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Boundary values and generalized normal derivatives of harmonic Dirichlet functions

By

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Introduction. In the study of harmonic functions on an open Riemann surface it is often useful to consider a compactification of the Riemann surface, as one can define the boundary values and generalized normal derivatives at the ideal boundary.

It is known that every canonical potential, especially every harmonic measure assumes constant values quasi-everywhere on each component of Kuramochi boundary. Such a property for boundary behavior was first shown in Kusunoki [5] by using the generalized normal derivatives, and further investigated by Kusunoki-Mori [7] Ikegami [4] and Watanabe [11] in various methods. At the same time it was investigated by Kusunoki [5], [6] and Watanabe [12] that whether this boundary behavior would characterize those functions, but the problem is still not settled.

In this paper we shall deal with this problem from the viewpoint of generalized normal derivatives and show several equivalent statements with applications and examples concerning the boundary values and generalized normal derivatives.

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1. Let R be a hyperbolic Riemann surface and R^* be a resolutive compactification of R. We denote by Δ the ideal boundary of R^*

and by ω_a the harmonic measure on Δ with respect to a point $a\in R$. Particularly we fix $a_0\in R$ and write ω_{a_0} as ω. We denote by $L^2=L^2(\Delta)$ all functions on Δ square integrable with respect to $d\omega$ and by $\Gamma=\Gamma(R)$ the space of all square integrable differentials on R. L^2 and Γ are Hilbert spaces with the scalar products

$$(f, g)=(f, g)_\Delta=\int_\Delta f\bar{g}d\omega,$$

$$<\sigma, \tau> = <\sigma, \tau>_R=\int_R \sigma^*\bar{\tau},$$

respectively. Let Γ_χ be any subspace of Γ whose each element is a harmonic differential. We say that R^* is *Γ_χ-normal* if and only if for any harmonic function u with $du\in\Gamma_\chi$ there exists a resolutive function f_u such that

$$u(a)=\int_\Delta f_u d\omega_a=H_{f_u}(a),$$

holds for any $a\in R$. Such a function f_u is uniquely determined as an L^1-function and we shall call f_u the *boundary function* of u and write it $\hat{u}$ from now. Hereafter we shall use same terminologies and notations in Constantinescu-Cornea [2] and Ahlfors-Sario [1] without repetitions.

2. We shall use later the following fact (cf. Doob [3], Maeda [8]).

Lemma 1. *Let R^* be Γ_χ-normal and u be a harmonic function with $du\in\Gamma_\chi$, then $\hat{u}$ is in L^2. Especially if $u(a_0)=0$, we have*

$$\|\hat{u}\|^2=\int_\Delta|\hat{u}|^2 d\omega\leq M\|du\|^2,$$

here the constant M is independent of u.

Proof. It is sufficient to show the inequality for real u with $u(a_0)=0$. Let g_b be the Green function on R with pole at b and we set

$$P(b)=\frac{1}{2\pi}\int_R g_b \mathrm{d}^*\mathrm{d}(u^2),$$

then we know $P(b)(\neq\infty)$ is a potential. Now we take a constant $N_b>0$ so that $V_b=\{p\in R;\, g_b(p)>N_b\}$ becomes a parametric disk i.e. $e^{-(g_b+i^*g_b)}$ gives a conformal mapping from $\bar{V}_b$ to $\{z;\,|z|\leqq e^{-N_b}\}$. We can write $\mathrm{d}^*\mathrm{d}(u^2)=2\{u_x^2+u_y^2\}\mathrm{d}x\mathrm{d}y$ on V_b. We have

$$2\{u_x^2+u_y^2\}\leqq K<\infty \qquad \text{on } V_b,$$

and

$$\begin{aligned} P(b)&=\frac{1}{2\pi}\int_{V_b} g_b \mathrm{d}^*\mathrm{d}(u^2)+\frac{1}{2\pi}\int_{R-V_b} g_b \mathrm{d}^*\mathrm{d}(u^2)\\ &\leqq\frac{K}{2\pi}\int_{V_b} g_b \mathrm{d}x\mathrm{d}y+\frac{N_b}{\pi}\|\mathrm{d}u\|^2\\ &=\frac{K}{2\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^{e^{-N_b}}\int_0^{2\pi}\left(\log\frac{1}{r}\right)r\mathrm{d}r\mathrm{d}\theta+\frac{N_b}{\pi}\|\mathrm{d}u\|^2<\infty. \end{aligned}$$

