J. Math. Kyoto Univ. (JMKYAZ)
13-2 (1973) 323-371

Analyticity of solutions of hyperbolic mixed problems

By

Mikio TSUJI

(Received, June 1, 1972)

§1. Introduction

We consider the mixed problems for the first order hyperbolic systems in a quarter space $V=\{(t, x);\ t>0,\ x=(x_1, \dots, x_n)=(x', x_n),\ x'\in R^{n-1},\ x_n>0\}$

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=L(t)u(t, x)+f(t, x) \\ u(0, x)=g(x) \\ P(t, x')u(t, x)|_{x_n=0}=0 \end{cases}$$

where $L(t)=\sum_{i=1}^{n} A_i(t, x)\frac{\partial}{\partial x_i}+B(t, x)$, $A_i(t, x)$ and $B(t, x)$ are $N\times N$ matrices, and $P(t, x')$ is an $l\times N$ matrix. We assume that A_k and B are smooth with respect to (t, x), and that $P(t, x')$ is a smooth matrix of (t, x').

Assume that the coefficients of $L(t)$ and P are analytic with respect to (t, x) and (t, x') respectively, and that the Cauchy data $g(x)$ and the second member $f(t, x)$ are also analytic and they satisfy the compatibility conditions of infinite order, which we explain in §2. Then the Cauchy-Kowalewski theorem states that a solution $u(t, x)$ is analytic with respect to t, x for small t. The aim of this article is to show this property *in the large*, which we give as Theorem 2 in §2. The step-by-step reasoning could not be used directly for any t and x, because the size of each step (with respect to t) in the argument depends on the radius of convergence of the Cauchy data obtained by the previous step. Therefore, for the proof of Theorem 2 we need the additional informations about the solution $u(t, x)$ of (1.1), i.e., we assume the conditions C.1, C.2 and C.3, stated precisely in § 2, which assert that the solution $u(t, x)$ of (1.1) satisfies the usual energy inequalities and that it has the finite propagation property. Under these conditions, our proof is carried out by estimating the successive derivatives of $u(t, x)$. We could mention that this method was already used by S. Mizohata [4] for hyperbolic Cauchy problems. Compared with Cauchy problems, the difficulty of the mixed problems is the treatment of the normal derivatives of the solution $u(t, x)$ with respect to the boundary. At first, we estimate the tangential derivatives of $u(t, x)$. Next, from the fact that A_n is non-singular, we can estimate the normal derivatives of $u(t, x)$ from the equation (1.1). In this way we prove Theorem 2.

In the next section we define our notations and state our results. In § 3, we prove the regularity of the solution $u(t, x)$ of (1.1), which we state as Theorem 1 in § 2. In § 4 and § 5 we prove Theorem 2. In § 6, § 7 and § 8, we shall show that the symmetric hyperbolic systems with maximal non-negative boundary conditions satisfy the above conditions C.1, C.2 and C.3.

The auther wishes to express his sincere gratitude to Professor S. Mizohata and Mr. S. Miyatake for their valuable suggestions.

§ 2. Notations and results

Let R^n_+ be the set $\{(x_1, x_2, \dots, x_n)=(x', x_n);\ x_n>0,\ x'\in R^{n-1}\}$. We put $D_x=(D_1, D_2, \dots, D_n)$ where $D_i=\frac{\partial}{\partial x_i}$, $i=1, 2, \dots, n$, and $D_0=\frac{\partial}{\partial t}$. We remark that, although we put $D_0=\frac{\partial}{\partial t}$, we don't use x_0 as the time variable t in the following sections.

$H^s(R^n_+)$, $s=0, 1, 2, \dots$, is the set of functions defined in R^n_+ whose partial derivatives of order $\leqq s$ (in the sense of distribution) are all square integrable in R^n_+. For $u\in H^s(R^n_+)$, we define

$$||u||_s^2=\sum_{\alpha\leqq s}||D_x^\alpha u||^2,$$

$$\|u\|_{s,0}^2=\sum_{|\alpha|\leqq s,\alpha_n=0}\|D_x^\alpha u\|^2.$$

$\mathscr{E}_t^p(E)$ is the set of E-valued functions of t which are p-times continuously differentiable. For $u\in\mathscr{E}_t^s(L^2)\cap\mathscr{E}_t^{s-1}(H^1)\cap\cdots\cap\mathscr{E}_t^0(H^s)$, we define

$$|||u(t)|||_s=\sum_{i=0}^s\|D_0^i u(t)\|_{s-i},$$

$$|||u(t)|||_{s,0}=\sum_{i=0}^s\|D_0^i u(t)\|_{s-i,0}.$$

$C^s(R_+^n)$ is the set of functions defined in R_+^n whose partial derivatives of order $\leqq s$ are all continuous in R_+^n.

$\mathscr{B}^s(R_+^n)$ is the set of functions in $C^s(R_+^n)$ whose partial derivatives of order $\leqq s$ are all bounded.

Let $u(t,x)$ be a smooth solution of (1.1), then the given data should satisfy certain conditions. For example, if $u(t,x)$ is in $H^1((0,T)\times R_+^n)$, then

$$P(0,x')g(x)|_{x_n=0}=0. \tag{2.1}$$

We say that (2.1) is the compatibility condition of order zero. Similarly, if $u(t,x)$ is in $H^{m+1}((0,T)\times R_+^n)$, then

$$D_0^k(Pu)\Big|_{\substack{t=0\\x_n=0}}=0,\qquad k=0,1,\dots,m.$$

If we rewrite these by using $g(x)$ and $f(t,x)$, we get the compatibility conditions of order m as follows.

Definition 1. *The data $g(x)\in H^{m+1}(R_+^n)$ and $f(t,x)\in H^{m+1}((0,T)\times R_+^n)$ are said to satisfy the compatibility conditions of order m, if*

$$\sum_{i=0}^k\binom{k}{i}P^{(i)}(0,x')g^{(k-i)}(x)|_{x_n=0}=0,\qquad k=0,1,\dots,m,$$

where $P^{(i)}(0,x')=\frac{\partial^i P}{\partial t^i}(0,x')$, $g^{(0)}(x)=g(x)$ and $g^{(p+1)}(x)$ $(p\geqq 0)$ is defined successively by the formula

$$(2.2)\qquad g^{(p+1)}(x)=\sum_{i=0}^{p}\binom{p}{i}\left(\sum_{j=1}^{n}\frac{\partial^i A_j}{\partial t^i}(0, x)\frac{\partial}{\partial x_j}+\frac{\partial^i B}{\partial t^i}(0, x)\right)g^{(p-i)}(x)$$
$$+\frac{\partial^p f}{\partial t^p}(0, x).$$

Now we state the following conditions C.1, C.2 and C.3.

C. 1) *A_n is non-singular and rank $P=l$.*

We need the former of this condition to prove the regularity of solutions of (1.1) and the latter to reduce the general case where $P(t, x')$ is an $l\times N$ variable matrix to the constant case $P(t, x')=[E_l\ 0]$, where E_l is an $l\times l$ unit matrix, *i.e.*,

$$[E_l\ 0]=\begin{pmatrix} 1 & & & 0\cdots 0 \\ & 1 & & 0\ 0 \quad 0 \\ & & \ddots & \vdots \quad \vdots \\ 0 & & 1 & 0 \quad 0 \end{pmatrix}.$$

C. 2) *Let the coefficients of $L(t)$ be in $\mathscr{B}^s(V)$ and $P(t, x')$ in $\mathscr{B}^s(R^1_+\times R^{n-1})$ where s is large enough. For any $g(x)\in H^1(R^n_+)$ such that $Pg|_{x_n=0}=0$ and for any $f(t, x)\in\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$, there exists a unique solution $u(t, x)$ of (1.1) in $\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$ which satisfies the following energy inequalities*

$$(2.3)\qquad \|u(t)\|\leqq c_0\cdot e^{\mu_0 t}\|g\|+d_0\int_0^t e^{\mu_0(t-s)}\|f(s)\|ds,$$

$$(2.4)\qquad |||u(t)|||_1\leqq c_1\cdot e^{\mu_1 t}|||u(0)|||_1^{1)}+d_1\int_0^t e^{\mu_1(t-s)}|||f(s)|||_1 ds,$$

where $c_0, c_1; d_0, d_1; \mu_0, \mu_1$ are positive constants independent of $u(t, x)$, $g(x), f(t, x)$ and t.

We use C.2 to estimate the successive derivatives of the solution $u(t, x)$ of (1.1) for the proof of the regularity and analyticity.

C. 3) *There exists a convex cone $C=\{(t, x); t<-\lambda|x|, \lambda>0\}$ such that, for any point $(t_0, x_0)\in V$, the domain of dependence of the*

1) $|||u(0)|||_1$ is the value of $|||u(t)|||_1$ at $t=0$. Therefore, $|||u(0)|||_1=\|g\|_1+\|L(0)g+f(0)\|$, but $|||u(0)|||_1\neq|||g|||_1=\|g\|_1$.

point (t_0, x_0) *with respect to the equation* (1.1) *is contained in* $C+(t_0, x_0)=C_{(t_0,x_0)}$.

The condition C.3 means that, if $u(t, x)\in C^1(V\cap C_{(t_0,x_0)})$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t}=L(t)u+f(t, x) \\ u(0, x)=g(x) \\ Pu|_{x_n=0}=0, \end{cases}$$

and if $f(t, x)\equiv 0$ in $V\cap C_{(t_0,x_0)}$ and $g(x)\equiv 0$ in $\bar{V}\cap C_{(t_0,x_0)}\cap\{t=0\}$, then

$$u(t, x)\equiv 0 \quad \text{in} \quad V\cap C_{(t_0,x_0)}.$$

We shall show in §6, §7 and §8 that the symmetric hyperbolic systems with maximally non-positive boundary condition satisfy the contidions C.1, C.2 and C.3. K. Kajitani [2] and J. Rauch [7] state that the strictly hyperbolic first order systems with uniformly Lopatinski conditions satisfy the conditions C.1 and C.2.

We consider the following problem

$$(2.5)\qquad \begin{cases} \frac{\partial u_i}{\partial t}=L(t)u_i+\sum_{j=1}^{m} c_{ij}u_j+f_i \quad, & i=1, 2, \dots, m, \\ u_i(0, x)=g_i(x) \quad, & i=1, 2, \dots, m, \\ Pu_i|_{x_n=0}=0 \quad, & i=1, 2, \dots, m, \end{cases}$$

where $L(t)$ and P are the same ones given in (1.1). This problem will often appear in the following sections. We put $U={}^t({}^tu_1, {}^tu_2,\dots, {}^tu_m)$, $G={}^t({}^tg_1, {}^tg_2,\dots, {}^tg_m)$ and $F={}^t({}^tf_1, {}^tf_2,\dots, {}^tf_m)$. Then we have

Lemma 1. *Let the conditions* C.1, C.2 *and* C.3 *be satisfied for the problem* (1.1). *Assume* $c_{ij}\in\mathscr{B}^1(V)$ *for any* i *and* j. *Then there exists a unique solution* $U(t, x)$ *of* (2.5) *which satisfies the conditions* C.2 *and* C.3, *i.e.*,
(I) *For any* $g_i(x)\in H^1(R^n_+)$ *such that* $Pg_i|_{x_n=0}=0$ *and for any* $f_i(t, x)\in\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$, $i=1, 2, \dots, m$, *there exists a unique solution* $U(t, x)$ *of* (2.5) *in* $\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$ *which satisfies the following inequalities*

(2.6) $$||U(t)||\leqq c'_0 e^{\mu'_0 t}||U(0)||+d'_0\int_0^t e^{\mu'_0(t-s)}||F(s)||ds$$

(2.7) $$|||U(t)|||_1\leqq c'_1 e^{\mu'_1 t}|||U(0)|||_1+d'_1\int_0^t e^{\mu'_1(t-s)}|||F(s)|||_1 ds,$$

where $|||U(t)|||_k=\sum_{i=1}^{m}|||u_i(t)|||_k$ $(k=1, 2)$ *and* c'_0, c'_1, d'_0, d'_1, μ'_0 *and* μ'_1 *are positive constants.*

(II) *Let* C *be the cone stated in the condition* C.3. *The domain of dependence of the point* (t_0, x_0) *with respect to* (2.5) *is contained in* $C+(t_0, x_0)=C_{(t_0,x_0)}$.

We shall prove this lemma in § 3. Concerning the regularity of the solution, we have the following theorem.

Theorem 1. *Let the conditions* C.1 *and* C.2 *be satisfied. We assume that* A_n^{-1} *is bounded in* V, *and that* $P(t, x')$ *is constant outside a compact set in* $\bar{V}\cap\{x_n=0\}$. *Suppose that* $g(x)\in H^m(R^n_+)$ *and* $f(t, x)\in\mathscr{E}_t^m(L^2)\cap\cdots\cap\mathscr{E}_t^0(H^m)$ *satisfy the compatibility conditions of order* $(m-1)$ $(m\geqq 2)$, *then there exists a unique solution* $u(t, x)$ *of* (1.1) *in* $\mathscr{E}_t^m(L^2)\cap\cdots\cap\mathscr{E}_t^0(H^m)$ *which satisfies the following inequalities*

(2.8) $$|||u(t)|||_k\leqq c_k e^{\mu_k t}|||(0)|||_k$$
$$+d_k\int_0^t e^{\mu_k(t-s)}|||f(s)|||_k ds, \qquad k=0, 1,\dots, m,$$

where c_k, d_k *and* μ_k *are positive constants.*

Now we can state our main theorem on the analyticity of the solution of (1.1). Take a point (t_0, x_0) in V. Denote by C_0 the intersection of the cone $C_{(t_0,x_0)}$ with the initial plane $\{t=0\}\cap\bar{V}$. Then we have

Theorem 2. *Let* C.1, C.2 *and* C.3 *be satisfied. We assume that the coefficients of* $L(t)$ *and* $P(t, x')$ *are analytic in* V *and* $\bar{V}\cap\{x_n=0\}$ *respectively. Suppose that* $g(x)$ *and* $f(t, x)$ *are analytic in a neighborhood of* C_0 *and* $C_{(t_0,x_0)}\cap V$ *respectively, and that they satisfy the compatibility conditions of infinite order, then the solution* $u(t, x)$ *of* (1.1) *is analytic with respect to* (t, x) *at the point* (t_0, x_0).

We remark that, although we assumed the coefficients of $L(t)$ and $P(t, x')$ to be analytic in V and $\bar{V}\cap\{x_n=0\}$ respectively, it is sufficient for the proof of Theorem 2 to assume that these coefficients and P are analytic in a neighborhood of $V\cap C_{(t_0,x_0)}$ and $\bar{V}\cap\{x_n=0\}\cap C_{(t_0,x_0)}$ respectively.

We confined ourselves here to the case of half-space. However, if we taked account of the finite propagation property and used a suitable local transformation of independent variables, we could obtain the same results in a general domain $[0, T]\times\Omega$, where Ω is open in R^n whose boundary is compact and analytic.

