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On the uniqueness of solutions of stochastic differential equations with boundary conditions

By

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It is very useful to formulate the boundary problems of Markov processes by means of stochastic differential equations.

Recently, S. Watanabe showed in [9] [10], the existence and the uniqueness of solutions of stochastic differential equations with boundary conditions in the case that all coefficients are Lipschitz continuous. On the other hand, D. W. Stroock and S. R. S. Varadhan in [8] formulated this problem as sub-martingale problem and proved that if diffusion processes with non-degenerated continuous coefficients have boundary conditions with Lipschitz continuous oblique derivatives, then the existence and the uniqueness of solutions of the sub-martingale problems are valid.

We will refer to N. E. Karoui [3] as for the equivalence of sub-martingale problems and stochastic differential equations with boundary conditions.

We will also refer to S. Nakao [6] who showed that there exist solutions if all coefficients are continuous.

In the present paper, we will be concerned with the case that the boundary conditions have non-degenerated second order term with continuous coefficients and solve the uniqueness problem.

Finally, the authors would like to express their hearty gratitude

to Professor S. Watanabe for his valuable advices.

Let $\bar{D}=R_n^+=\{x=(x^1, x^2, \cdots, x^n)\in R^n: x^1\geqq 0\}$ $D=\{x\in\bar{D}: x^1>0\}$ and $\partial D=\{x\in\bar{D}: x^1=0\}$.

Let α, b, σ, γ and ρ be given as follows:

$$\begin{aligned}
&\alpha=(\alpha_{ij}) \quad && i,j=1,\cdots,n: \ \bar{D}\to R^n\otimes R^n \ ^{1)}\\
&b=(b_i) && i=1,\cdots,n: \ \bar{D}\to R^n\\
&\sigma=(\sigma_{ij}) && i,j=2,\cdots,n: \ \partial D\to R^{n-1}\otimes R^{n-1}\\
&\gamma=(\gamma_i) && i=2,\cdots,n: \ \partial D\to R^{n-1}\\
&\rho: \partial D\to[0,\infty).
\end{aligned}$$

It is assumed that they are all bounded and Borel measurable. We consider a stochastic differential equation of the following form;

$$(1)\quad \begin{cases}
dx_t^1=\alpha^1(x_t)1_D(x_t)dB_t+b_1(x_t)1_D(x_t)dt+d\varphi_t\\
dx_t^i=\alpha^i(x_t)1_D(x_t)dB_t+b_i(x_t)1_D(x_t)dt\\
\qquad\qquad +\sigma^i(x_t)1_{\partial D}(x_t)dM_t+\gamma_i(x_t)1_{\partial D}(x_t)d\varphi_t \quad i=2,3,\cdots,n\\
1_{\partial D}(x_t)dt=\rho(x_t)1_{\partial D}(x_t)d\varphi_t,
\end{cases}$$

where

$$B_t=(B_t^1,\cdots,B_t^n),\ M_t=(M_t^2,\cdots M_t^n),$$

$$\alpha^i(x_t)1_D(x_t)dB_t=\sum_{j=1}^n\alpha_{ij}(x_t)1_D(x_t)dB_t^j \qquad i=1,2,\cdots,n$$

and

$$\sigma^i(x_t)1_{\partial D}(x_t)dM_t=\sum_{j=2}^n\sigma_{ij}(x_t)1_{\partial D}(x_t)dM_t^j \qquad i=2,\cdots,n.$$

1_D, $1_{\partial D}$ are indicator functions of D, ∂D.

To be precise, by a solution of the equation (1), we mean a family of stochastic processes $\mathfrak{X}=\{x_t=(x_t^1, x_t^2, \cdots, x_t^n), B_t=(B_t^1, B_t^2, \cdots, B_t^n), M_t=(M_t^2, \cdots, M_t^n), \varphi_t\}$ defined on a standard probability space in the sense of K. Ito [1] with a right continuous increasing family of Borel fields $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, i.e. $\mathcal{F}_{t+0}\equiv\bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}=\mathcal{F}_t$, such that

(i) with probability one, they are all continuous in t, $B_0=0$, $M_0=0$, $\varphi_0=0$ and $x_t\in\bar{D}$ for each $t\geqq 0$,

(ii) they are all $\mathcal{F}_t$-adapted, i.e. for any t, they are $\mathcal{F}_t$-measurable,

1) $R^n\otimes R^n$ is the class of linear applications of R^n into R^n.

(iii) with probability one, φ_t is non-decreasing and φ_t increases only when $x_t^1=0$, i.e. $\varphi_t=\int_0^t 1_{\partial D}(x_s)d\varphi_s$,

(iv) (B_t, M_t) is a system of $\mathcal{F}_t$-martingales such that $\langle B^i, B^j\rangle_t =\delta_{ij}t$, $\langle B^i, M^j\rangle_t=0$ and $\langle M^i, M^j\rangle_t=\delta_{ij}\varphi_t$,

(v) $\mathfrak{X}=\{x_t, B_t, M_t, \varphi_t\}$ satisfies

$$(1)' \quad \begin{cases} x_t^1-x_0^1=\int_0^t \alpha^1(x_s)1_D(x_s)dB_s+\int_0^t b_1(x_s)1_D(x_s)ds+\varphi_t \\ x_t^i-x_0^i=\int_0^t \alpha^i(x_s)1_D(x_s)dB_s+\int_0^t b_i(x_s)1_D(x_s)ds \\ \qquad +\int_0^t \sigma^i(\tilde{x}_s)1_{\partial D}(x_s)dM_s+\int_0^t \gamma_i(\tilde{x}_s)1_{\partial D}(x_s)d\varphi_s \quad i=2, \cdots, n \\ \int_0^t 1_{\partial D}(x_s)ds=\int_0^t \rho(\tilde{x}_s)d\varphi_s, \end{cases}$$

where $\tilde{x}$ denotes the projection of x on ∂D,

$$\alpha^i(x_s)dB_s=\sum_{j=1}^n \alpha_{ij}(x_s)dB_s^j, \quad \sigma^i(\tilde{x}_s)dM_s=\sum_{j=2}^n \sigma_{ij}(\tilde{x}_s)dM_s^j$$

and the integrals by dB and dM are understood in the sense of stochastic integrals.

Remark 1. In $(1)'$, if $\rho\equiv 0$, it is called "non-sticky case". Then if $|\alpha^1(x)|\equiv(\sum_{j=1}^n \alpha_{1j}^2(x))^{1/2}\geq c$ $\forall x\in\bar{D}$, for some constant $c>0$, $(1)'$ is equivalent to the following equations;

$$\begin{cases} x_t^1-x_o^1=\int_0^t \alpha^1(x_s)dB_s+\int_0^t b_1(x_s)ds+\varphi_t \\ x_t^i-x_o^i=\int_0^t \alpha^i(x_s)dB_s+\int_0^t b_i(x_s)ds \\ \qquad +\int_0^t \sigma^i(\tilde{x}_s)dM_s+\int_0^t \gamma_i(\tilde{x}_s)d\varphi_s \qquad i=2, \cdots, n. \end{cases}$$

Then $\int_0^t 1_{\partial D}(x_s)ds=0$ holds automatically.

Assumption (C-I). There are positive constants μ_1, μ_2 such that

(a) $\alpha(x)$ is continuous on $\bar{D}$ and $\mu_1|\xi|^2\leq\sum_{i,j=1}^n [\alpha\alpha^*]_{ij}(x)\xi_i\xi_j\leq\mu_2|\xi|^2$

(b) $\sigma(\tilde{x})$ is continuous on ∂D and $\mu_1|\tilde{\xi}|^2 \leqq \sum_{i,j=2}^{n} [\sigma\sigma^*]_{ij}(\tilde{x})\xi_i\xi_j \leqq \mu_2|\tilde{\xi}|^2$ $\forall \xi \in R^n$.

Now, our result is summarized in the following.

Theorem 1. *Under Assumption (C-I), the solutions of the equation* (1) *is unique in the sense of probability law. That is, if* $\mathfrak{X} = (x_t, B_t, M_t, \varphi_t)$ *defined on* $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ *and* $\mathfrak{X}' = (x'_t, B'_t, M'_t, \varphi'_t)$ *defined on* $(\Omega', \mathcal{F}', P'; \mathcal{F}'_t)$ *such that* $x_0 = x$, *a.s. and* $x'_0 = x$ *a.s. are two solutions of* (1), *the probability law of the processes* $\{x_t\}$ *and* $\{x'_t\}$ *on the space* $\{W, B(W)\}$ *coincides, where* W *is the Fréchet space of all* $\bar{D}$*-valued continuous functions on* $[0, \infty)$ *with the compact uniform topology and* $B(W)$ *is the topological Borel field on* W.

First, we consider the special case $b \equiv 0$, $\gamma \equiv 0$, $\rho \equiv 1$ and then, we reduce the general case to this special case by means of drift transformations and time-changes.

For the sake of technical convenience we treat the following time-dependent case.

Assumption (C-II).

a) $\alpha(t, x) : [0, \infty) \times \bar{D} \to R^n \otimes R^n$

$\sigma(t, \tilde{x}) : [0, \infty) \times \partial D \to R^{n-1} \otimes R^{n-1}$

α and σ are continuous mappings.

b) $\alpha_{11}(t, x) \equiv 1$, $\alpha_{1i}(t, x) \equiv 0$ $(i \neq 1)$, $\alpha_{ij}(t, x) = \delta_{ij}$ for $\forall t \geqq {}^{\exists}T_0$ and $\sigma_{ij}(t, x) = \delta_{ij}$ for $\forall t \geqq T_0$.

c) $\sum_{i,j=1}^{n} |[\alpha\alpha^*](t, x)_{ij} - \delta_{ij}| < \varepsilon$, $\sum_{i,j=2}^{n} |[\sigma\sigma^*](t, x)_{ij} - \delta_{ij}| < \varepsilon$

and for every $\xi \in R^n$,

$$\mu_1|\xi|^2 \leqq \sum_{i,j=1}^{n} [\alpha\alpha^*]_{ij}(t, x)\xi_i\xi_j \leqq \mu_2|\xi|^2,$$

$$\mu_1|\tilde{\xi}|^2 \leqq \sum_{i,j=2}^{n} [\sigma\sigma^*]_{ij}(t, x)\xi_i\xi_j \leqq \mu_2|\tilde{\xi}|^2,$$

where ε depends only on μ_1, μ_2, T_0 and is chosen in the proof of Theorem 2 and Theorem 4.

