J. Math. Kyoto Univ. (JMKYAZ)
13-2 (1973) 373-380

On asymptotic behaviors of analytic mappings at Martin boundary points

Dedicated to Prof. Y Tôki on his 60th birthday

By

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(Received, June 8, 1972)

Introduction Let f be an analytic mapping of a hyperbolic Riemann surface R into a Riemann surface R', and R^* and R'^* Martin compactification and a metrizable compactification of R and R', respectively. For a point p of Martin boundary Δ of R^* we consider the full cluster set $C(f, p)$ on R'^*. If a point of R' is a limit of f along a path in R tending to Δ, we call the point an asymptotic value of f and the path an asymptotic path. We are going to prove that each point of the set $C(f, p) \cap R'$ is either assumed infinitely often by f in every neighborhood of p or belongs to one of the following three kinds of sets:

1) the set of points $\alpha \in C(f, p) \cap R'$ to which corresponds a *Koebe cotinuum* for α on Δ.
2) the set of points $\alpha \in C(f, p) \cap R'$ for which every neighborhood of p contains an asymptotic path.
3) the set of points $\alpha \in C(f, p) \cap R'$ any neighborhood of which contains a non-polar set of asymptotic values of f at points arbitrarily near p.

As a consequence of the above theorem we obtain the following: Let $C_E(f, p)$ be the boundary cluster set of f at p modulo E, where E is a set on Δ of harmonic measure 0 containing no continua. Then $\{C(f, p) - C_E(f, p)\} \cap R'$ is contained in $R(f, p)$ except a set of capacity 0, and every neighborhood of p contains asymptotic paths for the exceptional values.

In Section 5 we generalize the main theorem in the small of Collingwood-Cartwright [2] in the case of a meromorphic function in the unit circle, which is a generalization of the theorem of Gross-Iversen.

In Section 6 we consider the case where R' is the Riemann sphere and f is a Fatou mapping of a hyperbolic Riemann surface R, refering some results in the author's previous paper [4].

1. We consider the following three sets of points of $C(f, p)\cap R'$. Every points of $C(f, p)\cap R'$ belongs to at least one of them.

1') the set of points $\alpha\in C(f, p)\cap R'$ for which there exists a neighborhood U_0 of p such that, for any neighborhood V of α, any connected component of $f^{-1}(V)$ is not contained in U_0. We denote this set by $\Phi(f, p)$.

2') the set of points $\alpha\in C(f, p)\cap R'$ such that, for any neighborhood U of p, there exists a neighborhood V of α such that at least one connected component D of $f^{-1}(V)$ is contained in U and of class SO_{HB}. We denote this set by $\chi_0(f, p)$.

3') the set of points $\alpha\in C(f, p)\cap R'$ such that, for any neighborhood U of p, there exists a neighborhood V of α such that at least one connected component D of $f^{-1}(V)$ is contained in U and not of class SO_{HB}. We denote this set by $\chi_*(f, p)$.

2. First, we study the set $\Phi(f, p)$ of 1'). We call a non-degenerate continuum K on Δ a *Koebe continuum* for α if it satisfies the following conditions for a sequence of arcs K_n in R:

$$\lim_{n=\infty}\max_{q\in K} d(q, K_n)=0,$$

$$\lim_{n=\infty}\max_{z\in K_n} d(z, K)=0$$

and

$$\lim_{n=\infty}\max_{z\in K_n} d'(\alpha, f(z))=0,$$

where $d(\cdot, *)$ and $d'(\cdot, *)$ denote distances in R^* and R'^*, respectively.

Theorem 1. *To each point α of $\Phi(f, p)$ there corresponds a Koebe continuum for α.*

Proof. Let $\{z_n\}$ be the sequence for which $f(z_n)$ tends to α and V_n the $\frac{1}{n}$-neighborhood of α. Let D_n be the component of $f^{-1}(V_n)$ which contains the point z_n and K_n a Jordan curve in D_n which joins z_n and a point ζ_n on ∂U_0. Since R^* is a compact metric space we can choose a subsequence of $\{K_n\}$, which we denote by $\{K_n\}$ again, converging to a compact set K in the sense that

$$\lim_{n=\infty} \max_{q\in K} d(q, K_n)=0$$

and

$$\lim_{n=\infty} \max_{z\in K_n} d(z, K)=0$$

([1] p. 115). Since K_n is connected it is easily seen that K is connected. K does not degenerate into a single point because $z_n \in K_n$ tends to the point p and the sequence $\{\zeta_n\}$ contains a subsequence converging to a point on $\overline{\partial U_0}$. And K does not contain a point of R, otherwise K contains a continuum in R and f reduces to a constant. From the construction of $\{K_n\}$ and K, it is evident that

