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Projective modules, Grothendieck groups and the Jacobson-Cartier operator

By

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§ 1. **Introduction:** Let B be a commutative ring (with identity) of characteristic $p>0$, $d:B\rightarrow B$ a derivation satisfying a nonzero p-polynomial and $A=\ker d$. Let n be an integer$\geqslant 1$. In this paper, we study the set $\mathfrak{p}_n(B/A)$ of isomorphism classes of finitely generated projective A-modules of rank n, which, when tensored with B, become free. To do this, we define an action of $GL_n(B)$ on $M_n(B)$ and prove, (see theorem 3.5) that under suitable conditions on B, $\mathfrak{p}_n(B/A)$ is in a bijective correspondence with the set $H^1(M_n(B))$ of orbits (under this action) of elements on which the so called Jacobson-Cartier operator vanishes. For $n=1$, we have $\mathfrak{p}_1(B/A)=\ker(\operatorname{Pic} A\rightarrow \operatorname{Pic} B)$ and this case has already been considered by Shuen Yuan [4, Theorem 2.6].

We regard $M_n(B)$ as a subgroup of $M_{n+1}(B)$ by treating any element $c\in M_n(B)$ as the element $\begin{pmatrix} c & 0\\ 0 & 0\end{pmatrix}\in M_{n+1}(B)$ and write $M(B)=\bigcup_{n\geqslant 1} M_n(B)$. We use our theorem on the classification of $\mathfrak{p}_n(B/A)$ to get a bijection between the set $K^\circ(B/A)$ $(=\ker(K^\circ(A)\rightarrow K^\circ(B)))$ and the set $H^1(M(B))$ which is the direct limit of $H^1(M_n(B))$. (Here $K^\circ(A)$ and $K^\circ(B)$ denote respectively the Grothendieck groups of A and B.)

The operation of direct sum of matrices induces a group structure on $H^1(M(B))$ and Tr: $M(B)\rightarrow B$ gives rise to a homomorphism

$H^1(M(B)) \to H^1(B)$. We show in Proposition 5.8 that this map corresponds under our identification to the homomorphism det: $K^\circ(B/A) \to \mathfrak{p}_1(B/A)$.

In the last section, we make two remarks on logarithmic derivatives which are relevant to the material treated in the earlier sections. Further applications of these ideas such as the computation of Grothendieck groups and other questions related to these will be treated elsewhere.

For all standard results on projective modules, we refer to Bass [1]. Throughout all tensor product signs without subscripts will denote tensor product over A.

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§ 2. An action of $GL_n(B)$ on $M_n(B)$ and the trace map.

Let B be a commutative ring with identity and let d be a derivation of B into B. For any integer $n \geqslant 1$, we define a derivation (also denoted by d) of $M_n(B)$, by setting for $c=(c_{ij}) \in M_n(B)$, $d(c)=(d(c_{ij}))$. Let $A=\ker$ d, and $d^{(n)}$: $B^n \to B^n$ be the map defined by $d^{(n)}(b_1, \ldots, b_n)=(d(b_1), \ldots, d(b_n))$. Then it is clear that $d^{(n)}$ is an A-linear map which satisfies $d^{(n)} \circ c - c \circ d^{(n)} = d(c)$ for every $c \in M_n(B)$. We remark that the map η_n: $GL_n(B) \times M_n(B) \to M_n(B)$ defined for $\alpha \in GL_n(B)$, $c \in M_n(B)$ by

$$(\alpha, c) \longrightarrow \alpha c \alpha^{-1} + \alpha d(\alpha^{-1})$$

gives an action of $GL_n(B)$ on $M_n(B)$. In fact, clearly $\eta_n(1, c)=c$ and for α, $\beta \in GL_n(B)$, $\eta_n(\alpha\beta, c)=\alpha\beta c\beta^{-1}\alpha^{-1}+\alpha\beta d(\beta^{-1}\alpha^{-1})=\alpha\beta c\beta^{-1}\alpha^{-1}+\alpha\beta d(\beta^{-1})\alpha^{-1}+\alpha d(\alpha^{-1})=\eta_n(\alpha, \eta_n(\beta, c))$. In particular, if $n=1$, we have an action of the group $U(B)$ of units of B on B. Let Tr: $M_n(B) \to B$ denote the trace map. We shall show that this induces a map of the set of orbits of $M_n(B)$ on the set of orbits of B. To do this, we need some preliminary results, which will be used often.

Let M be a B-module. We shall be interested in A-endomorphisms f of M which satisfy

$$f(bm)=bf(m)+d(b)m \qquad (*)$$

for all $m\in M$, $b\in B$.

For example, if $M=B^n$, then $d^{(n)}: B^n\to B^n$ defined earlier satisfies $(*)$. Also, if N is any A-module, then the map $d\otimes 1_N: B\otimes N\to B\otimes N$ also satisfies $(*)$. If $f: M\to M$ satisfies $(*)$ and θ is any B-automorphism of M, then clearly $\theta\circ f\circ\theta^{-1}$ also satisfies $(*)$. If f_i, $i=1, 2$ are additive endomorphisms of M which satisfy $(*)$, then clearly f_1-f_2 is B-linear.

2.1. Proposition: *Let $A=\ker d$, N an A-module such that we have a B-isomorphism $\phi: B\otimes N\xrightarrow{\sim} B^n$. Let $c=\phi\circ(d\otimes 1_N)\circ\phi^{-1}-d^{(n)}$. Then c is B-linear so that it can be regarded as an element of $M_n(B)$ by choosing the canonical base for B^n. Further,*

$$\mathrm{Tr}(c)=\overset{n}{\wedge}\phi\circ(d\otimes 1_{\overset{n}{\wedge}N})\circ(\overset{n}{\wedge}\phi)^{-1}-d$$

(here we have identified $\overset{n}{\underset{B}{\wedge}}(B\otimes N)$ with $B\otimes\overset{n}{\wedge}N$).