Consider the following equation under Assumption (C–II).

$$(2)\qquad \begin{cases} x_t^1 - x_{t_0}^1 = \int_{t_0}^t 1_D(x_s)dB_s^1 + \varphi_t^{t_0} \\ x_t^i - x_{t_0}^i = \int_{t_0}^t \alpha^i(s, x_s)1_D(x_s)dB_s + \int_{t_0}^t \sigma^i(s, \tilde{x}_s)1_{\partial D}(x_s)dM_s \\ \qquad\qquad i = 2, \cdots, n \\ \int_{t_0}^t 1_{\partial D}(x_s)ds = \varphi_t^{t_0}. \end{cases}$$

It suffices to prove in the case $t_0 = 0$.

Let $\mathfrak{X} = (x_t, B_t, M_t, \varphi_t)$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a solution of (2).

Lemma 1. *Let $X(t)$ be a continuous martingale which is uniformly bounded on any bounded interval and $\psi(t) \geqq 0$ be a continuous increasing process such that $E\psi(t) < +\infty$ for each $t \geqq 0$. Then $X(t)\psi(t) - \int_0^t X(u)d\psi(u)$ is a martingale.*

The proof of Lemma 1 can be found in Stroock-Varadhan [8] Lemma 2.1.

Lemma 2. *For φ of the equation (2), $E\varphi_t < +\infty$ for each $t \geqq 0$ and $\lim_{t\to\infty} \varphi_t = +\infty$. a.s. (P).*

Proof. $x_t^1 - x_0^1 = \int_0^t 1_{(0,\infty)}(x_s^1)dB_s^1 + \varphi_t$.

Let ξ_t be a Brownian local time at the origin and $A_t = t + \xi_t$. It is known that φ_t has the same law as $\xi_{A_t^{-1}}$. Using this fact and the properties of Brownian local time, we can conclude this lemma (cf. Ito-McKean [2]).

Lemma 3. *Let $f(t, x)$ be a bounded continuous function on $[0, \infty) \times \overline{D}$ such that $f(t, x) = 0$ for $\forall t \geqq \exists T > 0$ and let $g(s, \omega)$, $h(s, \omega)$ be bounded non-anticipating functionals such that $g(t, \omega) = h(t, \omega) = 0$ for $\forall t \geqq \exists T > 0$. If*

$$X(t) = f(t, x(t)) - \int_0^t g(s, \omega)ds - \int_0^t h(s, \omega)d\varphi_s$$

is a martingale, then

$$(3)\qquad f(0, x_0)=\lambda E\left[\int_0^\infty e^{-\lambda\varphi_s} f(s, \tilde{x}(s))d\varphi_s\right]-E\left[\int_0^\infty e^{-\lambda\varphi_s} g(s, \omega)ds\right] -E\left[\int_0^\infty e^{-\lambda\varphi_s} h(s, \omega)d\varphi_s\right]$$

holds.

Proof. Noting $\int_0^\infty e^{-\lambda\varphi_s}d\varphi_s=\frac{1}{\lambda}$, because of $\lim_{t\to\infty}\varphi_t=\infty$ a.s., the integrals of the right hand of (3) are well-defined and by Lemma 1 $X(t)\int_0^t e^{-\lambda\varphi_s}d\varphi_s-\int_0^t e^{-\lambda\varphi_s}X(s)d\varphi_s$ is a martingale.

Therefore $\frac{1}{\lambda}E[X(\infty)]=E\left[\int_0^\infty e^{-\lambda\varphi_s}X(s)d\varphi_s\right]$. (3) can be obtained by arranging this equality.

Throughout the present paper we shall assume $p>n+2$. We will introduce two functionals as follows and obtain some L^p-estimates of them.

Let $\mu_\lambda[h]=E\left[\int_0^\infty e^{-\lambda\varphi_t}h(t, \tilde{x}_t)d\varphi_t\right]$ for any h defined on $[0, \infty)\times\partial D$ and $\nu_T[u]=E\left[\int_0^T 1_D(x_t)u(t, x_t)dt\right]$ for any u defined on $[0, T)\times\bar{D}$.

Theorem 2. *Under Assumption (C-II), we have*

$$(4)\qquad |\mu_\lambda[h]|\leqq C_\lambda\|h\|_{\sim p}\quad\text{[2)]}\quad \textit{for}\ {}^\forall h\in L^p\ ([0, \infty)\times\partial D)$$

$$(5)\qquad |\nu_T[u]|\leqq B_T\|u\|_{p,T}\quad \textit{for}\ {}^\forall u\in L^p\ ([0, T)\times\bar{D}),$$

where C_λ is uniformly bounded in $\lambda\geqq 1$.

In order to prove this theorem, we approximate $\{x_t\}$ in the following procedure. Let $\pi_m(s)$ be defined as follows:

$$\pi_m(s)=\begin{cases}\frac{k}{m} & \text{for } \frac{k}{m}\leqq s<\frac{k+1}{m},\ k=0, 1, \cdots, m^2-1\\ m & \text{for } s\geqq m^2.\end{cases}$$

2) The definitions of $\|\cdot\|_{\sim p}$ and $\|\cdot\|_{p,T}$ are stated in Appendix.

Let $\{x_t^{(m)}\}=\{x_t^{(m),1},\cdots,x_t^{(m),n}\}$ $m=1,2,\cdots$ be defined by

$$(6)\qquad \begin{cases} x_t^{(m),1}=x_t^1 \\ x_t^{(m),i}=x_0^i+\int_0^t 1_{(0,\infty)}(x_s^1)\alpha^i(\pi_m(s),\, x_{\pi_m(s)})dB_s \end{cases}$$

$$+\int_0^t 1_{\{0\}}(x_s^1)\sigma^i(\pi_m(s),\, \tilde{x}_{\pi_m(s)})dM_s \qquad i=2,3,\cdots,n.$$

Then,

$$(7)\qquad \int_0^t 1_{\{0\}}(x_s^{(m),1})ds=\varphi_t$$

is verified trivially.

Also, we can show by usual method

$$(8)\qquad P(\max_{0\leqq s\leqq t}|x_s^{(m)}-x_s|>\varepsilon)\to 0 \text{ as } m\to\infty \text{ for } {}^\forall\varepsilon>0.$$

Lemma 4. *If* u *of* $C_b^{1,2}([0,\infty)\times\bar{D})$[3] *satisfies* $u(t,x)=0$ *for* ${}^\forall t\geqq T>0$, *then,*

$$(9)\qquad u(0,x_0)$$

$$=-E\Big[\int_0^\infty e^{-\lambda\varphi_s}1_{(0,\infty)}(x_s^1)\Big[D_s+\sum_{i,j=1}^n\frac{1}{2}[\alpha\alpha^*]_{ij}(\pi_m(s),\, x_{\pi_m(s)})D_{ij}^2\Big]$$

$$u(s,x_s^{(m)})ds\Big]$$

$$+E\Big[\int_0^\infty e^{-\lambda\varphi_s}\Big[\lambda-D_s-\sum_{i,j=2}^n\frac{1}{2}[\sigma\sigma^*]_{ij}(\pi_m(s),\, \tilde{x}_{\pi_m(s)})D_{ij}^2-D_1\Big]$$

$$u(s,\tilde{x}_s^{(m)})d\varphi_s\Big].$$

Proof. Applying Ito's formula to (6) for u, we see that

$$u(t,x_t^{(m)})-\int_0^t 1_{(0,\infty)}(x_s^1)\Big[\sum_{i,j=1}^n\frac{1}{2}[\alpha\alpha^*]_{ij}(\pi_m(s),\, x_{\pi_m(s)})D_{ij}^2u(s,x_s^{(m)})\Big]ds$$

$$-\int_0^t\Big[\sum_{i,j=2}^n\frac{1}{2}[\sigma\sigma^*]_{ij}(\pi_m(s),\, \tilde{x}_{\pi_m(s)})D_{ij}^2u(s,\tilde{x}_s^{(m)})+D_1u(s,\tilde{x}_s^{(m)})\Big]d\varphi_s$$

$$-\int_0^t D_su(s,x_s^{(m)})ds$$

is a martingale. Therefore, we obtain (9) by making use of (7) and Lemma 3.

3) $C_b^{1,2}([0,\infty)\times\bar{D})$ denotes the class of functions which together with their first t-derivative and first two x-derivatives are bounded and continuous.

We define two sequences of functionals;

$$\mu_\lambda^{(m)}[h]=E\left[\int_0^\infty e^{-\lambda\varphi_t}h(t,\tilde{x}_t^{(m)})d\varphi_t\right] \qquad h\in C_0^\infty([0,\infty)\times\partial D)^{4)}$$

$$\nu_{\lambda,T}^{(m)}[u]=E\left[\int_0^T e^{-\lambda\varphi_t}1_{(0,\infty)}(x_t^1)u(t,x_t^{(m)})dt\right] \quad u\in C_0^\infty([0,T)\times\bar{D}).$$

Lemma 5. *There exist constants $C_{m,\lambda}$, $B_{m,T}$ such that*

$$(10) \qquad |\mu_\lambda^{(m)}[h]|\leqq C_{m,\lambda}\|h\|_{\sim p} \qquad \forall h\in C_0^\infty([0,\infty)\times\partial D)$$

$$(11) \qquad |\nu_{\lambda,T}^{(m)}[u]|\leqq B_{m,T}\|u\|_{p,T} \qquad \forall u\in C_0^\infty([0,T)\times\bar{D}),$$

where $C_{m,\lambda}$ is uniformly bounded in $\lambda\geqq 1$.

