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On the initial-value problems with data on a characteristic hypersurface

By

Yukiko HASEGAWA

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§ 1. Introduction

In this paper we study an initial-value problem for the equation;

$$(1.1) \qquad \sum_{i+|\alpha|\leq m} a_{i,\alpha}(x, y)\partial_x^i \partial_y^\alpha u = f(x, y), \quad x \in R^1, \quad y \in R^n,$$

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y^\alpha = \left(\frac{\partial}{\partial y_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{\alpha_n}$$

with data on $x=0$. In this article we treat our problem only within the class of real analytic functions, more precisely we assume that the coefficients, initial data and solutions are all real analytic in the neighborhood of the origin.

For the following initial data;

$$(1.2) \qquad \partial_x^i u(0, y) = u_i(y), \quad i = 0, 1, 2, \ldots, m-1,$$

it is well-known that under the assumption

$$(1.3) \qquad a_{m,0}(0, 0) \neq 0,$$

the problem (1.1)–(1.2) has always a unique solution (Cauchy-Kowalevski's theorem). But if we remove the condition (1.3), the problem (1.1)–(1.2) has not always a solution. In [2], we had treated the following initial-value problem;

$$\begin{cases} (1.4) \quad x a_{m,0}(x,y)\partial_x^m u + \sum_{|\alpha|=1} x a_{m-1,\alpha}(x,y)\partial_x^{m-1}\partial_y^\alpha u + a_{m-1,0}(x,y)\partial_x^{m-1}u \\ \qquad + \sum_{i\leq m-2,\, |\alpha|+i\leq m} a_{i,\alpha}(x,y)\partial_x^i\partial_y^\alpha u = f(x,y), \\ \qquad\qquad x\in R^1, \quad y\in R^{n-1}, \quad \sum_{|\alpha|=2} |a_{m-2,\alpha}(0,0)|\neq 0. \\ (1.5) \quad \partial_x^j u(0,y) = u_j(y), \quad j=0,1,2,\dots,m-2. \end{cases}$$

We proved in [2].

Theorem 1.1

In the case where $a_{m,0}(0,y)\equiv 0$, *if* $a_{m-1,0}(0,y)\neq 0$, *the problem* (1.4)–(1.5) *has not always a solution.*

Theorem 1.2

In the case where $a_{m,0}(0,0)\neq 0$, *and moreover for all nonnegative integers* p, *we have* $p a_{m,0}(0,0)+a_{m-1,0}(0,0)\neq 0$, *then there exists always a unique solution for the problem* (1.4)–(1.5).

Theorem 1.3

In the case where $a_{m,0}(0,0)\neq 0$, *and moreover if for some non-negative integer* p_0, *we have* $p_0 a_{m,0}(0,y)+a_{m-1,0}(0,y)\equiv 0$, *then a necessary and sufficient condition concerning the initial data for the existence of the solution of the problem* (1.4)–(1.5) *is the following compatibility condition*:

$$(1.6) \quad \sum_{\substack{k+s=p_0+m-1 \\ k\geq 1}} a_{m,0}^{(k)}(y)\frac{u_s(y)}{(s-m)!} + \sum_{|\alpha|=1}\sum_{k+s=p_0+m-2} a_{m-1,\alpha}^{(k)}(y)\frac{\partial_y^\alpha u_s(y)}{(s-m+1)!}$$
$$+ \sum_{\substack{k+s=p_0+m-1 \\ k\geq 1}} a_{m-1,0}^{(k)}(y)\frac{u_s(y)}{(s-m+1)!}$$
$$+ \sum_{\substack{i\leq m-2 \\ i+|\alpha|\leq m}}\sum_{k+s=p_0+i} a_{i,\alpha}^{(k)}(y)\frac{\partial_y^\alpha u_s(y)}{(s-i)!} = f^{(p_0)}(y),$$

where $a_{j,\beta}(x,y)=\sum_{k\geq 0} a_{j,\beta}^{(k)}(y)x^k$, $f(x,y)=\sum_{k\geq 0} f^{(k)}(y)x^k$ and $u_{m-1}, u_m, \dots, u_{m+p_0-2}$ are uniquely determined by the initial data $\{u_0, u_1, \dots, u_{m-2}\}$.

In this paper we treat more general equations than (1.4). Generalizing the above compatibility condition, we obtain similar results to the above theorems.

Let us introduce the following notion;

Definition 1.1

For the differential operator of the form $x^p a(x, y) \partial_x^q \partial_y^\alpha$ ($p \geqq 0$, $a(0, y) \not\equiv 0$), we call the pair of integers $(q\text{-}p, |\alpha|)$ its degree.