Proof. The fact that c is a B-linear map is immediate from the remarks preceding the proposition. To prove the formula for the trace of c, let $e_1, \ldots, e_n$ denote the canonical base for B^n. Then we obviously have

$$\begin{aligned}Tr(c)(e_1\wedge\cdots\wedge e_n)&=\sum_i e_1\wedge\cdots\wedge c(e_i)\wedge\cdots\wedge e_n\\ &=\sum_i e_1\wedge\cdots\wedge\phi_1(d\otimes 1)\circ\phi^{-1}(e_i)\wedge\cdots\wedge e_n.\end{aligned}$$

Let $\phi^{-1}(e_i)=\sum_j b_{ij}\otimes x_{ij}$, for $1\leqslant i\leqslant n$. Note that

$(\phi\circ(d\otimes 1)\circ\phi^{-1}(e_i)=\sum_j d(b_{ij})\phi(1\otimes x_{ij})$. Now

$$\begin{aligned}&(\overset{n}{\wedge}\phi\circ(d\otimes 1)\circ(\overset{n}{\wedge}\phi)^{-1}-d)(e_1\wedge\cdots\wedge e_n)\\ &=(\overset{n}{\wedge}\phi\circ(d\otimes 1))((\sum_j b_{1j}\otimes x_{1j})\wedge\cdots\wedge(\sum_j b_{nj}\otimes x_{nj}))\\ &=(\overset{n}{\wedge}\phi\circ(d\otimes 1))(\sum_{j_1,\cdots,j_n} b_{1j_1}\cdots b_{nj_n}\otimes x_{1j_1}\wedge\cdots\wedge x_{nj_n})\\ &=\overset{n}{\wedge}\phi(\sum_{j_1\cdots j_n} d(b_{1j_1}\cdots b_{nj_n})\otimes x_{1j_1}\wedge\cdots\wedge x_{nj_n})\end{aligned}$$

$$
\begin{aligned}
&= \overset{n}{\wedge}\phi\Big(\sum_{j_1\cdots j_n}\Big(\sum_i b_{1j_1}\wedge\cdots\wedge d(b_{ij_i})\ldots b_{nj_n}\Big)\otimes x_{1j_1}\wedge\cdots\wedge x_{nj_n}\Big)\\
&= \sum_i \Big(\sum_{j_1} b_{1j_1}\phi(1\otimes x_{1j_1})\wedge\cdots\wedge\sum_{j_1} d(b_{ij_i})\phi(1\otimes x_{ij_i})\wedge\cdots\wedge\sum_{j_n} b_{nj_n}\phi(1\otimes x_{nj_n})\Big)\\
&= \sum_i (e_1\wedge\cdots\wedge\phi\circ(d\otimes 1)\circ\phi^{-1}(e_1)\wedge\cdots\wedge e_n)\\
&= \mathrm{Tr}(c)(e_1\wedge\cdots\wedge e_n).
\end{aligned}
$$

2.2. Proposition: *For any* $\alpha\in GL_n(B)$,

$$\mathrm{Tr}(\alpha d(\alpha^{-1}))=(\det\alpha)d((\det\alpha)^{-1}).$$

Proof. Let β: $B\otimes A^n \xrightarrow{\sim} B^n$ be the canonical B-isomorphism given by $\beta(b\otimes(a_1, \ldots, a_n))=(ba_1, \ldots, ba_n)$. Applying the proposition 2.1

to $\alpha\circ\beta$: $B\otimes A^n\xrightarrow{\sim} B^n$, we have

$$\mathrm{Tr}((\alpha\circ\beta)\circ(d\otimes 1)\circ(\alpha\circ\beta)^{-1}-d^{(n)})=\overset{n}{\wedge}(\alpha\circ\beta)\circ(d\otimes 1)\circ(\overset{n}{\wedge}(\alpha\circ\beta))^{-1}-d.$$

Note that $\beta\circ(d\otimes 1)\circ\beta^{-1}=d^{(n)}$; therefore,

$$
\begin{aligned}
(\alpha\circ\beta)\circ(d\otimes 1)\circ(\alpha\circ\beta)^{-1}-d^{(n)} &= \alpha\circ(\beta\circ(d\otimes 1)\circ\beta^{-1})\circ\alpha^{-1}-d^{(n)}\\
&= \alpha\circ d^{(n)}\circ\alpha^{-1}-d^{(n)}\\
&= \alpha d(\alpha^{-1}),
\end{aligned}
$$

since $d(\alpha^{-1})=d^{(n)}\circ\alpha^{-1}-\alpha^{-1}\circ d^{(n)}$. Similaly, we have that

$$\overset{n}{\wedge}(\alpha\circ\beta)\circ(d\otimes 1)\circ(\overset{n}{\wedge}(\alpha\circ\beta))^{-1}-d=(\overset{n}{\wedge}\alpha)d((\overset{n}{\wedge}\alpha)^{-1})=(\det\alpha)d((\det\alpha)^{-1}).$$

This proves the proposition.

2.3. Corollary. *The map* Tr: $M_n(B)\to B$ *induces a map of the set of orbits of* $M_n(B)$ *(under the action of* $GL_n(B)$ *described earlier) into the set of orbits of* B *(under the action of* $U(B)$ *described earlier).*

Proof. Let $c'=\alpha c\alpha^{-1}+\alpha d^{-1}(\alpha^{-1})$ for $c,c'\in M_n(B)$, $\alpha\in GL_n(B)$. Then by Proposition 2.2,

$$\mathrm{Tr}c'=\mathrm{Tr}(\alpha c\alpha^{-1})+\mathrm{Tr}(\alpha d(\alpha^{-1}))=\mathrm{Tr}c+(\det\alpha)d((\det\alpha)^{-1}).$$

This proves the corollary.

§ 3. A classification of isomorphism classes of projective modules and the Jacobson-Cartier operator.

Let B be a commutative ring of prime characteristic $p>0$. Let $d: B\to B$ be a derivation and $A=\ker d$. Let $f(d)=0$, where $f(X)=a_0X+a_1X^p+\ldots+a_rX^{p^r}\in A[X]$ be a nonzero polynomial. Let P be a finitely generated projective A-module such that $B\otimes P\cong B^n$. Let $\phi: B\otimes P\cong B^n$ be a B-isomorphism. We define $c_\phi=\phi\circ(d\otimes 1)\circ\phi^{-1}-d^{(n)}$, where $d^{(n)}$ is defined as in §2. Then by definition the following diagram is commutative:

$$\begin{array}{ccc} B\otimes P & \xrightarrow[\sim]{\phi} & B^n \\ {\scriptstyle d\otimes 1}\downarrow & & \downarrow{\scriptstyle c_\phi+d^{(n)}} \\ B\otimes P & \xrightarrow[\sim]{\phi} & B^n \end{array}$$

By Proposition 2.1, it follows that $c_\phi\in M_n(B)$. Also note that

$$f(c_\phi+d^{(n)})=f(\phi\circ(d\otimes 1)\circ\phi^{-1})=\phi\circ f(d\otimes 1)\circ\phi^{-1}=0.$$

We wish to prove that for any $c\in M_n(B)$, $f(c+d^{(n)})$ is B-linear and therefore can be considered as an element of $M_n(B)$. In order to do this, we need some preliminaries.