Proof.

$$\mu_\lambda^{(m)}[h]=\sum_{j=0}^{m2-1}E\left[\int_{j/m}^{(j+1)/m}e^{-\lambda\varphi_s}h(s,\tilde{x}_s^{(m)})d\varphi_s\right]+E\left[\int_m^\infty e^{-\lambda\varphi_s}h(s,\tilde{x}_s^{(m)})d\varphi_s\right].$$

We shall prove only

$$\left|E\left[\int_m^\infty e^{-\lambda\varphi_s}h(s,\tilde{x}_s^{(m)})d\varphi_s\right]\right|\leqq C_{m,\lambda}\|h\|_{\sim p},$$

since other terms can be proved in a similar way. (6) is rewritten in the following form

$$(12) \quad \begin{cases} x_{m+t}^{(m),1}=x_m^{(m),1}+\int_0^t 1_{(0,\infty)}(x_{m+s}^{(m),1})dB_s^1(\theta_m)^{5)}+\varphi_{t+m}-\varphi_m \\ x_{m+t}^{(m),i}=x_m^{(m),i}+\alpha^i(m,x_m)\int_0^t 1_{(0,\infty)}(x_{m+s}^{(m),1})dB_s(\theta_m) \\ \qquad +\sigma^i(m,\tilde{x}_m)(M_{t+m}-M_m) \qquad i=2,\cdots,n. \end{cases}$$

Let $\hat{P}=P(\cdot|\mathcal{F}_m)$ be the regular conditional probability of P relative to $\mathcal{F}_m$ and $\hat{\mathcal{F}}_t=\mathcal{F}_{t+m}$, $\hat{B}_t=B_{t+m}-B_m$, $\hat{M}_t=M_{t+m}-M_m$, $\hat{\varphi}_t=\varphi_{t+m}-\varphi_m$ and $\hat{x}_t=x_{t+m}$. Then, $\hat{\mathfrak{X}}=(\hat{x}_t,\hat{B}_t,\hat{M}_t,\hat{\varphi}_t)$ on $(\Omega,\mathcal{F},\hat{P};\hat{\mathcal{F}}_t)$ is a solution of equation (2) with coefficients $\{\alpha(m,x_m),\sigma(m,x_m)\}$. However, we can regard $\{\alpha(m,x_m),\sigma(m,x_m)\}$ as constant coefficients, as far as we consider the probability space $(\Omega,\mathcal{F},\hat{P};\hat{\mathcal{F}}_t)$. Suppose $\mathfrak{X}=(x_t,B_t,M_t,\varphi_t)$ on $(\Omega,\mathcal{F},P;\mathcal{F}_t)$ is a solution of the equation (2) with constant coefficients (α,σ).

4) $C_0^\infty([0,\infty)\times\partial D)$ is the class of infinitely differeentiable functions having compact support.

5) θ is the shift operator.

Let $H^{[\alpha]}$ be the harmonic operator corresponding to $D_t+\frac{1}{2}\sum_{i,j=1}^{n}[\alpha\alpha^*]_{ij}D_{ij}^2$ and $V_\lambda^{[\alpha,\sigma]}$ be the resolvent operator corresponding to $D_t+\sum_{i,j=2}^{n}\frac{1}{2}[\sigma\sigma^*]_{ij}D_{ij}^2+\frac{\partial}{\partial n}H^{[\alpha]}$, where $\frac{\partial}{\partial n}=D_1$.

For each h of $C_0^\infty([0,\infty)\times\partial D)$, it is known that $H^{[\alpha]}V_\lambda^{[\alpha,\sigma]}h$ belongs to $C^\infty([0,\infty)\times\bar{D})$ and there exists constant T such that $H^{[\alpha]}V_\lambda^{[\alpha,\sigma]}h(t,x)=0$ for $^\forall t\geqq T$. Applying Ito's lemma for $H^{[\alpha]}V_\lambda^{[\alpha,\sigma]}h$, we see that

$$H^{[\alpha]}V_\lambda^{[\alpha,\sigma]}h(t,x_t)-\int_0^t\left[D_t+\frac{1}{2}\sum_{i,j=2}^{n}[\sigma\sigma^*]_{ij}D_{ij}^2+\frac{\partial}{\partial n}H^{[\alpha]}\right]V_\lambda^{[\alpha,\sigma]}h(s,\tilde{x}_s)d\varphi_s$$

is martingale.

Accordingly, by Lemma 3

$$H^{[\alpha]}V_\lambda^{[\alpha,\sigma]}h(0,x_0)=E\left[\int_0^\infty e^{-\lambda\varphi_s}h(s,\tilde{x}_s)d\varphi_s\right] \tag{13}$$

and

$$\left|E\left[\int_0^\infty e^{-\lambda\varphi_s}h(s,\tilde{x}_s)d\varphi_s\right]\right|=|H^{[\alpha]}V_\lambda^{[\alpha,\sigma]}h(0,x_0)|$$
$$\leqq\sup_{t,\tilde{x}}|V_\lambda^{[\alpha,\sigma]}h(t,\tilde{x})|\leqq C_\lambda\|h\|_{\sim p},$$

where C_λ is bounded in $\lambda\geqq 1$.

The last estimate is found in Theorem A. 3 and Theorem A. 4 in Appendix. It is uniform in $\alpha\in M[\mu_1;\mu_2]$ and $\sigma\in\widetilde{M}[\mu_1;\mu_2]$.

(cf. Appendix)

Hence, we have obtained the following estimate as to (12)

$$\left|E\left[\int_m^\infty e^{-\lambda\varphi_s}h(s,\tilde{x}_s^{(m)})d\varphi_s\,|\,\mathcal{F}_m\right]\right|=\left|\hat{E}\left[\int_0^\infty e^{-\lambda\hat{\varphi}_s}e^{-\lambda\varphi_m}h(m+s,\hat{\tilde{x}}_s^{(m)})d\hat{\varphi}_s\right]\right| \tag{14}$$
$$\leqq C_{m,\lambda}\|h\|_{\sim p}.$$

By an analogous argument we can obtain

$$\left|E\left[\int_{j/m}^{(j+1)/m}e^{-\lambda\varphi_s}h(s,\tilde{x}_s^{(m)})d\varphi_s\,|\,\mathcal{F}_{j/m}\right]\right|\leqq C_{m,\lambda}\|h\|_{\sim p}. \tag{15}$$

Taking expectation in (14) and (15), the first estimate in Lemma 5 follows.

Next, we shall estimate the functional $\nu_{\lambda,T}^{(m)}$. We may assume $\{\alpha, \sigma\}$ are constant matrices by the same arguments as $\mu_{\lambda}^{(m)}$.

Let us denote by $G_0^{[\alpha]}$ the minimal (absorbing) potential operator on D associated with $D_t+\sum_{i,j=1}^{n}\frac{1}{2}[\alpha\alpha^*]_{ij}D_{ij}^2$.

It is well-known that for each f of $C_0^\infty([0, T)\times\bar{D})$, $u=G_0^{[\alpha]}f$ belongs to $C^\infty([0, T)\times\bar{D})$ and satisfies the following equation.

$$(16)\qquad \begin{cases} -\left[D_t+\sum_{i,j=1}^{n}\frac{1}{2}[\alpha\alpha^*]_{ij}D_{ij}^2\right]u=f \\ u|_{\partial D}=0. \end{cases}$$

Therefore it follows easily from Ito's formula that

$$(17)\qquad u(t, x_t)+\int_0^t 1_{(0,\infty)}(x_s^1)f(s, x_s)ds-\int_0^t D_1u(s, \tilde{x}_s)d\varphi_s$$

is a matringale. Applying Lemma 3 to (17), we have

$$(18)\qquad u(0, x_0)=E\left[\int_0^T e^{-\lambda\varphi_s}1_{(0,\infty)}(x_s^1)f(s, x_s)ds\right] -E\left[\int_0^T e^{-\lambda\varphi_s}D_1u(s, \tilde{x}_s)d\varphi_s\right].$$

By Theorem A. 4 in Appendix, we have

$$|u(0, x_0)|=|G_0^{[\alpha]}f(0, x_0)|\leqq C_T\|f\|_{p,T}$$
$$|D_1u(s, \tilde{x})|=|D_1G_0^{[\alpha]}f(s, \tilde{x})|\leqq C_T\|f\|_{p,T} \qquad \text{for } \forall\tilde{x}\in\partial D.$$

Thus, it is immediately from (18) that

$$(19)\qquad \left|E\left[\int_0^T e^{-\lambda\varphi_s}1_{(0,\infty)}(x_s^1)f(s, x_s)ds\right]\right|\leqq B_T\|f\|_{p,T} \qquad \lambda\geqq 1.$$

By (19) and the similar arguments as the proof of (10), we can obtain the estimate (11).

Now, we will prove Theorem 2, by making use of the estimates in Lemma 5.

Let us denote by $H^{(0)}$ the harmonic extension operator of $D_t+\frac{1}{2}\sum_{i=1}^{n}D_{ii}^2$, and by $V_\lambda^{(0)}$ the resolvent operator associated with

$D_t+\frac{1}{2}\sum_{i=2}^{n} D_{ii}^2+\frac{\partial}{\partial n}H^{(0)}$.

We apply Lemma 4 to $H^{(0)}V_\lambda^{(0)}h$ $(h\in C_0^\infty([0, T)\times\partial D))$. Then noting $H^{(0)}V_\lambda^{(0)}h(t, \tilde{x})=0$ for $\forall t\geqq T$, we have

$$(20)\qquad H^{(0)}V_\lambda^{(0)}h(0, x_0)=E\Big[\int_0^T e^{-\lambda\varphi_s}\sum_{i,j=2}^{n}\frac{1}{2}[\delta_{ij}-[\sigma\sigma^*]_{ij}(\pi_m(s), \tilde{x}_{\pi_m(s)})]D_{ij}^2 V_\lambda^{(0)}h(s, \tilde{x}_s^{(m)})d\varphi_s\Big]$$
$$+E\Big[\int_0^T e^{-\lambda\varphi_s}h(s, \tilde{x}_s^{(m)})d\varphi_s\Big]$$
$$+E\Big[\int_0^T e^{-\lambda\varphi_s}1_{(0,\infty)}(x_s^1)\sum_{i,j=1}^{n}\frac{1}{2}[\delta_{ij}-[\alpha\alpha^*]_{ij}(\pi_m(s), x_{\pi_m(s)})]D_{ij}^2 H^{(0)}V_\lambda^{(0)}h(s, x_s^{(m)})ds\Big].$$

Noting $\|D_{ij}^2 V_\lambda^{(0)}h\|_{\sim p}\leqq C_\lambda\|h\|_{\sim p}$, $\|D_{ij}^2 H^{(0)}V_\lambda^{(0)}h\|_p\leqq C_\lambda\|h\|_{\sim p}$, $|H^{(0)}V_\lambda^{(0)}h(0, x_0)|\leqq C_\lambda\|h\|_{\sim p}$, (where C_λ is uniformly bounded in $\lambda\geqq 1$) and Assumption (C–II) (cf. Appendix, Theorem A. 2), we can obtain that

$$(21)\qquad \|\mu_\lambda^{(m)}\|_T\leqq\frac{\varepsilon}{2}C_\lambda\|\mu_\lambda^{(m)}\|_T+\frac{\varepsilon}{2}C_\lambda\|\nu_{\lambda,T}^{(m)}\|+C_\lambda,$$

where $\|\mu_\lambda^{(m)}\|_T$ is the functional norm of $\mu_\lambda^{(m)}$ on $L^p([0, T)\times\partial D)$ and $\|\nu_{\lambda,T}^{(m)}\|$ is that on $L^p([0, T)\times D)$.

