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On the classification of H-spaces of rank 2

By

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§1. Introduction

For a finite H-complex X, the classical Hopf theorem states that the rational cohomology $H^*(X; Q)$ is isomorphic to $\Lambda(x_1, \dots, x_l)$, the exterior algebra over Q with deg x_i odd. We call *l the rank of X* and (deg $x_1, \dots,$ deg x_l) *the type of X*.

In the present paper we will consider the homotopy type classification for 1-connected, finite H-complexes of rank 2. In the case $H_*(X; Z)$ has no 2-torsion, the classification has been given by Hilton-Roitberg [6] and Zabrodsky [21] as follows:

Theorem. *The complete list of homotopy types of* 1*-connected,* 2*-torsion free, finite H-complexes of rank* 2 *is the following*: $S^3 \times S^3$, $SU(3)$, E_k $(k=0, 1, 3, 4, 5)$, $S^7 \times S^7$, *where E_k is the principal S^3-bundle over S^7 with the characteristic class $k\omega \in \pi_7(BS^3) \cong Z_{12}$, ω a generator.*

Thus our object is to classify H-spaces of rank 2 with 2-torsion.

Let X be a 1-connected, finite H-complex of rank 2 such that $H^*(X; Z)$ has 2-torsion. According to J. R. Hubbuck [7], $H^*(X; Z_2) \cong H^*(G_2; Z_2)$ as Hopf algebras, where G_2 is the compact, exceptional Lie group of rank 2.

Let f: $V_{7,2} \to BS^3$ be the classifying map of G_2, φ: $V_{7,2} \to V_{7,2} \vee S^{11}$ the suitable shrinking map, and α a generator of $\pi_{11}(BS^3)$ suitably chosen. We denote by $G_{2,b}$ the principal S^3-bundle over $V_{7,2}$ induced by the composition $(f\vee g_b)\circ\varphi$: $V_{7,2}\to V_{7,2}\vee S^{11}\to BS^3$, where g_b represents ba, $b\in Z$. (For details see §5).

Then our result is

Theorem 5.1. *Let X be a 1-connected, finite H-complex of rank 2 such that $H_*(X;Z)$ has 2-torsion. Then X is homotopy equivalent to $G_{2,b}$ for some b. There are just 8 homotopy types of such H-complexes: $G_{2,i}$ for $-2\leq i\leq 5$.*

Then together with the result by Zabrodsky [21] we obtain

Main Theorem. *The complete list of homotopy types of 1-connected, finite H-complexes of rank 2 is the following: $S^3\times S^3$, $SU(3)$, E_k $(k=0,1,3,4,5)$, $S^7\times S^7$, $G_{2,i}$ $(-2\leq i\leq 5)$.*

The paper is organized as follows. The Hubbuck's theorem is introduced in §2. In §3 we determine the mod p homotopy types of S^3-bundles over S^{11}. Some results on homotopy, which will be needed in §5, are prepared in §4. The classification of the homotopy types of H-complexes of type (3, 11) are discussed and thoroughly determined in the section 5. Further, some additional properties of $G_{2,b}$ is studied. Namely $G_{2,b}$ is homotopy equivalent to a loop space if and only if $1+8b\not\equiv 0$ mod 3 and 5 (Theorem 5.8).

Throughout the paper, we use the following notations. For two complexes X and Y, $X\simeq Y$ denotes that X is homotopy equivalent to Y; $X\underset{p}{\simeq}Y$ denotes that X is p-equivalent to Y. (The direction of a p-equivalence is irrelevant, since all complexes under consideration are H-spaces mod 0, see [11]). $X^{(n)}$ stands for the n-skeleton of X and $\pi_i(X:p)$ the p-component of $\pi_i(X)$. We denote by $\mathcal{A}_p$ the mod p Steenrod algebra.

§2. H-spaces of rank 2 with 2-torsion

Let X be a simply connected, finite H-complex of rank 2 where $H_*(X;Z)$ has 2-torsion. Let G_2 be the compact, exceptional Lie group of rank 2.

Then the following theorem is due to J. R. Hubbuck [7].

Theorem 2.1. *$H^*(X;Z_2)$ is isomorphic as a Hopf algebra to $H^*(G_2;Z_2)$.*

From this theorem we deduce some facts for later use.

Theorem 2.2.

(i) *$H^*(X;Z_2)\cong H^*(G_2;Z_2)$ as $\mathcal{A}_2$-algebras, in particular, $Sq^4Sq^2H^3(X;Z_2)=0$.*

(ii) *$H^*(X;Z_p)\cong H^*(G_2;Z_p)$ for any odd prime p.*

Proof. (i) From Theorem 2.1 we have

$$H^*(X;Z_2)\cong Z_2[x_3]/[x_3^4]\otimes\Lambda(x_5),$$

where $\deg x_i=i$.