§3. Proof of Lemma 1 and Theorem 1

Proof of Lemma 1. We prove this lemma by successive approximation. Let us construct a series of functions $U^{(k)}(t, x)={}^t({}^tu_1^{(k)},\dots, {}^tu_m^{(k)})\in\mathcal{E}_t^0(H^1)\cap\mathcal{E}_t^1(L^2)$ $(k=0, 1, 2,\dots)$ as follows: Let $U^{(0)}(t, x)=0$ and we define $U^{(k)}(t, x)$ for $k\geqq 1$ successively by the formula

$$
(3.1)\quad \begin{cases}
\dfrac{\partial}{\partial t}u_i^{(k)}=L(t)u_i^{(k)}+\sum_{j=1}^m c_{ij}u_j^{(k-1)}+f_i\ , & i=1,\dots,m\\
u_i^{(k)}(0, x)=g_i(x) & ,\quad i=1,\dots,m\\
Pu_i^{(k)}|_{x_n=0}=0 & ,\quad i=1,\dots,m.
\end{cases}
$$

The existence of $U^{(k)}(t, x)$ $(k=1, 2, \dots)$ is assured by the condition C.2. The sequence $\{U^{(k)}(t, x)\}_{k=0,1,2,\dots}$ converges in $\mathcal{E}_t^1(L^2)\cap\mathcal{E}_t^0(H^1)$. In fact, since

$$
(3.1)'\quad \begin{cases}
\dfrac{\partial}{\partial t}(u_i^{(k+1)}-u_i^{(k)})=L(t)(u_i^{(k+1)}-u_i^{(k)})\\
\qquad\qquad +\sum_{j=1}^m c_{ij}(u_j^{(k)}-u_j^{(k-1)})\ , & i=1, 2,\dots, m\\
(u_i^{(k+1)}-u_i^{(k)})(0, x)=0 & ,\quad i=1, 2,\dots, m\\
P(u_i^{(k+1)})-u_i^{(k)})|_{x_n=0}=0 & ,\quad i=1, 2,\dots, m
\end{cases}
$$

for any $k\geqq 1$, the application of the inequality (2.4) to (3.1)$'$ gives

$$
e^{-\mu_1 t}\sum_{i=1}^m |||u_i^{(k+1)}(t)-u_i^{(k)}(t)|||_1
$$

$$\leqq K\int_0^t e^{-\mu_1 s}\sum_{i=1}^m |||u_i^{(k)}(s)-u_i^{(k-1)}(s)|||_1 ds$$

for $k\geqq 1$ where K is a positive constant independent of $U^{(k)}(t, x)$ and t. From the above inequality, we get

$$e^{-\mu_1 t}\sum_{i=1}^m |||u_i^{(k+1)}(t)-u_i^{(k)}(t)|||_1$$

$$\leqq \frac{(Kt)^k}{k!}\sup_{s\in[0,t]}(e^{-\mu_1 s}\sum_{i=1}^m |||u_i^{(1)}(s)|||_1), \qquad k=1, 2, \ldots,$$

which shows the convergence of $U^{(k)}(t, x)$ in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Denote its limit by $U(t, x)={}^t({}^tu_1, {}^tu_2,\ldots, {}^tu_m)$ and pass to the limit in (3.1), then we see that $U(t, x)$ is a unique solution of (2.5). For the proof of the energy inequalities (2.6) and (2.7), we apply (2.3) and (2.4) to each $u_i(t, x)$ and sum up the obtained inequalities from $i=1$ to m, then we get (2.6) and (2.7).

Next, we prove (II) of Lemma 1. Take a point (t_0, x_0) in V. Assume that $F(t, x)=0$ in $V\cap C_{(t_0,x_0)}$ and $G(x)\equiv 0$ in $V\cap C_{(t_0,x_0)}$, then the condition C.3 means that

$$U^{(k)}(t, x)=0 \qquad \text{in} \quad V\cap C_{(t_0,x_0)}, \quad k=1, 2, 3,\ldots ,$$

where $U^{(k)}(t, x)$ is defined by (3.1). As the solution $U(t, x)$ of (3.1) is the limit of the sequence $\{U^{(k)}(t, x)\}_{k=1,2,\ldots}$, we see that $U(t, x)=0$ in $V\cap C_{(t_0,x_0)}$. This completes the proof of Lemma 1. q.e.d.

Using the results of Lemma 1, we prove Theorem 1.

Proof of Theorem 1. It will be sufficient to prove our statements in the case $m=2$, because we can apply our reasoning to the case $m\geqq 2$ in the same way. We shall show in the appendix that the general case $P(t, x')$ can be reduced to the constant case $P(t, x')=[E_l\ 0]$ by a unitary transformation of unknown functions. Therefore, we consider (1.1) under the condition $P(t, x')=[E_l\ 0]$. We show that $D_iu(t, x)$ $(i=0, 1, \ldots, n-1)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Then, as $A_n(t, x)$ is non-singular, it follows that $D_nu(t, x)$ is also in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Let us put $U(t, x)={}^t({}^tu, {}^tD_0u, {}^tD_1u,\ldots, {}^tD_{n-1}u)$. Then, by the same reasoning as Lemma 1.1 of Lax and Phillips [3], we see that $U(t, x)$ belongs to $\mathscr{E}_t^0(L^2(R_+^n))$

$\cap \mathscr{E}^0_{x_n}(H^{-1}_\mu(R^+_1 \times R^{n-1})))^{1)}$ $(\mu>\mu_1)$ and it satisfies

$$(3.2)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=\tilde{L}(t)U+F(t, x) & \text{in distribution sense} \\ U(0, x)=G(x) & \text{in } L^2(R^n_+) \\ \tilde{P}U|_{x_n=0}=0 & \text{in } H^{-1}_\mu(R^1_+\times R^{n-1}) \end{cases}$$

where, if we denote by E_N an $N\times N$ unit matrix and put $A_0(t, x)=-E_N$,

$$\tilde{L}(t)=\begin{pmatrix} L(t) & & & \\ & L(t) & & \\ & & \ddots & \\ & & & L(t) \end{pmatrix}+\begin{pmatrix} 0 & 0 & \cdots & 0 \\ A_nD_0(A_n^{-1}B) & & & \\ A_nD_1(A_n^{-1}B) & & M_n & \\ \vdots & & & \\ A_nD_{n-1}(A_n^{-1}B) & & & \end{pmatrix},$$

$M_n=[A_nD_i(A_n^{-1}A_j)]_{0\leq i, j\leq n-1}$,

$$\tilde{P}=\begin{pmatrix} P & & & \\ & P & & \\ & & \ddots & \\ & & & P \end{pmatrix},\ G=\begin{pmatrix} g \\ L(0)g+f(0, x) \\ D_1g \\ \vdots \\ D_{n-1}g \end{pmatrix}\ \text{and}\ F=\begin{pmatrix} f \\ A_nD_0(A_n^{-1}f) \\ A_nD_1(A_n^{-1}f) \\ \vdots \\ A_nD_{n-1}(A_n^{-1}f) \end{pmatrix}.$$

Since $g(x)$ and $f(t, x)$ are in $H^2(R^n_+)$ and $\mathscr{E}^2_t(L^2)\cap\mathscr{E}^1_t(H^1)\cap\mathscr{E}^0_t(H^2)$ respectively and they satisfy the compatibility conditions of order 1, we see that $G(x)\in H^1(R^n_+)$, $\tilde{P}G|_{x_n=0}=0$ and $F(t, x)\in\mathscr{E}^0_t(H^1)\cap\mathscr{E}^1_t(L^2)$. Therefore, by using the results of Lemma 1, we see that there exists a unique solution $V(t, x)={}^t({}^tv, {}^tv_0, {}^tv_1,\dots, {}^tv_{n-1})$ of (3.2) in $\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$. Our assertion is to prove $D_iu=v_i$ for $i=0, 1,\dots, n-1$. Let us put $w=u-v$ and $w_i=D_iu-v_i$ $(i=0, 1,\dots, n-1)$, then $W={}^t({}^tw, {}^tw_0, {}^tw_1,\dots, {}^tw_{n-1})$ is in $\mathscr{E}^0_t(L^2)\cap\mathscr{E}^0_{x_n}(H^{-1}_\mu(R^n_+\times R^{n-1}))$ and satisfies

$$(3.3)\qquad \begin{cases} \dfrac{\partial W}{\partial t}=\tilde{L}(t)W & \text{in distribution sense} \\ W(0, x)=0 & \text{in } L^2(R^n_+) \\ \tilde{P}W|_{x_n=0}=0 & \text{in } H^{-1}_\mu(R^1_+\times R^{n-1}). \end{cases}$$

1) $H^{-1}_\mu(R^1_+\times R^{n-1})$ is the set of the functions $u(t, x')$ such that $e^{-\mu t}u(t, x')\in H^{-1}(R^1\times R^{n-1})$. $H^{-1}(R^1_+\times R^{n-1})$ is the dual space of $H^1_0(R^1_+\times R^{n-1})$.

Here we remark that, for the validity of the inequality (2.3), it suffices only to be $u(t, x)\in\mathscr{E}_t^0(L^2)$, and that the assumption $u(t, x)\in\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ is not necessary. We can prove this fact by using the mollifier with respect to (t, x') as the reasoning of M. Ikawa [1], p. 131. Since we use the same method in the proof of Theorem 4 in §7, we omit its process here. Applying the inequality (2.3) to (3.3), we get

$$||W(t)||\leqq K\int_0^t e^{\mu_0(t-s)}||W(s)||ds$$

where K is a positive constant independent of W and t. From this inequality, we have

$$e^{-\mu_0 t}||W(t)||\leqq\frac{(Kt)^m}{m!}\sup_{s\in[0,t]}(e^{-\mu_0 s}||W(s)||), \qquad m=1, 2,\dots .$$

Hence we obtain $W(t)=0$, i.e., $v_i=D_iu$ for any i, which implies $u(t, x)$ is in $\mathscr{E}_t^2(L^2)\cap\mathscr{E}_t^1(H^1)\cap\mathscr{E}_t^0(H^2)$.
Next, we prove the inequality (2.8) for $k=2$. Applying (2.7) to (3.2), we have

$$|||U(t)|||_1\leqq c_2' e^{\mu_2 t}|||U(0)|||_1+d_2'\int_0^t e^{\mu_2(t-s)}|||F(s)|||_1 ds \tag{3.4}$$

where c_2', d_2' and μ_2 are positive constants. As it holds

$$D_nu(t, x)=A_n^{-1}\{D_0u(t, x)-\sum_{i=1}^{n-1}A_i(D_iu)(t, x)-Bu(t, x)-f(t, x)\},$$

we have

$$||D_n^2u(t)||\leqq\text{const.}\{|||U(t)|||_1+||D_nf(t)||+||f(t)||\}. \tag{3.5}$$

And we see easily that it holds

$$\begin{aligned}||D_nf(t)||&\leqq\int_0^t||D_nD_0f(s)||ds+||D_nf(0)|| \\ &\leqq\int_0^t||D_nD_0f(s)||ds+||D_n(L(0)g+f(0))|| \\ &\qquad+||D_n(L(0)g)||\end{aligned} \tag{3.6}$$

$$\leqq \int_0^t \|D_n D_0 f(s)\| ds + \text{const.} ||| u(0) |||_2$$

and

(3.7) $$\|f(s)\| \leqq \int_0^t \|f'(s)\| ds + \|L(0)g + f(0)\| + \|L(0)g\|$$

$$\leqq \int_0^t \|f'(s)\| ds + \text{const.} ||| u(0) |||_1.$$

Recalling that $||| U(t) |||_1^2 + \|D_n^2 u(t)\|^2 = ||| u(t) |||_2^2$, and combining (3.4), (3.5), (3.6) and (3.7), we get the inequality (2.8) in the case $k=2$. The proof of Theorem 1 is complete. q.e.d.

§4. Proof of Theorem 2 (I)

As will be seen in the appendix, we can reduce the general case $P(t, x')$ to the constant case $[E_l\ 0]$ by a unitary transformation of unknown functions which is analytic in a neighborhood of $C_{(t_0,x_0)} \cap V$. Hence we can put $P(t, x') = [E_l\ 0]$ without loss of generality.

We can see from Theorem 1 that the solution of (1.1) in this case is C^∞ in a neighborhood of $C_{(t_0,x_0)} \cap V$. The aim of this section is to define a series of functions $v(t, x)$, $v_0(t, x)$, $v_1(t, x), \dots, v_n(t, x)$, $v_{ij}(t, x)$, ..., which are equal to u, $D_0 u$, $D_1 u, \dots, D_n u$, $D_{ij} u$, ... respectively in a neighborhood of $C_{(t_0,x_0)} \cap V$. For this purpose, we extend the set $C_{(t_0,x_0)}$ in the following way; Let $S_{(t_0,x_0)}$ be a small ball in V with center (t_0, x_0). Denote by $\mathscr{D}$ the set

$$\Big(\bigcup_{(t,x)\in S_{(t_0,x_0)}} C_{(t,x)}\Big) \cap V,$$

and by $\mathscr{D}_0$ the set $\bar{\mathscr{D}} \cap \{t=0\}$. Here we take the ball $S_{(t_0,x_0)}$ so small that $f(t, x)$ and the coefficients of $L(t)$ are analytic in neighborhood of $\mathscr{D}$ and that $g(x)$ is also analytic in a neighborhood of $\mathscr{D}_0$. We choose a function $\alpha(t, x) \in C_0^\infty(R^{n+1})$ which takes the value 1 in a neighborhood of $\mathscr{D}$ and whose support is contained in the domains of analyticity of $g(x)$, $f(t, x)$ and the coefficients of $L(t)$. We denote $\alpha(x) = \alpha(0, x)$.

Now we define $v(t, x)$ by the solution of

$$
(4.1)\quad \begin{cases} \dfrac{\partial v}{\partial t}=L(t)v+\alpha(t, x)f(t, x) \\ v(0, x)=\alpha(x)g(x) \\ Pv|_{x_n=0}=0 \end{cases}
$$

We know from the conditions C.1 and C.2 that the solution $v(t, x)$ of (4.1) exists uniquely in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ with the property that $v(t, x)=u(t, x)$ in $\mathscr{D}$, because the initial value of $v(t, x)$ coincides with that of u is a neighborhood of $\mathscr{D}_0$.

Next, we operate D_i $(i=0, 1, \dots, n-1)$ on the both sides of the equation (1.1), then

$$
(4.2)\quad \begin{cases} \dfrac{\partial}{\partial t}(D_iu)=L(t)(D_iu)+(D_iL(t))u+D_if\,, & i=0, 1,\dots, n-1 \\ (D_iu)(0, x)=g_i(x) & ,\quad i=0, 1,\dots, n-1 \\ P(D_iu)|_{x_n=0}=0 & ,\quad i=0, 1,\dots, n-1 \end{cases}
$$

where $(D_iL(t))=\sum_{j=1}^{n}(D_iA_j)\frac{\partial}{\partial x_j}+(D_iB)$ and $g_i(x)$ is an initial value of $(D_iu)(t, x)$, i.e., $g_0(x)=L(0)g+f(0)$ and $g_i(x)=D_ig(x)$, $i=1, 2,\dots, n-1$. $D_nu(t, x)$ does not satisfy (4.2), because $P(D_nu)|_{x_n=0}$ is not necessarily equal to zero. However it is represented by a linear combination of u, D_iu $(i=0, 1,\dots, n-1)$ and f as follows

$$
(4.2)'\qquad D_nu=A_n^{-1}\{D_0u-\sum_{i=1}^{n-1}A_i(D_iu)-Bu-f\}.
$$

Taking account of these, we define $v_i(t, x)$ $(i=0, 1,\dots, n)$ in the following way: $v_i(t, x)$ $(i=0, 1,\dots, n-1)$ are solutions of

$$
(4.3)\quad \begin{cases} \dfrac{\partial}{\partial t}v_i=L(t)v_i+\alpha(t, x)\sum_{j=1}^{n}(D_iA_j)v_j+\alpha(t, x)(D_iB)v \\ \qquad\qquad +\alpha(t, x)D_if, \quad i=0, 1,\dots, n-1 \\ v_i(0, x)=\alpha(x)g_i(x) \quad ,\quad i=0, 1,\dots, n-1 \\ Pv_i|_{x_n=0}=0 \quad ,\quad i=0, 1,\dots, n-1 \end{cases}
$$

where $v_n(t, x)$ which appears in the lower order terms of the right hand side of (4.3) is replaced by a function

$$(4.3)' \qquad \alpha(t, x)A_n^{-1}\{v_0-\alpha(t, x)\sum_{j=1}^{n-1} A_j v_j-\alpha(t, x)Bv-\alpha(t, x)f\}.$$

This is a system of equations for unknown functions $v_0, v_1, \ldots, v_{n-1}$. Now we can apply Lemma 1 to such a system. For this purpose it is necessary to be

$$(4.4) \qquad \begin{cases} \alpha(x)g_i(x)\in H^1(R_+^n)\ , & i=0, 1,\ldots, n-1, \\ P(\alpha(x)g_i(x))|_{x_n=0}=0, & i=0, 1,\ldots, n-1. \end{cases}$$

The former of (4.4) is trivial. The latter is shown in the following way. As the compatibility condition of order 0 means $Pg|_{x_n=0}=0$, it follows

$$P(D_i g)|_{x_n=0}=0, \qquad i=1, 2, \ldots, n-1.$$

The compatibility conditions of order 1 assert moreover

$$P(L(0)g+f(0))|_{x_n=0}=0.$$

From the definition of $g_i(x)$ $(i=0, 1, \ldots, n-1)$, we get the latter of (4.4). Hence we see by Lemma 1 that there exists a unique solution $\{v_0, v_1, \ldots, v_{n-1}\}$ of (4.3) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ which satisfies

$$v_i(t, x)=D_i u(t, x) \quad \text{in} \quad \mathscr{D}, \qquad i=0, 1,\ldots, n-1.$$

Now we define $v_n(t, x)$ by the function (4.3)$'$. It is easy to see that $v_n(t, x)=D_n u(t, x)$ in $\mathscr{D}$ and $v_n(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$.