Let us denote by $G_0^{(0)}$ the minimal potential operator on D associated with $D_t+\sum_{i=1}^{n}\frac{1}{2}D_{ii}^2$.

Applying Lemma 4 to $u=G_0^{(0)}f$ $(f\in C_0^\infty([0, T)\times\bar{D}))$, we have

$$(22)\qquad G_0^{(0)}f(0, x_0)=E\Big[\int_0^T e^{-\lambda\varphi_s}1_{(0,\infty)}(x_s^1)f(s, x_s^{(m)})ds\Big]$$
$$+E\Big[\int_0^T e^{-\lambda\varphi_s}1_{(0,\infty)}(x_s^1)\sum_{i,j=1}^{n}\frac{1}{2}[\delta_{ij}-[\alpha\alpha^*]_{ij}(\pi_m(s), x_{\pi_m(s)})]D_{ij}^2 G_0^{(0)}f(s, x_s^{(m)})ds\Big]$$
$$-E\Big[\int_0^T e^{-\lambda\varphi_s}D_1 G_0^{(0)}f(s, \tilde{x}_s^{(m)})d\varphi_s\Big].$$

Note that $|G_0^{(0)}f(0, x)|\leqq C_T\|f\|_{p,T}$, $\|D_{ij}^2 G_0^{(0)}f\|_{p,T}\leqq C_T\|f\|_{p,T}$ and

$|D_1 G_0^{(0)} f(s, \tilde{x})| \leqq C_T \|f\|_{p,T}$ hold (cf. Appendix, Theorem A. 4).

Accordingly, we see

$$(23) \qquad \|\nu_{\lambda,T}^{(m)}\| \leqq \frac{\varepsilon}{2} C_T \|\nu_{\lambda,T}^{(m)}\| + 2C_T \qquad \forall \lambda \geqq 1.$$

Choose $\varepsilon > 0$ to be $\varepsilon C_{T_0} < 2$ and $\varepsilon C_\lambda < 2$ for $\forall \lambda \geqq 1$. Then there exist $\bar{C}_\lambda$ and $\bar{B}_{T_0}$ such that are independent of m and

$$(24) \qquad \|\mu_\lambda^{(m)}\|_{T_0} \leqq \bar{C}_\lambda, \quad \|\nu_{\lambda,T_0}^{(m)}\| \leqq \bar{B}_{T_0} \qquad (\lambda \geqq 1).$$

Now, remember $\alpha(t, x) = \{\delta_{ij}\}_{i,j=1,\cdots,n}$ $\sigma(t, \tilde{x}) = \{\delta_{ij}\}_{i,j=2,\cdots,n}$ for $t \geqq T_0$. Accordingly, repeating a similar argument as the proof of Lemma 5, we can obtain the following estimates.

$$(25) \qquad \left| E\left[\int_{T_0}^{\infty} e^{-\lambda\varphi_s} h(s, x_s^{(m)}) d\varphi_s\right]\right| \leqq \bar{\bar{C}}_\lambda \|h\|_{\sim p}$$

$$(26) \qquad \left| E\left[\int_{T_0}^{T} e^{-\lambda\varphi_s} 1_{(0,\infty)}(x_s^1) u(s, x_s^{(m)}) ds\right]\right| \leqq \bar{\bar{B}}_T \|u\|_{p,T}$$

for $\forall T > T_0$; fixed,

where $\bar{\bar{C}}_\lambda$, $\bar{\bar{B}}_T$ are independent of m and $\lambda \geqq 1$.

From (24), (25) and (26), we have

$$(27) \qquad |\mu_\lambda^{(m)}[h]| \leqq (\bar{C}_\lambda + \bar{\bar{C}}_\lambda) \|h\|_{\sim p}$$

$$(28) \qquad |\nu_{\lambda,T}^{(m)}[u]| \leqq (\bar{B}_{T_0} + \bar{\bar{B}}_T) \|h\|_{p,T}.$$

Noting (8), (4) in Theorem 2 follows immediately from (27) for $h \in C_0^\infty([0, \infty) \times \partial D)$.

On the other hand, from (8) and (28)

$$\left| E\left[\int_0^T e^{-\lambda\varphi_t} 1_{(0,\infty)}(x_t^1) u(t, x_t) dt\right]\right| \leqq (B_{T_0} + B_T) \|u\|_{p,T}.$$

Therefore, (5) in Theorem 2 is also obvious for $\forall u \in C_0^\infty([0, T) \times D)$, since $\varphi_t \leqq t$, a.s. (P). Moreover, it is easy to extend $C_0^\infty([0, \infty) \times \partial D)$ and $C_0^\infty([0, T) \times \bar{D})$ to $L^p([0, \infty) \times \partial D)$ and $L^p([0, T) \times \bar{D})$, respectively.

Let $A = D_t + \sum_{i,j=1}^{n} \frac{1}{2} [\alpha\alpha^*](t, x)_{ij} D_{ij}^2$ and denote by G^A the minimal (absorbing) potential operator associated with A and by H^A the harmonic extention operator associated with A (cf. Appendix).

For each $u\in L^p([0, T)\times\bar{D})$ ($T<+\infty$: arbitrarily fixed), $u=G^A f$ belongs to $W_{p,T}^{(1,2)}([0, T)\times\bar{D})$[6] and satisfies the following equation;

$$(29)\qquad \begin{cases} -Au=f \\ u|_{\partial D}=0, \end{cases}$$

as to H^A,

(30) $H^A h=H^{(0)}h+G^A D_\varepsilon H^{(0)}h$ holds for each $h\in C_0^{1,2}([0, \infty)\times\bar{D})$,

where $D_\varepsilon=\sum_{i,j=1}^{n}\frac{1}{2}([\alpha\alpha^*]_{ij}(t, x)-\delta_{ij})D_{ij}^2$.

Theorem 3. *Fix any $T<+\infty$. For each h of $C_0^{1,2}([0, \infty)\times\partial D)$ and $g\in L^p([0, T)\times\bar{D})$, put $u=G^A g+H^A h$. Then*

$$(31)\quad u(t, x_t) -\int_0^t 1_{(0,\infty)}(x_s^1)\Big(D_t+\frac{1}{2}\sum_{i,j=1}^{n}[\alpha\alpha^*]_{ij}(s, x_s)D_{ij}^2\Big)u(s, x_s)ds -\int_0^t\Big(D_t+\frac{1}{2}\sum_{i,j=2}^{n}[\sigma\sigma^*]_{ij}(s, \tilde{x}_s)D_{ij}^2+\frac{\partial}{\partial n}\Big)u(s, x_s)d\varphi_s$$

is a martingale.

Proof. It suffices to prove for $u=G^A g$, since $H^{(0)}h$ belongs to $C^{(1,2)}([0, \infty)\times\bar{D})$, by (30).

Since u belongs to $W_{p,T}^{(1,2)}([0, T)\times\bar{D})$ and $u|_{\partial D}=0$, it is not difficult to show that there is a sequence $\{u_n\}$ of $C^\infty([0, T)\times\bar{D})$ such that each u_n vanishes on ∂D and $\{u_n\}$ converges to u in $W_{p,T}^{(1,2)}([0, T)\times\bar{D})$.

Let us denote by $X_n(t)$, $X(t)$ the processes given by (31) corresponding to u_n, u respectively. It is sufficient to prove $E|X_n(t)-X(t)|\to 0$ as $n\to\infty$, since $X_n(t)$ is evidently a martingale by Ito's formula. Now,

$$E|X(t)-X_n(t)|\leq\sup_{t,x}|u(t, x)-u_n(t, x)| +E\Big[\int_0^t 1_{(0,\infty)}(x_s^1)\Big|\Big(D_s+\sum_{i,j=1}^{n}\frac{1}{2}[\alpha\alpha^*]_{ij}(s, x_s)D_{ij}^2\Big)(u-u_n)(s, x_s)\Big|ds\Big]$$

6) The definition of $W_{p,T}^{(1,2)}([0, T)\times\bar{D})$ appears in Appendix.

$$+E\left[\int_0^t\left|\frac{\partial}{\partial n}(u-u_n)(s,\tilde{x}_s)\right|d\varphi_s\right]=I_1+I_2+I_3\,.$$

Clearly I_1, I_3 converges to 0 as $n\to\infty$, because there exists a constant C_p such that

$$\sup_{t,x}|u(t,x)|\leqq C_p\|u\|_{p,T}^{(1,2)},\quad \sup_{t,\tilde{x}}\left|\frac{\partial}{\partial n}u(t,x)\right|\leqq C_p\|u\|_{p,T}^{(1,2)}$$

(cf. Appendix (A. 38) (A. 39)).

Theorem 2 implies $|I_2|\leqq C\|u-u_n\|_{p,T}^{(1,2)}$ for a constant C. Therefore, $E|X(t)-X_n(t)|\to 0$ as $n\to\infty$.

Let

$$L_0=D_t+\sum_{i=2}^{n}\frac{1}{2}D_{ii}^2+\frac{\partial}{\partial n}H^{(0)}$$

and

$$L=D_t+\sum_{i,j=2}^{n}\frac{1}{2}[\sigma\sigma^*]_{ij}(t,\tilde{x})D_{ij}^2+\frac{\partial}{\partial n}H^A\,.$$

Lemma 6. *There exists a unique solution v in $W_p^{(1,2)}([0,\infty)\times\partial D)$ of the following equation*;

(32) $\qquad (\lambda-L)v=h,$

for any h of $L^p([0,\infty)\times\partial D)$, $\lambda\geqq 1$. Moreover if we denote the solution by $V_\lambda h$, there exists a constant C_λ such that

(33) $\qquad \sup_{t,\tilde{x}}|V_\lambda h(t,\tilde{x})|\leqq C_\lambda\|h\|_{\sim p}\,.$

Proof.

$$\|(L-L_0)V_\lambda^{(0)}h\|_{\sim p}\leqq\sup_{t,\tilde{x}}\sum_{i,j=2}^{n}\frac{|[\sigma\sigma^*]_{ij}(t,\tilde{x})-\delta_{ij}|}{2}\sum_{i,j=2}^{n}\|D_{ij}^2V_\lambda^{(0)}h\|_{\sim p}$$

$$+\left\|\frac{\partial}{\partial n}[H^A-H^{(0)}]\,V_\lambda^{(0)}h\right\|_{\sim p}=I_1+I_2\,.$$

By Assumption (C-II) and Theorem A. 2 in Appendix, $I_1<\frac{\varepsilon}{2}C\|h\|_{\sim p}$

$$I_2=\left\|\frac{\partial}{\partial n}(H^A-H^{(0)})\,V_\lambda^{(0)}h\right\|_{\sim p}=\left\|\frac{\partial}{\partial n}G^A D_\varepsilon H^{(0)}\,V_\lambda^{(0)}h\right\|_{\sim p}\leqq\frac{\varepsilon}{2}\bar{C}\|h\|_{\sim p}$$

(by Theorems A. 1 and A. 3 in Appendix).