From the relation $x_3^2=Sq^3x_3=Sq^1Sq^2x_3$ it follows that $Sq^2x_3=x_5$. Thus $H^*(X;Z_2)\cong H^*(G_2;Z_2)$ as $\mathcal{A}_2$-algebras. The element $Sq^4Sq^2x_3$ is trivial, since it is primitive. (ii) By (i) X is of type (3,11). Then apparently $H^*(X;Z)$ has no p-torsions for $p>3$ by Theorem 4.7 of [3]. Assume that X has 3-torsion. Then we can easily see again by Theorem 4.7 of [3] that

$$H^*(X;Z_3)\cong\Lambda(x_3, x_3')\otimes Z_3[x_4]/[x_4^3] \text{ with } x_4=\beta x_3.$$

Now consider an Adem relation

$$\beta\mathcal{P}^2=\mathcal{P}^2\beta-\mathcal{P}^1\beta\mathcal{P}^1 \qquad (2.1)$$

and an (unstable) secondary operation ϕ associated with (2.1). Then ϕ is well defined on x_4, since $\beta x_4=\beta\mathcal{P}^1x_4=0$. So we can apply Theorem

1.1 of [22] and obtain an indecomposable element $\phi(x_4)$ in $H^{12}(X;Z_3)$, which is a contradiction. So $H^*(X;Z)$ has no 3-torsion.

q. e. d.

As a corollary we have

Corollary 2.3. *Let Y be a simply connected, finite H-complex of rank 2. Then $H^*(Y;Z)$ has 2-torsion if and only if Y is of type* (3, 11).

§3. Homotopy type mod odd of S^3-bundles over S^{11}

The notion "homotopy type mod p" means the classification by p-equivalences. Remark that the p-equivalence is an equivalence relation, since all spaces we shall consider are H-spaces mod 0 (see [11]).

Let us determine the homotopy types mod p, p odd, of S^3-bundles over S^{11}. Such bundles are classified by $\pi_{11}(BSO(4))\cong\pi_{10}(SO(4))$. Since $SO(4)\cong SO(3)\times S^3$, we have

$$\pi_{10}(SO(4))\cong\pi_{10}(SO(3))\oplus\pi_{10}(S^3)\cong Z_{15}\oplus Z_{15}.$$

We represent an element of $\pi_{10}(SO(4))$ by a pair (n,m) with $n,m\in Z_{15}$. We denote by $B(n,m)$ the bundle corresponding to $(n,m)\in\pi_{10}(SO(4))$. Note that for any S^3-bundle B over S^{11}, there exists a S^3-bundle B' over S^{11} with the characteristic class $\chi'\in\pi_{10}(SO(4):p)$ such that $B\underset{p}{\simeq}B'$. Thus to determine the homotopy types mod p, it is enough to consider the bundles classified by $\pi_{10}(SO(4):p)$.

Before stating a theorem let us recall the result due to James-Whitehead. Consider a sequence:

$$\pi_{13}(S^{10})\xrightarrow{(\pi_*\chi)_*}\pi_{13}(S^3)\xleftarrow{J}\pi_{10}(SO(3))\xrightarrow{i_*}\pi_{10}(SO(4))$$

for $\chi\in\pi_{10}(SO(4))$, where $\pi:SO(4)\to S^3$ is the projection. Denote by $G(\chi)$ the subgroup $i_*\circ J^{-1}\circ(\pi_*\chi)_*(\pi_{13}(S^{10}))$ of $\pi_{10}(SO(4))$. For a subset S of $\pi_{10}(SO(4))$, $\{S\}_\chi$ means the coset of S modulo $G(\chi)$. Then the following is a special case of the James-Whitehead theorem [9].

Proposition 3.1. *Let B_1 and B_2 be total spaces of S^3 bundles over S^{11} with characteristic classes χ_1 and χ_2 in $\pi_{10}(SO(4))$ respectively. Then $B_1 \simeq B_2$ if and only if $\pi_*\chi_1 = \pm\pi_*\chi_2$ and $\{\pm\chi_1\}_{\chi_1} = \{\pm\chi_2\}_{\chi_2}$.*

The following is a main result in this section:

Theorem 3.2. *The complete list of the homotopy types mod p of S^3-bundles over S^{11} is the following*

(i) $B(0,0)$ *for any prime* $p \geq 7$,

(ii) $B(0,0)$ *and* $B(0,3)$ *for* $p=5$,

(iii) $B(0,0)$, $B(0,5)$ *and* $B(5,0)$ *for* $p=3$.

Further, all but $B(5,0)$ are H-spaces mod p for the respective p.

Proof. First we show the last statement that all representatives except $B(5,0)$ are H-spaces mod p. In fact $B(0,0)=S^3\times S^{11}$ is an H-space mod p for any odd prime p ([1]). Also by [10] we have $B(0,5)\underset{3}{\simeq}G_2$ and $B(0,3)\underset{5}{\simeq}G_2$, whence $B(0,5)$ is an H-space mod 3 and $B(0,3)$ is an H-space mod 5. Now we prove the theorem dividing it into three cases:

[Case i) $p\geq 7$]. Clearly the homotopy type mod p is unique, i.e., $B(0,0) = S^3\times S^{11}$, since $\pi_{10}(SO(4):p)=0$.

[Case ii) $p=5$]. An element of $\pi_{10}(SO(4):5)\cong Z_5\oplus Z_5$ is represented by (n,m) with $n\equiv 0$ (3) and $m\equiv 0$ (3). If $m\not\equiv 0$ (15), there is an integer r with $(r,5)=1$ such that $(n,m)=r(n',3)$. So $B(n,m)\underset{5}{\simeq}B(n',3)$ for some n'. Now we apply Proposition 3.1. We get that $(\pi_*\chi)_*=0$ for any $\chi\in\pi_{10}(SO(4):5)$, since $\pi_{13}(S^{10}:5)=0$, and hence

$$G(\chi)=i_*(Z_5)=\{(n,0): n\equiv 0\ (3)\}.$$

Therefore by Proposition 3.1 we obtain that $B(n,m)\simeq B(n',m)$ for any n and n'. So there are only two representatives: $B(0,0)$ and $B(0,3)$. But apparently $B(0,0)$ is not 5-equivalent to $B(0,3)$.