Definition 1.2

For the differential operator $A = x^p a(x, y) \partial_x^q \partial_y^\alpha$ and $A' = x^{p'} a'(x, y) \partial_x^{q'} \partial_y^{\alpha'}$ ($p, p' \geqq 0$, $a(0, y) \not\equiv 0$, $a'(0, y) \not\equiv 0$), we say that the degree of A is higher than that of A' if and only if $q\text{-}p > q'\text{-}p'$ or $q\text{-}p = q'\text{-}p'$, $|\alpha| > |\alpha'|$. And two operators A and A' are of the same degree if $q\text{-}p = q'\text{-}p'$, $|\alpha| = |\alpha'|$.

For the coefficients of (1.1), when $\partial_x^k a_{i\alpha}(0, y) \equiv 0$ $k = 0, 1, \dots, p\text{-}1$, and $\partial_x^p a_{i\alpha}(0, y) \not\equiv 0$, we can write $a_{i\alpha}(x, y) = x^p a_{pi\alpha}(x, y)$ ($a_{pi\alpha}(0, y) \not\equiv 0$). We rewrite
(1.1) in the following form;

$$\sum_{q+|\alpha| \leqq m} x^p a_{pq\alpha}(x, y) \partial_x^q \partial_y^\alpha u = f(x, y), \qquad a_{pq\alpha}(0, y) \not\equiv 0, \quad p = p(q, \alpha). \tag{1.7}$$

Definition 1.3

We define that the degree of the differential operator $\sum_{p,q,\alpha} x^p a_{pq\alpha}(x, y) \partial_x^q \partial_y^\alpha$ is as the highest degree of the terms in the summation.

§ 2. Statements of Theorems

Let the degree of (1.7) be (m', r), we suppose $m' \geqq 0$ then (1.7) can be written in the following form;

$$\sum_{j=0}^{m-r-m'} \Big(x^j \partial_x^{m'+j} \sum_{|\alpha|=r} b_{j\alpha}(y) \partial_y^\alpha\Big) u = \tilde{\sum} x^p a_{pq\alpha}(x, y) \partial_x^q \partial_y^\alpha u + f(x, y), \tag{2.1}$$

where $\tilde{\Sigma}$ means the summation of the terms whose degree is lower than (m', r), namely the summation of the terms where p, q, α satisfy $q\text{-}p<m'$ or $q\text{-}p=m'$, $|\alpha|<r$. Let us remark that the degree of the equation which we treated in [2] is $(m\text{-}1, 0)$.

At first we consider under the following assumption;

$$\text{(H)} \qquad \sum_{|\alpha|=r} |b_{m-r-m', \alpha}(0)| \neq 0 \quad \text{in (2.1).}$$

This assumption (H) corresponds to the hypothesis $a_{m,0}(0, 0)\neq 0$ in the theorem 1.2 and 1.3.

For the simplicity we denote (2.1) by

$$\text{(2.2)} \qquad \sum_{j=0}^{m-r-m'} (x^j \partial_x^{m'+j} L_j(y, \partial_y))u = \Gamma(x, y, \partial_x, \partial_y)u + f,$$

where $L_k(y, \partial_y) = \sum_{|\alpha|=r} b_{k,\alpha}(y)\partial_y^\alpha$, $k=0, 1, \ldots, m+r-m'$,

and $\Gamma(x, y, \partial_x, \partial_y) = \tilde{\Sigma} x^p a_{pq\alpha}(x, y)\partial_x^q \partial_y^\alpha$.

Let us consider the following initial data;

$$\text{(2.3)} \qquad \partial_x^i u(0, y) = u_i(y), \quad i=0, 1, \ldots, m'-1.$$

Let

$$\text{(2.4)} \qquad u(x, y) = \sum_{j>0} u_j(y) x^j/j!$$

be the formal solution of the problem (2.2)-(2.3). Then we have

$$\text{(2.5)} \qquad \left(\sum_{j=0}^{m-r-m'} \frac{L_j(y, \partial_y)}{(k-j)!}\right) u_{k+m'}(y) = \frac{1}{k!}\partial_x^k (\Gamma \varphi_{k+m'})\Big|_{x=0} + \frac{f_k}{k!},$$
$$k=0, 1, 2, \ldots,$$

where $\varphi_n = \sum_{j=0}^{n} u_j(y)x^j/j!$, $f = \sum_{j>0} f_j(y)x^j/j!$ and $1/s!=0$ when $s<0$.