If we choose ε in Assumption (C–II) to be $\left(\frac{C+\bar{C}}{2}\right)<1$, then

$$V_\lambda = V_\lambda^{(0)} \sum_{n\geq 0} [(L-L_0)V_\lambda^{(0)}]^n \tag{34}$$

is well-defined as a bounded operator on $L^p([0,\infty)\times\partial D)$.

Noting that $v=V_\lambda^{(0)}h$ is unique solution in $W_p^{(1,2)}([0,\infty)\times\partial D)$ of $(\lambda-L_0)v=h$, we can show easily that $v=V_\lambda h$ is unique solution of (32).

(33) is immediate from (34) and Theorem A. 2 (A. 14) in Appendix.

Theorem 4. *Under Assumption (C–II), the uniqueness for the stochastic differential equation* (2) *is valid.*

Proof. Suppose that $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and $\mathfrak{X}'=(x'_t, B'_t, M'_t, \varphi'_t)$ on $(\Omega', \mathcal{F}', P'; \mathcal{F}'_t)$ are solutions of (2). Fix any $T>0$, let

$$h\in C_0^\infty([0,\infty)\times\partial D) \quad \text{and} \quad f\in C_0^\infty([0,T)\times\bar{D}).$$

Applying Theorem 3 to $H^A V_\lambda^{(0)}h$, we see that

$$H^A V_\lambda^{(0)}h(t,x_t)-\int_0^t LV_\lambda^{(0)}h(s,\tilde{x}(s))d\varphi_s$$

is a martingale. Taking expectation

$$H^A V_\lambda^{(0)}h(0,x_0)=E\left[\int_0^\infty e^{-\lambda\varphi_s}[I-(L-L_0)V_\lambda^{(0)}]h(s,\tilde{x}_s)d\varphi_s\right]. \tag{35}$$

Further

$$G^A f(t,x_t)+\int_0^t 1_{(0,\infty)}(x_s^1)f(s,x_s)ds-\int_0^t \frac{\partial}{\partial n}G^A f(s,\tilde{x}_s)d\varphi_s$$

is a martingale and so

$$\begin{aligned} G^A f(0,x_0) &= E\left[\int_0^T 1_{(0,\infty)}(x_s^1)f(s,x_s)ds\right] \\ &\quad -E\left[\int_0^T \frac{\partial}{\partial n}G^A f(s,\tilde{x}_s)d\varphi_s\right]. \end{aligned} \tag{36}$$

(35) implies $H^A V_\lambda^{(0)} h(0, x_0) = \mu_\lambda([I-(L-L_0)V_\lambda^{(0)}]h)$.

Since it is easy to check that $\{[I-(L-L_0)V_\lambda^{(0)}]h : h \in C_0^\infty([0,\infty) \times \partial D)\}$ is dense in $L^p([0,\infty)\times\partial D)$, and hence there exists a sequence $\{v_n = [I-(L-L_0)V_\lambda^{(0)}]h_n : h_n \in C_0^\infty([0,\infty)\times\partial D)\}$ which converges to a given v of $L^p([0,\infty)\times\partial D)$. Thus

$$H^A V_\lambda v_n(0, x_0) = H^A V_\lambda^{(0)} h_n(0, x_0) = \mu_\lambda[v_n].$$

According to Theorem 2 and Lemma 6, $H^A V_\lambda$ and μ_λ are bounded linear functional on $L^p([0,\infty)\times\partial D)$. So we see

$$H^A V_\lambda v(0, x_0) = \mu_\lambda[v] \qquad \forall v \in L^p([0,\infty)\times\partial D). \tag{37}$$

Therefore

$$E\left[\int_0^\infty e^{-\lambda\varphi_t} v(t, x_t) d\varphi_t\right] = E'\left[\int_0^\infty e^{-\lambda\varphi'_t} v(t, x'_t) d\varphi'_t\right] \tag{38}$$
$$\forall v \in L^p([0,\infty)\times\partial D), \qquad \forall \lambda \geqq 1.$$

However, since both sides of (38) are analytic in $\lambda > 0$, (38) holds for each $\lambda > 0$. Hence

$$E\left[\int_0^T v(s, \tilde{x}_s) d\varphi_s\right] = E'\left[\int_0^T v(s, \tilde{x}'_s) d\varphi'_s\right] \tag{39}$$
$$\text{for } \forall v \in C_b{}^{7)} \cap L^p([0,\infty)\times\partial D) \text{ and } \forall T > 0.$$

From (36) and (39), we conclude

$$E\left[\int_0^T v(s, x_s) ds\right] = E'\left[\int_0^T v(s, x'_s) ds\right] \tag{40}$$
$$\text{for } \forall v \in C_b \cap L^p([0,\infty)\times\bar{D}).$$

Now, we can complete the proof of Theorem 4 by standard arguments.

Theorem 4'. *Suppose that $\alpha(t, x)$ and $\sigma(t, x)$ satisfy the following conditions.*

(a) *There is a constant strictly positive definite $(n\times n)$ matrix a such that*

$$\sum_{i,j=1}^n |[\alpha\alpha^*]_{ij}(t, x) - a_{ij}| < \varepsilon.$$

7) C_b is the class of bounded continuous functions.

(b) *There is a constant strictly positive definite* $(n-1\times n-1)$ *matrix* τ *such that*

$$\sum_{i,j=2}^{n} |[\sigma\sigma^*]_{ij}(t, \tilde{x}) - \tau_{ij}| < \varepsilon.$$

(c) $\alpha(t, x)$ *and* $\sigma(t, \tilde{x})$ *are continuous in* $(t, x) \in [0, \infty) \times \overline{D}$ *and* $(t, \tilde{x}) \in [0, \infty) \times \partial D$ *respectively.*

Moreover there is a constant $0 < T_0 < +\infty$ *such that*

$$\alpha\alpha^*(t, x) = a \qquad \text{for } \forall t \geqq T_0$$

$$\sigma\sigma^*(t, \tilde{x}) = \tau \qquad \text{for } \forall t \geqq T_0 .$$

Then, if we choose ε *sufficiently small, the uniqueness of the solution of the stochastic differential equation* (2) *is valid.*

The proof is essentially same as the proof of Theorem 4, if we use some estimates of Theorem A. 3 in Appendix.

Theorem 5. *Suppose that* α *and* σ *satisfy Assumption* (C-I). *Let* b *and* γ *be bounded measurable functions on* $\overline{D}$ *and* ∂D, *respectively and* $\rho \equiv 1$. *Then the uniqueness of the solution of* (1) *is valid.*

Proof. First we shall prove this theorem in the case that $b \equiv 0$, $\gamma \equiv 0$ and $\alpha_{11} \equiv 1$ $\alpha_{1i} \equiv 0$ $(i \neq 1)$. Fix point x_0 of $\overline{D}$.

Let $U_n(x_0) = \{x \in \overline{D};\ |x - x_0| < n\}$ and

$$\tau_n = \inf\{t \geqq 0 : x_t \in U_n(x_0)\} \wedge n.$$

For any $\tilde{x} \in U_n(x_0) \cap \partial D$ and any $\varepsilon > 0$, there is a neighborhood of $\tilde{x}$;

$$N(\tilde{x}) = \{y \in \overline{D};\ |y - \tilde{x}| < \delta\} \qquad \exists \delta > 0,$$

such that

$$\sum_{i,j=1}^{n} |[\alpha\alpha^*]_{ij}(y) - [\alpha\alpha^*]_{ij}(\tilde{x})| < \varepsilon \quad \text{for } \forall y \in N(\tilde{x})$$

and

$$\sum_{i,j=2}^{n} |[\sigma\sigma^*]_{ij}(y) - [\sigma\sigma^*]_{ij}(\tilde{x})| < \varepsilon \qquad \text{for } \forall y \in N(\tilde{x}) \cap \partial D.$$

Theorem 4′ implies that the stochastic differential equation (1) has the unique solution up to the first exit time from $N(x)$ or n, if we choose ε sufficiently small.

On the other hand it is known that the uniqueness of (1) is valid up to the first hitting time to the boundary, if the solution starts at an interior point.

$U_n(x_0)\cap\partial D$ can be covered with finite number of $N(x)$. Therefore, by the standard arguments and taking regular conditional probabilities, we can prove that the uniqueness of (1) is valid up to τ_n.

Accordingly, in order to complete the proof of this case, it suffices to note $\lim\tau_n=\infty$ a.s. (P). But it follows easily from the boundedness of α and σ.

Now we can extend to the general case, when $\alpha_{11}\equiv 1$, $\alpha_{1i}\equiv 0$ $(i\neq 1)$ is not assumed, by means of a transformation of Brownian motion and a time change (cf. Watanabe [9]).

Next, suppose that $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ is a solution of (1) with $[\alpha, b, \sigma, \gamma, \rho\equiv 1]$.

Let $\widetilde{P}$ be the probability measure on $(\Omega, \mathcal{F})$ such that for each t and each $B\in\mathcal{F}_t$,

$$\widetilde{P}(B)=\int_B \exp\Big[-\int_0^t \alpha^{-1}\cdot b(x_s)1_D(x_s)dB_s-\frac{1}{2}\int_0^t|\alpha^{-1}\cdot b|^2(x_s)1_D(x_s)ds \\ -\int_0^t\sigma^{-1}\cdot\gamma(x_s)dM_s-\frac{1}{2}\int_0^t|\sigma^{-1}\cdot\gamma|^2(\tilde{x}_s)d\varphi_s\Big]dP(\omega).$$

Then

$$\widetilde{\mathfrak{X}}=\Big[x_t,\ \widetilde{B}_t=B_t+\int_0^t\alpha^{-1}\cdot b(x_s)1_D(x_s)ds,\ \widetilde{M}_t=M_t+\int_0^t\sigma^{-1}\cdot\gamma(x_s)d\varphi_s,\ \varphi_t\Big]$$

is a solution on $(\Omega, \mathcal{F}, \widetilde{P}; \mathcal{F}_t)$ which corresponds to $[\alpha, b\equiv 0, \sigma, \gamma\equiv 0, \rho\equiv 1]$. Therefore, it is reduced to the previous case.