$$\lim_{n=\infty} \max_{z\in K_n} d'(\alpha, f(z))=0.$$

3. Next we consider a point α of the set $\chi_0(f, p)$ of 2'). For any small neighborhood U of p, there is a neighborhood V of α for which at least one component D of $f^{-1}(V)$ is contained in U and of class SO_{HB}. Then the restriction f_D of f on D is of type Bl on V. So, the set $V-f(D)$ is of (Green) capacity 0 and each point of which is an asymptotic value along a curve tending to the ideal boundary of D. And we have the following

Theorem 2. *The set $\chi(f, p)=\chi_0(f, p)-R(f, p)$ is of capacity 0. And, for a point α of $\chi(f, p)$, every neighborhood of p contains an asymptotic path of f for α.*

Proof. Since α does not belong to $R(f, p)$, we can choose a neighborhood U_0 of p so that $f(U_0)$ does not contain α. And since α belongs to $\chi_0(f, p)$ we can take a nieghborhood V of α so that there exists a connected component D of $f^{-1}(V)$ which is contained in U_0 and of class SO_{HB}. Then, the restriction f_D of f on D is of type Bl and assumes every value on V except for a set of capacity 0. Since α is an omitted value and $f_D(D)$ is dense in V, it is proved, by a standard technic, that α is an asymptotic value of f along a path in U_0. And, since $\alpha \in \chi_0(f, p)$, this is true for all neighborhood $U \subset U_0$. Now let $\{U_n\}$ be a sequence of 1/n-neighborhoods of p. Then, $\chi(f, p) = \bigcup_n (\chi_0(f, p) - f(U_n))$. Let $\alpha \in \chi_0(f, p) - f(U_n)$ and take a neighborhood V such that a component D of $f^{-1}(V)$ is contained in U_n and of class SO_{HB}. f_D covers V except a set of capacity 0. $\chi_0(f, p) - f(U_n)$ is covered by a countable number of such V as above. Hence $\chi_0(f, p) - f(U_n)$ is of capacity 0. Consequently, $\chi(f, p) = \chi_0(f, p) - R(f, p)$ is of capacity 0.

4. Now we consider the set $\chi_*(f, p)$. Let α be a point of $\chi_*(f, p)$, then for any neighborhood U of p, there exists a neighborhood V of α such that at least one connected component D of $f^{-1}(V)$ is contained in U and not of class SO_{HB}. We consider the set $\Delta_1(D)$ of the ideal boundary points of R at which $R-D$ is thin. Since $D \notin SO_{HB}$ $\Delta_1(D)$ is of harmonic measure positive. And since f_D is a Fatou mapping f has fine limits almost everywhere on $\Delta_1(D)$. The set of those fine limits is contained in $\overline{V}$ and not polar by the theorem of Lusin-Privalov. We notice that a fine limit of f at an ideal boundary point is also an asymptotic value at the point (cf. [3]), and have the following

Theorem 3. *Every neighborhood of a point of $\chi_*(f, p)$ contains a non-polar set of asymptotic values of f at ideal boundary points in an arbitrary neighborhood of p.*

As a result of the above three theorems we obtain the following corollary. It is a generalization of Noshiro's theorem ([7]) generalized by McMillan ([6]) in the case of arbitrary plane regions (cf. also [5]).

Corollary. *Let E be a set on Δ of harmonic measure* 0 *containing no continua, and $C_E(f, p)$ the boundary cluster set of f at p modulo E. Then the set $\{C(f, p)-C_E(f, p)\}\cap R'$ is contained in $R(f, p)$ except a set of capacity* 0. *And every neighborhood of p contains asymptotic paths for the exceptional values.*

5. Now we study relations between the sets $C(f, p)$, $R(f, p)$, $\chi(f, p)$, $\chi_*(f, p)$ and $\Phi(f, p)$, which are generalization of Collingwood-Cartwright's boundary theorems in the case of meromorphic functions in the unit disk.

First we prove the following Theorem 4 which shows that $R(f, p)$ is dense in $\operatorname{int} C(f, p)$. Operations of closures, interior and complement are taken in the metrizable compactification R'^*.