From the assumption (H), there exists some integer k_0' such that for any k $(>k_0')$ we have the following relation;

$$\text{(2.6)} \qquad \sum_{|\alpha|=r} \left| \sum_{j=0}^{m-r-m'} \frac{b_{j\alpha}(0)}{(k-j)!} \right| \neq 0.$$

Let k_0 be the minimum number which has the above property.

Now there exists $u_{k+m'}(y)$ $(k\geqslant k_0)$ which satisfies (2.5) (assuming that $u_{m'}, u_{m+1}, \dots, u_{k+m'-1}$ are already defined). In fact, let us consider that (2.5) is a differential equation whose unknown function is $u_{k+m'}(y)$. And we write (2.5) in the following form:

$$\left(\sum_{j=0}^{m-r-m'}\frac{L_j(y,\partial_y)}{(k-j)!}\right)u_{k+m'}(y)-\sum_{\substack{q-p=m'\\|\alpha|<r}}a_{pq\alpha}(0,y)\frac{\partial_y^\alpha}{(k-p)!}u_{k+m'}(y)=\mathfrak{L}(u_0, u_1, \dots, u_{k+m'-1}). \tag{2.7}$$

In view of (2.6), (2.7) is Kowalevskian type for some direction. So there exist $u_{k+m'}(y)$ which satisfies (2.5).

If the problem (2.2)-(2.3) has at least one solution, there exists $\{u_{k+m'}(y);\ 0\leqslant k<k_0\}$ which satisfies (2.5). Now this is necessary for the existence of the solution of the problem (2.2)-(2.3). We shall prove that this also a sufficient condition. Namely,

Theorem 2.1

If there exists $\{u_{k+m'}(y);\ k=0, 1, \dots, k_0\text{-}1\}$ *which satisfies* (2.5), *the problem* (2.2)-(2.3) *has at least one solution in a neighborhood of the origin.*

Concerning the initial data (1.2), we have

Corollary

In the case where $k_0\leqslant m\text{-}m'$, *a necessary and sufficient condition for the existence of the solution of the problem* (2.2)-(1.2) *is that the initial data* $\{u_1(y), u_2(y), \dots, u_{m-1}(y)\}$ *satisfies* (2.5).

In the case where $k_0>m\text{-}m'$, *a necessary and sufficient condition for the existence of the solution of the problem* (2.2)-(2.1) *is that there exists* $\{u_m, u_{m+1}, \dots, u_{m'+k_0-1}(y)\}$ *such that a set* $\{u_0, u_1, \dots, u_{m'+k_0-1}\}$ *satisfies* (2.5).

Remark 2.1

When $r=0$ and $k_0=0$, the problem (2.2)-(2.3) has always a unique solution.

Remark 2.2

When $r>0$, the solution is not unique.

Let us consider the following initial-value problem (2.8)-(2.9), where (2.8) does not satisfy the assumption (H).

$$\begin{cases} (2.8) \quad \partial_x^{m-1}u = \sum_{\substack{q+|\alpha|\leq m,\, p\leq 2 \\ q-p\leq m-2}} x^p a_{pq\alpha}(x, y)\partial_x^q \partial_y^\alpha u + f(x, y), \\ (2.9) \quad \partial_x^i u(0, y) = u_i(y), \quad i=0, 1, \dots, m-2. \end{cases}$$

The degree of the differential operator (2.8) is (m-1. 0). And the degree of the right-hand side of (2.8) is at most (m-2, 2). Let

$$(2.10) \qquad u(x, y) = \sum_{k\geq 0} u_k(y) x^k / k!$$

be the formal solution of the problem (2.8)-(2.9). Then we have the similar relation to (2.5),

$$(2.11) \qquad \frac{1}{k!} u_{k+m-1}(y) = F(u_0, u_1, \dots, u_{k+m-2}) + \frac{f_k}{k!}, \quad k=0, 1, 2, \dots.$$

So the formal solution (2.10) is uniquely defined. But generally this formal solution does not always converge in a neighborhood of the origin. Namely

Theorem 2.2

If for some integer $s\geq 2$, *we have*

$$(2.12) \qquad \begin{aligned} &\sum_{|\nu|<s} \left| \frac{\partial^{|\nu|} h(0, y, 1, 0)}{\partial \eta^\nu} \right| \equiv 0, \\ &\sum_{|\nu|=s} \left| \frac{\partial^{|\nu|} h(0, 0, 1, 0)}{\partial \eta^\nu} \right| \neq 0, \end{aligned}$$

then there exists an initial data (2.9) *such that the problem* (2.8)-(2.9) *has no analytic solution in any neighborhood of the origin, where* $h(x, y, \xi, \eta) = \sum_{q+|\alpha|=m} x^q a_{pq\alpha}(x, y)\xi^q \eta^\alpha$.