Let Γ be a ring with identity, not necessarily commutative and let $d: \Gamma\to\Gamma$ be a derivation. We denote by $\Gamma\{X, d\}$ the ring generated by Γ and an indeterminate X which satisfies the relation $X\gamma-\gamma X=d(\gamma)$ for every $\gamma\in\Gamma$. We call $\Gamma\{X, d\}$ the *Öre-extension* of Γ with respect to d. We need the following

3.1. Lemma. *Let Γ be a (not necesarily commutative) ring with identity and d: $\Gamma\to\Gamma$ a derivation. Let $\Gamma\{X, d\}$ denote the Öre-extension of Γ with respect to d. Then for any $\gamma\in\Gamma$ and any integer $i\geqslant 0$, we have*

$$(\gamma+X)^i=\sum_{0\leqslant r\leqslant i}\binom{i}{r}a_r(\gamma)X^{i-r},$$

where $a_r(\gamma)=a_r\in\Gamma$ *with* $a_0=1$ *and for* $r\geqslant 1$, $a_r=\gamma a_{r-1}+d(a_{r-1})$.

Proof. We prove the proposition by induction on i. If $i=0$, the assertion is clear. Assume that the result has already been proved for $i-1$. We have

$$\begin{aligned}(\gamma+X)^i&=(\gamma+X)\{(\gamma+X)^{i-1}\}\\&=(\gamma+X)\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}a_r(\gamma)X^{i-1-r}\\&=\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}\gamma a_r(\gamma)X^{i-1-r}+\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}Xa_r(\gamma)X^{i-1-r}.\end{aligned}$$

Substituting $Xa_r=a_rX+d(a_r)$ in the second term on the right hand side, we get

$$\begin{aligned}&\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}Xa_r(\gamma)X^{i-1-r}\\&\quad=\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}a_rX^{i-r}+\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}d(a_r)X^{i-r-1}\ \text{so that}\end{aligned}$$

$$\begin{aligned}(\gamma+X)^i&=\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}(\gamma a_r+d(a_r))X^{i-r-1}+\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}a_rX^{i-r}\\&=\sum_{1\leqslant r\leqslant i}\binom{i-1}{r-1}a_rX^{i-r}+\sum_{0\leqslant r\leqslant i-1}\binom{i-1}{r}a_rX^{i-r}\\&=\sum_{0\leqslant r\leqslant i}\left(\binom{i-1}{r-1}+\binom{i-1}{r}\right)a_rX^{i-r}\\&=\sum_{0\leqslant r\leqslant i}\binom{i}{r}a_rX^{i-r},\end{aligned}$$

and this proves the lemma.

3.2. Corollary. *Let* Γ *be a ring of prime characteristic* $p>0$. *Then for* $\gamma\in\Gamma$, *we have*

$$(\gamma+X)^{p^r}=X^{p^r}+a_{p^r}(\gamma),\ a_{p^r}(\gamma)\in\Gamma.$$

This is immediate from the above lemma, since $\binom{p^r}{s}=0$ for all s with $0<s<p^r$.

Taking Γ to be a commutative ring of prime characteristic $p>0$, we have

3.3. Corollary. *Let $\Gamma=B$ be a commutative ring of prime characterestic $p>0$. Then for any $b\in B$,*

$$a_{p^i}(b)=(a_{p^{i-1}}(b))^p+(d^{p^{i-1}})^{p-1}(a_{p^{i-1}}(b)), \text{ for } i\geqslant 1.$$

Proof. By Corollary 3.2,

$$\begin{aligned} a_{p^i}(b)&=(b+d)^{p^i}-d^{p^i}\\ &=((d+d)^{p^{i-1}})^p-d^{p^i}\\ &=(a_{p^{i-1}}(b)+d^{p^{i-1}})^p-d^{p^i}\\ &=(a_{p^{i-1}}(b))^p+(d^{p^{i-1}})^{p-1}(a_{p^{i-1}}(b)), \text{ by [2, p. 201].}\end{aligned}$$

We apply Corollary 3.2 to the particular case where $\Gamma=M_n(B)$ and d: $M_n(B)\to M_n(B)$ is the derivation $(b_{ij})\mapsto(d(b_{ij}))$. We get that, for any x which satisfies $xc-cx=d(c)$, for $c\in M_n(B)$, and $r\geqslant 0$

$$(c+x)^{p^r}=x^{p^r}+a_{p^r}(c)$$

with $a_{p^r}(c)\in M_n(B)$. Since $d^{(n)}$ satisfies (*), we can specialise x to $d^{(n)}$ and obtain

$$(c+d^{(n)})^{p^r}=(d^{(n)})^{p^r}+a_{p^r}(c),$$

so that

$$\begin{aligned} f(c+d^{(n)})&=\sum_{0\leqslant i\leqslant r} a_i\{(d^{(n)})^{p^i}+a_{p^i}(c)\}\\ &=\sum_{0\leqslant i\leqslant r} a_i a_{p^i}(c)\ ,\end{aligned}$$

since $f(d^{(n)})=a_0 d^{(n)}+\ldots+a_r(d^{(n)})^{p^r}=0$. Thus $f(c+d^{(n)})$ is a B-linear map and hence can be considered as an element of $M_n(B)$. The map $c\mapsto f(c+d^{(n)})=\sum_{0\leqslant i\leqslant r} a_i a_{pi}(c)$ of $M_n(B)$ into itself is called the *Jacobson-Cartier operator*, and is denoted by δ_n. Note that $\delta_n(c_\phi)=f(c_\phi+d^{(n)})=0$.

Thus, given any finitely generated projective A-module P with $B\otimes P\xrightarrow{\sim}B^n$, we have an element $c_\phi\in M_n(B)$ which satisfies $\delta_n(c_\phi)=0$. Let $B\otimes P\xrightarrow[\sim]{\phi'}B^n$ by any other B-isomorphism and $c_{\phi'}\in M_n(B)$ the corresponding element. In view of the commutativity of the following diagram,

$$\begin{tikzcd}
& B\otimes P \arrow[rr, "d\otimes 1"] \arrow[d, "\phi"] \arrow[ddl, "\phi'"'] & & B\otimes P \arrow[d, "\phi"] \arrow[ddl, "\phi'"'] \\
& B^n \arrow[rr, "c_\phi + d^{(n)}"] \arrow[dl, "\phi'\cdot\phi^{-1}"] & & B^n \arrow[dl, "\phi'\cdot\phi^{-1}"] \\
B^n \arrow[rr, "c_{\phi'} + d^{(n)}"'] & & B^n &
\end{tikzcd}$$

we have $c_{\phi'}+d^{(n)}=\theta\circ(c+d^{(n)})\circ\theta^{-1}$, where $\theta=\phi'\circ\theta^{-1}$: $B^n\to B^n$ is a B-isomorphism and hence belongs to $M_n(B)$ by choosing the canonical base for B^n. We now have

$$\begin{aligned}c_{\phi'}+d^{(n)}&=\theta\circ c_\phi\circ\theta^{-1}+\theta\circ d^{(n)}\circ\theta^{-1}\\&=\theta\circ c_\phi\circ\theta^{-1}+d^{(n)}+\theta d(\theta^{-1})\end{aligned}$$