We take a C_0^∞-function f on R such that $f=0$ on $R-V_b$, $f=1$ on $V_1=\{p\in R;\, g_b(p)>N_1(>N_b)\}$. Since fu^2 is a C_0^∞-function, by well known formula (cf. [2], p 34) we get

$$u^2(a)=fu^2(a)=-\frac{1}{2\pi}\int_R g_a \mathrm{d}^*\mathrm{d}(fu^2)\,,$$

for $a\in V_1$. We set

$$h(c)=-\frac{1}{2\pi}\int_R g_c \mathrm{d}^*\mathrm{d}\{(1-f)u^2\}$$

for $c\in R$, then $h(c)$ is harmonic on V_1. This shows that $-P=fu^2+h$ is a C^∞-function on V_1 and Laplacian $\Delta(-P)=\Delta(fu^2)=\Delta(u^2)$ on V_1, therefore $P+u^2$ is harmonic on V_1. Since b and V_1 are arbitrary, $P+u^2$ is a harmonic majorant of u^2. Also $\int_\Delta \hat{u}^2\mathrm{d}\omega_a$ is the least harmonic majorant of u^2 for variable a and

$$\|\hat{u}\|^2=\int_\Delta |\hat{u}|^2 d\omega\leq P(a_0)+u^2(a_0)=P(a_0)$$

$$=\frac{1}{2\pi}\int_{V_{a_0}} g_{a_0}\mathrm{d}^*\mathrm{d}(u^2)+\frac{1}{2\pi}\int_{R-V_{a_0}} g_{a_0}\mathrm{d}^*\mathrm{d}(u^2)$$

$$=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{e^{-N_{a_0}}}\int_0^{2\pi}\left(\log\frac{1}{r}\right)k(r,\theta)r\mathrm{d}r\mathrm{d}\theta+\frac{1}{2\pi}\int_{R-V_{a_0}} g_{a_0}\mathrm{d}^*\mathrm{d}(u^2)$$

where $k(r,\theta)=u_x^2+u_y^2$. Since $k(r,\theta)$ is subharmonic, $K(r)=\int_0^{2\pi}k(r,\theta)\,\mathrm{d}\theta$ is a monotone increasing function, and for $\alpha=e^{-N_{a_0}}$

$$\int_0^\alpha K(r)\mathrm{d}r=\frac{1}{\alpha}\left\{\int_0^{\alpha/2}(\alpha-r)K(r)\mathrm{d}r+\int_{\alpha/2}^{\alpha}(\alpha-r)K(r)\mathrm{d}r+\int_0^\alpha rK(r)\mathrm{d}r\right\}$$

$$\leqq\frac{1}{\alpha}\left\{\int_0^{\alpha/2}(\alpha-r)K(\alpha-r)\mathrm{d}r+\int_{\alpha/2}^{\alpha}rK(r)\mathrm{d}r+\int_0^\alpha rK(r)\mathrm{d}r\right\}$$

$$\leqq\frac{3}{\alpha}\int_0^\alpha rK(r)\mathrm{d}r.$$

Now we may assume $r\log\frac{1}{r}\leqq N_{a_0}e^{-N_{a_0}}$, hence it follows that

$$\|\hat{u}\|^2\leqq\frac{3N_{a_0}}{\pi}\int_0^{e^{-N_{a_0}}}\int_0^{2\pi}k(r,\theta)r\mathrm{d}r\mathrm{d}\theta+\frac{1}{2\pi}\int_{R-V_{a_0}} g_{a_0}\mathrm{d}^*\mathrm{d}(u^2)$$

$$\leqq\frac{3N_{a_0}}{\pi}\|\mathrm{d}u\|^2+\frac{N_{a_0}}{\pi}\|\mathrm{d}u\|^2\leqq M\|\mathrm{d}u\|^2.$$

3. We denote by $H=H(R)$ the Hilbert space of all HD-functions on R with the scalar product

$$\langle\!\langle u, v\rangle\!\rangle=\langle\mathrm{d}u,\mathrm{d}v\rangle+u(a_0)\overline{v(a_0)},$$

and by $H_0(\Gamma_\chi)$ a Hilbert subspace of H such that $u\in H_0(\Gamma_\chi)$ vanishes at a_0 and $\mathrm{d}u$ is in Γ_χ. Since $\langle\!\langle u, v\rangle\!\rangle=\langle\mathrm{d}u,\mathrm{d}v\rangle$ in $H_0(\Gamma_\chi)$, we use the scalar product $\langle\mathrm{d}u,\mathrm{d}v\rangle$ instead of $\langle\!\langle u, v\rangle\!\rangle$ in $H_0(\Gamma_\chi)$.