[Case iii) $p=3$]. By the same argument as in the Case ii), we can see that the candidates for the representatives of the homotopy type mod 3 are $B(0,0)$, $B(0,5)$ and $B(5,0)$. We shall show that they are actually of the distinct homotopy type mod 3. Clearly neither $B(0,0)$ nor $B(5,0)$ is 3-equivalent to $B(0,5)$. For they are not 3-equivalent on the 11-skeleton. The following lemma then completes the proof.

In fact, the lemma indicates that $B(0,0)$ is not 3-equivalent to $B(5,0)$, since $B(0,0)$ is an H-space mod 3.

Lemma 3.3. $B(5,0)$ *admits no H-structures* mod 3.

Proof. Assume that $B(5,0)$ admits an H-structure mod 3. So by definition ([12]), there exists a map μ: $B(5,0)\times B(5,0)\to B(5,0)$ such that $f=\mu(\ ,*)=\mu(*,\ )$: $B(5,0)\to B(5,0)$ is a 3-equivalence, where $*$ is a base point of $B(5,0)$. Then $\mu|B(5,0)\vee B(5,0)=f\circ\pi$, where π: $B(5,0)\vee B(5,0)\to B(5,0)$ is the canonical projection. Therefore $f_*[\alpha,\beta]=0$ for $\alpha\in\pi_n(B(5,0))$ and $\beta\in\pi_m(B(5,0))$, and hence the Whitehead product $[\alpha,\beta]$ is of order prime to 3. Since $B(5,0)$ has a cross-section, we have $B(5,0)^{(11)}\simeq S^3\vee S^{11}$ and i_*: $\pi_n(S^3)\to\pi_n(B(5,0))$ is a monomorphism, where i_* is factored as $\pi_n(S^3)\xrightarrow{i_{1*}}\pi_n(S^3\vee S^{11})\xrightarrow{i_{2*}}\pi_n(B(5,0))$. Let $\varphi\in\pi_{13}(B(5,0)^{(11)})$ be the attaching element of the top cell. Then by [8] we obtain $\varphi=ki_{1*}\circ J(\alpha_2)+[\sigma_3,\sigma_{11}]$, where α_2 is a generator of $\pi_{10}(SO(3):3)$, $k\not\equiv 0$ (3) and σ_i: $S^i\to S^3\vee S^{11}$ is the canonical inclusion ($i=3,11$). Since $i_{2*}\varphi=0$, we deduce that $ki_{,*}J(\alpha_2)=ki_{2*}i_{1*}J(\alpha_2)=-i_{2*}[\sigma_3,\sigma_{11}]$ is of order prime to 3. But this contradicts to the fact that α_2 is a generator of $\pi_{10}(SO(3):3)$, since i_* and J are monomorphisms on the 3-component and since $k\not\equiv 0$ (3). q.e.d.

We end this section with

Corollary 3.4. *Every principal S^3-bundle over S^{11} is an H-space* mod *p, for any odd p.*

§4. Some results on homotopy

The results in this section will be used in the next section. Let G_2

be the compact, exceptional Lie group of rank 2. Let $V_{7,2}=SO(7)/SO(5)$ be the Stiefel manifold. Then we have the principal bundle

$$(4.1) \qquad S^3 \rightarrow G_2 \xrightarrow{p} V_{7,2}.$$

Denote by $M^n=S^{n-1}\underset{2}{\cup} e^n$ the mapping cone of a map: $S^{n-1}\rightarrow S^{n-1}$ of degree 2. We have cellular decompositions: $V_{7,2}=M^6\cup e^{11}$, $G_2^{(9)}=p^{-1}(M^6)=S^3\cup e^5\cup e^6\cup e^8\cup e^9$ the 9-skeleton of G_2 and $G_2=G_2^{(9)}\cup e^{11}\cup e^{14}$. Let $S^{n-1}\xrightarrow{i} M^n \xrightarrow{q} S^n$ be the cofibering.

Lemma 4.1. *Let $h: S^{10}\rightarrow M^9$ be a map such that $q\circ h: S^{10}\rightarrow S^9$ is essential, and let $K=M^9\cup CM^{10}$ be the mapping cone of $h\circ q: M^{10}\rightarrow S^{10}\rightarrow M^9$. Then there exists a map $f:K\rightarrow G_2^{(9)}$ such that $f_*:\pi_i(K)\rightarrow\pi_i(G_2^{(9)})$ is a mod 2 isomorphism for $3<i<13$. The inclusion $S^3\rightarrow G_2^{(9)}$ is a p-equivalence for any odd prime p.*