so that

$$c_{\phi'}=\theta\circ c_\phi\circ\theta^{-1}+\theta d(\theta^{-1})$$

and c_ϕ, $c_{\phi'}$ define the same orbit for the action of $GL_n(B)$ on $M_n(B)$ described in §2. Thus, to each finitely generated projective A-module P with $B\otimes P\overset{\phi}{\simeq}B^n$, we have associated an orbit of $M_n(B)$, which we denote by $c(P)$. We show that if $P\simeq P'$, then $c(P)=c(P')$. Let f: $P\simeq P'$ be an A-isomorphism. Set $\phi'=\phi\circ(1\otimes f^{-1})$. Then $B\otimes P\overset{\phi'}{\simeq}B^n$ and hence $c(P')$ is the orbit of $c_{\phi'}$. Now

$$\begin{aligned}c_{\phi'}&=\phi'\circ(d\dot{\otimes}1)\circ\phi'^{-1}-d^{(n)}\\&=\phi\circ(1\otimes f^{-1})\circ(d\otimes 1)\circ(1\otimes f)\circ\phi^{-1}-d^{(n)}\\&=\phi\circ(d\otimes 1)\circ\phi^{-1}-d^{(n)}\\&=c_\phi.\end{aligned}$$

Thus $c(P')=c(P)$.

Let $\mathfrak{p}_n(B/A)$ be the set of isomorphism classes $[P]$ of all finitely generated projective A-modules P such that $P\otimes B\simeq B^n$. The above considerations show that we have a map of $\mathfrak{p}_n(B/A)$ into the set $H^1(M_n(B))$ of orbits of elements c of $M_n(B)$ such that $\delta_n(c)=0$.

3.4. Definition. Let B, A be as before. We say that the ring extension B/A is of *Galois type* if B is a finitely generated projective A-module and $\mathrm{Hom}_A(B, B)=B[d]$.

If B/A is of Galois type then d satisfies a p-polynomial $f(X)=\sum_{0\leqslant i\leqslant r} a_i X^{p^i}$, $a_i\in A$ and a_r is a non-zero idempotent such that the canonical map $B\{X, d\}/f(X)\to\mathrm{Hom}_A(B,B)$ is an isomorphism [4, Theorem 2.4].

We prove the following

3.5. Theorem: *The map* $\Theta_n:\ \mathfrak{p}_n(B/A)\to H^1(M_n(B))$ *given by* $[P]\to c(P)$ *is injective. It is a bijection, if* B/A *is of Galois type.*

Proof. We first check that the map in question is injective. Let P,P' be finitely generated projective A-modules such that $c(P)=c(P')$. If $B\otimes P\overset{\phi}{\rightrightarrows}B^n$, $B\otimes P'\overset{\phi'}{\rightrightarrows}B^n$, there exists by definition an element $\alpha\in GL_n(B)$ such that

$$c_{\phi'}=\alpha\circ c_\phi\circ\alpha^{-1}+\alpha d(\alpha^{-1}).$$

Or, equivalently,

$$(c_{\phi'}+d^{(n)})\circ\alpha=\alpha\circ(c_\phi+d^{(n)})$$

i.e.
$$(\phi'\circ(d\otimes 1)\circ\phi'^{-1})\circ\alpha=\alpha\circ(\phi\circ(d\otimes 1)\circ\phi^{-1})$$

or
$$(d\otimes 1)(\phi'^{-1}\circ\alpha\circ\phi)=(\phi'^{-1}\circ\alpha\circ\phi)(d\otimes 1),$$

i.e. the diagram

$$\begin{tikzcd} B\otimes P \arrow[r, "d\otimes 1"] \arrow[d, "\phi'^{-1}\circ\alpha\circ\phi"'] & B\otimes P \arrow[d, "\phi'^{-1}\circ\alpha\circ\phi"] \\ B\otimes P' \arrow[r, "d\otimes 1"] & B\otimes P' \end{tikzcd}$$

is commutative, with the vertical maps being B-isomorphisms. We therefore have an induced A-isomorphism $\ker(d\otimes 1_P)\cong\ker(d\otimes 1_P)$. Since P is A-projective, $\ker\,(d\otimes 1_P)\cong A\otimes P\cong P$ and similarly, $\ker\,(d\otimes 1_{P'})\cong P'$. We thus have an A-isomorphism $P\cong P'$ i.e. $[P]=[P']$ which proves the injectivity of the map.

Let us now assume that B/A is of Galois type and prove the surjectivity of the map. Let $\bar{c}\in H^1(M_n(B))$ with $c\in M_n(B)$ as a representative. Define

$$P=\{x\in B^n\,|\,(c+d^{(n)})(x)=0\}.$$

We will show that $[P]\in\mathfrak{p}_n(B/A)$ and that $[P]$ maps onto $\bar{c}$. We note that since c is B-linear and $d^{(n)}$ satisfies $(*)$ of §2, it follows that $c+d^{(n)}$ satisfises $(*)$. We have therefore an A-algebra homomorphism

$$\bar{\mu}\colon B\{X,d\}\longrightarrow \mathrm{End}_A(B^n)$$

such that $\mu(X)=c+d^{(n)}$ and $\mu/B=$identity. Since $f(c+d^{(n)})=\delta_n(c)=0$ $(\bar{c}\in H^1(M_n(B))$, we have an induced homomorphism

$$\mu\colon B\{X,d\}/f(X)\longrightarrow \mathrm{End}_A(B^n).$$

Since B/A is of Galois type, we have an isomorphism

$$\xi\colon B\{X,d\}/f(X)\longrightarrow \mathrm{End}_A(B),$$

of A-algebras, so that, we have an A-algebra homomorphism

$$\bar{\mu}\circ\xi^{-1}\colon\ \mathrm{End}_A(B)\longrightarrow \mathrm{End}_A(B^n).$$

In otherwords, B^n becomes an $\mathrm{End}_A(B)$-module with the action of $d\in\mathrm{End}_A\,(B)$ on B^n given by

$$d*x=\mu(X)x=(c+d^{(n)})(x),\qquad x\in B^n.$$

By Morita-equivalence [1, p. 69] applied to the pair $(A,\ \mathrm{End}_A\,B)$, we have a B-isomorphism,