We assume that R^* is Γ_χ-normal from now on to the section 5. Let $L_0=\{f\in L^2(\Delta);\int_\Delta f\mathrm{d}\omega=0\}$. Now we define *$\Gamma_\chi$-generalized normal derivative* of $u\in HD$ by the function f in L_0 for which $\langle\mathrm{d}v,\mathrm{d}u\rangle=\int_\Delta\hat{v}\bar{f}\mathrm{d}\omega$ holds for any $v\in H_0(\Gamma_\chi)$. $N(\Gamma_\chi)$ denotes the all HD-functions having Γ_χ-generalized normal derivatives. Let $N_0(\Gamma_\chi)=H_0(\Gamma_\chi)\cap N(\Gamma_\chi)$, and $L_0(\Gamma_\chi)$ be the closure of $\{f \in L_0;\ H_f \in H_0(\Gamma_\chi)\}$ in $L^2(\Delta)$, which is a Hilbert subspace of $L^2(\Delta)$. Here we have

Lemma 2. *For any $f \in L_0$ there exists a unique $u \in N_0(\Gamma_\chi)$ such that f is a Γ_χ-generalized normal derivative of u.*

Proof. Consider a mapping $v \to \int_\Delta \hat{v}\bar{f}\mathrm{d}\omega$, $v \in H_0(\Gamma_\chi)$, then this is a bounded linear functional, because by Schwarz's inequality and Lemma 1

$$\left|\int_\Delta \hat{v}\bar{f}\mathrm{d}\omega\right|^2 \leqq \int_\Delta |\hat{v}|^2\mathrm{d}\omega \cdot \int_\Delta |\bar{f}|^2\mathrm{d}\omega \leqq M\|\mathrm{d}v\|^2\|f\|^2.$$

Hence by Riesz theorem there exists a unique $u \in H_0(\Gamma_\chi)$ such that $<\mathrm{d}v, \mathrm{d}u> = \int_\Delta \hat{v}\bar{f}\mathrm{d}\omega$ for any $v \in H_0(\Gamma_\chi)$, which is to be proved.

For given f we write above u as u_f. By this Lemma we can define a mapping A_χ from L_0 to $H_0(\Gamma_\chi)$; $A_\chi(f) = u_f$, and a mapping B_χ from $H_0(\Gamma_\chi)$ to $L^2(\Delta)$; $B_\chi(u) = \hat{u}$.
Let $A_{\chi\chi'}$ denotes the restriction of A_χ to $L_0(\Gamma_{\chi'})$, when R^* is also $\Gamma_{\chi'}$-normal. We set

$$F_{\chi\chi'} = A_{\chi\chi'}B_{\chi'}:\ H_0(\Gamma_{\chi'}) \longrightarrow H_0(\Gamma_\chi),$$

$$G_{\chi\chi'} = B_\chi \circ A_{\chi\chi'}:\ L_0(\Gamma_{\chi'}) \longrightarrow L_0(\Gamma_\chi).$$

We shall treat these mappings for any fixed Γ_χ and $\Gamma_{\chi'} = \Gamma_\chi$, so from now on we shall omit the index and write $A(=A_\chi)$, $F(=F_{\chi\chi})$ and so on. From definitions directly we have

Lemma 3.

$$<\mathrm{d}u, \mathrm{d}(Af)> = \int_\Delta \hat{u}\bar{f}\mathrm{d}\omega = \int_\Delta B(u)\bar{f}\mathrm{d}\omega = (B(u), f),$$

$$<\mathrm{d}v, \mathrm{d}(Fu)> = \int_\Delta \hat{v}\bar{\hat{u}}\mathrm{d}\omega = \int_\Delta B(v)\overline{B(u)}\mathrm{d}\omega = (B(v), B(u)),$$

$$<\mathrm{d}(Af), \mathrm{d}(Ag)> = \int_\Delta B \circ A(f)\bar{g}\mathrm{d}\omega = \int_\Delta G(f)\bar{g}\mathrm{d}\omega = (G(f), g).$$

Lemma 4. (1) A, B, F and G are bounded linear operators, and all are injective.
(2) F and G are positive definite self-adjoint operators.