We operate $D_i D_j$ on the both sides of (1.1). If $i\neq n$ and $j\neq n$, then

$$(4.5) \qquad \begin{cases} \dfrac{\partial}{\partial t}(D_i D_j u)=L(t)(D_i D_j u)+(D_i L(t))(D_j u)+(D_j L(t))(D_i u) \\ \qquad\qquad +(D_i D_j L(t))u+(D_i D_j f) \\ (D_i D_j u)(0, x)=g_{ij}(x) \\ P(D_i D_j u)|_{x_n=0}=0 \end{cases}$$

where $g_{ij}(x)$ is an initial value of $(D_iD_ju)(t, x)$, i.e., $g_{00}(x)=g^{(2)}(x)$, $g_{i0}(x)=g_{0i}(x)=D_ig^{(1)}(x)$ $(i=1, 2,\dots, n-1)$, $g_{ij}(x)=D_iD_jg(x)$ $(i, j=1, 2,\dots, n-1)$. The functions $g^{(1)}(x)$ and $g^{(2)}(x)$ are defined by (2.2) in §2.

If $i=n$ or $j=n$, $D_iD_ju(t, x)$ does not satisfy (4.5). However $D_iD_nu(t, x)$ is represented by a linear combination of $u, D_1u, \dots, D_nu, D_iD_0u, D_iD_1u,\dots, D_iD_{n-1}u$ and D_if. In fact, operating D_i to (1.1), we get

$$(4.5)' \qquad D_iD_nu=A_n^{-1}\{D_iD_0u-\sum_{j=1}^{n-1}A_j(D_iD_ju)-B(D_iu) \\ -(D_iL(t))u-D_if\}, \qquad i=0, 1,\dots, n.$$

We remark that $D_iD_ju=D_jD_iu$ in a neighborhood of $C_{(t_0,x_0)}\cap V$.

Taking account of these, we define $v_{ij}(t, x)$ as follows: $v_{ij}(t, x)$ $(i, j=0, 1,\dots, n-1)$ are solutions of

$$(4.6) \qquad \begin{cases} \dfrac{\partial}{\partial t}v_{ij}=L(t)v_{ij}+\alpha(t, x)\sum_{k=1}^{n}(D_iA_kv_{jk}+D_jA_kv_{ik}) \\ \qquad +\alpha(t, x)\sum_{k=1}^{n}(D_iD_jA_k)v_k+\alpha(t, x)(D_iB)v_j \\ \qquad +\alpha(t, x)(D_jB)v_i+\alpha(t, x)(D_iD_jB)v+\alpha(t, x)D_iD_jf \\ v_{ij}(0, x)=\alpha(x)g_{ij}(x) \\ Pv_{ij}|_{x_n=0}=0 \end{cases}$$

where each $v_{in}(t, x)$ $(i\neq n)$ which appears in the lower order terms of the right hand side of (4.6) is replaced by a function

$$(4.6)'_i \qquad \alpha(t, x)A_n^{-1}\{v_{i0}-\alpha(t, x)\sum_{k=1}^{n-1}A_kv_{ik}-\alpha(t, x)\sum_{k=1}^{n}(D_iA_k)v_k \\ -\alpha(t, x)Bv_i-\alpha(t, x)(D_iB)v-\alpha(t, x)D_if\}.$$

This is a system of equations for unknown functions v_{ij} $(i\neq n$ and $j\neq n)$. Now we can apply Lemma 1 to (4.6). For this purpose it is necessary to be $\alpha(x)g_{ij}(x)\in H^1(R_+^n)$ and $P(\alpha(x)g_{ij}(x))|_{x_n=0}=0$ $(i, j=0, 1, \dots, n-1)$. This fact is seen by the compatibility conditions of order 2 which mean concretely

$$Pu\Big|_{\substack{t=0\\x_n=0}}=P(D_0u)\Big|_{\substack{t=0\\x_n=0}}=P(D_0^2u)\Big|_{\substack{t=0\\x_n=0}}=0.$$

Moreover we see that $v_{ij}(0, x)=(D_iD_ju)(0, x)$ in $\mathscr{D}$ and $v_{ij}(0, x)=v_{ji}(0, x)$ in $\bar{V}\cap\{t=0\}$. Therefore, by Lemma 1 we get a unique solution $\{v_{ij}(t, x);\ i, j=0, 1, \dots, n-1\}$ of (4.6) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ which satisfies

$$v_{ij}(t, x)=D_iD_ju(t, x) \quad \text{in} \quad \mathscr{D}, \qquad i, j=0, 1,\dots, n-1,$$

$$v_{ij}(t, x)=v_{ji}(t, x) \quad \text{in} \quad V, \qquad i, j=0, 1,\dots, n-1.$$

Now we define $v_{in}(t, x)$ by $(4.6)_i'$ for $i=0, 1,\dots, n-1$. Next, we define $v_{ni}(t, x)$ $(i\neq n)$ by $v_{ni}(t, x)=v_{in}(t, x)$, $v_{nn}(t, x)$ by $(4.6)_i'$ replaced i by n. We see that any $v_{ij}(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ and satisfies

$$v_{ij}(t, x)=D_iD_ju(t, x) \qquad \text{in} \qquad \mathscr{D},$$

$$v_{ij}(t, x)=v_{ji}(t, x) \qquad \text{in} \qquad V.$$

Assume that we could construct $v_{i_1\dots i_k}(t, x)$ in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ $(i_1,\dots, i_k=0, 1,\dots, n;\ k=1, 2,\dots, m-1)$ with the properties that

$$v_{i_1\dots i_k}(t, x)=D_{i_1}\dots D_{i_k}u(t, x) \qquad \text{in} \qquad \mathscr{D}$$

and

$$v_{i_1\dots i_k}(t, x)=v_{j_1\dots j_k}(t, x) \qquad \text{in} \qquad V$$

where $(j_1,\dots, j_k)$ is a permutation of $(i_1,\dots, i_k)$. Under these assumptions, we construct $v_{i_1\dots i_m}(t, x)$.

In general, we operate $D_{i_1}\dots D_{i_m}$ on the both sides of (1.1). If $n\not\in\{i_1,\dots, i_m\}$, we have

$$(4.7)\quad \begin{cases} \dfrac{\partial}{\partial t}(D_{i_1}\dots D_{i_m}u)=L(t)(D_{i_1}\dots D_{i_m}u) \\ \qquad +\sum_{k=1}^{n}\sum_{p=1}^{m}(D_{i_p}A_k)(D_{i_1}\dots \widehat{D_{i_p}}\dots D_{i_m}D_ku) \\ \qquad +\sum_{n=1}^{k}\sum_{p,q}(D_{i_p}D_{i_q}A_k)(D_{i_1}\dots \widehat{D_{i_p}}\dots \widehat{D_{i_q}}\dots D_{i_m}D_ku) \\ \qquad +\cdots+\sum_{k=1}^{n}(D_{i_1}\dots D_{i_m}A_k)(D_ku) \end{cases}$$

$$\left\{\begin{aligned}&\qquad+\sum_{p=1}^{m}(D_{i_p}B)(D_{i_1}\dots\widehat{D_{i_p}}\dots D_{i_m}u)+\cdots\cdots\\&\qquad+(D_{i_1}\dots D_{i_m}B)u+(D_{i_1}\dots D_{i_m}f)\\&(D_{i_1}\dots D_{i_m}u)(0,x)=g_{i_1\dots i_m}(x)\\&P(D_{i_1}\dots D_{i_m}u)|_{x_n=0}=0\end{aligned}\right.$$

where $g_{i_1\dots i_m}(x)$ is an initial value of $(D_{i_1}\dots D_{i_m}u)(t,x)$, i.e., if we use functions $g^{(k)}(x)$ $(k=0,1,\dots,m)$ defined by (2.2) in §2,

$$g_{\underbrace{0\dots0}_{j}i_1\dots i_{m-j}}(x)=D_{i_1}\dots D_{i_{m-j}}g^{(j)}(x)\qquad(i_1,\dots,i_{m-j}\neq0;\ j=0,1,\dots,m).$$

If $n\in\{i_1,\dots,i_m\}$, $(D_{i_1}\dots D_{i_m}u)(t,x)$ don't satisfy (4.7). However such $(D_{i_1}\dots D_{i_m}u)$(t, x) have the following properties. Let the number of n contained in $\{i_1,\dots,i_m\}$ be r. We take out the index i_j of the r indices which are equal to n, and operate $D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}$ on the both sides of (1.1), then we get

$$\begin{aligned}(D_{i_1}\dots D_{i_m}u)(t,x)=A_n^{-1}\{&D_0D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}u\\&-\sum_{k=1}^{n-1}A_k(D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}D_ku)\\&-\sum_{k=1}^{n}\sum_{p\neq j}(D_{i_p}A_k)(D_{i_1}\dots\widehat{D_{i_p}}\dots\widehat{D_{i_j}}\dots D_{i_m}D_ku)-\cdots\\&-\sum_{k=1}^{n}(D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}A_k)(D_ku)-B(D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}u)\\&-\sum_{p\neq j}(D_{i_p}B)(D_{i_1}\dots\widehat{D_{i_p}}\dots\widehat{D_{i_j}}\dots D_{i_m}u)-\cdots\\&-(D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}B)u-(D_{i_1}\dots\widehat{D_{i_j}}\dots D_{i_m}f)\}.\end{aligned}\tag{4.7'}$$

The equality (4.7)′ shows that $D_{i_1}\dots D_{i_k}u(t,x)$ in the right hand side of (4.7)′ have the following properties

(1) if $k=m$ where k is the number of the indices $i_1,\dots,i_k$, then the number of n contained in $\{i_1,\dots,i_k\}$ is $r-1$,

(2) if $0\leqq k\leqq m-1$, then the number of n contained in $\{i_1,\dots,i_k\}$ is r at most.

Here we remark that (4.7)$'$ holds for any $i_1,\dots, i_{j-1}, i_{j+1},\dots, i_m$, and that $D_{i_1} \dots D_{i_m}u(t, x)$ is invariant by permutation of $(i_1,\dots, i_m)$, i.e.,

$$D_{i_1} \dots D_{i_m}u(t, x)=D_{j_1} \dots D_{j_m}u(t, x)$$

if $(j_1,\dots, j_m)$ is a permutation of $(i_1,\dots, i_m)$.

We will construct $v_{i_1\dots i_m}(t, x)$ as invariant by permutation of $(i_1, \dots, i_m)$. Therefore, it is sufficient to construct $v_{i_1\dots i_{m-j}\underbrace{n\dots n}_{j}}(t, x)$ $(i_1,\dots, i_{m-j} \neq n;\ j=0, 1,\dots, m)$.

Taking account of these, we define $v_{i_1\dots i_m}(t, x)$ as follows: $v_{i_1\dots i_m}(t, x)$ $(i_1, \dots, i_m=0, 1, \dots, n-1)$ are solutions of

$$(4.8)\quad \begin{cases} \dfrac{\partial}{\partial t}v_{i_1\dots i_m}=L(t)v_{i_1\dots i_m}+\alpha(t, x)\sum_{k=1}^{n}\sum_{p=1}^{m}(D_{i_p}A_k)v_{i_1\dots\hat{i}_p\dots i_m k} \\ \quad +\alpha(t, x)\sum_{k=1}^{n}\sum_{p,q}(D_{i_p}D_{i_q}A_k)v_{i_1\dots\hat{i}_p\dots\hat{i}_q\dots i_m k}+\cdots \\ \quad \cdots+\alpha(t, x)\sum_{k=1}^{n}(D_{i_1} \dots D_{i_m}A_k)v_k \\ \quad +\alpha(t, x)\sum_{p=1}^{m}(D_{i_p}B)v_{i_1\dots\hat{i}_p\dots i_m}+\cdots\cdots \\ \quad +\alpha(t, x)(D_{i_1} \dots D_{i_m}B)v+\alpha(t, x)(D_{i_1} \dots D_{i_m}f)(t, x) \\ v_{i_1\dots i_m}(0, x)=\alpha(x)g_{i_1\dots i_m}(x) \\ Pv_{i_1\dots i_m}|_{x_n=0}=0 \end{cases}$$

where each $v_{i_1\dots i_{m-1}n}(t, x)$ $(i_1,\dots, i_{m-1}\neq n)$ which appears in the lower order terms of the right hand side of (4.8) is replaced by a function

$$(4.8)'_{i_1\dots i_{m-1}}\quad \alpha(t, x)A_n^{-1}\Big\{v_{i_1\dots i_{m-1}0}-\alpha(t, x)\sum_{k=1}^{n-1}A_k v_{i_1\dots i_{m-1}k}$$
$$-\alpha(t, x)\sum_{k=1}^{n}\sum_{p=1}^{m-1}(D_{i_p}B)v_{i_1\dots\hat{i}_p\dots i_{m-1}k}-\cdots\cdots$$
$$-\alpha(t, x)\sum_{k=1}^{n}(D_{i_1} \dots D_{i_{m-1}}A_k)v_k-\alpha(t, x)Bv_{i_1\dots i_{m-1}}$$
$$-\alpha(t, x)\sum_{p=1}^{m-1}(D_{i_p}B)v_{i_1\dots\hat{i}_p\dots i_{m-1}}-\cdots\cdots$$
$$-\alpha(t, x)(D_{i_1} \dots D_{i_{m-1}}B)v-\alpha(t, x)(D_{i_1} \dots D_{i_{m-1}}f)\Big\}.$$

This is a system of equations for functions $v_{i_1\dots i_m}(t, x)$ $(i_1,\dots, i_m\neq n)$. Since $g(x)$ and $f(t, x)$ satisfy the compatibility conditions of order $(m-1)$, and since $g(x)$ and $f(0, x)$ are analytic on the support of $\alpha(x)$, we see that

$$\alpha(x)g_{i_1\dots i_m}(x)\in H^1(R^n_+), \qquad i_1, \dots, i_m=0, 1, \dots, n-1,$$

$$P(\alpha(x)g_{i_1\dots i_m}(x))|_{x_n=0}=0, \qquad i_1, \dots, i_m=0, 1, \dots, n-1.$$

Hence, applying Lemma 1 to (4.8), we can get a unique solution $\{v_{i_1\dots i_m}(t, x);\ i_1,\dots, i_m\neq n\}$ of (4.8) in $\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$ which satisfy

$$v_{i_1\dots i_m}(t, x)=D_{i_1}\dots D_{i_m}u(t, x) \qquad \text{in} \qquad \mathscr{D}$$

and

$$v_{i_1\dots i_m}(t, x)=v_{j_1\dots j_m}(t, x) \qquad \text{in} \qquad V$$

where $(j_1, \dots, j_m)$ is a permutation of $(i_1, \dots, i_m)$. Now we define $v_{i_1\dots i_{m-1}n}(t, x)$ $(i_1,\dots, i_{m-1}\neq n)$ by the function $(4.8)'_{i_1\dots i_{m-1}}$, and define $v_{i_1\dots i_k n i_{k+1}\dots i_{m-1}}$ $(k=0, 1, \dots, m-2)$ by

$$v_{i_1\dots i_k n i_{k+1}\dots i_{m-1}}=v_{i_1\dots i_{m-1}n}.$$

Then we see by $(4.8)'_{i_1\dots i_{m-1}}$ that $v_{i_1\dots i_{m-1}n}(t, x)$ is invariant by permutation of $(i_1, \dots, i_{m-1})$.