§ 3. Proof of Theorem 2.1

At first we consider the following fairly simple equation:

$$(3.1)\qquad \left(\sum_{j=0}^{m-r-m'} x^j\partial_x^{m'+j}\partial_{y_1}^r\right)u=\bar{\sum} x^p a_{pq\alpha}(y)\partial_x^q\partial_y^\alpha u+f(y),$$

where $\bar{\sum}$ means the summation of the terms where p, q, α satisfy

$q+|\alpha|\leqslant m$, $q-p<m'$, $p\leqslant p_0$ or $q+|\alpha|\leqslant m$, $q-p=m'$, $|\alpha|\leqslant r$, $\alpha_1<r$,

$p\leqslant p_0$ (where p_0 is some constant).

Let us consider the following Goursat data*);

$$(3.2)\qquad \begin{cases} \partial_x^i u(0,y)=0, & i=0,1,\dots,m'-1,\\ \partial_{y_1}^i u(x,y)|_{y_1=0}=0, & i=0,1,\dots,r-1.\end{cases}$$

We shall show

Lemma

The Goursat problem (3.1)-(3.2) (*when* $r=0$, *this is Cauchy problem*) *has a unique solution* $u(x,y)$ *in a neighborhood of the origin.*

Proof

Let

$$(3.3)\qquad u(x,y)=\sum_{j\geqslant m'} u_j(y)x^j/j!$$

be the formal solution of (3.1)-(3.2). Then we have

$$(3.4)\qquad \left\{\sum_{j=0}^{m-r-m'} 1/(k-j)!\right\}\partial_{y_1}^r u_{k+m'}(y)$$
$$=\bar{\sum} a_{pq\alpha}(y)\partial_y^\alpha u_{k+q-p}(y)/(k-p)!+\delta_0^k f(y),\quad k\geqslant 0,$$

where $1/j!=0$ when $j<0$, and δ_0^k is Kronecker's δ. For to be brief, we denote (3.4) by

*) when $r=0$, we consider that (3.2) is Cauchy data $\partial_x^i u(0,y)=0$, $i=0,1,\dots,m'$-1.

$$(3.5)\qquad \partial_{y_1}^{r} u_{k+m'}(y)=\frac{1}{N(k,m-r-m')}\{\bar{\Sigma} a_{pq\alpha}(y)\partial_y^{\alpha} u_{k+q-p}(y)/(k-p)!+\delta_0^k f(y)\}$$

where $N(k,s)=\sum_{j=0}^{s} 1/(k-j)!$. The formal solution (3.3) is uniquely defined by (3.2) and (3.5). Now, we want to show the convergence and the analyticity of (3.3).

Let $M/\left(1-\frac{y_1}{\rho}\right)\left(1-\frac{y_2+\dots+y_n}{\rho}\right)$ be a common majorant of $f(y)$ and $a_{pq\alpha}(y)$ which appear in (3.5). Let us consider the following equation;

$$(3.6)\qquad \partial_{y_1}^{r} U_{k+m'}(y)=\frac{M}{N(k,m-r-m')\left(1-\frac{y_1}{\rho}\right)\left(1-\frac{y_2+\dots+y_n}{\rho}\right)}\times\{\bar{\Sigma}\,\partial_y^{\alpha} U_{k+q-p}(y)/(k-p)!+\delta_0^k\},\quad k=0,1,2,\dots.$$

and the following initial data ($r\geqq 1$);

$$(3.7)\qquad \partial_{y_1}^{i} U_{k+m'}(y)|_{y_1=0}=0,\quad i=0,1,\dots,r-1,\quad k=0,1,2,\dots.$$

Then $\{U_{k+m'}(y);\ k=0,1,2,\dots\}$ are determined successively as a solution of the problem (3.6)-(3.7). It is easy to see that we have

$$(3.8)\qquad U_{k+m'}(y)\gg u_{k+m'}(y),\qquad k=0,1,2,\dots.$$

Let us consider the following Taylar expansion.