Proof. It is clear that A and B are linear. and by Lemma 1 B is continuous. Also A is continuous, because by Lemma 1, 3 and Schwarz's inequality

$$\|\mathrm{d}(Af)\|^2 = <\mathrm{d}(Af), \mathrm{d}(Af)> = \int_\Delta B\circ A(f)\bar{f}\mathrm{d}\omega$$

$$\leqq \left\{\int_\Delta |B\circ A(f)|^2\mathrm{d}\omega \cdot \int_\Delta |\bar{f}|^2\mathrm{d}\omega\right\}^{1/2}$$

$$= \|B\circ A(f)\|\cdot\|f\| \leqq \sqrt{M}\|\mathrm{d}(Af)\|\cdot\|f\|.$$

It follows $\|\mathrm{d}(Af)\| \leqq \sqrt{M}\|f\|$. And $F = A\circ B$, $G = B\circ A$ are linear and continuous. Let $A(f)=0$, then

$$0 = <\mathrm{d}u, \mathrm{d}(Af)> = \int_\Delta \hat{u}\bar{f}\mathrm{d}\omega,$$

for any $u \in H_0(\Gamma_\chi)$. Since $B[H_0(\Gamma_\chi)]$ is dense in $L_0(\Gamma_\chi)$, $f=0$. This shows A is injective. Also B is injective. Hence F and G are injective. This completes the proof of (1). Next F and G are self-adjoint, because

$$<\mathrm{d}u, \mathrm{d}(Fv)> = \int_\Delta \hat{u}\bar{\hat{v}}\mathrm{d}\omega = \int_\Delta \overline{\hat{v}\bar{\hat{u}}}\mathrm{d}\omega$$

$$= \overline{<\mathrm{d}v, \mathrm{d}(Fu)>} = <\mathrm{d}(Fu), \mathrm{d}v>$$

and
$$(Gf, g) = \int_\Delta G(f)\bar{g}\mathrm{d}\omega = <\mathrm{d}(Af), \mathrm{d}(Ag)> = \overline{<\mathrm{d}(Ag), \mathrm{d}(Af)>}$$

$$= \int_\Delta \overline{G(g)\bar{f}}\mathrm{d}\omega = \overline{(Gg, f)} = (f, Gg).$$

If we set $u=v$ or $f=g$, we know that F and G are positive definite.

(q.e.d)

For generalized normal derivatives we state next

Proposition 1. *$F^n[H_0(\Gamma_\chi)]$ is dense in $H_0(\Gamma_\chi)$, where F^n means n-times iterates of F.*

Proof. If v is in $H_0(\Gamma_\chi)$, and $<dv, d(Fu)> =0$ for any $u \in H_0(\Gamma_\chi)$, then by Lemma 3 $\int_\Delta \hat{v}\bar{\hat{u}} d\omega = <dv, d(Fu)> =0$. Particularly we choose $u=v$, then $0=\int_\Delta |\hat{v}|^2 d\omega$, hence $\hat{v}=0$ and $v=0$. This shows that $F[H_0(\Gamma_\chi)]$ is dense in $H_0(\Gamma_\chi)$. Similarly $F^{k+1}[H_0(\Gamma_\chi)]$ is dense in $F^k[H_0(\Gamma_\chi)]$, and $F^n[H_0(\Gamma_\chi)]$ is dense in $H_0(\Gamma_\chi)$ which completes the proof.

Since $H_0(\Gamma_\chi) \supset N_0(\Gamma_\chi) = A_\chi(L_0) \supset A_\chi[L_0(\Gamma_\chi)] \supset F[H_0(\Gamma_\chi)]$, we have

Corollary 1.1 *The set of all $H_0(\Gamma_\chi)$-functions which have Γ_χ-generalized normal derivatives is dense in $H_0(\Gamma_\chi)$.*

4. Next we shall use the following basic theorem for Banach space (cf. [9], p 61).

Theorem. *Let B and E be Banach spaces, T be a bounded linear operator from B to E, then the image T(B) is either of the first category in E or E itself. Moreover if T is injective and surjective, then the inverse of T is well defined and continuous.*