Theorem 4.

$$\operatorname{int} C(f, p)\subset\overline{R(f, p)}.$$

Proof. We prove the relation

$$\operatorname{int}(R(f, p)^c)\subset\overline{C(f, p)^c}.$$

Then, by taking the complement of the both sides, we obtain the theorem. Let α be an interior point of $R(f, p)^c$, that is, $\alpha\in R'^*-\overline{R(f, p)}$ and $V(\alpha)$ an arbitrary neighborhood of α contained in $R(f, p)^c$. If $\operatorname{int}(R(f, p)^c)$ is empty, the theorem is trivial. We put $X_n=f(U_n)$ for the $1/n$-neighborhood of p. Then, X_n^c is closed in R'^* and $R(f, p)^c=\bigcup_n X_n^c$. If $X_n^c(=\overline{X_n^c})$ does not contain an interior point $\bigcup_n X_n^c$ is of first Baire category. And, since R'^* is compact metric space, the set $\bigcup_n X_n^c$ of first category does not contain an interior point. This contradicts our assumption that $R(f, p)^c$ contains the interior point α. Hence there exists $X_{n_0}^c$ containing interior points.

We may assume $X_{n_0}^c\cap V(\alpha)$ contains an interior point by the same way as above by taking $\alpha_0\in V(\alpha)\cap R'$ and $\overline{V(\alpha_0)}$ as R'^*. Then, $X_{n_0}^c\cap V(\alpha)$ contains a domain D, and

$$C(f,\ p)^c=(\bigcap_n \overline{X}_n)^c=\bigcup_n \overline{X}_n^c\supset \overline{X}_{n_0}^c\supset D.$$

Every neighborhood of α contains a domain such as D. We conclude $\alpha\in\overline{C(f,\ p)^c}$. This proves the theorem.

Analogously to Collingwood-Cartwright, we obtain the following

Theorem 5.

$$\text{fr}R(f,\ p)\cup\text{fr}C(f,\ p)=\overline{R(f,\ p)^c}\cap C(f,\ p),\ \text{ and}$$

$$\overline{R(f,\ p)^c}\cap C(f,\ p)\cap R'\subset\overline{\chi(f,\ p)}\cup\overline{\chi_*(f,\ p)}\cup\overline{\Phi(f,\ p)}.$$

where X^c *and* fr$X$ *means respectively the complement and the frontier of* X *with respect to* R'^*.

Proof. By the preceding theorem,

$$\text{int}\,(R(f,\ p)^c)\subset\overline{C(f,\ p)^c}=C(f,\ p)^c\cup\text{fr}C(f,\ p)$$

and so,

$$C(f,\ p)\cap\text{int}\,(R(f,\ p)^c)\subset\text{fr}C(f,\ p).$$

Since $\text{int}\,(R(f,\ p)^c)\subset(\text{fr}R(f,\ p))^c$ we have

$$(*)\qquad C(f,\ p)\cap\text{int}\,(R(f,\ p)^c)\subset\text{fr}C(f,\ p)\cap(\text{fr}R(f,\ p))^c.$$

Since

$$\begin{aligned}(\text{fr}R(f,\ p))^c&\subset\text{int}\,R(f,\ p)\cup\text{int}\,(R(f,\ p)^c)\\&\subset\text{int}\,C(f,\ p)\cup\text{int}\,(R(f,\ p)^c),\end{aligned}$$

we have

$$\begin{aligned}(**)\qquad \text{fr}C(f,\ p)\cap(\text{fr}R(f,\ p))^c&\subset\text{fr}C(f,\ p)\cap\text{int}\,(R(f,\ p)^c)\\&\subset C(f,\ p)\cap\text{int}\,(R(f,\ p)^c).\end{aligned}$$

By $(*)$ and $(**)$ we have

$$C(f,\ p)\cap\text{int}\,(R(f,\ p)^c)=\text{fr}C(f,\ p)\cap(\text{fr}R(f,\ p))^c.$$

Therefore,

$$\overline{R(f, p)^c} \cap C(f, p) = \{\mathrm{fr}R(f, p) \cup \mathrm{int}(R(f, p)^c)\} \cap C(f, p)$$

$$= \{\mathrm{fr}R(f, p) \cap C(f, p)\} \cup \{\mathrm{int}(R(f, p)^c) \cap C(f, p)\}$$

$$= \mathrm{fr}R(f, p) \cup \{\mathrm{int}(R(f, p)^c) \cap C(f, p)\}$$

$$= \mathrm{fr}R(f, p) \cup \{\mathrm{fr}C(f, p) \cap (\mathrm{fr}R(f, p))^c\}$$

$$= \{\mathrm{fr}R(f, p) \cup \mathrm{fr}C(f, p)\} \cap \{\mathrm{fr}R(f, p) \cup (\mathrm{fr}R(f, p))^c\}$$

$$= \mathrm{fr}R(f, p) \cup \mathrm{fr}C(f, p),$$

which proves the first equality of the theorem.