$$(3.9)\qquad U_{k+m'}(y)=\sum_{j\geq r} U_{k+m'}^{(j)}(y')y_1^j/j!,\qquad k=0,1,2,\dots$$

where $y'=(y_2,y_3,\dots,y_n)$.
From (3.6) and (3.9) we obtain

$$(3.10)\qquad \frac{U_{k+m'}^{(j)}(y')}{(j-r)!}=\frac{U_{k+m'}^{(j-1)}(y')}{\rho(j-r-1)!}+\frac{B(y')}{N(k,m-r-m')}\times\left\{\bar{\Sigma}\frac{\partial_{y'}^{\alpha'} U_{k+q-p}^{(j-r+\alpha_1)}(y')}{(k-p)!(j-r)!}+\delta_0^k\right\}\quad j\geqq r,\ k=0,1,2,\dots,$$

where $B(y')=M/\left(1-\frac{y_2+\ldots+y_n)}{\rho}\right)$ and $\partial_{y'}^{\alpha'}=\left(\frac{\partial}{\partial y_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial y_n}\right)^{\alpha_n}$.
Then we have

$$(3.11)\qquad |\partial_{y'}^{\nu} U_k^{(j)}(y')| \leqslant A\frac{(j+(m-r-m'+1)k+|\nu|)!}{(k!)^{m-m'-r}}\times C^{2j+(2m+2)k+|\nu|},$$
$$|y'|\leqslant a,$$

for any ν and for any k and j, if we choose the constant C, A sufficient large and constant a small. Concerning the proof of (3.11), refer to [2] p. 366∼.

From (3.11), we see easily that $U(x, y)$ defined by

$$(3.12)\qquad U(x, y)=\sum_k U_k(y)x^k/k!=\sum_{\substack{k\geqslant m'\\ j\geqslant r}} U_k^{(j)}(y')x^k y_1^j/k!j!$$

has the estimates of the form

$$(3.13)\qquad |\partial_x^q \partial_{y_1}^{\alpha_1}\partial_{y'}^{\nu} U(x, y)|\leqslant (q+\alpha_1+|\nu|)!\, C'^{q+\alpha_1+|\nu|} A'$$

for $|x|\leqslant\delta$, $|y_1|\leqslant\delta'$, $|y'|\leqslant a$ if we choose δ, δ' small and C', A' large. So the formal solution defined by (3.3) which is majorated by $U(x,y)$ converges in a neighborhood of the origin. The proof of Lemma is thus complete.

Proof of Theorem **2.1**

Without loss of generality, we can assume that $b_{m-r-m', r, 0\cdots 0}(y)$. (the coefficient of $x^{m-r-m'}\ \partial_{y_1}^r \partial_x^{m-r}$) is not zero at $y=0$.
In fact, from the assumption (H), there exists an $\eta\in C^n$ such that

$$(3.14)\qquad \sum_{|\alpha|=r} b_{m-r-m', \alpha}(0)\eta^\alpha=1.$$

So there exists some i such that $\eta_i\neq 0$. Without loss of generality we can assume $\eta_1\neq 0$. Let us consider the following change of variables;

$$(3.15)\qquad \begin{cases} z_1=\langle y, \eta\rangle=y_1\eta_1+y_2\eta_2+\ldots+y_n\eta_n \\ z_i=y_i,\ i=2, 3, \ldots, n \end{cases}$$

The coefficient of $x^{m-r-m'}\partial_{z_1}^r\partial_x^{m-r}$ is $\sum_{|\alpha|=r} b_{m-r-m',\alpha}(0)\eta^\alpha$ at $y=z=0$. The above change of variables does not change the degree of (2.1).

Then there exists some integer $k_1(\geqslant k_0)$ such that for any k $(\geqslant k_1)$ we have

$$\sum_{j=0}^{m-r-m'}\frac{b_{j,r,0\dots0}(0)}{(k-j)!}\neq 0. \tag{3.16}$$

Now, we rewrite (2.1) in the following form;

$$\sum_{j=0}^{m-r-m'} x^j\partial_x^{m'+j}A_j(y)\partial_{y_1}^r u=\bar{\sum} x^p a_{qp\alpha}(x,y)\partial_x^q\partial_y^\alpha u+f(x,y). \tag{3.17}$$

Differentiating (3.17) k_1 times with respect to x, we have

$$\sum_{j=0}^{m-r-m'} x^j\partial_x^{m'+k_1+j}B_j(y)\partial_{y_1}^r u=\bar{\sum}' x^p\tilde{a}_{pq\alpha}(x,y)\partial_x^q\partial_y^\alpha u+\tilde{f}(x,y), \tag{3.18}$$

where $\bar{\sum}'$ means the summation of the termes where p, q, α satisfy $q\text{-}p<m'+k_1$, $p<p_0$, $q+|\alpha|\leqslant m+k_1$ or $q\text{-}p=m'+k_1$, $|\alpha|\leqslant r$, $\alpha_1<r$, $p<p_0$, $q+|\alpha|\leqslant m+k_1$.