Now we can complete the proof of Theorem 1 as follows.
The uniqueness of (1) with $[\alpha, b, \sigma, \gamma, \rho\equiv 0]$ is reduced to the case $[\alpha, b, \sigma, \gamma, \rho\equiv 1]$, and the general case $[\alpha, b, \sigma, \gamma, \rho]$ is also reduced to the case $[\alpha, b, \sigma, \gamma, \rho\equiv 0]$ by time changes, cf. Watanabe [10].

Remark 2. The uniqueness implies that the solution x_t of the equation (1) defines a unique diffusion process (a strong Markov process with continuous paths) on $\bar{D}$.

It is the diffusion process, whose infinitesimal generator is given, roughly speaking, by the differential operator A with the domain characterized by $Lf=\rho\cdot Af$ on ∂D, where

$$A=\sum_{i,j=1}^{n}\frac{1}{2}[\alpha\alpha^*]_{ij}(x)D_{ij}^2+\sum_{i=1}^{n}b_i(x)D_i$$

$$L=\sum_{i,j=2}^{n}\frac{1}{2}[\sigma\sigma^*]_{ij}(\tilde{x})D_{ij}^2+\sum_{i=2}^{n}\gamma_i(\tilde{x})D_i+\frac{\partial}{\partial n}.$$

Remark 3. If the coefficients of the stochastic differential equations (1) are time-dependent, our problem remains open.

Appendix

Let $D=\{x\in R^n;\ x_1>0\}$. Given $u\in C_0^\infty([0,T)\times\bar{D})$, define the following norm.

$$\|u\|_{p,T}^{(1,2)}=\|u\|_{p,T}+\|D_t u\|_{p,T}+\sum_{|\alpha|\leq 2}\|D_x^\alpha u\|_{p,T},$$

where $\|\cdot\|_{p,T}$ is the ordinary L_p-norm on $[0,T)\times\bar{D}$. Denote by $W_p^{(1,2)}([0,T)\times\bar{D})$ the completion of $C_0^\infty([0,T)\times\bar{D})$ with respect to $\|\cdot\|_{p,T}^{(1,2)}$.

If $T=\infty$, we drop the subscript T. In order to avoid confusion, we use $\|\cdot\|_{\sim p}$ to denote ordinary L_p norm on $[0,\infty)\times\partial D$. $W_p^{(1,2)}[(0,T)\times\partial D)$ is defined in an analogous way. Define

$$h_{x_1}^{(0)}(t,\tilde{y})=1_{\{t\geq 0\}}\frac{x_1}{t(2\pi t)^{n/2}}\exp\{-(x_1^2+|\tilde{y}|^2)/2t\}\qquad x_1>0,\ y\in D,$$

where $\tilde{y}$ is the projection of y on ∂D, and let

(A. 1) $$H^{(0)}f(s,x)=H_{x_1}^{(0)}f(s,\tilde{x})=\int_s^\infty dt\int_{\partial D}d\tilde{y}\,h_{x_1}^{(0)}(t-s,\tilde{x}-\tilde{y})f(t,\tilde{y}).$$

Then $H^{(0)}f$ is the space-time harmonic extension for $\left(\frac{\partial}{\partial s}+\frac{1}{2}\Delta\right)$ of f in $[0,\infty)\times\partial D$.

It is easily checked that $\{H_{x_1}^{(0)}\}_{x_1>0}$ is a semi-group. Let $\frac{\partial}{\partial n}H^{(0)}$ denote its generator. Then

$$\widehat{\left(\frac{\partial}{\partial n}H^{(0)}\right)f}(\mu, \tilde{\theta}) = -(2i\mu + |\tilde{\theta}|^2)^{1/2}\hat{f}(\mu, \tilde{\theta}), \tag{A.2}$$

where the symbol "$\wedge$" denotes the Fourier transformation. Denoting by $R_\lambda^{(0)}$ the resolvent operators $\left(\lambda - \frac{\partial}{\partial n}H^{(0)}\right)^{-1}$ associated with $\{H_{x_1}^{(0)}\}_{x_1>0}$;

$$R_\lambda^{(0)} = \int_0^\infty e^{-\lambda x_1} H_{x_1}^{(0)} dx_1 . \tag{A.3}$$

We know the following estimates about $R_\lambda^{(0)}$ (cf. Stroock-Varadhan [8])

$$\|R_\lambda^{(0)} f\|_{\sim p} \leqq \frac{1}{\lambda} \|f\|_{\sim p} \qquad 1 \leqq p \leqq \infty \tag{A.4}$$

$$\sum_{j=2}^{n} \|D_j R_\lambda^{(0)} f\|_{\sim p} \leqq C\|f\|_{\sim p} \qquad 1 < p < \infty, \tag{A.5}$$

where the constant C is independent of $\lambda \geqq 0$.

Next we introduce the following L_p-type norms on $[0, \infty) \times \partial D$.

$$\langle\!\langle f \rangle\!\rangle_{\sim p, x} = \left[\int_0^\infty dt \int_{\partial D}\int_{\partial D} \frac{|f(t, x) - f(t, y)|^p}{|x-y|^{n+p-2}} dxdy\right]^{1/p}$$

$$\langle\!\langle f \rangle\!\rangle_{\sim p, t} = \left[\int_{\partial D} dx \int_0^\infty\int_0^\infty \frac{|f(t, x) - f(s, x)|^p}{|t-s|^{(p+1)/2}} dtds\right]^{1/p}$$

$$|\!|\!| f |\!|\!|_{\sim p} = \|f\|_{\sim p} + \langle\!\langle f \rangle\!\rangle_{\sim p, x} + \langle\!\langle f \rangle\!\rangle_{\sim p, t} .$$

Lemma A.1. *Let (X, m) be a σ-finite measure space. Fix α; $l < \alpha < p + l$. For each function $f(z: x)$ of $L^p[X \times R^l; dmdx]$ such that $D_j f \in L^p[X \times R^l: dmdx]$, the following inequality holds.*

$$\left[\int_X m(dz) \int_{R^l}\int_{R^l} \frac{|f(z: x) - f(z: y)|^p}{|x-y|^\alpha} dxdy\right]^{1/p} \tag{A.6}$$
$$\leqq C_1(\varepsilon) \sum_{j=1}^{l} \|D_j f\|_{L^p[X\times R^l]} + C_2(\varepsilon) \|f\|_{L^p[X\times R^l]} ,$$

where $C_1(\varepsilon)$ can be chosen arbitrarily small and $C_2(\varepsilon)$ depends on $C_1(\varepsilon)$.

Proof. Let us put

$$I_1=\int_X m(dz)\iint_{|x-y|>\varepsilon}\frac{|f(z:x)-f(z:y)|^p}{|x-y|^\alpha}dxdy$$

$$I_2=\int_X m(dz)\iint_{|x-y|<\varepsilon}\frac{|f(z:x)-f(z:y)|^p}{|x-y|^\alpha}dxdy$$

for any $\varepsilon>0$.

Noting

$$|f(z:x)-f(z:y)|^p\leqq C_p(|f(z:x)|^p+|f(x:y)|^p)$$

we can observe $I_1\leqq C\cdot\varepsilon^{l-\alpha}\|f\|^p_{L^p[X\times R^l]}$.

Next we will estimate I_2;

$$|f(z:x)-f(z:y)|^p=\left|\sum_{j=1}^l\int_{y_j}^{x_j}D_jf(z:x_1,\cdots,x_{j-1},u,y_{j+1},\cdots,y_l)du\right|^p$$

$$\leqq\sum_{j=1}^l|x_j-y_j|^{p-1}\left|\int_{y_j}^{x_j}|D_jf(z:x_1,\cdots,x_{j-1},u,y_{j+1},\cdots,y_l)|^p du\right|$$

$$I_2\leqq\sum_{j=1}^l\int_X m(dz)\iint_{|x-y|<\varepsilon}dxdy\frac{|x_j-y_j|^{p-1}}{|x-y|^\alpha}\left|\int_{y_j}^{x_j}|D_jf(z:x_1,\cdots,u,y_{j+1},\cdots,y_l)|^p du\right|$$

$$\leqq\sum_{j=1}^l\int_X m(dz)\iint_{|\tilde{x}-\tilde{y}|<\varepsilon}dxdy\int_{|v|<\varepsilon}dv\frac{|v|^{p-1}}{(|x-y|^2+|v|^2)^{\alpha/2}}$$

$$\int_{R^1}dy_j\left|\int_{y_j}^{y_j+v}|D_jf(z:x_1,\cdots,u,\cdots,y_l)|^p du\right|$$

$$\leqq\sum_{j=1}^l\int_X\int_{R^l}|D_jf(z:x)|^p m(dz)dx\int_{|\tilde{u}|<\varepsilon,|v|<\varepsilon}dvdu\frac{|v|^p}{(|u|^2+|v|^2)^{\alpha/2}}$$

$$\leqq C\cdot\varepsilon^{p+l-\alpha}\sum_{j=1}^l\|D_jf\|^p_{L^p[X\times R^l]}.$$

Therefore, we can put $C_1(\varepsilon)=C^{1/p}\cdot\varepsilon^{(p+l-\alpha)/p}$, $C_2(\varepsilon)=C^{1/p}\cdot\varepsilon^{(l-\alpha)/p}$.

Theorem A. 1.

(A. 7) $$\sum_{i,j=1}^n\|D^2_{ij}H^{(0)}f\|_p\leqq C\|f\|^{(1,2)}_{\sim p}\qquad p>1,$$

where $$\|f\|^{(1,2)}_{\sim p}=\|f\|_{\sim p}+\sum_{i=2}^n\|D_if\|_{\sim p}+\sum_{i,j=2}^n\|D^2_{ij}f\|_{\sim p}+\|D_tf\|_{\sim p}.$$

Proof. The following estimate is known (cf. Ladyzhenskaya & others [5], Chap. IV).