Proof. Let F be the 3-connective fibre space over $G_2^{(9)}$. Then we have a fibering:

$$F \xrightarrow{i} G_2^{(9)} \xrightarrow{\pi} K(Z,3).$$

Since $H^*(G_2^{(9)}; Z_2)=\{1, x_3, x_5=Sq^2x_3, x_3^2, x_3x_5, x_3^3\}$, we have that $\pi^*: H^*(Z,3;Z_2)\rightarrow H^*(G_2^{(9)};Z_2)$ is an epimorphism with Ker $\pi^*=\sum_{i\geq 10} H^i(Z,3;Z_2)+\{Sq^4Sq^2u\}$, u being the fundamental class. It follows that there exists a transgressive element a of $H^8(F;Z_2)$ whose transgression image is $\tau(a)=Sq^4Sq^2u$. Then $\tau(Sq^1a)=Sq^5Sq^2u=(Sq^2u)^2$ and $\tau(Sq^2Sq^1a)=Sq^2(Sq^2u)^2=(Sq^3u)^2=u^4$. Furthermore, a spectral sequence argument leads us to conclude $H^*(F;Z_2)=\{1, a, Sq^2a, b, Sq^2Sq^1a, c,\ldots\}$, where $b\in H^{10}(F;Z_2)$ with $\tau(b)=u^2Sq^2u$, $c\in H^{14}(F;Z_2)$ and … denote higher dimensional elements ($d_4(1\otimes c)=Sq^2u\otimes b$). This follows from the fact that Ker π^* is generated by $\{Sq^4Sq^2u, (Sq^2u)^2, u^2Sq^2u, u^4, Sq^8Sq^4Sq^2u, \ldots\}$ as a right $H^*(Z,3;Z_2)$-module and that the lowest dimensional relation is $(Sq^2u)^2u^2=(u^2Sq^2u)Sq^2u$. Since $\tau(Sq^1b)=Sq^1(u^2Sq^2u)=u^2Sq^3u=u^4=\tau(Sq^2Sq^1a)$ and since $\tau(Sq^2a)=Sq^2Sq^4Sq^2u=Sq^6Sq^2u+Sq^5Sq^3u=0$, we have that $Sq^1b=Sq^2Sq^1a$ and $Sq^2a=0$. Take a *CW* complex K' with minimum cells 2-equivalent to F, and so we may take $K'=M^9\underset{g}{\cup} CM^{10}\cup e^{14}\cup\cdots$. Consider the attaching map $g: M^{10}\to M^9$. $g|S^9$ cannot cover the 9-cell of M^9 essentially. Then the relation $Sq^2 a=0$ shows that $g|S^9$ is homotopic to zero. Thus we may choose g as the composition $h\circ q: M^{10}\to S^{10}\to M^9$. Besides, the relation $Sq^2(Sq^1 a)=Sq^1 b$ shows that $q\circ h: S^{10}\to M^9\to S^9$ is essential. Let K be the 11-skeleton of K' and f the composition of the restriction of the 2-equivalence $K\to F$ and the inclusion $i: F\to G_2^{(9)}$. Then clearly $f_*: \pi_i(K)\to\pi_i(G_2^{(9)})$ is a mod 2 isomorphism for $3<i<13$. The assertion of the second half follows from that $i^*: H^*(G_2^{(9)}; Z_p)\cong H^*(S^3; Z_p)$ for all odd prime p.

q.e.d.

Lemma 4.2.

(i) $[M^5, G_2^{(9)}]=[M^6, G_2^{(9)}]=0$;

$[M^8, G_2^{(9)}]\cong Z_2$ *generated by the class of* $(f|S^8)\circ q$;

$[M^9, G_2^{(9)}]\cong Z_4$ *generated by the class of* $f|M^9$.

(ii) $\pi_{10}(G_2^{(9)})\cong Z_{120}$; *there exists an exact sequence*

$$0\to\pi_{10}(S^3)\to\pi_{10}(G_2^{(9)})\to\pi_{10}(M^6)\to 0;$$

The image of the composition $[M^9, G_2^{(9)}]\otimes\pi_{10}(M^9)\to\pi_{10}(G_2^{(9)})$ *is isomorphic to* Z_4.

Proof. (i) Since $[M^n, X]$ is a Z_4-group, we deduce that $f_*: [M^n, K]\to[M^n, G_2^{(9)}]$ is an isomorphism for $4<n<13$. Obviously $[M^n, M^9]\cong[M^n, K]$ for $n\leq 8$, in particular $[M^8, K]\cong[M^8, M^9]\cong Z_2$ generated by $\{i\circ q\}$. We have an exact sequence $[M^9, M^{10}]\xrightarrow{(h\circ q)_*}[M^9, M^9]\to[M^9, K]\to[M^9, M^{11}]=0$, where $[M^9, M^{10}]\cong Z_2$ is generated by the class $\{i\circ q\}$ and $[M^9, M^9]\cong Z_4$ by [13]. Then $(h\circ q)_*\{i\circ q\}=0$, since $q_*\{i\}=\{q\circ i\}=0$. So $[M^9, K]\cong[M^9, M^9]\cong Z_4$ generated by the class of the inclusion (identity). Thus (i) is proved.

(ii) By Lemma 4.1, the odd component of $\pi_{10}(G_2^{(9)})$ is isomorphic to $\pi_{10}(S^3)\cong Z_{15}$ and the 2-component of that is isomorphic to $\pi_{10}(K)$. It is a classical result of Barratt-Paechter that $\pi_{10}(M^9)\cong Z_4$ generated by the class of h (for a proof see [13]). Since the top cell of K is attached to $K^{(10)}=M^9 \vee S^{10}$ by the sum of h and the map of degree 2, we obtain that $\pi_{10}(K^{(10)}) \cong Z_8$ and it is generated by the class of the identity of S^{10} twice of which is the class of h. Thus $\pi_{10}(G_2^{(9)}) \cong Z_{120}$. Consider the exact sequence

$$\pi_{11}(M^6) \rightarrow \pi_{10}(S^3) \xrightarrow{i_*} \pi_{10}(G_2^{(9)}) \xrightarrow{p_*} \pi_{10}(M^6) \xrightarrow{\partial} \pi_9(S^3),$$

Since $H^*(M^6; Z_p)$ is trivial for all odd primes p, $\pi_{10}(M^6)$ has only 2-torsion. Since $\pi_{10}(S^3) \cong Z_{15}$ and $\pi_9(S^3) \cong Z_3$, ∂ is trivial. Since $\pi_{10}(G_2^{(9)}) \cong Z_{120}$, we obtain a short exact sequence in the lemma. The second half of (ii) is clear from (i). q.e.d.