For the equation (3.18), let us consider the following Goursat data;

(3.19)
$$\begin{cases} \partial_x^i u(0,y)=u_i(y) \quad i=0,1,\dots,m'+k_1-1 \\ \quad \text{when } i=0.1,\dots,m'-1,\ u_i(y) \text{ is the same as (2.3)}, \\ \quad \text{when } i=m',\dots,m'+k_1-1,\ u_i(y) \text{ is one which satisfies (2.5)}, \\ \partial_{y_1}u(x,y)|_{y_1=0}=v_i(x,y'), \quad i=0,1,\dots,r-1 \\ \quad \text{where } v_i(x,y')=\partial_{y_1}^i\varphi_{m'+k_1-1}(x,y)|_{y_1=0}+x^{m'+k_1}\psi_i(x,y') \\ \quad \text{and where } \varphi_p(x,y)=\sum_{k=0}^{p}u_k(y)x^k/k!,\ \psi_i \text{ is arbitrary.} \end{cases}$$

We shall show that the problem (3.18)-(3.19) has a unique solution in a neighborhood of the origin. Without loss of generality, we can assume that the Goursat data (3.19) are all zero.

Let

$$u(x,y)=\sum_{j\geqslant m'+k_1}u_j(y)x^j/j! \tag{3.30}$$

be the formal solution of (3.18)-(3.19). Form (3.18) and (3.20), we have the following similar relation to (3.4)

$$\left\{\sum_{j=0}^{m-r-m'} B_j(y)/(k-j)!\right\}\partial_{y_1}^r u_{k+m'+k_1}=\cdots. \tag{3.21}$$

On the other hand, from (3.16) we have

$$\sum_{j=0}^{m-r-m'} B_j(0)/(\mathrm{k}-j)!\neq 0, \qquad k\geqq 0. \tag{3.22}$$

In view of the recurrence formulas between $u_{m'+k_1}$, $u_{m'+k_1+1},\dots,$ we choose a majorant $C(y)$ such that

$$\frac{1}{\sum_{j=0}^{m-r-m'} B_j(y)/(k-j)!}\ll\frac{C(y)}{\sum_{j=0}^{m-r-m'} 1/(k-j)!} \tag{3.23}$$

for all non-negative integers k. Such a function $C(y)$ exists because of (3.22).

Let us consider the following Goursat problem (3.24)-(3.25).

$$\sum_{j=0}^{m-r-m'} x^j\partial_x^{m'+k_1+j}\partial_{y_1}^r U=\frac{MC(y)}{\left(1-\frac{x}{\rho}\right)\left(1-\frac{y_1+\dots+y_n}{\rho}\right)} \times\{\bar{\sum}' x^p\partial_x^q\partial_y^\alpha U+1\}, \tag{3.24}$$

where $M/\left(1-\frac{x}{\rho}\right)\left(1-\frac{y_1+\dots+y_n}{\rho}\right)$ is a common majorant of $\tilde{f}$ and $\tilde{a}_{pq\alpha}(x,y)$ which appear in (3.18).

$$\begin{cases} \partial_x^i U(0,y)=0, & i=0,1,\dots,m'+k_1-1,\\ \partial_{y_1}^i U(x,y)|_{y_1=0}=0, & i=0,1,\dots,r-1. \end{cases} \tag{3.25}$$

Then the formal solution

$$U(x,y)=\sum_{j\geqq m'+k_1} U_j(y)x^j/j! \tag{3.26}$$

of the problem (3.24)-(3.25) is a majorant of the formal solution of the problem (3.18)-(3.19).

On the other hand (3.24) is a special case of (3.1). In fact, (3.24) can be written in the form

$$(3.27)\qquad \sum_{j=0}^{m-r-m'} x^j \partial_x^{m'+k_1+j} \partial_{y_1}^r U = \frac{x}{\rho} \sum_{j=0}^{m-r-m'} x^j \partial_x^{m'+k_1+j} \partial_{y_1}^r U + \frac{MC(y)}{\left(1-\dfrac{y_1+\dots+y_n}{\rho}\right)} \{\bar{\sum}' x^p \partial_x^q \partial_y^\alpha U + 1\}.$$

Owing to the Lemma, we see that (3.26) represents a convergent series in a neighborhood of the origin. This proves that the problem (3.18)-(3.19) has a unique solution in a neighborhood of the origin. In view of the particular Goursat data (3.19), the solution of the problem (3.18)-(3.19) is also solution of the problem (2.2)-(2.3). Let us remark that when $r>0$, the solution is not unique because of arbitrariness of ψ_i in (3.19).