By Theorem 1, 2 and 3 we obtain

$$\overline{R(f, p)^c} \cap C(f, p) \cap R' \subset \overline{\chi(f, p)} \cup \chi_*(f, p) \cup \Phi(f, p)$$

hence, we have

$$\overline{R(f, p)^c} \cap C(f, p) \cap R' \subset \overline{\chi(f, p)} \cup \overline{\chi_*(f, p)} \cup \overline{\Phi(f, p)}$$

which completes the proof.

Corollary.

$$\mathrm{int}\ R(f, p) \supset C(f, p) - \overline{\chi(f, p)} \cup \overline{\chi_*(f, p)} \cup \overline{\Phi(f, p)}$$

that is, if $\alpha \in C(f, p) - \overline{\chi(f, p)} \cup \overline{\chi_*(f, p)} \cup \overline{\Phi(f, p)}$ *then* α *is an interior point of* $R(f, p)$.

6. In the case where R' is the Riemann sphere S we have proved [4] that if f is a Fatou mapping $C(f, p) - C^*(f, p)$ is open and contained in $R(f, p)$ except a set of capacity 0, where $C^*(f, p)$ is the essential cluster set of the fine boundary function of f.

First we prove the following

Lemma. *Let* p *be a regular minimal point of* Δ. *If* $S - R(f, p)$ *is of positive capacity, then* f *is a Fatou mapping in a neighborhood* U *of* p, *that is, restriction of* f *on each connected component of* $R \cap U$

is a Fatou mapping.

Proof. Let E_n be the closed set of points which f does not assume in the neighborhood U_n of p. Then $R(f, p)^c = \bigcup_n E_n$. If $R(f, p)^c$ is of positive capacity there exists an E_n of positive capacity and the restriction f_{U_n} of f on each component of $R \cap U_n$ is a Fatou mapping. And f_U is a Fatou mapping because $U \subset U_n$.

As stated above, $C(f, p) - C^*(f, p)$ is open and contained in $R(f, p)$ except a set F of capacity 0, and each point $\alpha \in F$ belongs either to $\chi(f, p)$ or to $\Phi(f, p)$. If, for a sufficiently small neighborhood V of α, there exists a component D of $f^{-1}(V)$ which is contained in U of p, then α belongs to $\chi(f, p)$, otherwise to $\Phi(f, p)$. If f is not a Fatou mapping, $R(f, p)^c$ is of capacity 0 and $C(f, p)$ is total by the above lemma.

Let E be a set on Δ of harmonic measure 0 containing no continua and J the family of the sets as E. Put $C^{*\prime}(f, p) = \bigcap_{E \in J} C_E(f, p)$, where $C_E(f, p)$ is the boundary cluster set modulo E. Then $\mathrm{fr}C(f, p) \subset C^*(f, p) \subset C^{*\prime}(f, p)$, and $C(f, p) - C^{*\prime}(f, p)$ is open and $C(f, p) - C^{*\prime}(f, p) \subset C(f, p) - C^*(f, p) \subset R(f, p)$ except a set of capacity 0. $C^*(f, p)$ is the union of $\overline{\chi_*(f, p)}$ and the subset of $\overline{\Phi(f, p)}$ for each point of which $\Delta_1(f^{-1}(V))$ is of harmonic measure positive for any small V.

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References

[1] Alexandroff, P. und Hopf, H.: Topologie. Springer (1935).
[2] Collingwood, E. F. and Cartwright, M. L.: Boundary theorems for a function meromorphic in the unit circle. Acta Math. **87** (1952), 83–146.
[3] Constantinescu, C. und Cornea, A.: Ideale Ränder Riemannscher Flächen. Springer (1963).
[4] Fuji'i'e, T.: Cluster sets at ideal boundary points. J. Math. Kyoto Univ. **12** (1972), 17–31.
[5] Kuramochi, Z.: Cluster sets of analytic functions in open Riemann surfaces with regular metrics I. Osaka Math. J. **11** (1959), 83–90.
[6] McMillan, J. E.: Boundary properties of analytic functions. J. Math. and Mech. **17** (1967), 407–419.
[7] Noshiro, K.: On the theory of the cluster sets of analytic functions. J. Fac. Sci. Hokkaido Univ. **6** (1937), 217–231.