If $i_1, \dots, i_{m-2}\neq n$, we define $v_{i_1\dots i_{m-2}nn}$ by the function $(4.8)'_{i_1\dots i_{m-1}}$ replaced i_{m-1} by n. Any $v_{i_1\dots i_m}(t, x)$ whose number of n contained in $\{i_1, \dots, i_m\}$ is 2 is defined by $v_{i_1\dots i_m}(t, x)=v_{j_1\dots j_{m-2}nn}(t, x)$ where $(j_1, \dots, j_{m-2}, n, n)$ is a permutation of $(i_1,\dots, i_m)$. Since $(4.8)'_{i_1\dots i_{m-1}}$ shows that $v_{i_1\dots i_{m-2}nn}$ is invariant by permutation of $(i_1,\dots, i_{m-2})$, the above construction of $v_{i_1\dots i_m}$ is well-defined.

Similarly, by induction on the number of n contained in $\{i_1,\dots, i_m\}$, and by using $(4.8)'_{i_1\dots i_{m-1}}$, we can construct all $v_{i_1\dots i_m}(t, x)$ in $\mathscr{E}^1_t(L^2)\cap\mathscr{E}^0_t(H^1)$ which satisfy

$$v_{i_1\dots i_m}(t, x)=D_{i_1}\dots D_{i_m}u(t, x) \qquad \text{in} \qquad \mathscr{D}$$

and

$$v_{i_1\dots i_m}(t, x)=v_{j_1\dots j_m}(t, x) \quad \text{in} \quad V$$

where $(j_1,\dots, j_m)$ is a permutation of $(i_1,\dots, i_m)$.

§5. Proof of Theorem 2 (II)

In this section we estimate a series of functions $v, v_0, v_1,\dots, v_n, v_{ij}, \dots$, defined in §4. Before we proceed to the main subject, we state the notations. For any u and v in $\boldsymbol{C}^N$, we write $u\cdot v=\sum_{i=1}^{N} u_i v_i$ and $|u|=\sqrt{\bar{u}\cdot u}$. For any $N\times N$ matrix $C=[c_{ij}]$, we denote

$$|C|=\sup_{i,j}|c_{ij}|.$$

Then there exists a positive constant σ depending only on N such that

$$|Cu|\leqq\sigma|C|\cdot|u|, \qquad \text{any} \qquad u\in\boldsymbol{C}^N.$$

At first, we recall that the coefficients $A_k(t, x)$, $B(t, x)$ and the second member $f(t, x)$ are analytic on the support of $\alpha(t, x)$, and that $A_n(t, x)$ is non-singular, then it follows

$$\begin{aligned}
&|\alpha(t, x)D_{i_1}\dots D_{i_p}A_k(t, x)|\leqq p!a^pK, \qquad p=0, 1, 2,\dots,\\
(5.1)\qquad &|\alpha(t, x)D_{i_1}\dots D_{i_p}B(t, x)|\leqq p!a^pK, \qquad p=0, 1, 2,\dots,\\
&\|\alpha(t, x)D_{i_1}\dots D_{i_p}f(t, x)\|_{L^2(R^n_+)}\leqq p!a^pK, \qquad p=0, 1, 2,\dots,\\
&|\alpha(t, x)A_n^{-1}(t, x)|\leqq K
\end{aligned}$$

for any $(t, x)\in V$. Let us $2K\sigma=\gamma_1$ and assume $\gamma_1\geqq 2$ by taking K large.

Next, we set

$$(5.2)\qquad \phi_0(t)=\|v(t)\|,$$

$$\phi_m(t)=\sum_{i_1\dots i_m=0}^{n}\|v_{i_1\dots i_m}(t)\|, \qquad m=1, 2,\dots$$

and

$$(5.3)\qquad \phi_{m-p,p}(t)=\sum_{i_1,\dots,i_{m-p}=0}^{n-1}\|v_{i_1\dots i_{m-p}\underbrace{n\dots n}_{p}}(t)\|_{L^2(R^n_+)}, \qquad p=0, 1,\dots, m.$$

Since $g(x)$ and $f(t, x)$ are analytic in a neighborhood of the supports of $\alpha(x)$ and $\alpha(t, x)$ respectively, $(D_{i_1}\dots D_{i_m}u)(0, x)=g_{i_1\dots i_m}(x)$, defined in (4.7), are analytic in a neighborhood of the support of $\alpha(x)$. Hence we get

$$(5.4)\qquad \phi_{m,0}(0)=\sum_{i_1,\dots,i_m=0}^{n-1}\|\alpha(x)g_{i_1\dots i_m}(x)\|\leqq m!\rho^m A, \qquad m=0, 1, 2,\dots .$$

Under these conditions, we want to prove

Lemma 2. *Assume that* $\mu_0\geqq 1$, $c_0\geqq 1$, $d_0\geqq 1$[1], $an\ \geqq 1$ *and* $A\geqq 1$, *and take a positive constant* ρ *as larger than* $2an\{and_0\gamma_1(1+\gamma_1^2)^2+\gamma_1^2\}$, *then we have*

$$(5.5)\qquad \phi_m(t)\leqq m!\rho_0^m A_0, \qquad m=0, 1, 2, \dots,$$

where $A_0=3(1+\gamma_1^2)(c_0A+d_0\gamma_1)e^{\mu_0 t}$ *and* $\rho_0=2(1+3\gamma_1^2)\rho(1+t)exp\{and_0\gamma_1(1+\gamma_1^2)t\}$.

Hereafter we put $\theta=and_0\gamma_1(1+\gamma_1^2)$ and $\xi=3(1+\gamma_1^2)(c_0A+d_0\gamma_1)$.

Remark. When all or some of the positive numbers μ_0, c_0, d_0, an and A are smaller than 1, we can replace them by 1.

The proof of this lemma is divided into three steps.

The first step. We state an elementary fact without proof. *Let* $\varphi(t)$, $\psi(t)$ *and* $h(t)$ *be non-negative continuous functions defined on* $[0, \infty)$. *If they satisfy*

$$\varphi(t)\leqq h(t)+\int_0^t\psi(s)ds+d\int_0^t\varphi(s)ds, \qquad \text{any } t>0,$$

where d *is a positive constant, then we get*

$$\varphi(t)\leqq h(t)+d\int_0^t e^{d(t-s)}h(s)ds+\int_0^t e^{d(t-s)}\psi(s)ds$$

for any $t>0$.

The second step. Under the same assumptions as Lemma 2, we obtain the following estimate

1) The positive numbers μ_0, c_0 and d_0 are already determined in (2.3) in §2.

(5.6) $$\phi_{m,0}(t)+\phi_{m-1,1}(t)\leqq m!(\rho(1+t)e^{\theta t})^m A_0, \qquad m=1, 2, \dots .$$

We prove (5.6) by induction on m. First, we consider (4.1). Applying the inequality (2.3) to (4.1), we get

$$\|v(t)\|\leqq c_0 e^{\mu_0 t}\|\alpha(x)g(x)\|+d_0\int_0^t e^{\mu_0(t-s)}\|\alpha(s)f(s)\|ds.$$

Substituting (5.1) and (5.4) into this inequality, it follows

(5.7) $$\phi_0(t)\leqq k_0 e^{\mu_0 t},\ k_0=c_0 A+\frac{d_0\gamma_1}{2\mu_0}<c_0 A+d_0\gamma_1.$$

Next, we proceed to the equations (4.3) and (4.3)′. We consider in (4.3)

$$f_i(t, x)=\alpha(t, x)\sum_{j=1}^n (D_i A_j)v_j+\alpha(t, x)\,(D_i B)v$$
$$+\alpha(t, x)\ (D_i f), \qquad i=0, 1, \dots, n-1,$$

as the second members of (1.1). Applying (2.3) to (4.3), we obtain

$$\|v_i(t)\|\leqq c_0 e^{\mu_0 t}\|v_i(0)\|+d_0\int_0^t e^{\mu_0(t-s)}\|f_i(s)\|ds$$
$$\leqq c_0 e^{\mu_0 t}\|v_i(0)\|+d_0\int_0^t e^{\mu_0(t-s)}\{\sigma\sum_{j=1}^n|\alpha D_i A_j|\cdot\|v_j\|$$
$$+\sigma|\alpha D_i B|\cdot\|v\|+\|\alpha D_i f\|\}ds$$
$$\leqq c_0 e^{\mu_0 t}\|v_i(0)\|+aK\sigma d_0\int_0^t e^{\mu_0(t-s)}(\sum_{j=1}^n\|v_j\|+\|v\|+1)ds.$$

Summing up these inequalities from $i=0$ to $n-1$, we get

(5.8) $$\phi_{1,0}(t)\leqq c_0 e^{\mu_0 t}\phi_{1,0}(0)+and_0\gamma_1\int_0^t e^{\mu_0(t-s)}\{\phi_{1,0}(s)$$
$$+\phi_{0,1}(s)+\phi_0(s)+1\}ds.$$

From (4.3)′ we have

$$\|v_n(t)\|\leqq\sigma|\alpha A_n^{-1}|\,\{\|v_0\|+\sigma\sum_{1}^{n-1}|\alpha A_j|\cdot\|v_j\|+\sigma|\alpha B|\,\|v\|+\|\alpha f\|\}$$

$$\leqq (K\sigma)^2\{\sum_{j=0}^{n-1}\|v_j\|+\|v\|+1\},$$

which implies

$$\phi_{0,1}(t)\leqq\gamma_1^2\{\phi_{1,0}(t)+\phi_0(t)+1\}. \tag{5.9}$$

Combining (5.8) and (5.9), and substituting (5.7), we get

$$\begin{aligned}\phi_{1,0}(t)+\phi_{0,1}(t)&\leqq(1+\gamma_1^2)\phi_{1,0}(t)+\gamma_1^2\phi_0(t)+\gamma_1^2\\ &\leqq(1+\gamma_1^2)c_0e^{\mu_0t}\phi_{1,0}(0)+\gamma_1^2(1+k_0)e^{\mu_0t}\\ &+and_0\gamma_1(1+\gamma_1^2)\int_0^t e^{\mu_0(t-s)}\{1+k_0e^{\mu_0s}+\phi_{1,0}(s)\\ &+\phi_{0,1}(s)\}ds,\end{aligned}$$

from which it follows

$$\begin{aligned}e^{-\mu_0t}\{\phi_{1,0}(t)+\phi_{0,1}(t)\}&\leqq(1+\gamma_1^2)c_0A\rho+\gamma_1^2(1+k_0)\\ &+\theta\int_0^t e^{-\mu_0s}(1+k_0e^{\mu_0s})ds+\theta\int_0^t e^{-\mu_0s}\{\phi_{1,0}(s)+\phi_{0,1}(s)\}ds,\end{aligned} \tag{5.10}$$

where $\theta=and_0\gamma_1(1+\gamma_1^2)$. Applying the result of the first step to (5.10), we get

$$\begin{aligned}\phi_{1,0}(t)+\phi_{0,1}(t)&\leqq e^{(\theta+\mu_0)t}\{\rho c_0A(1+\gamma_1^2)+\gamma_1^2(1+k_0)\}\\ &+\theta\int_0^t e^{(\mu_0+\theta)(t-s)}(1+k_0e^{\mu_0s})ds\\ &\leqq e^{(\theta+\mu_0)t}(1+\gamma_1^2)\ (1+\rho)k_0,\end{aligned}$$

which means the inequality (5.6) in the case $m=1$.

Now we pass to general m. Assuming that the inequality (5.6) is true for $\phi_{1,0}(t)+\phi_{0,1}(t), \ldots, \phi_{m-1,0}(t)+\phi_{m-2,1}(t)$, we show that (5.6) is true for $\phi_{m,0}(t)+\phi_{m-1,1}(t)$.

From (4.8), we have

$$\|v_{i_1\ldots i_m}(t)\|\leqq c_0e^{\mu_0t}\|v_{i_1\ldots i_m}(0)\|+d_0\int_0^t e^{\mu_0(t-s)}\times \tag{5.11}$$

$$\times\{aK\sigma\sum_{k=1}^{n}\sum_{p=1}^{m}\|v_{i_1\dots\hat{i}_p\dots i_m k}\|$$

$$+2!a^2K\sigma\sum_{k=1}^{n}\sum_{p,q}\|v_{i_1\dots\hat{i}_p\dots\hat{i}_q\dots i_m k}\|+\cdots\cdots$$

$$+m!\,a^mK\sigma\sum_{k=1}^{n}\|v_k\|+aK\sigma\sum_{p=1}^{m}\|v_{i_1\dots\hat{i}_p\dots i_m}\|+\cdots\cdots$$

$$+m!\,a^mK\sigma\|v\|+m!\,a^mK\sigma\}ds,\quad i_1,\dots,i_m\neq n.$$

Summation of (5.11) with respect to $i_1,\dots,i_m$ from 0 to $n-1$ gives

$$\phi_{m,0}(t)\leqq c_0e^{\mu_0 t}\phi_{m,0}(0)+d_0\int_0^t e^{\mu_0(t-s)} \tag{5.12}$$

$$\times\{K\sigma\sum_{k=1}^{m}k!\tbinom{m}{k}(an)^k(\phi_{m+1-k,0}(s)+\phi_{m-k,1}(s))$$

$$+K\sigma\sum_{k=1}^{m}k!\tbinom{m}{k}(an)^k\phi_{m-k,0}(s)+m!(an)^mK\sigma\}ds$$

$$\leqq c_0e^{\mu_0 t}\phi_{m,0}(0)+d_0\gamma_1\int_0^t e^{\mu_0(t-s)}$$

$$\times\{\sum_{k=1}^{m}k!\tbinom{m}{k}(an)^k(\phi_{m+1-k,0}(s)+\phi_{m-k,1}(s))$$

$$+m!(an)^m(1+\phi_0(s))\}ds.$$

On the other hand, from $(4.8)'_{i_1\dots i_{m-1}}$ we have

$$\|v_{i_1\dots i_{m-1}n}\|\leqq K\sigma\{\|v_{0i_1\dots i_{m-1}}\|+K\sigma\sum_{k=1}^{n-1}\|v_{i_1\dots i_{m-1}k}\| \tag{5.13}$$

$$+aK\sigma\sum_{k=1}^{n}\sum_{p=1}^{m-1}\|v_{i_1\dots\hat{i}_p\dots i_{m-1}k}\|+\cdots\cdots$$

$$+(m-1)!a^{m-1}K\sigma\sum_{k=1}^{n}\|v_k\|+K\sigma\|v_{i_1\dots i_{m-1}}\|$$

$$+aK\sigma\sum_{p=1}^{m-1}\|v_{i_1\dots\hat{i}_p\dots i_{m-1}}\|+\cdots\cdots$$

$$+(m-1)!a^{m-1}K\sigma\|v\|+(m-1)!a^{m-1}K\sigma\}.$$

Summing up (5.13) with respect to $i_1,\dots,i_{m-1}$ from 0 to $n-1$, we get

(5.14) $$\phi_{m-1,1}(t) \leqq K\sigma\{K\sigma\phi_{m,0}(t) + K\sigma\sum_{k=1}^{m-1} k!\binom{m-1}{k}(an)^k(\phi_{m-k,0}(t)+\phi_{m-k-1,1}(t)) + K\sigma\sum_{k=0}^{m-1} k!\binom{m-1}{k}(an)^k\phi_{m-k-1,1}(t) + (m-1)!(an)^{m-1}(1+\phi_0(t))\}.$$