Remark

When $r=0$, (3.19) and (3.25) are initial data and the problem (3.18)-(3.19) and (3.24)-(3.25) are initial-value problems.

§4 Proof of Theorem 2.2

From the assumption we can rewrite (2.8) in the following form.

$$(4.1)\qquad \begin{cases} \partial_x^{m-1} u = x^2 a_{m,0}(x,y) \partial_x^m u + x \sum_{|\alpha|\leq 1} a_{m-1,\alpha}(x,y) \partial_x^{m-1} \partial_y^\alpha u \\ \qquad + x\{ \sum_{|\alpha|=2} a_{m-2,\alpha}(x,y) \partial_x^{m-2} \partial_y^\alpha \\ \qquad + \dots + \sum_{|\alpha|=s-1} a_{m-s+1,\alpha}(x,y) \partial_x^{m-s+1} \partial_y^\alpha \} u \\ \qquad + \sum_{\substack{m\geq i\geq s \\ |\alpha|=i}} a_{m-i,\alpha}(x,y) \partial_x^{m-i} \partial_y^\alpha u \\ \qquad + \sum_{\substack{i\leq m-2 \\ i+|\alpha|\leq m-1}} a_{i,\alpha}(x,y) \partial_x^i \partial_y^\alpha u + f(x,y), \\ \text{where } \sum_{|\alpha|=s} |a_{m-s,\alpha}(0,0)| \neq 0, \quad s\geq 2. \end{cases}$$

Now we consider the formal solution of (4.1),

$$(4.2)\qquad u(x,y) = \sum_{j\geq m-2} u_j(y) x^j / j!.$$

We want to show that we can choose an initial data $u_{m-2}(y)$ such that the formal solution (4.2) never converges in any neighborhood of the origin.

Substituting (4.2) into (4.1), we have

$$
\begin{aligned}
(4.3)\quad & \sum_j u_j(y)x^{j-m+1}/(j-m+1)!=\sum_j a_{m,0}(x,y)u_j(y)x^{j-m+2}/(j-m)! \\
& +\sum_{|\alpha|\leq 1}\sum_j a_{m-1,\alpha}(x,y)\partial_y^\alpha u_j(y)x^{j-m+2}/(j-m+1)! \\
& +\{\sum_{|\alpha|=2}\sum_j a_{m-2,\alpha}(x,y)\partial_y^\alpha u_j(y)x^{j-m+2+1}/(j-m+2)! \\
& +\dots+\sum_{|\alpha|=s-1}\sum_j a_{m-s+1,\alpha}(x,y)\partial_y^\alpha u_j(y)x^{j-m+s-1+1}/(j-m+s-1)!\} \\
& +\sum_{\substack{m>i>s\\ |\alpha|=i}}\sum_j a_{m-i,\alpha}(x,y)\partial_y^\alpha u_j(y)x^{j-m+i}/(j-m+i)! \\
& +\sum_{\substack{i\leq m-2\\ i+|\alpha|\leq m-1}}\sum_j a_{i,\alpha}(x,y)\partial_y^\alpha u_j(y)x^{j-i}/(j-i)!+f(x,y).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(4.4)\quad u_k(y)= & L_{k,1}(y,\partial_y)u_{k-1}+L_{k,2}(y,\partial_y)u_{k-2}+\dots+L_{k,s-1}(y,\partial_y)u_{k-s+1} \\
& +\sum_{|\alpha|=s} a_{m-s,\alpha}(0,y)\partial_y^\alpha u_{k-s+1}+\tilde{L}_{k,s+1}(y,\partial_y)u_{k-s} \\
& +\dots+\tilde{L}_{k,m}(y,\partial_y)u_{k-m+1} \\
& +\tilde{\tilde{L}}_{k,m}(y,\partial_y)\{u_{k-m},u_{k-m-1},\dots,u_{m-2}\}+f_k,\quad k\geqq m-1
\end{aligned}
$$

where $L_{k,i}(y,\partial_y)$, $\tilde{L}_{k,i}(y,\partial_y)$ and $\tilde{\tilde{L}}_{k,i}(y,\partial_y)$ are differential operator of order $\leqq i$.

