_i\}$ of M and local diffeomorphisms f_i defining a deformation $g(t)$ of g_0. For $i, j \in J$ such that $U_i \cap U_j \neq \phi$, a local transformation $f_i(t)^{-1}f_j(t)$ is considered as a 1-parameter family of local automorphisms of g_0 continuously dependent to t and its germ at $t=0$ is a section of the sheaf $[A(g_0) \times t]$ over $U_i \cap U_j$.

Let $\psi(x, t)$ be a local transformation of $M\times I$ such that $\psi(x, 0)$ is identity and $\psi(x, t)$ for any fixed t is a local C^{r+1}-transformation of $M\times t$ which transforms $\mathfrak{A}(g_0)$ onto itself and such that partial derivatives of $\psi(x, t)$ of any order ($\leqq r+1$) with respect to x are continuous on $M\times I$. Let $[N(g_0)\times t]$ denote the whole of germs of such local transformations at every point of $M\times 0$. Then $[N(g_0)\times t]$ is a sheaf of group and $[N(g_0)\times t]\supset[A(g_0)\times t]$. Therefore, a system

$$\{\text{germs of } f_i(t)^{-1}f_j(t) \text{ at } t=0;\ i, j\in J \text{ such that } U_i\cap U_j\neq\phi\}$$

is a $[N(g_0)\times t]$-valued 1-cocycle of the nerve of $\{U_i\}$. This cocycle is coboudary, if and only if the germ of $g(t)$ is equivalent to a deformation having the equivalent infinitesimal automorphisms. Let Ω denote the correspondence

$$H^1(M, [A(g_0)\times t]) \longrightarrow H^1(M, [N(g_0)\times t])$$

induced by the injection $A(g_0)\to N(g_0)$. Then we have

Theorem 2. *A cohomology class* $\mathbf{g}$ *of* $H^1(M, [A(g_0)\times t])$ *corresponds to a class of germ of a deformation having the equivalent infinitesimal automorphisms, if and only if* $\Omega\cdot\mathbf{g}$ *is coboundary in* $H^1(M, [N(g_0)\times t])$. *Any such class is represented by a unique germ of a curve in* $\mathcal{V}$.

Since $d\tau''(T_e(C)+T_{g_0}(\mathcal{V}))=\{\bar{\delta}_{g_0}\theta+\dot{v};\ \theta\in T_e(C),\ \dot{v}\in T_{g_0}(\mathcal{V})\}$, the tangent vector of a differentiable curve in $\tau''(U_e\times\mathcal{V})$ at $t=0$ is $\bar{\delta}_{g_0}\theta+\dot{v}$. Conversely, for any $\theta\in T_e(C)$ and any $\dot{v}\in T_{g_0}(\mathcal{V})$, a vector $\bar{\delta}_{g_0}\theta+\dot{v}$ is tangent to a differentiable curve in $\tau''(U\times\mathcal{V})$ at $t=0$. Here, from the definition of C and $\mathcal{V}$,

$$\{\bar{\delta}_{g_0}\theta+\dot{v};\ \theta\in T_e(C),\ \dot{v}\in T_{g_0}(\mathcal{V})\}=\{\bar{\delta}_{g_0}\Gamma(\mathfrak{T})+\Gamma(\delta_{g_0}p'\mathfrak{N}(\tilde{g}_0))\}.$$

Each element of $\bar{\delta}_{g_0}\Gamma(\mathfrak{T})+\Gamma(\delta_{g_0}p'\mathfrak{N}(\tilde{g}_0))$ is called an *infinitesimal deformation of* g_0 *having the equivalent automorphisms*. Thus we have

Theorem 3. *Every infinitesimal deformation having the equivalent infinitesimal automorphisms can be extended to a deformation having the equivalent infinitesimal automorphisms.*

The whole of equivalent classes of infinitesimal deformations of g_0 is a linear space $\Gamma(\delta_{g_0}\mathfrak{T})/\bar{\delta}_{g_0}\Gamma(\mathfrak{T})$ which is isomorphic to $H^1(M, \mathfrak{A}(g_0))$. Since $\mathfrak{A}(g_0) \subset p'\mathfrak{N}(\tilde{g}_0)$, we have a homomorphism $\omega' : H^1(M, \mathfrak{A}(g_0)) \to H^1(M, p'\mathfrak{N}(\tilde{g}_0))$. Ker $\omega'(=\mathcal{K}$ in Introduction) is the whole of equivalent classes of infinitesimal deformations having the equivalent infinitesimal automorphisms and this is a linear space with the dimension of the manifold $\mathcal{V}$, which is equal to $[\dim. N(\tilde{g}_0) - \dim. A(\tilde{g}_0) - \dim. N(g_0) + \dim. A(g_0)]$. Then we have

Theorem 4. *If* $\omega' : H^1(M, \mathfrak{A}(g_0)) \to H^1(M, p'\mathfrak{N}(\tilde{g}_0))$ *is injective, that is, if* $[\dim. N(\tilde{g}_0) - \dim. A(\tilde{g}_0) - \dim. N(g_0) + \dim. A(g_0)] = 0$, *then every deformations of* g_0 *having the equivalent infinitesimal automorphisms are trivial.*

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