$$B \otimes \mathrm{Hom}_{\mathrm{End}_A(B)}(B, B^n) \overset{\sim}{\to} B^n$$

induced by $(b, g) \mapsto b.g(1)$. We assert that under this isomorphism $\mathrm{Hom}_{\mathrm{End}_A(B)}(B, B^n)$ gets mapped onto P. For if $g \in \mathrm{Hom}_{\mathrm{End}_A(B)}(B, B^n)$, we have $(c+d^{(n)})(g(1))=\mu(X).\, g(1)=g(d(1))=g(0)=0$ and conversely, if $x \in B^n$ with $(c+d^{(n)})(x)=0$, the element $g \in \mathrm{Hom}_{\mathrm{End}_A(B)}(B, B^n)$ defined by $g(1)=x$ gets mapped onto x. Thus P is isomorphic to $\mathrm{Hom}_{\mathrm{End}_A(B)}(B, B^n)$. Hence P is A-projective. Further $B \otimes P \overset{\sim}{\to} B \otimes \mathrm{Hom}_{\mathrm{End}\,A(B)}(B, B^n) \overset{\sim}{\to} B^n$. Let ϕ: $B \otimes P \to B^n$ be the B-isomorphism induced by the inclusion map $P \hookrightarrow B^n$. Since for $x \in P$, $b \in B$, $\phi \circ (d \otimes 1)(b \otimes x)=\phi(d(b) \otimes x)=d(b) \cdot x$ and $(c+d^{(n)})\phi(b \otimes x)=(c+d^{(n)})(b \cdot x)$ $=b(c+d^{(n)}(X))+d(b)x=d(b)x$, it follows that $c=\phi \circ (d \otimes 1) \circ \phi^{-1}-d^{(n)}$ and hence P maps onto c. Thus the map in question is surjective. This proves the theorem.

With the same notation and hypothesis of the theorem, let $\mathrm{Pic}(B/A)$ $(=\mathfrak{p}_1(B/A))$ denote the set of isomorphism classes of rank 1 projective A-modules such that $B \underset{A}{\otimes} P \overset{\sim}{\to} B$. We then have the following corollary which was proved by Shuen Yuan [4, Theorem 2.6].

3.6. Corollary. *The map* Θ_1: $\mathrm{Pic}\,(B/A) \to H^1(B)$ *given by* $[P] \to c(P)$ *is a bijection.*

§ 4. The Kernel of $\mathbf{K}^\circ(\mathbf{A}) \to \mathbf{K}^\circ(\mathbf{B})$.

Let B be any commutative ring with identity, d:$B \to B$ be a derivation and $n \geqslant 1$ any integer. Consider the map j : $M_n(B) \to M_{n+1}(B)$ given by $c \mapsto \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}=c \oplus 0$ where 0 denotes the 1×1 matrix with 0 as its entries. Clearly j is a monomorphism of additive groups. We regard $M_n(B)$ as a subgroup of $M_{n+1}(B)$ through j and denote $M(B)=\bigcup_{n \geq 1} M_n(B)$ and $GL_n(B)$ as a subgroup of $GL_{n+1}(B)$ by the monomorphism i: $\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}=\alpha \oplus 1$ and denote $GL(B)=\bigcup_{n \geqslant 1} GL_n(B)$. We remark that the diagram

$$\begin{tikzcd} GL_n(B)\times M_n(B) \arrow[r, "\eta_n"] \arrow[d, "i\times j"'] & M_n(B) \arrow[d, "j"] \\ GL_{n+1}(B)\times M_{n+1}(B) \arrow[r, "\eta_{n+1}"] & M_{n+1}(B) \end{tikzcd}$$

is commutative, where η_n is defined as in §2. In fact,

$$\begin{aligned} \eta_{n+1}\circ(i\times j)(a,c) &= \eta_{n+1}((a\oplus 1),\ (c\oplus 0)) \\ &= (a\oplus 1)(c\oplus 0)(a^{-1}\oplus 1)+(a\oplus 1)d(a^{-1}\oplus 1) \\ &= (aca^{-1}\oplus 0)+(ad(a^{-1})\oplus 0) \\ &= (aca^{-1}+ad(a^{-1}))\oplus 0 \\ &= j(aca^{-1}+ad(a^{-1})) \\ &= j\circ\eta_n(a,c). \end{aligned}$$

We thus have a map η: $GL(B)\times M(B)\to M(B)$ which defines an action of $GL(B)$ on $M(B)$.

Let now B be a ring of prime characteristic $p>0$ and let d satisfy a polynomial $f(X)=a_0X+a_1X^p+\ldots+a_rX^{p^r}\in A[X]$. Let δ_n: $M_n(B)\to M_n(B)$ denote the Jacobson-Cartier operator defined in §3.

4.1. Lemma. *The diagram*

$$\begin{tikzcd} M_n(B) \arrow[r, "\delta_n"] \arrow[d, "j"'] & M_n(B) \arrow[d, "j"] \\ M_{n+1}(B) \arrow[r, "\delta_{n+1}"] & M_{n+1}(B) \end{tikzcd}$$

is commutative. In other words, δ_n *induces a map* δ: $M(B)\to M(B)$.

Proof: For $c\in M_n(B)$, we have $\delta_{n+1}\circ j(c)=\delta_{n+1}(c\oplus 0)=f((c\oplus 0)+d^{(n+1)})$ $=f((c+d^{(n)})\oplus d)=f(c+d^{(n)})\oplus f(d)=\delta_n(c)\oplus 0=j\delta_n(c)$.

4.2. Corollary. *The map* $j: M_n(B) \to M_{n+1}(B)$ *induces a map* $H^1(j): H^1(M_n(B)) \to H^1(M_{n+1}(B))$.

Let $\mathfrak{p}_n(B/A)$ denote the set of isomorphism classes of finitely generated projective A-modules such that $B \otimes P \to B^n$. Let k: $\mathfrak{p}_n(B/A) \to \mathfrak{p}_{n+1}(B/A)$ denote the map $[P] \to [P \oplus A]$.

4.3. Proposition. *The diagram*

$$\begin{array}{ccc}
\mathfrak{p}_n(B/A) & \xrightarrow{\Theta_n} & H^1(M_n(B)) \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle H^1(j)} \\
\mathfrak{p}_{n+1}(B/A) & \xrightarrow{\Theta_{n+1}} & H^1(M_{n+1}(B))
\end{array}$$

is commutative.

Proof. Let $[P] \in \mathfrak{p}_n(B/A)$ and let ϕ: $B \otimes P \xrightarrow{\sim} B^n$ be a B-isomorphism. Let $c_\phi = \phi \circ (d \otimes 1) \circ \phi^{-1} - d^{(n)}$ so that $c(P)$ is represented by c_ϕ. Now $H^1(j)(c(P))$ is represented by $c_\phi \oplus 0$. We need therefore to show that $\Theta_{n+1}([P \oplus A])$ is represented by $c_\phi \oplus 0$. We have the B-isomorphism

$$B \otimes (P \oplus A) \overset{\lambda}{\xrightarrow{\sim}} (B \otimes P) \oplus B \overset{\phi \oplus 1}{\xrightarrow{\sim}} B^{n+1}.$$