Combining (5.12) and (5.14), we have

(5.15) $$\begin{aligned}\phi_{m,0}(t)+\phi_{m-1,1}(t) &\leqq (1+\gamma_1^2)\phi_{m,0}(t) \\ &+\gamma_1^2(m-1)!(an)^{m-1}(1+\phi_0(t)) \\ &+\gamma_1^2\sum_{k=1}^{m-1} k!\binom{m-1}{k}(an)^k(\phi_{m-k,0}(t)+\phi_{m-k-1,1}(t)) \\ &\leqq c_0(1+\gamma_1^2)e^{\mu_0 t}\phi_{m,0}(0)+(m-1)!(an)^{m-1}(1+\phi_0(t))\gamma_1^2 \\ &+\gamma_1^2\sum_{k=1}^{m-1} k!\binom{m-1}{k}(an)^k(\phi_{m-k,0}(t)+\phi_{m-k-1,1}(t)) \\ &+d_0\gamma_1(1+\gamma_1^2)\int_0^t e^{\mu_0(t-s)}\{m!(an)^m(1+\phi_0(s)) \\ &+\sum_{k=2}^{m} k!\binom{m}{k}(an)^k(\phi_{m-k+1,0}(s)+\phi_{m-k,1}(s)\}ds \\ &+anmd_0\gamma_1(1+\gamma_1^2)\int_0^t e^{\mu_0(t-s)}\{\phi_{m,0}(s)+\phi_{m-1,1}(s)\}ds\ .\end{aligned}$$

We apply the result of the first step to (5.15) and use

$$\begin{aligned}&\int_0^t e^{(\mu_0+\theta m)(t-s)}(\phi_{m-k,0}(s)+\phi_{m-1-k,1}(s))ds \\ &\leqq e^{(\mu_0+\theta m)t}\xi\cdot\rho^{m-k}(1+t)^{m-k+1}(m-k-1)!, \qquad k=1,2,\dots,m-1,\end{aligned}$$

which is lead by the fact that $\phi_{m-k,0}(t)+\phi_{m-k-1,1}(t)$ $(k=1, 2, \dots, m-1)$ satisfy (5.6). And moreover, using the estimates (5.4) of $\phi_{m,0}(0)$, we finally get

(5.16) $$\phi_{m,0}(t)+\phi_{m-1,1}(t)\leqq m!(\rho(1+t)e^{\theta t})^m\xi e^{\mu_0 t}K_m,$$

where

$$K_m=\frac{c_0A(1+\gamma_1^2)}{\xi}+(\gamma_1^2+\theta\gamma_1^2+\theta)\sum_{k=1}^{m}\left(\frac{an}{\rho}\right)^k.$$

As

$$\rho\geqq 2\,an\{and_0\gamma_1(1+\gamma_1^2)^2+\gamma_1^2\}=2\,an(\gamma_1^2+\theta\gamma_1^2+\theta)$$

and

$$\xi=3(1+\gamma_1^2)\ (c_0A+d_0\gamma_1),$$

we see that

$$K_m<\frac{1}{3}+\frac{1}{2}\sum_{k\geqq 0}\left(\frac{an}{\rho}\right)^k<1.$$

This completes the proof of (5.6).

The third step. Let l be a fixed positive integer, and r, R be functions of t which satisfy $r\geqq\rho(1+t)e^{\theta t}$ and $R\geqq A_0=\xi e^{\mu_0 t}$. Assume that

(5.17) $$\sum_{i=0}^{l-1}\phi_{m-i,i}(t)\leqq m!r^mR,\qquad m=1,2,\dots,$$

where $\phi_{ij}(t)=0$ if $i<0$ or $j<0$, then we get

(5.18) $$\sum_{i=0}^{l}\phi_{m-i,i}(t)\leqq(1+3\gamma_1^2)m!r^mR,\qquad m=1,2,\dots.$$

For this proof, we return to the inequality (5.13). From the definition of $v_{i_1\dots i_m}(t,x)$, we see that (5.13) is true for any $i_1,\dots,i_{m-1}$. So we put $i_{m-l+1}=\dots=i_{m-1}=n$ and sum up (5.13) with respect to $i_1, i_2,\dots, i_{m-l}$ from 0 to $n-1$, then

(5.19) $$\phi_{m-l,l}(t)\leqq\gamma_1^2\{\phi_{m+1-l,l-1}(t)+(m-1)!(an)^{m-1}(1+\phi_0(t))$$
$$+\sum_{k=1}^{m-1}k!\binom{m-1}{k}(an)^k\sum_{i=0}^{l}\phi_{m-k-i,i}(t)\},\qquad l=1,2,\dots,m.$$

Put

$$\lambda_{m-l,l}=\phi_{m-l,l}(t)/m!r^m R, \qquad l=0, 1,\dots, m.$$

and, if $i<0$ or $j<0$,

$$\lambda_{i,j}=0.$$

Since

$$\frac{1}{m!r^m R}\sum_{i=0}^{l}\phi_{m-i,i}(t)\leqq 1+\lambda_{m-l,l},$$

it follows from (5.19)

$$\begin{aligned}(5.20)\qquad \lambda_{m-l,l}&\leqq\gamma_1^2\Big\{1+\frac{1}{anm}\Big(\frac{an}{r}\Big)^m \\ &\quad+\sum_{k=1}^{m-1}k!\tbinom{m-1}{k}(an)^k\frac{(m-k)!}{m!r^k}(1+\lambda_{m-k-l,l})\\ &\leqq\gamma_1^2\Big\{1+\frac{1}{anm}\Big(\frac{an}{r}\Big)^m+\sum_{k-1}^{m-1}\Big(\frac{an}{r}\Big)^k(1+\lambda_{m-k-l,l})\\ &\leqq\gamma_1^2\Big\{\sum_{k=0}^{m}\Big(\frac{an}{\rho}\Big)^m+\sum_{k=1}^{m-l}\Big(\frac{an}{\rho}\Big)^k\lambda_{m-k-l,l}\Big\}\\ &\leqq 2\gamma_1^2+\gamma_1^2\sum_{k=1}^{m-l}\Big(\frac{an}{\rho}\Big)^k\lambda_{m-k-l,l},\end{aligned}$$

because $\rho>2an(\gamma_1^2+\theta\gamma_1^2+\theta)>2an$.

We solve (5.20) by induction on $m-l$, then

$$(5.21)\qquad \lambda_{m-l,l}\leqq\lambda_{0,l}(1+\gamma_1^2)^m\Big(\frac{an}{\rho}\Big)^m+2\gamma_1^2\sum_{k=0}^{m-1}(1+\gamma_1^2)^k\Big(\frac{an}{\rho}\Big)^k.$$

On the other hand, we put $m=l$ in (5.19) and substitute (5.17) into (5.19), then

$$\begin{aligned}\phi_{0,l}(t)\leqq\gamma_1^2\{&l!r^l R+(l-1)!(an)^{l-1}k_0e^{\mu_0 t}\\ &+\sum_{k=1}^{l-1}k!\tbinom{l-1}{k}(an)^k(l-k)!r^{l-k}R\}\end{aligned}$$

$$\leqq \gamma_1^2 l!r^l R \sum_{k=0}^{l}\left(\frac{an}{r}\right)^k < 2\gamma_1^2 l!r^l R,$$

which means

(5.22) $$\lambda_{0,l} \leqq 2\gamma_1^2.$$

Substituting (5.22) into (5.21), then we have

$$\lambda_{m-l,l} \leqq 2\gamma_1^2 \sum_{k=0}^{m}(1+\gamma_1^2)^k\left(\frac{an}{\rho}\right)^k \leqq 2\gamma_1^2 \sum_{k=0}^{m}\left(\frac{1}{2\theta}\right)^k$$

$$\leqq 3\gamma_1^2,$$

which implies

$$\phi_{m-l,l}(t) \leqq 3\gamma_1^2 m!r^m R.$$

Thus we get (5.18). This completes the proof of the 3rd step.

Proof of Lemma 2. Using the results of the 2nd and 3rd steps, we have

$$\sum_{i=0}^{m}\phi_{m-i,i}(t) \leqq m!(1+3\gamma_1^2)^m(\rho(1+t)e^{\theta t})^m A_0$$

$$\leqq m!\left(\frac{1}{2}\rho_0\right)^m A_0.$$

Since

$$\phi_m(t) = \sum_{i=0}^{m}\binom{m}{i}\phi_{m-i,i}(t) < 2^m \sum_{i=0}^{m}\phi_{m-i,i}(t),$$

we have

$$\phi_m(t) < m!\rho_0^m A_0.$$

This completes the proof of Lemma 2. q.e.d.

Now we prove Theorem 2. For this purpose we mention Sobolev's well known lemma: There exists a positive constant $c(n)$ depending only on the dimention of the space such that

$$\sup_{x\in R^n}|u(x)|\leqq c(n)\sum_{|\alpha|\leqq[\frac{n}{2}]+1}\|D_x^\alpha u\|$$

where $\|\cdot\|$ denotes the norm of $L^2(R^n)$.

Let S be the ball with center (t_0, x_0) whose radius is so small that S is contained in $\mathscr{D}$. Let $\beta(t, x)$ be a function in $C_0^\infty(R^{n+1})$ with the properties that $0\leqq\beta(t, x)\leqq 1$, and that it takes the value 1 on the set S, and that its support is contained in V, then

$$\sup_{(t,x)\in S}|u(t, x)|\leqq\sup_{(t,x)\in B}|\beta(t, x)u(t, x)|$$

$$\leqq C\sum_{|\alpha|\leqq[\frac{n+1}{2}]+1}\|D_{t,x}^\alpha u\|_{L^2(B)}$$

where $B=$the support of $\beta(t, x)$ and C depends only on β and n. Since $v_{i_1\dots i_m}(t, x)$ coincides with $D_{i_1}\dots D_{i_m}u(t, x)$ in $\mathscr{D}$, we have

$$\sum_{i_1,\dots,i_m=0}^{n}\sup_S|D_{i_1}\dots D_{i_m}u(t, x)|$$

$$\leqq\text{const.}\sum_{i_1,\dots,i_m}\sum_{|\alpha|\leqq[\frac{n+1}{2}]+1}\|D_{t,x}^\alpha D_{i_1}\dots D_{i_m}u\|_{L^2(B)}$$

$$\leqq\text{const.}\sum_{i_1,\dots,i_m,i_{m+1},\dots,i_{m+[\frac{n+1}{2}]+1}}\|v_{i_1\dots i_m i_{m+1}\dots i_{m+[\frac{n+1}{2}]+1}}\|_{L^2(B)}$$

$$\leqq\text{const.}\left(m+\left[\frac{n+1}{2}\right]+1\right)!\rho_0^{m+[\frac{n+1}{2}]+1}A_0.$$

This inequality shows that $u(t, x)$ is analytic with respect to (t, x) in S. The proof of Theorem 2 is complete. q.e.d.

§6. Symmetric hyperbolic systems I

From now on, we consider the symmetric hyperbolic systems with maximally non-positive boundary conditions, stated below, which are the typical examples to which we can apply our preceding results. We prove in §6 and §7 that the solutions of these problems satisfy the condition C.2, and in §8 that they satisfy C.3.

We consider the mixed problems (1.1) under the following conditions.

B.1) *The coefficients of* $L(t)$ *are in* $\mathscr{B}^2(\bar{V})$. $A_i(t, x)$, $i=1, 2,\dots, n$, *are hermitian and* $|\det A_n(t, x)|\geqq\delta>0$ *where* δ *is independent of* t *and* x.

B.2) $P(t, x')$ *is an* $l\times N$ *matrix which satisfies the following properties*

1. $P(t, x')$ *is in* $\mathscr{B}^2(\overline{R^1_+}\times R^{n-1})$ *and it is constant outside a compact set in* $\overline{R^1_+}\times R^{n-1}$,
2. *rank* $P(t, x')=l$ *for each* $t\geqq 0$, $x'\in R^{n-1}$,
3. $\ker P(t, x')$ *is maximal non-positive for* $L(t)$ *at each* $t\geqq 0$, $x'\in R^{n-1}$, $x_n=0$, i.e., *we assume that*

$$u\cdot\overline{(-A_n u)}\leqq 0,\ u\in\ker P,\ t\geqq 0,\ x'\in R^{n-1},\ x_n=0,$$

and that $\ker P(t, x')$ *is not properly contained in any other subspace having this property.*

In this section we treat only the case where the coefficients of $L(t)$ and $P(t, x')$ are independent of t, and in the next section the case where they depend on t.

We define the domain $\mathscr{D}(L)$ of L by the graph norm closure of the set $\{u(x)\in H^1(R^n_+);\ Pu|_{x_n=0}=0\}$ where the graph norm of u is defined by $\|Lu\|+\|u\|$. As $\ker P$ is maximal non-positive for L, there exists a positive constant μ_0 which satisfies

$$(6.1)\qquad (Lu, u)+(u, Lu)\leqq 2\mu_0\|u\|^2,\qquad u\in\mathscr{D}(L),$$

because for any $u\in H^1(R^n_+)\cap\mathscr{D}(L)$ it holds

$$\begin{aligned}(Lu, u)+(u, Lu)&=(u, \Big(B+B^*-\sum_{i=1}^n\frac{\partial A_i}{\partial x_i}\Big)u)\\ &\quad-\int_{R^{n-1}}u(x', 0)\cdot\overline{A_n(x', 0)u(x', 0)}dx'\\ &\leqq\sigma\cdot\sup_{x\in R^{n-1}}\Big|B+B^*-\sum_{i=1}^n\frac{\partial A_i}{\partial x_i}\Big|\cdot\|u\|^2.\end{aligned}$$

Using Theorem 3.2 of Lax and Phillips [3], we see that L generates a unique semi-group $T(t)=e^{Lt}$, $t\geqq 0$, in $L^2(R^n_+)$ which satisfies

$$\|T(t)\|\leqq e^{\mu_0 t},\qquad \text{any } t\geqq 0.$$

Using this fact, we prove

Lemma 3. *If* $g(x)\in\mathscr{D}(L)$ *and* $f(t,x)\in\mathscr{E}_t^1(L^2)$, *there exists a unique solution* $u(t,x)$ *of* (1.1) *in* $\mathscr{E}_t^1(L^2)\cap\mathscr{D}(L)$ *which satisfies the following inequalities*

$$\|u(t)\|\leqq e^{\mu_0 t}\|g\|+\int_0^t e^{\mu_0(t-s)}\|f(s)\|ds, \tag{6.2}$$

$$\|u'(t)\|\leqq e^{\mu_0 t}\|Lg+f(0)\|+\int_0^t e^{\mu_0(t-s)}\|f'(s)\|ds. \tag{6.2$'$}$$

Proof. The solution $u(t,x)$ is represented as follows

$$u(t,x)=T(t)g+\int_0^t T(t-s)f(s)ds. \tag{6.3}$$

The differentiation of (6.3) with respect to t gives

$$u'(t,x)=T(t)(Lg+f(0))+\int_0^t T(t-s)f'(s)ds. \tag{6.3$'$}$$

We obtain (6.2) and (6.2)′ from (6.3) and (6.3)′. q.e.d.

Theorem 3. *For any* $g(x)\in H^1(R_+^n)\cap\mathscr{D}(L)$ *and for any* $f(t,x)\in\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ *there exists a unique solution* $u(t,x)$ *of* (1.1) *in* $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)\cap\mathscr{D}(L)$ *which satisfies* (6.2), (6.2)′ *and the following inequalities*

$$|||u(t)|||_{1,0}\leqq e^{\mu_1 t}|||u(0)|||_{1,0}+d\int_0^t e^{\mu_1(t-s)}|||f(s)|||_{1,0}ds, \tag{6.4}$$

$$|||u(t)|||_1\leqq c_1 e^{\mu_1 t}|||u(0)|||_1+d_1\int_0^t e^{\mu_1(t-s)}|||f(s)|||_1 ds, \tag{6.5}$$

where μ_1, c_1, d *and* d_1 *are positive constant independent of* $u(t,x)$, $g(x)$, $f(t,x)$ and t.