(A. 8) $$\sum_{i,j=1}^{n}\|D_{ij}^{2}H^{(0)}f\|_{p}\leqq C\Big(\sum_{i=2}^{n}\langle\!\langle D_{i}f\rangle\!\rangle_{\sim p,x}+\Big[\int_{\partial D}dx\int_{0}^{\infty}\int_{0}^{\infty}\frac{|f(t,x)-f(s,x)|^{p}}{|t-s|^{1/2+p}}dtds\Big]^{1/p}\Big).$$

Noting

$$\sum_{i=2}^{n}\|D_{i}f\|_{\sim p}\leqq C[\|f\|_{\sim p}+\sum_{i,j=2}^{n}\|D_{ij}^{2}f\|_{\sim p}],$$

(A. 7) is immediate from (A. 6) and (A. 8).

Lemma A. 2. *If* $2<p<\infty$ *and* $\lambda\geqq 1$,

(A. 9) $$\|D_{t}R_{\lambda}^{(0)}f\|_{\sim p}\leqq C^{1}(\varepsilon)\|D_{t}f\|_{\sim p}+C^{2}(\varepsilon)\|f\|_{\sim p},$$

where constant $C^{1}(\varepsilon)$ *can be chosen arbitrary small and* $C^{2}(\varepsilon)$ *depends on* $C^{1}(\varepsilon)$.

Proof. By the analogous calculation to (A. 8),

(A. 10) $$\|D_{t}R_{\lambda}^{(0)}f\|_{\sim p}\leqq C_{\lambda}\Big[\int_{\partial D}dx\int_{0}^{\infty}\int_{0}^{\infty}\frac{|f(t,x)-f(s,x)|^{p}}{|t-s|^{p+1/2}}dtds\Big]^{1/p},$$

where C_{λ} is bounded in $\lambda\geqq 1$. Therefore (A. 9) follows from (A. 10) and Lemma A. 1.

Lemma A. 3. If $2<p<\infty$,

(A. 11) $$\Big\|\frac{\partial}{\partial n}H^{(0)}f\Big\|_{\sim p}\leqq C_{1}(\varepsilon)\|D_{t}f\|_{\sim p}+C\sum_{j=2}^{n}\|D_{j}f\|_{\sim p}+C_{2}(\varepsilon)\|f\|_{\sim p},$$

where $C_{1}(\varepsilon)$ *can be chosen arbitrarily small and* $C_{2}(\varepsilon)$ *depends on* $C_{1}(\varepsilon)$.

Proof. From (A. 2), we have

$$\widehat{\Big(\frac{\partial}{\partial n}H^{(0)}\Big)^{2}f}(\mu,\tilde{\theta})=(2i\mu+|\tilde{\theta}|^{2})\cdot\hat{f}(\mu,\tilde{\theta})$$

$$\Big\|\Big(\frac{\partial}{\partial n}H^{(0)}\Big)^{2}f\Big\|_{\sim p}\leqq C(\|D_{t}f\|_{\sim p}+\sum_{i,j=2}^{n}\|D_{ij}^{2}f\|_{\sim p}).$$

Noting $\Big(I-\frac{\partial}{\partial n}H^{(0)}\Big)R_{1}^{(0)}f=f$,

$$\left\|\left(\frac{\partial}{\partial n}H^{(0)}\right)f\right\|_{\sim p}\leqq\left\|R_1^{(0)}f-f-\left(\frac{\partial}{\partial n}H^{(0)}\right)f\right\|_{\sim p}+\|R_1^{(0)}f-f\|_{\sim p}$$
$$=\left\|\left(\frac{\partial}{\partial n}H^{(0)}\right)(R_1^{(0)}f-f)\right\|_{\sim p}+\|R_1^{(0)}f-f\|_{\sim p}$$
$$=\left\|\left(\frac{\partial}{\partial n}H^{(0)}\right)^2R_1^{(0)}f\right\|_{\sim p}+\|R_1^{(0)}f-f\|_{\sim p}$$
$$\leqq C(\|D_tR_1^{(0)}f\|_{\sim p}+\sum_{i,j=2}^{n}\|D_{ij}^2R_1^{(0)}f\|_{\sim p})+\|R_1^{(0)}f-f\|_{\sim p}.$$

Hence it is evident from (A. 5), (A. 9) and $D_{ij}R_1^{(0)}f=D_iR_1^{(0)}D_jf$.

Let $L_0=\frac{\partial}{\partial t}+\frac{\tilde{\Delta}}{2}+\frac{\partial}{\partial n}H^{(0)}$, where $\tilde{\Delta}$ denotes Laplace operator on $\partial D=R^{n-1}$, and denote by $\{V_\lambda^{(0)}\}$ resolvent operators $(\lambda-L_0)^{-1}$.

By Lemma A. 3 we can obtain the following essential estimates.

Theorem A. 2. *$\{V_\lambda^{(0)}\}_{\lambda>0}$ is resolvent operators on $L^p([0,\infty)\times\partial D)$ and the following estimates hold.*

(A. 12) $\|D_tV_\lambda^{(0)}f\|_{\sim p}\leqq C\|f\|_{\sim p}\qquad 2<p<\infty$

(A. 13) $\sum_{i,j=2}^{n}\|D_{ij}^2V_\lambda^{(0)}f\|_{\sim p}\leqq C\|f\|_{\sim p}.$

Furthe if $p>(n-1)/2$,

(A. 14) $\sup_{(t,\tilde{x})\in[0,\infty)\times\partial D}|V_\lambda^{(0)}f(t,\tilde{x})|\leqq C\|f\|_{\sim p},$

where above constants are independent of $\lambda\geqq1$.

Proof. It is easy to check that $V_\lambda^{(0)}$ is a convolution operator. Therefore, $V_\lambda^{(0)}$ is a bounded operator on $L^p([0,\infty)\times\partial D)$ with norm $1/\lambda$.

Let $B_\lambda=\left(\lambda-\frac{\partial}{\partial t}-\frac{\tilde{\Delta}}{2}\right)^{-1}$. Then the following estimates are well-known (cf. Stroock-Varadhan [8]).

(A. 15) $\|D_tB_\lambda f\|_{\sim p}\leqq C\|f\|_{\sim p}$

(A. 16) $\sum_{i,j=2}^{n}\|D_{ij}^2B_\lambda f\|_{\sim p}\leqq C\|f\|_{\sim p}.$

Noting $\left(\lambda-\frac{\partial}{\partial t}-\frac{\tilde{\Delta}}{2}\right)V_{\lambda}^{(0)}f=\frac{\partial}{\partial n}H^{(0)}V_{\lambda}^{(0)}f+f$, we have

$$V_{\lambda}^{(0)}f=B_{\lambda}\left(\frac{\partial}{\partial n}H^{(0)}\right)V_{\lambda}^{(0)}f+B_{\lambda}f. \tag{A. 17}$$

Note the following well-known fact.

$$\sum_{i=2}^{n}\|D_{i}f\|_{\sim p}\leqq C_{1}(\varepsilon)\sum_{i,j=2}^{n}\|D_{ij}^{2}f\|_{\sim p}+C_{2}(\varepsilon)\|f\|_{\sim p}, \tag{A. 18}$$

where $C_1(\varepsilon)$ can be chosen arbitrarily small.

Making use of Lemma A. 3, (A. 15) and (A. 18), we obtain the following two inequalities.

$$\|D_{t}V_{\lambda}^{(0)}f\|_{\sim p}\leqq C_{1}(\varepsilon)\left[\|D_{t}V_{\lambda}^{(0)}f\|_{\sim p}+\sum_{i,j=2}^{n}\|D_{ij}^{2}V_{\lambda}^{(0)}f\|_{\sim p}\right]+C_{2}(\varepsilon)\|f\|_{\sim p} \tag{A. 19}$$

$$\sum_{i,j=2}^{n}\|D_{ij}^{2}V_{\lambda}^{(0)}f\|_{\sim p}\leqq C_{1}(\varepsilon)\left[\|D_{t}V_{\lambda}^{(0)}f\|_{\sim p}+\sum_{i,j=2}^{n}\|D_{ij}^{2}V_{\lambda}^{(0)}f\|_{\sim p}\right]+C_{2}(\varepsilon)\|f\|_{\sim p} \quad \text{for } \lambda\geqq 1,\ 2<p<\infty. \tag{A. 20}$$

We remark that $\|D_{t}V_{\lambda}^{(0)}f\|_{\sim p}<\infty$, $\|D_{ij}^{2}V_{\lambda}^{(0)}f\|_{\sim p}<+\infty$ for each $f\in C_{0}^{\infty}([0,\infty)\times\partial D)$, because $V_{\lambda}^{(0)}$ is a convolution operator, and so, $D_{t}V_{\lambda}^{(0)}f=V_{\lambda}^{(0)}D_{t}f$, $D_{ij}^{2}V_{\lambda}^{(0)}f=V_{\lambda}^{(0)}D_{ij}^{2}f$.

Moreover we may assume $C_1(\varepsilon)<1/2$. Therefore (A. 12), (A. 13) are immediate from (A. 19) and (A. 20). (A. 14) is clear by Sobolev's lemma.

Let $M[\mu_1;\mu_2]$ be the collection of all $n\times n$ matrices which satisfy

i) $\alpha_{11}=1$ $\alpha_{1j}=0$ $(j\neq 1)$

ii) $\mu_1|\xi|^2\leqq\langle[\alpha\alpha^*]\xi,\xi\rangle\leqq\mu_2|\xi|^2$ for any ξ of R^n $(\mu_1>0,\mu_2>0)$

and $\widetilde{M}[\mu_1;\mu_2]$ be the collection of all $(n-1)\times(n-1)$ matrices which satisfy

ii)′ $\mu_1|\tilde{\xi}|^2\leqq\langle[\sigma\sigma^*]\tilde{\xi},\tilde{\xi}\rangle\leqq\mu_2|\tilde{\xi}|^2$ for any $\tilde{\xi}$ of R^{n-1},

where an element of $M[\mu_1;\mu_2]$ (resp. $\widetilde{M}[\mu_1;\mu_2]$) is denoted by $\alpha=(\alpha_{ij})_{i,j=1,\cdots,n}$ (resp. $\sigma=(\sigma_{ij})_{i,j=2,\cdots,n}$).

Here, we note that for every positive definite matrix a, there exists a matrix α such that $\alpha_{1i}=0$ $(i\neq 1)$, $\alpha\alpha^*=a$.