Let $e_1, \ldots, e_n, e_{n+1}$ be the canonical base of B^{n+1} such that $e_1, \ldots, e_n$ is the canonical base of B^n. For $i \leqslant n$, we have

$$\begin{aligned}
c_{(\phi \oplus 1) \circ \lambda}(e_i) &= ((\phi \oplus 1) \circ \lambda \circ (d \otimes 1) \circ \lambda^{-1} \circ (\phi^{-1} \oplus 1) - d^{(n+1)})(e_i) \\
&= (\phi \circ (d \otimes 1) \circ \phi^{-1})(e_i) \\
&= (\phi \circ (d \otimes 1) \circ \phi^{-1} - d^{(n)})(e_i) \\
&= c_\phi(e_i).
\end{aligned}$$

Also

$$\begin{aligned}
c_{(\phi \oplus 1) \circ \lambda}(e_{n+1}) &= ((\phi \oplus 1) \circ \lambda \circ (d \otimes 1) \circ \lambda^{-1} \circ (\phi^{-1} \oplus 1) - d^{(n+1)})(e_{n+1}) \\
&= (\phi \oplus 1) \circ \lambda \circ (d \otimes 1) \circ \lambda^{-1}(0, e_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&=(\phi\oplus 1)\circ\lambda\circ(d\otimes 1)(1\otimes(0,e_{n+1}))\\
&=(\phi\oplus 1)\circ\lambda(0,0)\\
&=0
\end{aligned}
$$

so that $c_{(\phi\oplus 1)\circ\lambda}=c_{\phi}\oplus 0$. Since $c_{(\phi\oplus 1)\circ\lambda}$ represents $\Theta_{n+1}([P\oplus A])$, the proposition follows.

Let $H^1(M(B))$ denote the set of orbits $\bar{c}$ of elements c of $M(B)$ (under the action of $GL(B)$) which satisfy $\delta(c)=0$. We remark that for every $n\geqslant 1$ the inclusion $M_n(B)\hookrightarrow M(B)$ induces a map $H^1(M_n(B))\to H^1(M(B))$ and in fact $H^1(M(B))=\varinjlim H^1(M_n(B))$.

Let now $A=\ker d$. We have a homomorphism $K^\circ(A)\to K^\circ(B)$, where $K^\circ(A)$ (respectively $K^\circ(B)$) denotes the Grothendieck group of A (respectively B). Let $K^\circ(B/A)=\ker(K^\circ(A)\to K^\circ(B))$. Let P be a projective A-module such that $P-A^n\in K^\circ(B/A)$. By definition $B\otimes P-B^n=0$ in $K^\circ(B)$ so that there exists an integer $m\geqslant 0$ such that

$$B\otimes(P\oplus A^m)\cong(B\otimes P)\oplus B^m\cong B^{m+n}$$

Thus, $[P\oplus A^m]\in\mathfrak{p}_{m+n}(B/A)$ and we have the element $\Theta_{m+n}([P\oplus A^m])$ $\in H^1(M_{m+n}(B))$ and this defines in view of the map $H^1(M_n(B))\to H^1(M(B))$ an element of $H^1(M(B))$. We assert that this assignment gives rise to a map $K^\circ(B/A)\to H^1(M(B))$. Let $P-A^n=P'-A^{n'}\in K^\circ(B/A)$. Let $m,m'\geqslant 0$ be integers such that $[P\oplus A^m]$, $[P'\oplus A^{m'}]$ are respectively in $\mathfrak{p}_{m+n}(B/A)$, $\mathfrak{p}_{m'+n'}(B/A)$. The equation $P-A^n=P'-A^{n'}$ implies that there exists an integer k (which we assume without loss of generality to be greater than both m and m') such that $P\oplus A^{n'}\oplus A^k\cong P'\oplus A^n\oplus A^k$. Thus by results of §3, it follows that $\Theta_{n+n'+k}([P\oplus A^{n'+k}])=\Theta_{n'+n+k}([P'\oplus A^{n+k}])$. Note that for any $k\geqslant m$, in view of Proposition 4.3, $\Theta_{m+n}([P\oplus A^m])$ and $\Theta_{m+k+n}([P\oplus A^k])$ define the same element of $H^1(M(B))$ and similarly for P' so that the elements of $H^1(M(B))$ corresponding to $P-A^n$ and $P'-A^{n'}$ are the same and we have a well defined map Ψ: $K^\circ(B/A)\to H^1(M(B))$ given by $\Psi(P-A^n)=$class of $\Theta_{m+n}([P\oplus A^m])$ where $[P\oplus A^m]\in\mathfrak{p}_{m+n}(B/A)$.

4.4. Theorem. *The map Ψ: $K^\circ(B/A)\to H^1(M(B))$ is injective. If B/A is of Galois type, then Ψ is a bijection.*

Proof. We first prove that Ψ is injective. Let $P-A^n$, $P'-A^{n'}\in K^\circ(B/A)$ be such that $\Psi(P-A^n)=\Psi(P'-A^{n'})$. Let$(P\oplus A^m)\otimes B\to B^{n+m}$, $(P'\oplus A^{m'})\otimes B\to B^{n'+m'}$. This means by definition that there exist integers k, $k'\geqslant 0$ which we can assume to be greater respectively than n' and n such that $c(P\oplus A^m)\oplus 0_k=c(P'\oplus A^{m'})\oplus 0_{k'}$ where 0_k (respectively $0_{k'}$) denote the matrix of order k (respectively k') all whose entries are zero. This implies that $c(P\oplus A^{m+k})=c(P'\oplus A^{m'+k'})$. Since by theorem 3.5, Θ_{m+n+k}: $\mathfrak{p}_{m+n+k}(B/A)\to H^1(M_{n+m+k}(B))$ is injective, it follows that $[P\oplus A^{m+k}]=[P'\oplus A^{m'+k'}]$ which implies that $P\oplus A^{n'}\oplus A^{m+k-n'}\simeq P'\oplus A^n\oplus A^{m'+k'-n}$. Since $m+n+k=m'+n'+k'$, we have that $P-A^n=P'-A^{n'}$ in $K^\circ(B/A)$ which shows Ψ is injective.

We now assume that B/A is of Galois type and show that Ψ is surjective. Let $c\in M_n(B)$ represent some element of $H^1(M(B))$. By theorem 3.5, there exists an element $[P]\in\mathfrak{p}_n(B/A)$ such that $\Theta_n([P])=$orbit of c in $M_n(B)$. Then, clearly $P-A^n\epsilon K^\circ(B/A)$ and $\Psi(P\text{-}A^n)$ is the given element of $H^1(M(B))$. This proves the therorem.

§ 5. The map Tr : $H^1(M(B))\to H^1(B)$.

In this section, we continue with our previous notation. In §2 we have seen that Tr: $M_n(B)\to B$ maps an orbit in $M_n(B)$ to an orbit in B. Here we shall prove that Tr maps $H^1(M(B))$ into $H^1(B)$ and that this map is a homomorphism for a natural group structure on $H^1(M(B))$ which we shall define.

To do this we need a few lemmas. The first lemma and its corollary are most probably well-known, but we include it for lack of proper reference.