This shows that $F[H_0(\Gamma_\chi)]$ and $G[L_0(\Gamma_\chi)]$ are either of the first category in $H_0(\Gamma_\chi)$ and $L_0(\Gamma_\chi)$ or identical with $H_0(\Gamma_\chi)$ and $L_0(\Gamma_\chi)$ respectively. If $B[H_0(\Gamma_\chi)]$ is of the second category in $L_0(\Gamma_\chi)$, then $B[H_0(\Gamma_\chi)]$ is $L_0(\Gamma_\chi)$ and B^{-1} is continuous i.e. for a suitable $K>0$,

$$||du|| = ||d(B^{-1}\hat{u}|| \leq K||\hat{u}||.$$

Here we have

Lemma 5. *The following* (1), (2), (3) *are equivalent;*

(1) $B[H_0(\Gamma_\chi)] = L_0(\Gamma_\chi)$

(2) $A[L_0(\Gamma_\chi)] = H_0(\Gamma_\chi)$

(3) $F[H_0(\Gamma_\chi)] = H_0(\Gamma_\chi)$

Proof. (1)$\Rightarrow$(2) and (3): If $B[H_0(\Gamma_\chi)]=L_0(\Gamma_\chi)$, then for any $f\in L_0(\Gamma_\chi)$ there exists a $u\in H_0(\Gamma_\chi)$ such that $B(u)=f$. By the continuity of B^{-1}, Schwarz's inequality and Lemma 3

$$\|f\|^2=\int_\Delta B(u)\bar{f}d\omega=<du,\ d(Af)>$$

$$\leqq \|du\|\|d(Af)\|\leqq K\|B(u)\|\|d(Af)\|.$$

It follows that $\|f\|\leqq K\|d(Af)\|$. For any $v\in H_0(\Gamma_\chi)$ by Proposition 1 there exists a sequence $\{f_n\}$ in $L_0(\Gamma_\chi)$ such that $\{d(Af_n)\}$ converges to dv in $H_0(\Gamma_\chi)$. Then $\{d(Af_n)\}$ is a Cauchy sequence and also $\{f_n\}$ is a Cauchy sequence in $L_0(\Gamma_\chi)$, because $\|f_n-f_m\|\leqq K\|d(A(f_n-f_m))\|$. So let f_n converge to $f\in L_0(\Gamma_\chi)$, then $A(f)=v$. Hence $A[L_0(\Gamma_\chi)]=H_0(\Gamma_\chi)$. Moreover since $B[H_0(\Gamma_\chi)]=L_0(\Gamma_\chi)$, $F[H_0(\Gamma_\chi)]=A\circ B[H_0(\Gamma_\chi)]=A[L_0(\Gamma_\chi)]=H_0(\Gamma_\chi)$.

(2)$\Rightarrow$(1): If $A[L_0(\Gamma_\chi)]=H_0(\Gamma_\chi)$, by above Theorem $\|f\|\leqq K'\|d(Af)\|$ for any $f\in L_0(\Gamma_\chi)$ and for any $u\in H_0(\Gamma_\chi)$ there exists a $f\in L_0(\Gamma_\chi)$ such that $A(f)=u$. Thus

$$\|du\|^2=<du,\ d(Af)>=(B(u),f)$$

$$\leqq \|B(u)\|\cdot\|f\|\leqq K'\|B(u)\|\cdot\|d(Af)\|.$$

It follows that $\|du\|\leqq K'\|B(u)\|$. Since $B[H_0(\Gamma_\chi)]$ is dense in $L_0(\Gamma_\chi)$ we get the conclusion by the similar argument as above ((1)$\Rightarrow$(2)).

(3)$\Rightarrow$(2): Since $B[H_0(\Gamma_\chi)]\subset L_0(\Gamma_\chi)$, if $H_0(\Gamma_\chi)=F[H_0(\Gamma_\chi)]=A\circ B[H_0(\Gamma_\chi)]$, then $H_0(\Gamma_\chi)\subset A[L_0(\Gamma_\chi)]$. This completes the proof of Lemma 5.

Remark Let u be in $H_0(\Gamma_{he})$ and u_χ be the orthogonal projection of u onto $H_0(\Gamma_\chi)$. Suppose that u_χ has a Γ_χ-generalized normal derivative f, then for any $v\in H_0(\Gamma_\chi)$

$$<dv,\ du>=<dv,\ du_\chi>=\int_\Delta \hat{v}\bar{f}d\omega=\int_\Delta \hat{v}\bar{f}_1 d\omega,$$

f_1 being the orthogonal projection of f onto $L_0(\Gamma_\chi)$. This implies that if all functions in $H_0(\Gamma_\chi)$ have Γ_χ-generalized normal derivatives then all functions in HD have Γ_χ-generalized normal derivatives, which belong to $L_0(\Gamma_\chi)$.