Remark. In (6.4), the coefficients of $e^{\mu_1 t}|||u(0)|||_{1,0}$ is 1. This fact is indispensable to use the method of the Cauchy's polygonal line for the proof of the existence of the solution of (1.1) in the case where

$L(t)$ depends on t. In §7 we shall give this reason in detail.

Proof. As in the preceding sections, we proceed with our discussions under the condition $P(t', x)=[E_l\ 0]$. As $A_n(t, x)$ is non-singular, we investigate the differentiability of the solution $u(t, x)$ with respect to t and $x'=(x_1, \ldots, x_{n-1})$. At first, we assume the following additional condition

(A) $g(x)$ and $f(t, x)$ are in $H^2(R^n_+)$ and $\mathscr{E}^2_t(L^2)\cap\mathscr{E}^1_t(H^1)$ respectively and they satisfy the compatibility conditions of order 1.

Since the condition (A) implies that $Lg+f(0)\in\mathscr{D}(L)$ and $f'(t)\in\mathscr{E}^1_t(L^2)$, we see from (6.3)′ that $D_0u(t, x)$ is in $\mathscr{E}^1_t(L^2)\cap\mathscr{D}(L)$. Differentiating (1.1) with respect to $x_i(i=1, 2, \ldots, n-1)$, and using Lemma 1.1 of Lax and Phillips [3], we see that $(D_iu)(t, x)$ $(i=1,\ldots, n-1)$ are in $\mathscr{E}^0_t(H^{-1}(R^n_+))$ and $\mathscr{E}^0_{x_n}(H^{-2}(R^{n-1}))$ at each $t\geqq 0$, and that they saitsfy

$$(6.6)\qquad \begin{cases} \dfrac{\partial}{\partial t}(D_iu)=L(D_iu)+\displaystyle\sum_{j=1}^{n}(D_iA_j)\frac{\partial u}{\partial x_j}+(D_iB)u \\ \qquad\qquad +(D_if), \qquad i=1, 2,\ldots, n-1 \\ (D_iu)(0, x)=(D_ig)(x) \qquad , \qquad i=1, 2,\ldots, n-1 \\ P(D_iu)|_{x_n=0}=0 \qquad , \qquad i=1, 2,\ldots, n-1 \end{cases}$$

where the first of (6.6) holds in the distribution sense. Let us put $U(t, x)={}^t({}^tu, {}^tD_0u, {}^tD_1u,\ldots, {}^tD_{n-1}u)$, then $U(t, x)$ is in $\mathscr{E}^0_t(H^{-1}(R^n_+)$ and $\mathscr{E}^0_{x_n}(H^{-2}(R^{n-1}))$ at each $t\geqq 0$. Substituting the relation

$$(6.6)'\qquad D_nu(t, x)=A_n^{-1}\{D_0u-\sum_{j=1}^{n-1}A_jD_ju-Bu-f\}$$

into (6.6), we see that $U(t, x)$ satisfies

$$(6.7)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=\tilde{L}U+F & \text{in the distribution sense} \\ U(0, x)=G(x) & \text{in } H^{-1}(R^n_+) \\ \tilde{P}U|_{x_n=0}=0 & \text{in } H^{-2}(R^{n-1}) \end{cases}$$

where $\tilde{L}$, $\tilde{P}$, F and G are defined by (3.2) in §3.

Similarly, we define the domain $\mathscr{D}(\tilde{L})$ of $\tilde{L}$ by the graph norm closure of $\{u(x)\in H^1(R^n_+);\ \tilde{P}u|_{x_n=0}=0\}$. Since the condition B.2 assures that ker $\tilde{P}$ is also maximal non-negative for $\tilde{L}$, $\tilde{L}$ also generates a semi-group $\tilde{T}(t)=e^{\tilde{L}t}$, $t\geqq 0$, which satisfies

$$\|\tilde{T}(t)\|\leqq e^{\mu_1 t}, \qquad \text{any } t\geqq 0.$$

The above positive constant μ_1 is determined by the relation

$$(6.8) \qquad (\tilde{L}u, u)+(u, \tilde{L}u)\leqq 2\mu_1\|u\|^2, \qquad \text{any } u\in\mathscr{D}(\tilde{L}).$$

As the condition (A) implies that $G(x)$ is in $H^1(R^n_+)\cap\mathscr{D}(\tilde{L})$ and $F(t, x)$ is in $\mathscr{E}^1_t(L^2)$, we get a unique solution $V(t, x)$ of (6.7) in $\mathscr{E}^1_t(L^2)\cap\mathscr{D}(\tilde{L})$, which satisfies

$$(6.9) \qquad \|V(t)\|\leqq e^{\mu_1 t}\|G\|+\int_0^t e^{\mu_1(t-s)}\|F(s)\|ds.$$

Our assertion is to show $U(t, x)=V(t, x)$. For this purpose we apply the method used by K. Kajitani [2]. We put $U-V$ by W, then W is in $\mathscr{E}^0_t(H^{-1}(R^n_+))$ and $\mathscr{E}^0_{x_n}(H^{-2}(R^{n-1}))$ at each $t\geqq 0$, and it satisfies

$$(6.10) \qquad \begin{cases} \dfrac{\partial W}{\partial t}=\tilde{L}W & \text{in the distribution sense} \\ W(0, x)=0 & \text{in } H^{-1}(R^n_+) \\ \tilde{P}W|_{x_n=0}=0 & \text{in } H^{-2}(R^{n-1}). \end{cases}$$

We write the Laplace transform of $u(t, x)$ with respect to t by $\hat{u}(\tau, x)$, i.e.,

$$\hat{u}(\tau, x)=\int_0^\infty e^{-\tau t}u(t, x)dt, \quad \tau=\mu+i\sigma,\ \mu>0.$$

If we take the real part μ of τ as larger than μ_1, we can perform the Laplace transform of $u(t, x)$, $D_0u(t, x)$ and $V(t, x)$ because (6.2), (6.2)′ and (6.9) hold. Moreover, since

$$\|D_iu(t)\|_{H^{-1}(R^n_+)}\leqq\|u(t)\|_{L^2(R^n_+)}, \qquad i=1, 2,\dots, n-1,$$

we can also perform the Laplace transform of $D_iu(t, x)$ $(i=1,\dots, n-1)$ in the distribution sense. Hence we can perform the Laplace transform of $W(t, x)$ and from (6.10) we get

$$(6.11) \qquad (\tau-\tilde{L})\hat{W}(\tau, x)=0 \quad \text{in } H^{-1}(R_+^n).$$

We define Λ_τ by $(|\tau|^2-\Delta_{x'})^{-\frac{1}{2}}$, where $\Delta_{x'}=D_1^2+\cdots+D_{n-1}^2$. Then we see that $\Lambda_\tau\hat{W}(\tau, x)$ is in $\mathscr{D}(\tilde{L})$ and satisfies

$$(6.11)' \qquad (\tau-\tilde{L})\ (\Lambda_\tau\hat{W})=[\Lambda_\tau, \tilde{L})\hat{W} \quad \text{in } L^2(R_+^n)$$

where $[\Lambda_\tau, \tilde{L}]=\Lambda_\tau\tilde{L}-\tilde{L}\Lambda_\tau$.
Using (6.8), we get

$$(6.12) \qquad \|(\tau-\tilde{L})U\|\geqq(\operatorname{Re}\tau-\mu_1)\|U\|,\ U\in\mathscr{D}(\tilde{L}),\ \operatorname{Re}\tau>\mu_1,$$

because

$$\begin{aligned}\|(\tau-\tilde{L})U\|\cdot\|U\| &\geqq \operatorname{Re}((\tau-\tilde{L})U, U)\\ &=\operatorname{Re}\tau(U, U)-\operatorname{Re}(\tilde{L}U, U)\\ &\geqq(\operatorname{Re}\tau-\mu_1)\|U\|^2.\end{aligned}$$

Applying (6.12) to (6.11)′, we have

$$(6.13) \qquad (\operatorname{Re}\tau-\mu_1)\|\Lambda_\tau\hat{W}(\tau)\|\leqq\|[\Lambda_\tau, \tilde{L}]\hat{W}(\tau)\|.$$

Moreover, we get

$$(6.14) \qquad \|[\Lambda_\tau, \tilde{L}]\hat{W}(\tau)\|\leqq K\|\Lambda_\tau\hat{W}(\tau)\|$$

where K is independent of τ. We explain this fact. If we put

$$\tilde{L}=\sum_{i=1}^n\tilde{A}_i(x)D_i+\tilde{B}(x)=\tilde{L}_1+\tilde{A}_n(x)D_n,$$

we see that

$$(6.15) \qquad [\Lambda_\tau, \tilde{L}]\hat{W}=[\Lambda_\tau, \tilde{L}_1]\hat{W}+[\Lambda_\tau, \tilde{A}_n]D_n\hat{W}.$$

From (6.11), we have

$$(6.16) \qquad D_n\hat{W}(\tau, x)=\tilde{A}_n{}^{-1}(\tau-\tilde{L}_1)\hat{W}(\tau, x).$$

Substituting (6.16) into (6.15), we get

$$[\Lambda_\tau, \tilde{L}]\hat{W}=[\Lambda_\tau, \tilde{L}_1]\hat{W}+[\Lambda_\tau, \tilde{A}_n]\tilde{A}_n^{-1}(\tau-\tilde{L}_1)\hat{W}.$$

Calculating the commutators $[\Lambda_\tau, \tilde{L}_1]$ and $[\Lambda_\tau, \tilde{A}_n]$, we get (6.14). Accordingly, taking the complex number τ as $\mathrm{Re}\,\tau>\mu_1+K$ and using (6.13) and (6.14), we have

$$(\Lambda_\tau\hat{W})(\tau, x)=0, \text{ i.e., } \hat{W}(\tau, x)=0,$$

which implies

$$v_i(t, x)=D_i u(t, x) \quad \text{in } V, \qquad i=0, 1,\dots, n.$$

Hence we see that $u(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$.

Next, we prove the inequalities (6.4) and (6.5). Since $U(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{D}(\tilde{L})$, it follows

$$\begin{aligned}\frac{d}{dt}\|U(t)\|^2&=2\|U(t)\|\frac{d}{dt}\|U(t)\|\\&=2\mathrm{Re}(\tilde{L}U(t), U(t))+2\mathrm{Re}(U(t), F(t))\\&\leqq 2\mu_1\|U(t)\|^2+2\|U(t)\|\cdot\|F(t)\|,\end{aligned}$$

from which it follows

$$\|U(t)\|\leqq e^{\mu_1 t}\|U(0)\|+\int_0^t e^{\mu_1(t-s)}\|F(s)\|ds. \tag{6.17}$$

As $\|U(t)\|=|\!|\!|u(t)|\!|\!|_{1,0}$, (6.17) means the inequality (6.4).
From (6.6)′ we get

$$\|D_n u(t)\|\leqq \text{const.}(|\!|\!|u(t)|\!|\!|_{1,0}+\|f(t)\|). \tag{6.18}$$

Combining (6.17) and (6.18) and using (3.7), we get (6.5).

At last, we remove the additional assumption (A). Let $g(x)\in H^1(R_+^n)\cap\mathscr{D}(L)$ and $f(t, x)\in\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Then we can choose the functions $g_m(x)$ and $f_m(t, x)$, $m=1, 2,\dots$, such that $g_m(x)$ and $f_m(t, x)$ satisfy (A) and

$$\begin{aligned}g_m&\longrightarrow g &&\text{in } H^1(R_+^n),\\ f_m&\longrightarrow f &&\text{in } \mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1).\end{aligned}$$

Denote by $u_m(t, x)$ the solution of (1.1) for the Cauchy data $g_m(x)$ and the second member $f_m(t, x)$. Then the energy inequality (6.5) gives

$$|||u_m(t)-u_{m'}(t)|||_1 \leqq c_1 e^{\mu_1 t}|||u_m(0)-u_{m'}(0)|||_1$$
$$+d_1\int_0^t e^{\mu_1(t-s)}|||f_m(s)-f_{m'}(s)|||_1 ds,$$

which shows that $\{u_m(t, x)\}$ converges in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Denote its limit by $u(t, x)$, then we see that $u(t, x)$ is the required solution of (1.1). This completes the proof of Theorem 3. q.e.d.

§7. Symmetric hyperbolic systems II

In this section we consider the case where the coefficients of $L(t)$ and $P(t, x')$ depend on t. Of course, we assume the conditions B.1 and B.2 stated in §6. Using the results of Theorem 3, we prove

Theorem 4. *For any* $g(x)\in H^1(R_+^n)\cap\mathscr{D}(L(0))$ *and for any* $f(t, x)\in\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$, *there exists a unique solution* $u(t, x)$ *of* (1.1) *in* $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)\cap\mathscr{D}(L(t))$ *which satisfies the following energy inequalities*

(7.1) $$\|u(t)\|\leqq e^{\mu_0 t}\|g\|+\int_0^t e^{\mu_0(t-s)}\|f(s)\|ds,$$

(7.2) $$|||u(t)|||_1\leqq c_1 e^{\mu t_1}|||u(0)|||_1+d_1\int_0^t e^{\mu_1(t-s)}|||f(s)|||_1 ds.$$

Before the proof of Theorem 4, we state the lemma which is necessary for the following discussions.

Lemma 4. *If* $u(t, x)\in\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)\cap\mathscr{D}(L(t))$ *is a solution of* (1.1), *then* $u(t, x)$ *satisfies the energy inequalities* (7.1) *and* (7.2).

Proof. Assume that $u(t, x)$ is in $\mathscr{E}_t^2(L^2)\cap\mathscr{E}_t^1(H^1)\cap\mathscr{E}_t^0(H^2)$, then we can prove (7.1) and (7.2) in the same manner as §6. Next, we remove the additional condition by using the mollifier with respect to

(t, x'). As we shall use the same method in the proof of Theorem 4, we omit this process here. q.e.d.

Proof of Theorem 4. We prove only the existence of a solution of (1.1) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Similarly we discuss under the condition $P(t, x')=[E_l\ 0]$. Assume that, if $g(x)\equiv 0$ and $f(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ and $f(0, x)\equiv 0$, there exists a unique solution of (1.1) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Then we see that Theorem 4 is true. We explain this fact. First, we assume that $g(x)\in H^3(R_+^n)$ and $f(t, x)\in\mathscr{E}_t^2(L^2)\cap\mathscr{E}_t^1(H^1)\cap\mathscr{E}_t^0(H^2)$, and that they satisfy the compatibility conditions of order 1. Put $v(t, x)=u(t, x)-g(x)-(L(0)g+f(0, x))t$, then $v(t, x)$ satisfies

$$
\left\{
\begin{array}{l}
\dfrac{\partial v}{\partial t}=L(t)v+F(t, x)\\
v(0, x)=0\\
Pv|_{x_n=0}=0
\end{array}
\right.
\tag{7.3}
$$

where

$$F(t, x)=f(t)+L(t)g+tL(t)\,(L(0)g+f(0))-L(0)g-f(0).$$

Since $F(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ and $F(0, x)\equiv 0$, we see by the assumption that there exists a unique solution $v(t, x)$ of (7.3) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Hence there exists a unique solution $u(t, x)$ of (1.1) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ for the above-mentioned $g(x)$ and $f(t, x)$. Next, we remove the additional assumption on g and f. For any $g(x)\in H^1(R_+^n)\cap\mathscr{D}(L(0))$ and $f(t, x)\in\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$, we can choose the sequences $\{g_m(x)\}$ in $H^3(R_+^n)$ and $\{f_m(t, x)\}$ in $\mathscr{E}_t^2(L^2)\cap\mathscr{E}_t^1(H^1)\cap\mathscr{E}_t^0(H^2)$ such that $g_m(x)$ and $f_m(t, x)$ satisfy the compatibility conditions of order 1 and

$$
\begin{aligned}
g_m(x)&\longrightarrow g(x) && \text{in } H^1(R_+^n),\\
f_m(t, x)&\longrightarrow f(t, x) && \text{in } \mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1).
\end{aligned}
$$

Denote by $u_m(t, x)$ the solution of (1.1) for the Cauchy data g_m and the second member $f_m(t, x)$. Using (7.2), we see that $\{u_m(t, x)\}$ converges in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$. Denote its limit by $u(t, x)$, then we see that $u(t, x)$ is the required solution of (1.1). Therefore it suffices to prove the existence of a solution of (1.1) in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ on the assumption that $g(x)\equiv 0$ and $f(t, x)$ is in $\mathscr{E}_t^1(L^2)\cap\mathscr{E}_t^0(H^1)$ and $f(0, x)\equiv 0$.