Lemma 4.3.

(i) $\pi_{10}(G_2)=\pi_{13}(G_2)=0$.

(ii) *The attaching class of the* 11-*cell in* $G_2^{(11)}=G_2^{(9)} \cup e^{11}$ *is a generator* ω *of* $\pi_{10}(G_2^{(9)}) \cong Z_{120}$.

(iii) *Let* $\pi: G_2^{(9)} \rightarrow M^9=G_2^{(9)}/G_2^{(6)}$ *be the projection. Then* $\pi_*(\omega)=\gamma$ *a generator of* $\pi_{10}(M^9) \cong Z_4$.

Proof. (i) is computed in [10]. Then (ii) follows easily from the exact sequence

$$\pi_{11}(G_2^{(11)}, G_2^{(9)}) \rightarrow \pi_{10}(G_2^{(9)}) \rightarrow \pi_{10}(G_2^{(11)}),$$

where $\pi_{10}(G_2^{(11)})=\pi_{10}(G_2)=0$.

(iii) follows easily from Lemma 4.2. q.e.d.

Remark 4.4. *The above lemma implies that the cokernel of the Hurewicz homomorphism*: $\pi_{11}(G_2) \rightarrow H_{11}(G_2; Z)$ *is isomorphic to* Z_{120}.

§5. Classification of H-spaces of type (3, 11)

Let f: $V_{7,2} \rightarrow BS^3$ be the classifying map of G_2. Let $\varphi: V_{7,2} \rightarrow V_{7,2} \vee S^{11}$ be the map pinching the equator $S^{10} \times \frac{1}{2}$ in $V_{7,2}=M^6 \cup CS^{10}$. Let α be a generator of $\pi_{11}(BS^3) \cong \pi_{10}(S^3)$ which corresponds to 8ω

under the monomorphism: $\pi_{10}(S^3)\to\pi_{10}(G_2^{(9)})\cong Z_{120}$ (see Lemma 4.2). For each integer b, let $g_b: S^{11}\to BS^3$ represent $b\alpha$ and let $G_{2,b}$ be the principal S^3-bundle over $V_{7,2}$ induced by the composition

$$f_b=(f\vee g_b)\circ\varphi: V_{7,2}\to V_{7,2}\vee S^{11}\to BS^3.$$

For example, $G_2=G_{2,0}$.

One of the main results of this section is the following:

Theorem 5.1. (i) *Each* 1*-connected H-complex of type* (3, 11) *has the homotopy type of* $G_{2,b}$ *for some* b.
(ii) $G_{2,b}$ *and* $G_{2,b'}$ *are homotopy equivalent if and only if* $b\equiv b'$ (15) *or* $b+b'\equiv 11$ (15).
(iii) *There are just* 8 *homotopy types of such H-complexes*: $G_{2,i}$ *for* $-2\leq i\leq 5$.

Before proving the theorem we prepare the following five lemmas. In the following we assume by Corollary 2.3 that every 1-connected H-complex of type (3, 11) has a cell structure

$$X\simeq S^3\cup e^5\cup e^6\cup e^8\cup e^9\cup e^{11}\cup e^{14}.$$

Lemma 5.2. *Let* X *be a* 1*-connected H-complex of type* (3, 11). *Then* $X^{(9)}$ *is homotopy equivalent to* $G_2^{(9)}$, *and hence* $X^{(11)}$ *is homotopy equivalent to* $H_k=G_2^{(9)}\cup_{k\omega} e^{11}$ *for some odd integer* k.

Proof. Let $j: S^3\to G_2^{(9)}$ be the inclusion. The obstructions to extending j over $X^{(9)}$ lie in $[M^5, G_2^{(9)}]$ and $[M^8, G_2^{(9)}]$. We obtain an extension $\bar{j}: X^{(6)}\to G_2^{(9)}$, since $[M^5, G_2^{(9)}]=0$ by Lemma 4.2. Next consider the Puppe sequence:

$$[X^{(9)}, G_2^{(9)}]\xrightarrow{i^*}[X^{(6)}, G_2^{(9)}]\xrightarrow{\varphi^*}[M^8, G_2^{(9)}]$$

associated with the cofibration

$$M^8\xrightarrow{\varphi}X^{(6)}\xrightarrow{i}X^{(9)}$$

where φ is the attaching map and i is the inclusion. To extend $\bar{j}$ over $X^{(9)}$, it suffices to show $\varphi^*(\bar{j})=0$. Assume that $\varphi^*(\bar{j})\neq 0$. Then by Lemma 4.2 $\varphi^*(\bar{j})=(f\,|\,S^8)\circ q$. It is not so difficult to see that $Sq^4Sq^2x_3$ is non-trivial in $H^*(X^{(9)};Z_2)$, which contradicts to Theorem 2.2. Thus j has an extension over $X^{(9)}$ which is clearly a homotopy equivalence from the structure of the cohomology. Therefore $X^{(11)}=G_2^{(9)}\cup_{k\omega} e^{11}$ for some integer k. The assertion that k is odd follows easily from the Z_2-cohomology structure. q.e.d.