5.1. Lemma. *Let B be a commutative ring (with 1) of prime characteristic $p>0$. Let f: $M_n(B)\to B$ be an additive homomorphism which satisfies* 1) *$f(\alpha\beta)=f(\beta\alpha)$, ii) $f(b\alpha)=b^pf(\alpha)$ for all $b\in B$, α, $\beta\in M_n(B)$. Then there exists a $\lambda\in B$ such that $f(\alpha)=\lambda(\mathrm{Tr}\ (\alpha))^p$.*

Proof. Let e_{ij} denote the $n\times n$ matrix which has 1 as its i, j^{th} entry and zero elsewhere. We remark that $f(e_{ij})=0$ for $i\neq j$. In fact $f(e_{ij})=f(e_{ii}\,e_{ij})=f(e_{ij}\,e_{ii})=f(0)=0$. Next, note that for any i, j, $f(e_{ii})=f(e_{jj})$; in fact $f(e_{ii})=f(e_{ij}\,e_{ji})=f(e_{ji}\,e_{ij})=f(e_{jj})$. Let $f(e_{ii})=\lambda$. We have $a=\sum_{i,j} a_{ij}\,e_{ij}\Rightarrow f(a)=\sum_{i,j} f(a_{ij}\,e_{ij})=\sum_{i,j} a_{ij}^p f(e_{ij})=\lambda\sum_i a_{ii}^p=\lambda(\mathrm{Tr}\ a)^p$.

5.2. Corollary: *For any $c\in M_n(B)$, we have* $\mathrm{Tr}(c^p)=(\mathrm{Tr}\ c)^p$.

Proof. Let f: $M_n(B)\to B$ be defined by $f(c)=\mathrm{Tr}(c^p)$. By Jacobson formula [3, P. 189], we have for c, $c'\in M_n(B)$ $(c+c')^p=c^p+c'^p+$sum of $(p-1)$-fold Lie-brackets, so that $f(c+c')=\mathrm{Tr}((c+c')^p)=\mathrm{Tr}(c^p+c'^p)=\mathrm{Tr}(c^p)+\mathrm{Tr}(c'^p)=f(c)+f(c')$. Also it is easily seen that $f(cc')=f(c'c)$ for c, $c'\in M_n(B)$. It is clear that $f(bc)=b^p f(c)$ so that f satisfies all the conditions of the lemma so that there exists $\lambda\in B$ such that $f(c)=\lambda(\mathrm{Tr}(c))^p$ for all $c\in M_n(B)$. Take now $c=c_{11}$. We have $f(e_{11})=\mathrm{Tr}(e_{11}^p)=\mathrm{Tr}(e_{11})=1$; thus $\lambda=1$ and the corollary is proved.

Let $c\in M_n(B)$. We regard c and $d^{(n)}$ as elements of $\mathrm{End}_A(B^n)$. We have by Jacobson's formula [3, P. 189]

$$(c+d^{(n)})^p=c^p+(d^{(n)})^p+\sum_{1\leqslant j\leqslant p-1} s_j(c,d^{(n)}),$$

where $j\,s_j(c, d^{(n)})$ is the co-efficient of x^{j-1} in $[\dots[c, d^{(n)}+cx], \dots, d^{(n)}+cx]$, where x is a commuting indeterminate.

5.3. Lemma: *Let* B *be a commutative ring of characteristic $p>0$. Then* $\mathrm{Tr}(s_1(c, d^{(n)}))=d^{p-1}(\mathrm{Tr}\ c)$ *and* $\mathrm{Tr}(s_j(c, d^{(n)}))=0$ *for* $j>1$.

Proof. Let $j=1$, we have $s_1(c, d^{(n)})=[\dots\underbrace{[c, d^{(n)}], \dots, d^{(n)}]}_{p-1\text{ times}}=(d^{(n)})^{p-1}(c)$ so that $\mathrm{Tr}(s_1(c, d^{(n)}))=\mathrm{Tr}((d^{(n)})^{p-1}(c))=d^{p-1}(\mathrm{Tr}\ c)$. Let $j>1$; we note that in any $(p-1)$-fold Lie bracket occuring in the co-efficient of x^{j-1}, c occurs at least twice, so that the Lie bracket is of the form $(d^{(n)})^i[c, c']$ for some $c'\in M_n(B)$ and $i\geqslant 0$. Thus $\mathrm{Tr}(s_j(c, d^{(n)}))$ is the sum of the terms which are of the form $d^i(\mathrm{Tr}\,[c, c'])$ so that for $j>1$, $\mathrm{Tr}(s_j(c, d^{(n)}))=0$.

5.4. Lemma. *For any integer $n \geqslant 1$, the diagram*

$$\begin{array}{ccc} M_n(B) & \xrightarrow{\ \delta_n\ } & M_n(B) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ B & \xrightarrow{\ \delta_1\ } & B \end{array}$$

is commutative.

Proof. We have to show that for any $c \in M_n(B)$,

$$\mathrm{Tr}(f(c+d^{(n)}))=f(\mathrm{Tr}(c)+d).$$

We know by results of §3 that $f(c+d^{(n)})=\sum_{0\leqslant i\leqslant r} a_i\alpha_{p^i}(c)$, where $f(X)=a_0X+a_1X^p+\ldots+a_rX^{p^r}\in A[X]$, so that it is enough to show that Tr $(\alpha_{p^i}(c))=\alpha_{p^i}(\mathrm{Tr}\ c)$, for $0\leqslant i\leqslant r$. We prove this by induction on i. For $i=0$, this is obvious. Assume that $i>1$ and the result holds for $i-1$. We have

$$\begin{aligned}\mathrm{Tr}(\alpha_{p^i}(c)) &= \mathrm{Tr}((c+d^{(n)})^{p^i}-(d^{(n)})^{p^i}) \\ &= \mathrm{Tr}(((c+d^{(n)})^{p^{i-1}})^p-(d^{(n)})^{p^i}) \\ &= \mathrm{Tr}((\alpha_{p^{i-1}}(c))^p+\sum_{1\leqslant j\leqslant p-1} s_j\alpha({}_{p^{i-1}}(c),(d^{(n)})^{p^i})) \\ &= (\mathrm{Tr}(\alpha_{p^{i-1}}(c)))^p+(d^{p^{i-1}})^{p-1}(\mathrm{Tr}(\alpha_{p^{i-1}}(c))\end{aligned}$$

by using (5.2) and (5.3).

$$\begin{aligned} &= (\alpha_{p^{i-1}}(\mathrm{Tr}\ c))^p+(d^{p^{i-1}})^{p-1}(\alpha_{p^{i-1}}(\mathrm{Tr}\ c)) \\ &= \alpha_{p^i}(\mathrm{Tr}\ c)\end{aligned}$$

by Corollary (3.3). This completes the proof of the lemma.