Proposition 2. *Any HD-function has a Γ_χ-generalized normal derivative if and only if there exists a constant $K>0$ such that $\|du\| \leqq K\|\hat{u}\|$ for any $u \in H_0(\Gamma_\chi)$.*

Proof. If any HD-function has a Γ_χ-generalized normal derivative, from above remark we have $A[L_0(\Gamma_\chi)] = H_0(\Gamma_\chi)$ and by Lemma 5 $B[H_0(\Gamma_\chi)] = L_0(\Gamma_\chi)$. It follows that $\|du\| \leqq K\|\hat{u}\|$ for any $u \in H_0(\Gamma_\chi)$. Conversely if this inequality is satisfied, we know that B is surjective by the similar argument as proof of Lemma 5, hence every function in $H_0(\Gamma_\chi)$ has Γ_χ-generalized normal derivative and we get the conclusion by above remark.

Remark 1. This inequality is independent of the compactification R^* as far as R^* is Γ_χ-normal. In fact $\|\hat{u}\|^2$ is the value at a_0 of the least harmonic majorant of $|u|^2$.

Remark 2. If $R \in O^n_{HD}$, R^* be Γ_{he}-normal, then all HD-functions have Γ_{he}-generalized normal derivatives. Because $H_0(\Gamma_{he})$ is of finite dimension and $F[H_0(\Gamma_{he})]$ is dense in $H_0(\Gamma_{he})$, it follows that $F[H_0(\Gamma_{he})] = H_0(\Gamma_{he})$.

Let $C(\Delta)$ be a Banach space with supremum norm which consists of all finite continuous functions on Δ. The compactification R^* is called *regular* if and only if $B[HD] \cap C(\Delta)$ is dense in $C(\Delta)$, here $f \in B[HD] \cap C(\Delta)$ means that f is in $C(\Delta)$ and $H_f \in HD$. Note that if R^* is regular, $L_0(\Gamma_{he}) = L_0$. In fact since $C(\Delta)$ is dense in L^2, $B[HD] \cap C(\Delta)$ is dense in L^2. For any $f \in L_0(\subset L^2)$ there exist $f_n \subset B[HD] \cap C(\Delta)$ which converge to f in L^2. And $f_n - H_{f_n}(a_0) \subset L_0(\Gamma_{he})$ converge to f in L_0. By definition of $L_0(\Gamma_{he})$ $L_0(\Gamma_{he}) = L_0$.

Corollary 2.1 (*H. Tanaka* [10]) *Let $R \in 0^n_{HD} - 0^{n-1}_{HD} - 0^n_{HB}$*

(1) *Let u belong to the class $HB(R)$ but not to the class $HD(R)$. If Γ_{he}-normal compactification R^* of R is regular, then u can not be represented by harmonic measure of R^*.*

(2) *Martin's compactification R_M^* of R is not regular.*

Proof. (1) Suppose we can represent $u-u(a_0)$ by harmonic measure: $u(a)-u(a_0)=\int_{\Delta} f d\omega_a$. Since $u-u(a_0)$ is bounded, f belongs to $L_0=L_0(\Gamma_{he})=B[H_0(\Gamma_{he})]$. On the other hand $u-u(a_0)$ does not belong to $H_0(\Gamma_{he})$. This is a contradiction. (2) In Martin's compactification all HB-functions are represented by harmonic measure.

5. We consider here in what case all HD-functions have Γ_χ-generalized normal derivatives. At first we show a simple example. Let R^* be Γ_{he}- or Γ_{hm}-normal and e be a boundary point of Kerékjártó-Stoïlow's compactification R_s^* of R and $\Delta_e=\cap(\overline{U\cap R})$ (U; neighborhood of e in R_s^* and closure is taken in R^*).

Example 1. If the Kerékjártó-Stoïlow's boundary consists of an infinite number (countable) of components, then harmonic measures (hence HD-functions) have not always Γ_{hm}-generalized normal derivatives hence Γ_{he}-generalized normal derivatives.