The reason why we reduce the general case to the case where $g(x)\equiv 0$ and $f(0, x)\equiv 0$ will be explained at the end of this proof.

We use the method of the Cauchy's polygonal line. Let

$$\Delta_k:\ 0=t_0<t_1<t_2<\cdots<t_k=T$$

be the subdivision of $[0, T]$ into k parts of equal length. $u_k(t, x)$ is the Cauchy's polygonal line for this subdivision, which is constructed as follows: Let $u_{k,i}(t, x)$, defined for $t\in[t_i, t_{i+1}]$, be the solution of

$$(7.4)\qquad \begin{cases} \dfrac{\partial}{\partial t}u_{k,i}=L(t_i)u_{k,i}+f(t, x) & i=0, 1,\dots, k-1 \\ u_{k,i}(t_i)=u_{k,i-1}(t_i) & i=0, 1,\dots, k-1 \\ Pu_{k,i}|_{x_n=0}=0 & i=0, 1,\dots, k-1 \end{cases}$$

where $u_{k,-1}(t_0)=u_{k,-1}(0)\equiv 0$. We define $u_k(t, x)$ for $t\in[0, T]$ by

$$u_k(t, x)=u_{k,i}(t, x) \qquad \text{for} \qquad t\in[t_i, t_{i+1}].$$

The existence of such $u_{k,i}(t, x)$ $(i=0, 1,\dots, k-1)$ is assured by Theorem 3, because the compatibility conditions of order 0 is satisfied at each t_i. Consequently we see

$$u_k(t, x)\in\mathscr{E}_t^0(H^1), \qquad t\in[0, T]$$

and

$$u_k(t, x)\in\mathscr{E}_t^1(L^2), \qquad t\neq t_i \qquad (i=1, 2,\dots, k-1),$$

which means

$$u_k(t, x)\in H^1((0, T)\times R_+^n).$$

We show that the sequence $\{u_k(t, x)\}$ is bounded in $H^1((0, T)\times R_+^n)$. For this purpose, it is sufficient to prove that $\{D_i u_k(t, x)\}$ $(i=0, 1,\dots, n-1)$ are bounded in $L^2((0, T)\times R_+^n)$, because A_n is non-singular. Applying (6.4) to (7.4), we get

$$
(7.5) \qquad |||u_{k,i}(t)|||_{1,0} \leqq e^{\mu_1(t-t_i)}|||u_{k,i}(t_i)|||_{1,0} + d\int_{t_i}^{t} e^{\mu_1(t-s)}|||f(s)|||_{1,0}ds, \qquad t\in[t_i, t_{i+1}],
$$
$$
i=0, 1,\dots, k-1.
$$

Substituting into (7.5) the following inequalities

$$
\begin{aligned}
|||u_{k,i}(t_i)|||_{1,0} &= \|u_{k,i}(t_i)\|_{1,0} + \|L(t_i)u_{k,i}(t_i) + f(t_i)\| \\
&\leqq \|u_{k,i-1}(t_i)\|_{1,0} + \|L(t_{i-1})u_{k,i-1}(t_i) + f(t_i)\| \\
&\quad + \|(L(t_i) - L(t_{i-1}))u_{k,i-1}(t_i)\| \\
&\leqq |||u_{k,i-1}(t_i)|||_{1,0} + \text{const.}(t_i - t_{i-1})\|u_{k,i-1}(t_i)\|_1 \\
&\qquad (i=1, 2,\dots, k-1)
\end{aligned}
$$

and

$$
\|D_n u_{k,i-1}(t_i)\| \leqq \text{const.}(|||u_{k,i-1}(t_i)|||_{1,0} + \|f(t_i)\|) \qquad (i=1,\dots, k-1),
$$

we get from (7.5)

$$
\begin{aligned}
(7.6) \qquad |||u_{k,i}(t)|||_{1,0} &\leqq e^{\mu_1(t-t_i)}(1+K_1(t_i-t_{i-1}))|||u_{k,i-1}(t_i)|||_{1,0} \\
&\quad + K_2 e^{\mu_1(t-t_i)}\|f(t_i)\|(t_i - t_{i-1}) \\
&\quad + d\int_{t_i}^{t} e^{\mu_1(t-s)}|||f(s)|||_{1,0}ds, \quad t\in[t_i, t_{i+1}],
\end{aligned}
$$

where K_1 and K_2 are positive constants independent of f, u_k and t. Taking account of $g(x)\equiv 0$ and $f(0, x)\equiv 0$, and making the induction from (7.6), we get

$$
\begin{aligned}
(7.7) \qquad |||u_k(t)|||_{1,0} &\leqq K_2 \sum_{i=1}^{p} e^{(\mu_1+K_1)(t-t_i)}\|f(t_i)\|(t_i - t_{i-1}) \\
&\quad + de^{K_1 t}\int_0^t e^{\mu_1(t-s)}|||f(s)|||_{1,0}ds, \qquad t\in[t_p, t_{p+1}].
\end{aligned}
$$

As A_n is non-singular, it follows

(7.8) $$\|D_n u_k(t)\| \leqq \text{const.}(|||u_k(t)|||_{1,0} + \|f(t)\|).$$

Combining (7.7) and (7.8), we get

(7.9) $$|||u_k(t)|||_1 \leqq \text{const.} e^{K_1 t} \int_0^t e^{\mu_1(t-s)} |||f(s)|||_{1,0} ds$$
$$+ \text{const.} \|f(t)\| + \varepsilon_k,$$

where ε_k is non-negative and by the definition of the definite integral it tends to zero when k increases infinitely. The inequality (7.9) shows that $\{u_k(t, x)\}$ is bounded in $H^1((0, T) \times R^n_+)$, i.e., weakly compact there. Therefore there exists a subsequence $\{k_p\}_{p=1,2,\dots}$ of $\{k\}_{k=1,2,\dots}$ and $u(t, x)$ in $H^1((0, T) \times R^n_+)$ such that

$$u_{k_p}(t, x) \longrightarrow u(t, x), \text{ weakly in } H^1((0, T) \times R^n_+)$$

when p increases infinitely. It is easily seen that $u(t, x)$ satisfies

(7.10) $$\begin{cases} \dfrac{\partial u}{\partial t} = L(t)u(t, x) + f(t, x) & \text{in } \mathscr{D}'((0, T) \times R^n_+) \\ u(0, x) = 0 & \text{in } L^2(R^n_+) \\ Pu|_{x_n=0} = 0 & \text{in } H^{\frac{1}{2}}((0, T) \times R^{n-1}). \end{cases}$$

At last, we prove that $u(t, x)$ belongs to $\mathscr{E}^1_t(L^2) \cap \mathscr{E}^0_t(H^1)$. For this purpose we use the mollifier with respect to (t, x'). Let $\rho(t)$ be a non-negative function in $C^\infty_0(R^1)$ such that its support is contained in $[-2, -1]$ and

$$\int_{-\infty}^{\infty} \rho(t) dt = 1.$$

We define a mollifier $\rho_{\delta(\overset{*}{t}, x')}$ for $u(t, x) \in H^1((0, T) \times R^n_+)$ by

$$u_\delta(t, x) = (\rho_{\delta(\overset{*}{t}, x')} u)(t, x)$$
$$= \int \rho_\delta(s) \prod_{i=1}^{n-1} \rho_\delta(y_i) u(t-s, x'-y', x_n) ds dy'$$

where $\rho_\delta(t) = \frac{1}{\delta} \rho\left(\frac{t}{\delta}\right)$. Let δ be $0 < \delta < \delta_0$ where δ_0 is a small positive constant. Then $u_\delta(t, x)$ is defined for $t\in[0, T-2\delta_0]$ and belongs to $\mathscr{E}_t^\infty(H^1(R_+^n))$ and satisfies

$$(7.11)\qquad \begin{cases} \dfrac{\partial}{\partial t}u_\delta = L(t)u_\delta + C_\delta u + f_\delta(t, x) \\ u_\delta(0, x) = g_\delta(x) \\ Pu_\delta|_{x_n=0} = 0 \end{cases}$$

where

$$C_\delta u = [\rho_{\delta(t,x')}\overset{*}{}, L(t)]u, \qquad f_\delta(t, x) = (\rho_{\delta(t,x')}\overset{*}{} f)(t, x)$$

$$g_\delta(x) = (\rho_{\delta(t,x')}\overset{*}{} u)(0, x).$$

Applying (7.2) to (7.11), we have

$$(7.12)\qquad |||u_\delta(t) - u_{\delta'}(t)|||_1 \leqq c_1 e^{\mu_1 t} |||u_\delta(0) - u_{\delta'}(0)|||_1 + d_1 \int_0^t e^{\mu_1(t-s)} |||C_\delta u + f_\delta - C_{\delta'} u - f_{\delta'}|||_1 ds.$$

We prove that the right hand side of (7.12) tends to zero when δ and δ' tend to zero. Since it is well known that

$$\int_0^t |||C_\delta u(s)|||_1 ds \longrightarrow 0,$$

$$\int_0^t |||f_\delta(s) - f(s)|||_1 ds \longrightarrow 0$$

when δ tends to zero, we show only

$$|||u_\delta(0)|||_1 \longrightarrow 0$$

when $\delta\to 0$. Using the assumptions that $f(t, x)\in\mathscr{E}_t^1(L^2)$ and $f(0, x) \equiv 0$, we see from (7.9) that

$$(7.13)\qquad \int_0^\delta |||u(s)|||_1^2 ds \leqq K_0\delta^2,$$

where K_0 is a positive constant independent of δ. Since

$$\|u_\delta(t)\|^2 \leqq \int \rho_\delta(s) \|u(t-s)\|^2 ds,$$

we get

$$(7.14) \qquad |||u_\delta(t)|||_1^2 = \|u_\delta(t)\|^2 + \sum_{i=0}^{n} \|(D_i u_\delta)(t)\|^2$$

$$\leqq \int \rho_\delta(s)(\|u(t-s)\|^2 + \sum_{i=0}^{n} \|(D_i u)(t-s)\|^2) ds$$

$$= \int \rho_\delta(s) |||u(t-s)|||_1^2 ds.$$

Putting $t=0$ in (7.14) and substituting (7.13) into (7.14), we have

$$|||u_\delta(0)|||_1^2 \leqq \int \rho_\delta(s) |||u(-s)|||_1^2 ds$$

$$\leqq \text{const.} \frac{1}{\delta} \int_\delta^{2\delta} |||u(s)|||_1^2 ds$$

$$\leqq \text{const.}\ \delta,$$

which means that $|||u_\delta(0)|||_1$ tends to zero when $\delta \to 0$.

Consequently we can see that $\{u_\delta(t, x)\}$ is a Cauchy's sequence in $\mathscr{E}_t^1(L^2) \cap \mathscr{E}_t^0(H^1)$, therefore its limit $u(t, x)$ is also in $\mathscr{E}_t^1(L^2) \cap \mathscr{E}_t^0(H^1)$ and $u(0, x) = 0$.

We state the reason why we reduce to the case where $g(x) \equiv 0$ and $f(0, x) \equiv 0$. For the boundedness of $\{u_k(t, x)\}$ in $H^1((0, T) \times R_+^n)$, it is not necessary to be $g(x) \equiv 0$ and $f(0, x) \equiv 0$. But, when we prove that the solution $u(t, x) \in H^1((0, T) \times R_+^n)$ of (1.1) belongs to $\mathscr{E}_t^1(L^2) \cap \mathscr{E}_t^0(H^1)$, we need in (7.12) that

$$|||u_\delta(0) - u_{\delta'}(0)|||_1 \to 0$$

when δ and δ' tend to zero. For this purpose, it is not sufficient only to be $u(t, x) \in H^1((0, T) \times R_+^n)$, i.e., we need tha additional informations for $u(t, x)$. So we reduced to the above-mentioned case. This completes the proof of Theorem 4. q.e.d.

§8. Finiteness of the propagation speed

In this section we show that the solution given by Theorem 4 has a finite speed of propagation. Let $\lambda_1(t, x; \xi) \leqq \lambda_2(t, x; \xi) \leqq \dots \leqq$

$\lambda_N(t, x; \xi)$ be the roots of the characteristic equation of $\frac{\partial}{\partial t} - L(t)$

$$\det[\lambda I - \sum_{i=1}^{n} A_i(t, x)\xi_i] = 0$$

for $(t, x) \in V$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in R^n$. Denote

$$\lambda_{\max} = \sup_{\substack{|\xi|=1, (t,x)\in V \\ i=1,2,\dots,N}} |\lambda_i(t, x; \xi)|. \tag{8.1}$$

By the condition B.1, we see that $\lambda_{\max}$ is finite. For each $(t_0, x_0) \in V$, we denote by $C_{(t_0,x_0)}$ the backward cone with a vertex (t_0, x_0) defined by

$$\{(t, x);\ \ |x - x_0| < \lambda_{\max}(t_0 - t)\}.$$

Then we have

Theorem 5. *Let $u(t, x)$ be a C^1-solution of (1.1) defined in $V \cap C_{(t_0,x_0)}$. If $g(x)$ is zero in $C_{(t_0,x_0)} \cap \{(0, x);\ x \in R^n_+\}$ and $f(t, x)$ is zero in $V \cap C_{(t_0,x_0)}$, then $u(t, x)$ is identically zero in $C_{(t_0,x_0)} \cap V$.*

The proof is divided in two parts.

Lemma 5. (*local uniqueness*) *Let $u(t, x) \in C^1$ defined in $D_\varepsilon = \{(t, x) \in \bar{V};\ t + |x - x_0|^2 \leqq \varepsilon,\ t \geqq 0\}$ where $x_0 \in R^n_+$. If $u(t, x)$ satisfies*

$$\begin{cases} \dfrac{\partial u}{\partial t} = L(t)u & \text{in } D_\varepsilon \cap V \\ u(0, x) = 0 & \text{in } D_\varepsilon \cap \bar{V} \cap \{t = 0\} \\ Pu|_{x_n=0} = 0 & \text{in } D_\varepsilon \cap \bar{V} \cap \{x_n = 0\}, \end{cases}$$

then $u(t, x)$ is identically zero in $D_\varepsilon \cap V$.