5.5. Corollary. *The map* Tr: $M_n(B)\to B$ *induces a map* Tr: $H^1(M_n(B))\to H^1(B)$ *as also a map* $H^1(M(B))\to H^1(B)$.

Proof. This first assertion is clear from Corollary 2.3 and lemma 5.4. The second assertion is obvious.

5.6. Proposition. *For any integer $n \geqslant 1$, the diagram*

$$\begin{array}{ccc} \mathfrak{p}_n(B/A) & \xrightarrow{\Theta_n} & H^1(M_n(B)) \\ {\scriptstyle \overset{n}{\wedge}} \downarrow & & \downarrow {\scriptstyle \mathrm{Tr}} \\ \mathrm{Pic}(B/A) & \xrightarrow{\Theta_1} & H^1(B) \end{array}$$

is commutative.

Proof. Let $[P] \in \mathfrak{p}_n(B/A)$ and let $\phi\colon B \otimes P \xrightarrow{\sim} B^n$ be a B-isomorphism. If $c_\phi = \phi \circ (d \otimes 1) \circ \phi^{-1} - d^{(n)}$, we know that $\Theta_n([P]) = c(P) =$ orbit of c_ϕ. Now

$$\begin{aligned} \mathrm{Tr}(\Theta_n([P])) &= \mathrm{Tr}(c(P)) \\ &= \text{orbit of } \mathrm{Tr}(c_\phi) \\ &= \text{orbit of } (\overset{n}{\wedge}\phi \circ (d \otimes 1_{\overset{n}{\wedge} P}) \circ \overset{n}{\wedge}\phi^{-1} - d) \quad \text{(by Proposition 2.1)} \\ &= \Theta_1([\overset{n}{\wedge} P]). \end{aligned}$$

This proves the proposition.

We know that we have a bijection $\Psi^{-1}\colon H^1(M(B)) \to K^\circ(B/A)$. Since $K^\circ(B/A)$ is an additive group (under the operation induced by direct sum), we can define an abelian group structure on $H^1(M(B))$ by setting for orbits $\bar{c}, \bar{c}' \in H^1(M(B))$ with c, c' as representative, $\bar{c} + \bar{c}' = \Psi(\Psi^{-1}(\bar{c}) + \Psi^{-1}(\bar{c}'))$.

5.7. Proposition: *For $\bar{c}, \bar{c}' \in H^1(M(B))$ with c, c' as representatives, we have $\bar{c} + \bar{c}' = \overline{c \oplus c'}$*

Proof. Let c, c' be representatives of $\bar{c}, \bar{c}'$. Let $P_c, P_{c'}$ be projective modules corresponding to c, c' respectively and $\phi\colon B \oplus P_c \xrightarrow{\sim} B^m$, $\phi'\colon B \otimes P_{c'} \xrightarrow{\sim} B^n$ be B-isomorphisms such that $c = \phi \circ (d \otimes 1) \circ \phi^{-1} - d^{(m)}$, $c' = \phi' \circ (d \otimes 1) \circ \phi'^{-1} - d^{(n)}$. We have isomorphisms

$$B \otimes (P_c \oplus P_{c'}) \overset{\lambda}{\xrightarrow{\sim}} (B \otimes P_c) \oplus (B \otimes P_{c'}) \xrightarrow{\phi \oplus \phi'} B^{m+n}$$

To prove the proposition, it is enough to show that if we choose the canonical base for B^{m+n}, the matrix of the B-linear map

$$T=(\phi\oplus\phi')\circ\lambda\circ(d\otimes 1)\circ\lambda^{-1}\circ(\phi'\oplus\phi'^{-1})-d^{(m+n)}$$

is $c\oplus c'$. Let $e_1, \ldots, e_m, e_{m+1}, \ldots, e_n$ be the canonical base for B^{m+n} such that $e_1, \ldots, e_m$ is the canonical base for B^m and $e_{m+1}, \ldots, e_{m+n}$ the canonical base for B^n. But it is clear that for $1\leqslant i\leqslant m$, $T(e_i)=c(e_i)$ for $m+1\leqslant i\leqslant m+n$, $T(e_i)=c'(e_i)$. This proves the proposition.

The above proposition shows that $H^1(M(B))$ becomes an additive group under an operation induced by the operation of direct sum of matrices. On the other hand, it is easy to see that $H^1(B)$ is an abelian group under the operation induced by the usual addition in B. It is clear that Tr: $H^1(M(B))\to H^1(B)$ is a homomorphism.

5.8. Proposition. *The diagram*

$$\begin{array}{ccc} K^\circ(B/A) & \xrightarrow{\ \Psi\ } & H^1(M(B)) \\ {\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Pic}(B/A) & \xrightarrow{\ \Theta_1\ } & H^1(B) \end{array}$$

is commutative.

Proof. Follows from Proposition 5.6.

§ 6. Some remarks on logarithmic derivatives.

As usual B denotes a commutative ring of prime characteristic $p>0$, d: $B\to B$ a derivation and $A=\ker d$.

6.1. If $c\in M_n(B)$ is a logarithmic derivative, i.e. $c=\alpha d(\alpha^{-1})$ for some $\alpha\in GL_n(B)$, then clearly $c\oplus 0_k$ is a logarithmic derivative for any k, in fact $c\oplus 0_k=\beta d(\beta^{-1})$ where $\beta=\alpha\oplus 1_k$. However, if $c\in M_n(B)$ $(n>1)$ is such that $c\oplus 0_k$ is a logarithmic derivative for some k, then c may not be a logarithmic derivative. We remark that this is however true if $n=1$.

Let $b \in B$ such that $b \oplus 0_k = \alpha d(\alpha^{-1})$ for some $\alpha \in GL_{k+1}(B)$. Let $\alpha^{-1} = (t_{ij})_{1 \leqslant i,j \leqslant k+1}$. The above relation gives $t_{i1}b = d(t_{i1})$ for all i with $1 \leqslant i \leqslant k+1$, $d(t_{ij}) = 0$ for $j \geqslant 2$. Multiplying these equations by the corresponding cofactors and adding up we get $(\det \alpha^{-1}).\ b = d(\det \alpha^{-1})$ which implies that b is a logarithmic derivative.

6.2. Let $f(X) = X^p$ be a polynomial satisfied by d and let there be an element $x \in B$ such that $d(x)$ is a unit in A. Then any $c \in M_n(A)$ such that $\delta_n(c) = 0$ is a logarithmic derivative.

Proof. Since $c \in M_n(A)$, we have by Corollary 3.2, $0 = \delta_n(c) = c^p$. If $d(x) = u \in U(A)$, let

$$\alpha^{-1} = u + uxc + \frac{ux^2c^2}{2!} + \ldots + \frac{ux^{p-1}c^{p-1}}{p-1!}.$$

It is easily seen that $c = \alpha d(\alpha^{-1})$.

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