Lemma 5.3. $(G_{2,b})^{(11)}\simeq H_{1+8b}$.

Proof. From the construction of the bundle $G_{2,b}$ we have a commutative diagram:

$$\begin{array}{ccccccc} G_2^{(9)} & \longrightarrow & G_{2,b} & \xrightarrow{\bar{\varphi}} & G_2\cup B_{b\alpha} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ M^6 & \longrightarrow & V_{7,2} & \xrightarrow[\varphi]{} & V_{7,2}\vee S^{11} & \xrightarrow[f\vee g_b]{} & BS_3 \end{array}$$

where $B_{b\alpha}$ is the S^3-bundle over S^{11} induced by bα, $G_2\cup B_{b\alpha}$ is the bundle induced by $f\vee g_b$ so that $G_2\cap B_{b\alpha}=S^3$ and two maps in the upper horizontal sequence are the inclusions. Remark that $(G_{2,b})^{(9)}=(G_2\cup B_\alpha)^{(9)}=G_2^{(9)}$. Therefore we obtain a commutative diagram:

$$\begin{array}{ccc} \pi_{11}(G_{2,b}, G_2^{(9)}) & \xrightarrow{\bar{\varphi}_*} & \pi_{11}(G_2\cup B_{b\alpha}, G_2^{(9)}) \\ & {\scriptstyle\partial}\searrow \quad \swarrow{\scriptstyle\partial'} & \\ & \pi_{10}(G_2^{(9)}) & \end{array}$$

where ∂ and ∂' are the boundary homomorphisms and $\pi_{11}(G_{2,b}, G_2^{(9)})\cong\pi_{11}(V_{7,2}, M^6)\cong Z$ and $\pi_{11}(G_2\cup B_{b\alpha}, G_2^{(9)})\cong\pi_{11}(V_{7,2}\vee S^{11}, M^6)\cong Z\oplus Z$. So for the generator $\iota\in\pi_{11}(G_{2,b}, G_2^{(9)})$, which is the class of the characteristic map of the 11-dimensional cell in $G_{2,b}$, we have that

$$\partial\iota=\partial'\bar{\varphi}_*(\iota)=\omega+b\alpha=(1+8b)\omega,$$

since $\bar{\varphi}^*$ is the map of type $(1,1)$. q.e.d.

Lemma 5.4. *Let k and k' be odd. Then $H_k \simeq H_{k'}$ if and only if $k \equiv \pm k'$* (mod 30).

Proof. To begin with we show
(5.1) every self homotopy equivalence of $G_2^{(9)}$ is homotopic to one of the following 8 maps:

$$f_t \colon G_2^{(9)} \xrightarrow{\varphi} G_2^{(9)} \vee M^9 \xrightarrow{1 \vee t\beta} G_2^{(9)}, \quad t=0, 1, 2, 3;$$

$$\bar{f}_t \colon G_2^{(9)} \xrightarrow{\varphi} G_2^{(9)} \vee M^9 \xrightarrow{\varepsilon \vee t\beta} G_2^{(9)}, \quad t=0, 1, 2, 3,$$

where $\varphi \colon G_2^{(9)} \to G_2^{(9)} \vee M^9$ is the map shrinking $M^8 \times \frac{1}{2}$ in $G_2^{(9)} = G_2^{(6)} \cup CM^8$, 1 is the indentity of $G_2^{(9)}$, ε is an extension of the map of degree $-1 \colon S^3 \to S^3 \subset G_2^{(9)}$ and β is a generator of $[M^9, G_2^{(9)}] \cong Z_4$.

The existence of ε is proved similarly to that of Lemma 5.2. The 8 maps in the above induce isomorphisms of the integral cohomology ring, since $\beta^*(x_3 x_5) = \beta^*(x_3)\beta^*(x_5) = 0$ and $\beta^*(x_3^3) = 0$ for $\beta^* \colon H^*(G_2^{(9)}; Z_2) \to H^*(M^9; Z_2)$. Thus these maps are homotopy equivalences. Consider the Puppe exact sequence:

$$[M^9, G_2^{(9)}] \xrightarrow{\pi^*} [G_2^{(9)}, G_2^{(9)}] \xrightarrow{i^*} [G_2^{(6)}, G_2^{(9)}].$$

If f and g of $[G_2^{(9)}, G_2^{(9)}]$ satisfy $i^* f = i^* g$, then there exists an element $t\beta \in [M^9, G_2^{(9)}]$ such that $f = (g \vee t\beta) \circ \varphi$. A similar statement holds in the sequence:

$$[M^6, G_2^{(9)}] \xrightarrow{\pi_0^*} [G_2^{(6)}, G_2^{(9)}] \xrightarrow{i_0^*} [S^3, G_2^{(9)}],$$

where i_0^* is injective, since $[M^6, G_2^{(9)}] = 0$ by Lemma 4.2. Now let g be 1 or ε according as $i_0^* i^* f = i_0^* i^* 1$ or $i_0^* i^* f = i_0^* i^* \varepsilon$. Then it follows that $f = (g \vee t\beta) \circ \varphi = f_t$ or $\bar{f}_t$. Thus the proof of (5.1) is completed.