Proof. It suffices to prove for the case $x_0 = 0 \in R^n$. After Holmgren transformation

$$s = t + |x|^2$$

$$y_i = x_i \qquad (i = 1, 2, \dots, n),$$

$\tilde{u}(s,y)=u(t, x)$ satisfies

$$(8.2)\qquad \begin{cases} A_0(s, y)\dfrac{\partial \tilde{u}}{\partial s}=\tilde{L}(s)\tilde{u} & \text{in } \tilde{D}_\varepsilon \\ P\tilde{u}|_{y_n=0}=0 & \text{in } \overline{\tilde{D}_\varepsilon}\cap\{y_n=0\} \end{cases}$$

where $A_0(s, y)=I-2\sum_{k=1}^{n} A_k y_k$, $\tilde{L}(s)=\sum_{k=1}^{n} A_k\dfrac{\partial}{\partial y_k}+B$ and $\tilde{D}_\varepsilon=\{(s, y);\ \varepsilon\geqq s>|y|^2,\ y_n>0\}$. By extending $\tilde{u}(s, y)$ by zero in $[0, \varepsilon]\times R^n_+-\tilde{D}_\varepsilon$, we see $\tilde{u}(s, y)\in\mathscr{E}^1_s(L^2)\cap\mathscr{E}^0_s(H^1)$, because $\tilde{u}(s, y)=0$ on $s=|y|^2\leqq\varepsilon$. We extend also the domains of the coefficients of (8.2) to $[0, \varepsilon]\times R^n_+$, keeping the properties that $A_0(s, y)$ is positive definite and $A_i(s, y)$ $(i=1,\dots, n)$ are hermitian. Then $\tilde{u}(s, y)$ satisfies

$$\begin{cases} A_0\dfrac{\partial \tilde{u}}{\partial s}=\tilde{L}(s)\tilde{u} & \text{in } [0, \varepsilon]\times R^n_+ \\ \tilde{u}(0, y)=0 & \text{in } R^n_+ \\ P\tilde{u}|_{y_n=0}=0 & \text{in } [0, \varepsilon]\times R^{n-1}. \end{cases}$$

Here we define the norm $\|\tilde{u}\|_{A_0(s)}$ by $(A_0(s)\tilde{u}, \tilde{u})^{\frac{1}{2}}$. As $A_o(s)$ is hermitian, there exist positive constants B' and B suh that

$$\frac{d}{ds}\|\tilde{u}(s)\|^2_{A_0(s)}=(\tilde{L}(s)\tilde{u}, \tilde{u})+(\tilde{u}, \tilde{L}(s)\tilde{u})+\left(\frac{\partial A_0}{\partial s}\tilde{u}, \tilde{u}\right)$$

$$\leqq 2B'\|\tilde{u}(s)\|^2\leqq 2B\|\tilde{u}(s)\|^2_{A_0(s)},$$

which implies

$$\|\tilde{u}(s)\|_{A_0(s)}\leqq e^{Bs}\|\tilde{u}(0)\|_{A_0(0)}=0.$$

This completes the proof of Lemma 5. q.e.d.

Proof of Theorem 5. We use the method of sweeping out of F. John. Define for $0<\theta<\lambda^2_{\max}\cdot t^2_0$

$$\varphi_\theta(t, x)=(t-t_0)+\frac{1}{\lambda_{\max}}\sqrt{|x-x_0|^2+\theta}$$

and

$$K^{\theta}_{(t_0,x_0)}=\{(t, x);\ \varphi_{\theta}(t, x)=0\}.$$

After the change of variables

$$s=\varphi_{\theta}(t, x);\ y_i=x_i, \qquad i=1, 2,\dots, n,$$

the equation (1.1) is transformed to

$$\begin{cases} A_{\theta}\dfrac{\partial \tilde{u}}{\partial s}=L_{\theta}\tilde{u} \\ P\tilde{u}|_{y_n=0}=0 \end{cases}$$

where $\tilde{u}(s, y)=u(t, x)$, $L_{\theta}(s, y; D_y)=L(t, x; D_x)$ and

$$A_{\theta}=I-\frac{1}{\lambda_{\max}}\sum_{k=1}^{n}A_k\frac{x_k-x_k^0}{\sqrt{|x-x_0|^2+\theta}}.$$

We see that A_{θ} is positive definite and hermitian. Lemma 5 implies that, if $u(t, x)$ is zero on $K^{\theta}_{(t_0,x_0)}$, then $u(t, x)$ is zero in $S_{\theta}\cap\{\varphi_{\theta}>0\}$, where S_{θ} is a certain neighborhood of $K^{\theta}_{(t_0,x_0)}$. On the other hand, we see that

$$\bigcup_{0<\theta<\lambda_{\max}^2\cdot t_0^2} K^{\theta}_{(t_0,x_0)}\supset V\cap C_{(t_0,x_0)}.$$

Step by step using the result of Lemma 5, we can show that the solution is equal to zero in $C_{(t_0,x_0)}\cap V$. q.e.d.

Appendix

In the preceding sections we discussed the mixed problems (1.1) under the condition that $P(t, x')=[E_l\ 0]$. In this appendix we show that the general case where $P(t, x')$ is an $l\times N$ variable matrix may be reduced to the constant case $P(t, x')=[E_l\ 0]$ by a unitary transformation of unknown functions.

Theorem A. *Let* $P_i(x)=(p_{i1}(x), \dots, p_{iN}(x))$, $i=1, 2,\dots, l$, *be given complex N-vectors depending on the parameter x which varies in* R^n. *Suppose that* $P_i(x)$ *all belong to* $\mathscr{B}^m(R^n)$, *and that they are constant outside a compact set in* R^n, *and that* $P_i\cdot P_j=\delta_{ij}$ *where* δ_{ij} *is Kro-*

necker's delta. Then we can add to them complex N-vectors $P_{l+1}(x)$, ..., $P_N(x) \in \mathscr{B}^m(R^n)$ *such that* $\{P_1, \ldots, P_l, P_{l+1}, \ldots, P_N\}$ *constitutes an orthonormal basis of* $\boldsymbol{C}^N$.

We state two lemmata which are necessary for the proof of Theorem A.

Lemma A.1. *We assume the same conditions as Theorem A. Suppose that there exist vectors* $P_{l+1}, \ldots, P_N$ *in* $\mathscr{B}^0(R^n)$ *such that* $\{P_1, \ldots, P_l, P_{l+1}, \ldots, P_N\}$ *constitutes an orthonormal basis of* $\boldsymbol{C}^N$. *Then Theorem A holds, i.e., we can take* $P_{l+1}, \ldots, P_N$ *as in* $\mathscr{B}^m(R^n)$.

Proof. As $P_1, \ldots, P_l$ are constant outside a compact set in R^n, we can reconstruct $P_{l+1}, \ldots, P_N$ so as to be constant outside a compact set K in R^n. Here we put

$$P_i(x) = (p_{i1}, \ldots, p_{iN}), \qquad i = 1, 2, \ldots, N.$$

We take a function $\alpha(x) \in C_0^\infty(R^n)$ which takes the value 1 in a neighborhood of K. By Weierstrass' approximation theorem, we get the sequences of polynomials $\{p_{ij}^{(n)}(x)\}_{n=1,2,\ldots}$ $(i = l+1, \ldots, N;\ j = 1, 2, \ldots, N)$ such that, when n increases infinitely, $p_{ij}^{(n)}(x)$ converges to $p_{ij}(x)$ in $\mathscr{B}^0(B)$ for any i and j, where B is the support of $\alpha(x)$. We write $P_k^{(n)}(x) = \alpha(x)(p_{k1}^{(n)}, p_{k2}^{(n)}, \ldots, p_{kN}^{(n)}) + (1 - \alpha(x))P_k(x)$ $(k = l+1, \ldots, N)$. Then we can choose n_0 large enough so that

$$\left|\det \begin{pmatrix} P_1(x) \\ \vdots \\ P_l(x) \\ P_{l+1}^{(n_0)}(x) \\ \vdots \\ P_N^{(n_0)}(x) \end{pmatrix}\right| \geqq \frac{1}{2} \quad \text{in whole } R^n.$$

Applying Schmidt's orthogonalization to $P_1, P_2, \ldots, P_l, P_{l+1}^{(n_0)}, \ldots, P_N^{(n_0)}$, we get $P_{l+1}, \ldots, P_N$ with the required properties. q.e.d.

Next, we get an elementary lemma by Hadamard's inequality.

Lemma A.2. *Let* $C = [c_{ij}]$ *and* $C' = [c'_{ij}]$ *be* $N \times N$ *matrices. Assume* *that* $|c_{ij}-c'_{ij}|<\varepsilon$ *for any* i *and* j, *and that*

$$\sum_{j=1}^{N}|c_{ij}|^2=\sum_{j=1}^{N}|c'_{ij}|^2=1, \qquad i=1, 2,\dots, N.$$

Then it follows

$$|\det[C]-\det[C']|\leqq \varepsilon N^{3/2}.$$

Proof of Theorem A. By Lemma A.1, it suffices to construct $P_{l+1}, P_{l+2},\dots, P_N$ in $\mathscr{B}^0(R^n)$. Since it is easy to construct $\{P_{l+1}, \dots, P_N\}$ locally, we proceed to the global construction of them. We prove that, if we assume that $P_{l+1},\dots, P_N$ exist in $C_R=\{x\in R^n;\ |x|\leqq R\}$, then we can extend the defineition domain of them to $C_{R+\delta}$, keeping that they belong to $\mathscr{B}^0(C_{R+\delta})$ and $\{P_1,\dots, P_l, P_{l+1},\dots, P_N\}$ constitutes an orthonormal basis of $\boldsymbol{C}^N$ there, and, moreover, that we can take such δ as independent of R. As $P_1(x),\dots, P_l(x)$ are constant outside a compact set, their components $p_{ij}(x)$ $(i=1,\dots, l; j=1, 2,\dots, N)$ are uniformly continuous in R^n. Therefore, for any $\varepsilon>0$, there exists a positive constant δ such that, if $|x-x'|<\delta$, then

$$\text{(A.1)} \qquad |p_{ij}(x)-p_{ij}(x')|<\varepsilon \qquad (i=1, 2,\dots, l;\ j=1, 2,\dots, N).$$

Here we introduce polar coordinates (r, ω) in R^n. We extend the domain of $P_i(r, \omega)$ $(i=l+1,\dots, N)$ to $C_{R+\delta}$ by

$$P_i(r, \omega)=P_i(R, \omega) \qquad \text{in} \quad R\leqq r\leqq R+\delta.$$

Then it holds

$$\text{(A.2)} \qquad P_i\cdot\overline{P_i}=1 \qquad \text{in} \quad C_{R+\delta},\ i=1, 2,\dots, N.$$

Now we put $\varepsilon=1/2N^{3/2}$ and determine a positive constant δ for such ε by the uniform continuity of $p_{ij}(x)$, i.e., (A.1). Then, by Lemma A.2 we get

$$\left|\det\begin{bmatrix} P_1(x)\\ \vdots\\ P_l(x)\\ P_{l+1}(x)\\ \vdots\\ P_N(x)\end{bmatrix}\right|\geqq\frac{1}{2} \qquad \text{in whole } C_{R+\delta}.$$

Applying Schmidt's orthogonalization, we obtain $P_{l+1},\dots, P_N$ in $C_{R+\delta}$ with the required properties. From the method of the construction, we can easily see that δ is independent of R. Hence, repeating this process, we finally get $P_{l+1},\dots, P_N$ in whole R^n. q.e.d.

Corollary of Theorem A. *If $P_1(x),\dots, P_l(x)\in\mathscr{B}^m(R^n)$ are analytic in a bounded open set K in R^n, then we can construct $P_{l+1},\dots, P_N$ so as to be analytic in K.*

If we apply the method used in Lemma A.1, we can prove this. Although we considered Theorem A in R^n, if we restrict x to R^n_+, we obtain Theorem A replaced R^n by R^n_+.

Now we return to the mixed problem (1.1). Put

$$P(t, x')=\begin{bmatrix} P_1(t, x') \\ \vdots \\ P_l(t, x') \end{bmatrix}$$

where $P_i(t, x')=(p_{i1}, p_{i2},\dots, p_{iN})$, $i=1, 2,\dots, l$. Applying Schmidt's orthogonalization to $P_1,\dots, P_l$, we obtain the orthonormal vectors Q_1, $\dots, Q_l$. By Theorem A, we can add to them vectors $Q_{l+1},\dots, Q_N$ so that the system $\{Q_1,\dots, Q_l, Q_{l+1},\dots, Q_N\}$ constitutes an orthonormal basis of $\boldsymbol{C}^N$. Denote by $T(t, x')$ a unitary matrix

$$[Q_1^*,\dots, Q_l^*, Q_{l+1}^*,\dots, Q_N^*]$$

where $Q_i^*=\overline{{}^tQ_i}$, $i=1,\dots, N$. We perform the unitary transform of the unknown functions $u=Tv$, then $v(t, x)$ satisfies

$$\text{(A.3)}\quad \begin{cases} \dfrac{\partial v}{\partial t}=\displaystyle\sum_{j=1}^n T^*A_jT\frac{\partial v}{\partial x_i}+\Big(T^*BT-T^*\frac{\partial T}{\partial t}+T^*\sum_{i=1}^n A_i\frac{\partial T}{\partial x_i}\Big)v+T^*f \\ v(0, x)=T^*(0, x')g(x) \\ (PT)v|_{x_n=0}=0 \end{cases}.$$

From the method of the construction of $T(t, x')$ it follows

$$PT=\begin{bmatrix} (P_1, Q_1) & & & & 0 & \dots & 0 \\ & (P_2, Q_2) & 0 & & 0 & \dots & 0 \\ & * & \cdot & & \vdots & & \vdots \\ & & & \cdot \quad (P_l, Q_l) & 0 & \dots & 0 \end{bmatrix}$$

$$=[M_l\ 0].$$

As M_l is non-singular, $(PT)v|_{x_n=0}=0$ is equivalent to $[E_l\ 0]v|_{x_n=0}=0$. Therefore, we can consider the mixed problem (1.1) under the condition $P=[E_l\ 0]$ without loss of generality.

At last we remark on the compatibility conditions. If $g(x)$ and $f(t, x)$ are in $H^{m+1}(R^n_+)$ and $\mathscr{E}^{m+1}_t(L^2)\cap\cdots\cap\mathscr{E}^0_t(H^{m+1})$ respectively, and if they satisfy the compatibility conditions of order m with respect to the equation (1.1), then $\tilde{g}(x)=T^*(0, x')g(x)$ and $\tilde{f}(t, x)=T^*(t, x')$ $f(t, x)$ also satisfy the compatibility conditions of order m with respect to the equation (A.3) replaced $(PT)v|_{x_n=0}=0$ by $[E_l\ 0]v|_{x_n=0}=0$. In fact, since the compatibility conditions of order m mean concretely that, if $u(t, x)$ is a sufficiently smooth solution of (1.1), then

$$D_0^k(Pu)\Big|_{\substack{t=0\\ x_n=0}}=0, \qquad k=0, 1,\dots, m,$$

we see that a sufficiently smooth solution $v(t, x)$ of (A.3) satisfies

$$D_0^k([E_l 0]v)\Big|_{\substack{t=0\\ x_n=0}}=D_0^k(M_l^{-1}Pu)\Big|_{\substack{t=0\\ x_n=0}}$$

$$=\sum_{i=0}^{k}\binom{k}{i}(D_0^i M_l^{-1})D_0^{k-i}(Pu)\Big|_{\substack{t=0\\ x_n=0}}=0, \qquad k=0, 1,\dots, m.$$

This means that $\tilde{g}(x)$ and $\tilde{f}(t, x)$ satisfy the compatibility conditions of order m with respect to (A.3).

Department of Mathematics
Kyoto Sangyo University

References

[1] M. Ikawa; A mixed problems for hyperbolic equations of second order with a first order derivative boundary condition, Publ. R. I. M. Kyoto Univ. vol. **5**, 119–149.

[2] K. Kajitani; First Order hyperbolic mixed problems, J. Math. Kyoto Univ. vol. **11** (1971), 449–484.

[3] P.D. Lax and R. S. Phillips; Local boundary conditions for dissipative symmetric linear differential operators, Comm. Pure Appl. Math. vol. **13** (1960), 427–455.

[4] S. Mizohata; Analyticity of solutions of hyperbolic systems with analytic coefficients, Comm. Pure Appl. Math. vol. **14**, 547–559.

[5] S. Mizohata; Theory of partial differential equations, Iwanami, Tokyo, 1965 (in Japanese).

[6] S. Miyatake; An approach to hyperbolic mixed problems by singular integral operators, Jour. Math, Kyoto Univ. vol. **10**, (1970), 439–474.

[7] J. Rauch; $\mathscr{L}_2$ is a continuous initial condition for Kreiss' mixed problems, to appear.

[8] R. Sakamoto; Mixed problems for hyperbolic equations I, II, J. Math. Kyoto Univ. vol. **10** (1970), 349–373, 403–417.