For the simplicity, by changing the notation we write (4.4) in the following form:

$$
(4.5)\quad u_k(y)=\sum_{i\leq k-1} b_{k,i,\alpha}(y)\partial_y^\alpha u_i+\sum_{|\alpha|=s} a_{m-s,\alpha}(0,y)\partial_y^\alpha u_{k-s+1}.
$$

Concerning the suffix of $b_{k,i,\alpha}$, taking account of order of differential operator $L_{k,i}$, $\tilde{L}_{k,i}$ and $\tilde{\tilde{L}}_{k,i}$, we have

$$
(4.6)\quad si+(s-1)|\alpha|<sk.
$$

Using (4.5) successively we obtain

$$
\begin{aligned}
(4.7)\quad u_{m-2+j(s-1)}(y)= & (\sum_{|\alpha|=s} a_{m-s,\alpha}(0,y)\partial_y^\alpha)^j u_{m-2}(y) \\
& +\mathcal{L}_k(y,\partial_y)u_{m-2}(y)+\tilde{\mathcal{L}}_{sj}(y,\partial_y)(f_0,f_1,\dots,f_{m-2+j(s-1)}),
\end{aligned}
$$

where $\mathcal{L}_k$ and $\tilde{\mathcal{L}}_{sj}$ are differential operator of order k and sj respectively. Taking account of (4.5), (4.6) and (4.7) we have

$$s(m-2)+(s-1)k<s\{(m-2)+j(s-1)\}. \tag{4.8}$$

Then

$$k<sj. \tag{4.9}$$

On the other hand, by the assumption $\sum_{|\alpha|=s}|a_{m-s,\alpha}(0,0)|\neq 0$, there exists an $\eta\in \boldsymbol{C}^n$ such that

$$\sum_{|\alpha|=s} a_{m-s,\alpha}(0,0)\eta^{\alpha}=1. \tag{4.10}$$

Now we define

$$u_{m-2}(y)=\sum_{p\geq 0}\rho^{sp}e^{i\theta_{sp}}<y,\eta>^{sp} \tag{4.11}$$

where $\rho(0<\rho<1)$ is a fixed constant, and the arguments $\theta_{sj}(j=0, 1, 2, \ldots)$ are defined recurrently in the following manner:

At first

$$\Big(\sum_{|\alpha|=s} a_{m-s,\alpha}(0,0)\partial_y^{\alpha}\Big)^j(\rho^{sj}e^{i\theta_{sj}}<y,\eta>^{sj})=e^{i\theta_{sj}}(sj)!. \tag{4.12}$$

Note that the order of differential operator $\mathcal{L}_k$ is at most sj-1, we see that $\mathcal{L}_k(0,\partial_y)u_{m-2}(0)$ depends only on the terms $\sum_{p=0}^{j-1}\ldots$ in (4.11). So we define θ_{sj} by (assuing that $\theta_0, \theta_s, \theta_{2s},\ldots,\theta_{(j-1)s}$ are already defined):

$$\begin{aligned}\theta_{sj}=\arg\Big(&\mathcal{L}_k(0,\partial_y)\sum_{p=0}^{j-1}\rho^{sp}e^{i\theta_{sp}}<y,\eta>^{sp}|_{y=0}\\ &+\tilde{\mathcal{L}}_{sj}(0,\partial_y)(f_0,\ldots,f_{m-2+j(s-1)})|_{y=0}\Big)\end{aligned} \tag{4.13}$$

Thus we have

$$\begin{aligned}|u_{m-2+j(s-1)}(0)|&\geq\Big|\Big(\sum_{|\alpha|=s}a_{m-s,\alpha}(0,0)\partial_y^{\alpha}\Big)^j(\rho^{sj}e^{i\theta_{sj}}<y,\eta>^{sj})|_{y=0}\Big|\\ &=\rho^{sj}(sj)!.\end{aligned} \tag{4.14}$$

On the other hand

$$u(x, 0)=\sum_{p>m-2} u_p(0)x^p/p! \tag{4.15}$$

Then

$$\overline{\lim_{k\to\infty}}\ {}^{k}\sqrt{\frac{|u_k(0)|}{k!}} \gtrsim \overline{\lim_{k\to\infty}}\ (\rho^s)\sqrt{\frac{(sk)!}{\{(s-1)k\}!}}=+\infty. \tag{4.16}$$

Thus (4.2) can not converge in any neighborhood of the origin, which proves the Theorem.

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KYOTO UNIVERSITY

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