By taking inverse for each element, we obtain a self homotopy equivalence of G_2 such that it is of degree -1 on S^3. Then we may choose ε as a cellular approximation of this map.

We have

(5.2) *$H_k \simeq H_k'$ if and only if $f_{t*} k\omega = \pm k'\omega$* or *$\bar{f}_{t*} k\omega = \pm k'\omega$.*

In fact, the restriction of every homotopy equivalence on $G_2^{(9)}$ is either f_t or $\bar{f}_t$ for some t.

Here we recall a result due to Whitehead [20]:

$$\pi_{10}(G_2^{(9)}\vee M^9)\cong\pi_{10}(G_2^{(9)})\oplus\pi_{10}(M^9)\oplus\partial\pi_{11}(G_2^{(9)}\times M^9,\ G_2^{(9)}\vee M^9).$$

So we have

$$\begin{aligned} f_{t*}(k\omega)&=(1\vee t\beta)_*\circ\varphi_*(k\omega)\\ &=(1\vee t\beta)_*(k\omega+k\gamma+kx[\iota_3,\iota_8])\\ &=k\omega+kt\beta_*(\gamma)+ktx[\iota_3,\beta\iota_8]. \end{aligned}$$

Here $\beta_*(\gamma)=\pm 30\omega$ by Lemma 4.3. Further we have $[\iota_3,\beta\iota_8]=0$. In fact, $p_*[\iota_3,\beta\iota_8]=0$ for $p_*:\pi_{10}(G_2^{(9)})\to\pi_{10}(M^6)$ and hence $[\iota_3,\beta\iota_8]$ is of odd order, since $\pi_{10}(S^3)\cong Z_{15}$, while $[\iota_3,\beta\iota_8]$ is of order 2, as $2\beta\iota_8=0$. Thus we have $f_{t*}(k\omega)=k(1\pm 30t)\omega$, whence $k'\equiv k(1\pm 30t)$ (mod 120). Similarly one can obtain that $k'\equiv k(-1\pm 30t)$ (mod 120). Since k and k' are odd, we can deduce that H_k is homotopy equivalent to $H_{k'}$ if and only if $k\equiv\pm k'$ (mod 30). q.e.d.

Lemma 5.5. *Every $G_{2,b}$ is an H-space of type* (3, 11).

Proof. Since $V_{7,2}$ is p-equivalent to S^{11} for all odd primes p, $G_{2,b}$ is p-equivalent to a principal S^3-bundle over S^{11}, and hence $G_{2,b}$ is an H-space mod p by Corollary 3.4. For $p=2$, consider a complex $V=M^6\cup_{15\sigma}e^{11}$, where σ is the attaching map of e^{11} in $V_{7,2}$. Apparently there is a 2-equivalence $h: V\to V_{7,2}$, which has degree 15 on the 11-dimensional cell. Let $\varphi': V\to V\vee S^{11}$ be the shrinking map similar to φ. Thus by commutativity of the diagram:

$$\begin{array}{ccccc} V & \xrightarrow{\varphi'} & V\vee S^{11} & & \\ \downarrow & & \downarrow{\scriptstyle h\vee 15\iota_{11}} & & \\ V_{7,2} & \xrightarrow[\varphi]{} & V_{7,2}\vee S^{11} & \xrightarrow[f\vee g_b]{} & BS^3 \end{array}$$

and by the fact that $15\alpha=0$, we obtain that $G_{2,b}$ is 2-equivalent to G_2. Therefore $G_{2,b}$ is an H-space by Theorem 7.1 of [12]. q.e.d.

Lemma 5.6. *Let X and Y be* 1*-connected H-complexes of type* (3,11). *Then $X\simeq Y$ if and only if $X^{(11)}\simeq Y^{(11)}$.*

Proof. The necessity is clear. We show the sufficiency. First we prove for the case that $Y=G_{2,b}$. Let $r': X^{(11)}\to G_{2,b}^{(11)}$ be a homotopy equivalence. If we obtain an extension $r: X\to G_{2,b}$, it is easily checked to be a homotopy equivalence from the cohomology ring structures of X and $G_{2,b}$.

As is shown in the proof of Lemma 5.5, $G_{2,b}\underset{2}{\simeq}G_2$ and $G_{2,b}$ is p-equivalent to a principal S^3-bundle over S^{11} for odd p. Then by Theorem 3.2 and Lemma 5.3, we have $\pi_{13}(G_{2,b}:p)=0$ for $p\neq 3$, and if $\pi_{13}(G_{2,b})$ is non-trivial, it is isomorphic to Z_3 and $G_{2,b}\underset{3}{\simeq}S^3\times S^{11}$. If $\pi_{13}(G_{2,b})=0$, clearly we have an extension $r: X\to G_{2,b}$. Hence we assume $\pi_{13}(G_{2,b})=Z_3$. Then X is also 3-equivalent to $S^3\times S^{11}$. For $X^{(11)}\simeq G_{2,b}^{(11)}$ and X is an H-space. So the attaching element δ of e^{14} in X satisfies that $q\delta=q'f_*[\iota_3,\iota_{11}]$ for some integers q,q' with $qq'\not\equiv 0$ (3) and for some 3-equivalence $f: S^3\vee S^{11}\to X^{(11)}$. Since $G_{2,b}$ is an H-space, we have that $r'_*(q\delta)=r'_*(q'f_*[\iota_3,\iota_{11}])=0$ in $\pi_{13}(G_{2,b})=Z_3$ and hence $r'_*\delta=0$ in $\pi_{13}(G_{2,b})$. That is, there is an extension $r: X\to G_{2,b}$.