To show this example consider a rectangle

$$D(a, b)=\{(x, y);\ 0<x<a,\ 0<y<b\},$$

and a family of curves $C=\{\gamma\}$ such that each γ is parallel to x-axis and connects two vertical sides. Then the extremal length $\lambda(C)=a/b$. Let h be a harmonic function in $D(a, b)$ such that

$$h=0 \quad \text{on } x=0,\ 0<y<b, \quad h=1 \quad \text{on } x=\mathrm{a},\ 0<y<b.$$

then $\sqrt{h_x^2+h_y^2}$ is an admissible function, for

$$\int_\gamma \sqrt{h_x^2+h_y^2}\,\mathrm{d}s=\int_0^1 \sqrt{h_x^2+h_y^2}\,\mathrm{d}x \geqq \int_0^1 |h_x| dx=1$$

and $$b/a=1/\lambda(C) \leqq \iint_{D(a,b)} [h_x^2+h_y^2]\mathrm{d}x\mathrm{d}y=\|\mathrm{d}h\|^2.$$

Now let $\hat{\boldsymbol{C}}$ be the extended complex plane, and

$$E_n=\{x+iy;\ x=a_n,\ 0\leq y\leq b_n\}$$

where $\{a_n\}$ is a monotone sequence converging to 0 and $\{b_n\}$ is a sequence such that $b_{2n-1}=b_{2n}\geqq b_{2n+1}=b_{2n+2}$.
We set $R=\hat{C}-\overline{\cup E_n}$ and R^* be the closure of R in $\hat{C}$. Let W_{2n-1} be the harmonic function in R such that

$$\hat{W}_{2n-1}=1 \quad \text{on } E_{2n-1}, \quad 0 \quad \text{on } E_j (j\neq 2n-1)$$

The Dirichlet norm of W_{2n-1}:

$$\|dW_{2n-1}\|_R \geqq \|dW_{2n-1}\|_{D'} \geqq b_{2n}/(a_{2n-1}-a_{2n})$$

where $D'=\{x+iy;\ (x-a_{2n})+iy\in D(a_{2n-1}-a_{2n},\ b_{2n})\}$. While the L^2-norm of $\hat{W}_{2n-1}$: $\|\hat{W}_{2n-1}\|\leqq 1$. Now we choose a_n, b_n such that $b_{2n}/(a_{2n-1}-a_{2n})\to\infty$, (for instance $b_{2n}=1/n$, $a_n=1/n^2$, then $b_{2n}/(a_{2n-1}-a_{2n})=\frac{4n(2n-1)^2}{4n-1}\to\infty$). Then by Proposition 2 harmonic measures have not always Γ_{hm}-generalized normal derivatives.
Next we shall show that if R satisfies the condition:
(C) A Green function $g_a(b)$ *converges to* 0 *as* b *converges to the ideal boundary of* R, (this condition is independent of the choice of Green function)
then any $u\in H_0(\Gamma_\chi)$ has *a* Γ_χ*-generalized normal derivative if and only if the dimension of* $H_0(\Gamma_\chi)$ *is finite.*
First we note that under the condition (C) B is a compact operator (cf. [3]). In fact let u_n converge to 0 weakly in $H_0(\Gamma_\chi)$, then u_n converge to 0 uniformly on any compact set K, and

$$\|\hat{u}_n\|^2 \leqq \frac{1}{2\pi}\int_R g_{a_0} d^*d(u_n^2)$$
$$=\frac{1}{2\pi}\int_K g_{a_0} d^*d(u_n^2)+\frac{1}{2\pi}\int_{R-K} g_{a_0} d^*d(u_n^2)\,.$$

For any $\varepsilon>0$ there exists a compact set K scuh that $g_{a_0}(b)<\varepsilon$ for $b\in R-K$, and $\overline{\lim_{n\to\infty}}\|\hat{u}_n\|^2\leqq\overline{\lim_{n\to\infty}}\frac{\varepsilon}{\pi}\|du_n\|^2$. Since $\|du_n\|$ are uniformly bounded and ε is arbitrary, B is a compact operator. Since B is a compact operator, $F=A\circ B$, $G=B\circ A$ are so. Now let $\{s_n\}$ be the eigenvalues of F. Then by Hilbert-Schmidt expansion theorem there exists a complete orthonormal base $\{u_n\}$ which consists of eigenvectors for $\{s_n\}$ including multiplicity. Now $F(u_n)=s_n u_n$, hence $B\circ A\circ B(u_n)=s_n B(u_n)$ i.e. $G(\hat{u}_n)=s_n\hat{u}_n$.

$$\int_\Delta \hat{u}_i(\overline{\hat{u}_j/s_j})d\omega=<du_i,\ du_j>=\begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j.\end{cases}$$

Since boundary functions of $H_0(\Gamma_\chi)$ is dense in $L_0(\Gamma_\chi)$, $\{\hat{u}_n/\sqrt{s_n}\}$ forms an orthonormal base in $L_0(\Gamma_\chi)$.
By the fact that $\{s_n\}$ accumulates at most one point 0, and the eigenspace for each s_n is of finite dimension, and by Proposition 2, we know our assertion. Note that the above conclusion is valid under the condition that F is a compact operator.