For each $\alpha \in M[\mu_1; \mu_2]$, denote by $H^{[\alpha]}$ the space time harmonic extension operator corresponding to $D_t + \sum_{i,j=1}^{n} \frac{1}{2} [\alpha\alpha^*]_{ij} D_{ij}^2$ and for each α of $M[\mu_1; \mu_2]$ and σ of $\widetilde{M}[\mu_1; \mu_2]$, denote by $V_\lambda^{[\alpha,\sigma]}$ the resolvent operator associated with $D_t + \sum_{i,j=2}^{n} \frac{1}{2} [\sigma\sigma^*]_{ij} D_{ij}^2 + \frac{\partial}{\partial n} H^{[\alpha]}$.

Theorem A. 3. *$\{V_\lambda^{[\alpha,\sigma]}\}_{\lambda>0}$ is resolvent operators on $L^p([0,\infty) \times \partial D)$ and the following estimates hold.*

(A. 21) $\|D_t V_\lambda^{[\alpha,\sigma]} f\|_{\sim p} \leqq C \|f\|_{\sim p}$ $\quad 2 < p < +\infty$

(A. 22) $\sum_{i,j=2}^{n} \|D_{ij}^2 V_\lambda^{[\alpha,\sigma]} f\|_{\sim p} \leqq C \|f\|_{\sim p}$.

Further if $p > (n-1)/2$,

(A. 23) $\sup_{(t,\tilde{x}) \in [0,\infty) \times \partial D} | V_\lambda^{[\alpha,\sigma]} f(t, \tilde{x}) | \leqq C \|f\|_{\sim p}$,

where the above constants can be taken independent of $\lambda \geqq 1$ and $\alpha \in M[\mu_1; \mu_2]$, $\sigma \in \widetilde{M}[\mu_1; \mu_2]$.

Proof. Let $\{B_t\}$ be a n-dim. Brownian motion on $(\Omega, \mathcal{F}, P)$ such that $B_{t_0} = 0$ and $\tau = \inf\{t \geqq t_0, x_1 + B_t^1 = 0\}$. Then $H^{[\alpha]}$ can be represented in the following form;

$$H^{[\alpha]} f(t_0, x) = Ef(\tau, \tilde{x} + \widetilde{\alpha \cdot B_\tau})$$
$$= Ef(\tau, \tilde{x} + \tilde{\alpha} \cdot \widetilde{B}_\tau - x_1 \cdot b),$$

where $\tilde{\alpha} = \{\alpha_{ij}\}_{i,j=2,\cdots,n}$ $b = (\alpha_{21}, \cdots, \alpha_{n1})$,

(A. 24) $H^{[\alpha]} f(t_0, (x_1, \tilde{x}))$

$$= \int_{\partial D} \int_{t_0}^{\infty} h_{x_1}^{(0)}(t - t_0, \tilde{y}) f(t, x + \tilde{\alpha} \cdot \tilde{y} - x_1 \cdot b) dt d\tilde{y}.$$

Accordingly, the following formula can be proved easily.

(A. 25) $\frac{\partial}{\partial n} H^{[\alpha]} f(t_0, \tilde{x}) = \frac{\partial}{\partial n} H^{(0)} f^{(\alpha)}(t_0, \tilde{\alpha}^{-1} x) - \sum_{i=2}^{n} \alpha_{i1} D_i f(t, x)$,

where $f^{(\alpha)}$ is defined as $f^{(\alpha)}(t, y) = f(t, \tilde{\alpha} \cdot \tilde{y})$.

Therefore, according to Lemma A. 3, we can see

(A. 26) $$\left\|\frac{\partial}{\partial n}H^{[\alpha]}f\right\|_{\sim p}\leqq C_1(\varepsilon)\|D_t f\|_{\sim p}$$
$$+[C\sum_{i,j=2}^{n}|\alpha_{ij}|+\sum_{i=2}^{n}|\alpha_{i1}|]\sum_{i=2}^{n}\|D_i f\|_{\sim p}+C_2(\varepsilon)\|f\|_{\sim p}.$$

It is easy to check that there exists a constant $C(\mu_1, \mu_2)$ such that

$$\sum_{i,j=1}^{n}|\alpha_{ij}|\leqq C(\mu_1, \mu_2) \quad \text{for } \forall\alpha\in M[\mu_1; \mu_2]$$

$$\sum_{i,j=2}^{n}|\sigma_{ij}^{-1}|\leqq C(\mu_1, \mu_2) \quad \text{for } \forall\sigma\in\widetilde{M}[\mu_1; \mu_2].$$

Thus, from (A. 26), we have

(A. 27) $$\left\|\frac{\partial}{\partial n}H^{[\alpha]}f\right\|_{\sim p}\leqq C_1(\varepsilon)\|D_t f\|_{\sim p}$$
$$+C(\mu_1, \mu_2)\sum_{i=2}^{n}\|D_i f\|_{\sim p}+C_2(\varepsilon)\|f\|_{\sim p}.$$

Moreover by an analogous consideration for

$$B_\lambda^{[\alpha]}=\left[\lambda-D_t-\sum_{i=2}^{n}\frac{1}{2}[\sigma\sigma^*]_{ij}D_{ij}^2\right]^{-1},$$

we can obtain

(A. 28) $$\|D_t B_\lambda^{[\sigma]}f\|_{\sim p}\leqq C(\mu_1, \mu_2)\|f\|_{\sim p}$$

(A. 29) $$\sum_{i,j=2}^{n}\|D_{ij}^2 B_\lambda^{[\sigma]}f\|_{\sim p}\leqq C(\mu_1, \mu_2)\|f\|_{\sim p} \quad \text{for } \forall\lambda\geqq 1.$$

Remaining part of the proof is same as in Theorem A. 2.

Let $\alpha(t, x)=\{\alpha_{ij}(t, x)\}_{i,j=1,\cdots,n}$ be a $n\times n$ matrix which is continuous in $(t, x)\in[0, \infty)\times\overline{D}$ and satisfies

(A. 30) $$\sum_{i,j=1}^{n}|[\alpha\alpha^*]_{ij}(t, x)-\delta_{ij}|<\varepsilon \quad \text{and} \quad \alpha_{ij}(t, x)=\delta_{ij} \quad \forall t\geqq{}^{\exists}T_0.$$

If we choose ε sufficiently small, the minimal (absorbing) potential operator G^A associated with $A=D_t+\sum_{i,j=1}^{n}\frac{1}{2}[\alpha\alpha^*]_{ij}(t, x)D_{ij}^2$ is well-defined as a bounded operator which maps $L^p([0, T)\times\overline{D})$ into $W_{p,T}^{(1,2)}([0, T)\times D)$, where $T<+\infty$ (cf. Stroock-Varadhan [8]; Appendix).

Moreover let us denote by H^A the harmonic extention operator and by G_λ^A the resolvent operator associated with A.

In the following theorem, we summarize some important facts about G^A, H^A and G_λ^A.

Theorem A. 4. *If* $p>n+2$,

(A. 31) $\|D_{ij}^2 G^A f\|_{p,T} \leqq C_T \|f\|_{p,T}$

(A. 32) $\sup_{(t,\tilde{x})\in[0,T)\times\partial D} |D_1 G^A f(t, \tilde{x})| \leqq C_T \|f\|_{p,T}$

(A. 33) $\sup_{(t,x)\in[0,T)\times\bar{D}} |G^A f(t, x)| \leqq C_T \|f\|_{p,T}$

(A. 34) $\|D_{ij}^2 G_\lambda^A f(t, x)\|_p \leqq C_\lambda \|f\|_p$

(A. 35) $\sup_{(t,\tilde{x})\in[0,\infty)\times\partial D} |D_1 G_\lambda^A f(t, x)| \leqq C_\lambda \|f\|_p$

(A. 36) $\sup_{(t,x)\in[0,\infty)\times\bar{D}} |G_\lambda^A f(t, x)| \leqq C_\lambda \|f\|_p$

(A. 37) $H^A g(t_0, x_0) = E_{(t_0,x_0)}[g(\tau^{t_0}, x_{\tau^{t_0}})]$

for $\forall g \in W_p^{(1,2)}([0, \infty)\times\partial D)$,

where C_λ *is uniformly bounded in* $\lambda \geqq 1$ *and further* $(P_{(t_0,x_0)}, x_t)$ *is the A-diffusion staring at* (t_0, x_0) *and* $\tau^{t_0} = \inf\{t \geqq t_0 \colon x_t \in \partial D\}$.

Proof. Let $p>n+2$.

The following estimate is immediate from Sobolev's lemma and the trace theory.

(A. 38) $\sup_{(t,\tilde{x})\in[0,\infty)\times\partial D} |D_1 u(t, \tilde{x})| \leqq C \|u\|_p^{(1,2)}$

(A. 39) $\sup_{(t,x)\in[0,\infty)\times\bar{D}} |u(t, x)| \leqq C \|u\|_p^{(1,2)}$.

Therefore, (A. 32), (A. 33), (A. 35) and (A. 36) are obvious. For any g of $C_0^\infty([0, \infty)\times\partial D)$, (A. 37) holds (Stroock-Varadhan [8] Appendix). Note

$$H^A g = H^{(0)} g + G^A D_\varepsilon H^{(0)} g,$$

where

$$D_\varepsilon = \frac{1}{2} \sum_{i,j=1}^{n} ([\alpha\alpha^*]_{ij}(t, x) - \delta_{ij}) D_{ij}^2 .$$

For $\forall g \in W_p^{(1,2)}([0, \infty) \times \partial D)$, there exists a sequence $\{g_n\}$ of $C_0^\infty([0, \infty) \times \partial D)$ such that $\{g_n\}$ converges to g in $W_p^{(1,2)}([0, \infty) \times \partial D)$. It is clear that $H^A g_n$ converges to $H^A g$ with uniform norm by means of the result about Brownian motion.

Hence (A. 37) holds for any g of $W_p^{(1,2)}([0, \infty) \times \partial D)$.

Remark 4. All estimates in Theorem A. 4 are valid for

$$A = D_t + \sum_{i,j=1}^{n} \frac{1}{2} [\alpha\alpha^*]_{ij}(t, x) D_{ij}^2,$$

if there exists a strictly positive definite matrix a such that

$$\sum_{i,j=1}^{n} | [\alpha\alpha^*]_{ij}(t, x) - a_{ij} | < \varepsilon$$

and if we choose ε sufficiently small.

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