Now for general Y, we have that $Y^{(11)}\simeq H_k$ for some odd k with $1\leq k\leq 15$ by Lemma 5.2 and Lemma 5.4. Since either k or $-k$ is expressed as $1+8b$, we have $Y^{(11)}\simeq G_{2,b}^{(11)}$ by Lemma 5.3. Thus $Y\simeq G_{2,b}$ by the above argument. This completes the proof. q.e.d.

(*Proof of Theorem* 5.1.) (i) Let X be a 1-connected H-complex of type (3,11). Then $X^{(11)}\simeq H_k$ for some odd integer k with $1\leq k\leq 15$ by Lemmas 5.2 and 5.4. Since either k or $-k$ is expressed as $1+8b$ with $-2\leq b\leq 5$, we can see by virtue of Lemmas 5.3 and 5.4 that $X^{(11)}\simeq G_{2,b}^{(11)}$ for some b, $-2\leq b\leq 5$. Then by Lemmas 5.5 and 5.6 we obtain (i).

(ii) By Lemmas 5.4 and 5.6, $G_{2,b}\simeq G_{2,b'}$ if and only if $H_{1+8b}\simeq H_{1+8b'}$ if and only if $1+8b\equiv\pm(1+8b')$ (30) if and only if $b\equiv b'$ (15) or $b+b'\equiv 11$ (15).

(iii) follows directly from (ii) and Lemma 5.5. q.e.d.

As a corollary of the proof of Theorem 5.1 we have

Corollary 5.7. *Let X be a* 1*-connected H-complex of type* (3, 11). *Then*

(i) $X \underset{p}{\simeq} G_2$ *for any prime p with $p \neq 3$ or* 5.

(ii) $X \underset{5}{\simeq} G_2$ *or* $\underset{5}{\simeq} S^3 \times S^{11}$ *according as* $\mathcal{P}^1 x_3 \neq 0$ *or* $\mathcal{P}^1 x_3 = 0$ *in* $H^*(X; Z_p)$

(iii) $X \underset{3}{\simeq} G_2$ *or* $\underset{3}{\simeq} S^3 \times S^{11}$ *according as* $\phi x_3 \neq 0$ *or* $\phi x_3 = 0$ *in* $H^*(X; Z_p)$, *where ϕ is a secondary operation considered in* §2. (*ϕ is known to detect a generator of* $\pi_{10}(S^3 : 3) \cong Z_3$.)

The proof is left to the reader.

Theorem 5.8. *$G_{2,b}$ has the homotopy type of a loop space if and only if* $1+8b \not\equiv 0$ (p) *for* $p=3$ *and* 5, *i.e.,* $b=-1, 0, 2, 5$.

Proof. By Theorem 7.1 of [12], $G_{2,b}$ has the homotopy type of a loop space if and only if $(G_{2,b})_{(p)}$ does for any prime p. Clearly $(G_{2,b})_{(p)}$ is a loop space for $p \neq 3$ or 5, since $G_{2,b} \underset{p}{\simeq} G_2$ by Lemma 5.5 and Corollary 5.7. Note that $(S^3 \times S^{11})_{(p)}$, for $p=3$ and 5, is not of the homotopy type of a loop space. In fact, if so, there exists the classifying space $B(S^3 \times S^{11})_{(p)}$, and hence the $\mathcal{A}_p$-algebra structure of $H^*(B(S^3 \times S^{11})_{(p)}; Z_p) \cong Z_p[u_4, u_{12}]$ induces a contradiction. Therefore $(G_{2,b})_{(p)}$ is a loop space if and only if $(H_{1+8b})_{(p)}$ is a loop space if and only if

$$H^*(H_{1+8b}; Z_p) \cong \begin{cases} \Lambda(x_3, \phi x_3) & p=3 \\ \Lambda(x_3, \mathcal{P}^1 x_3) & p=5 \end{cases}$$

if and only if $1+8b \not\equiv 0$ mod 3 and 5. q.e.d.

§6. Appendix

For convenience, we list the following table for $G_{2,b}, -2 \leq b \leq 5$

	3-type	5-type	p-type ($p\neq 3,5$)	
-2	$S^3\times S^{11}$	$S^3\times S^{11}$	G_2	not loop
-1	G_2	G_2	G_2	loop
0	G_2	G_2	G_2	loop
1	$S^3\times S^{11}$	G_2	G_2	not loop
2	G_2	G_2	G_2	loop
3	G_2	$S^3\times S^{11}$	G_2	not loop
4	$S^3\times S^{11}$	G_2	G_2	not loop
5	G_2	G_2	G_2	loop

According to L. Smith [14], the type of a 1-connected, associative H-space of rank 2 is either (3, 3), (3, 5), (3, 7) or (3, 11). Then, using Theorem 7.1 of [12] together with Theorem 5.8 and the results of [15], [21], we obtain the following

Theorem 6.1. *A 1-connected, finite, associative H-complex of rank 2 is homotopy equivalent to one of the following*: $S^3\times S^3$, $SU(3)$, $E_1=Sp(2)$, E_5, $G_{2,0}=G_2$, $G_{2,-1}$, $G_{2,2}$, $G_{2,5}$.

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