6. We know that if harmonic function u satisfies $du\in\Gamma_{hm}$, then $\hat{u}$ is constant ω a.e. on each Δ_e (cf. [5], [7]). While according to *M.* Watanabe an *HD*-function which is constant ω a.e. on each Δ_e is not always a harmonic measure. Here we shall give another such an example.

Example 2. Let $R=\{z;\ 1/4<|z|<1\}$, K a generalized Cantor set on $\left[\frac{1}{3},\frac{1}{2}\right]$ whose linear measure is positive, and $E=\{(r,\theta);\ r\in K,\ 0\leq\theta\leq\pi\}$. Set $R'=R-E$. R^*, R'^* are Royden's compactifications of R, R' respectively. It is clear that $d(-\log|z|)\in\Gamma_{hm}(R)$, $*d(r\cos\theta)\in *\Gamma_{hse}(R)$. Since we have the orthogonal decomposition $\Gamma_h=\Gamma_{hm}+*\Gamma_{hse}$,

$$\begin{aligned}0&=<d(-\log|z|),\ *d(r\cos\theta)>_R\\&=-\iint_R \sin\theta drd\theta<-\iint_{R'}\sin\theta drd\theta\\&=<d(-\log|z|),\ *d(r\cos\theta)>_{R'}.\end{aligned}$$

While $*d(r\cos\theta)\in *\Gamma_{hse}(R')$, so above inequality implies $d(-\log|z|)\notin\Gamma_{hm}(R')$, but $-\log|z|$ is an *HD*-function in R' and constant on each connected component of the boundary.

7. In this section let R^* be Γ_{he}-normal and $A=A_{he}$. From Proposition 1 we know $N_0(\Gamma_{he})$ is dense in $H_0(\Gamma_{he})$. Here we consider the next problem:

Whether $H_0(\Gamma_\chi)\cap N_0(\Gamma_{he})$ is dense in $H_0(\Gamma_\chi)$ or not? At first

Lemma 6. *Function $f\in L_0$ is a Γ_{he}-generalized normal derivative of some v in $H_0(\Gamma_\chi^\perp)$ if and only if f belongs to $L_0(\Gamma_\chi)^\perp$, where $L_0(\Gamma_\chi)^\perp$ (resp. $\Gamma_\chi^\perp$) is the orthogonal complement of $L_0(\Gamma_\chi)$ in L_0 (resp. of Γ_χ in Γ_h).*

Proof. For any $u\in H_0(\Gamma_\chi)$ $\hat{u}$ belongs to $L_0(\Gamma_\chi)$.
Hence if f belongs to $L_0(\Gamma_\chi)^\perp$,

$$0=\int_\Delta \hat{u}\bar{f}d\omega=<du,\ d(Af)>.$$

It follows that $d(Af)$ belongs to $\Gamma_\chi^\perp$ and $A(f)\in H_0(\Gamma_\chi^\perp)$.
Conversely if f is a Γ_{he}-generalized normal derivative of v in $H_0(\Gamma_\chi^\perp)$, then for any $u\in H_0(\Gamma_\chi)$,

$$0=<du,\ dv>=\int_\Delta \hat{u}\bar{f}d\omega\,.$$

Since $B[H_0(\Gamma_\chi)]$ is dense in $L_0(\Gamma_\chi)$, it follows that $f\in L_0(\Gamma_\chi)^\perp$.

(q.e.d.)

Proposition 3. *The following* (1), (2) *are equivalent*;

(1) $H_0(\Gamma_\chi^\perp)\cap N_0(\Gamma_{he})$ *is dense in* $H_0(\Gamma_\chi^\perp)$.

(2) $B[H_0(\Gamma_\chi^\perp)]\cap L_0(\Gamma_\chi)=\{0\}$.

Proof. (1)$\Rightarrow$(2): Suppose there exists a f ($\neq 0$) in $B[H_0(\Gamma_\chi^\perp)]\cap L_0(\Gamma_\chi)$ and let u be in $H_0(\Gamma_\chi^\perp)$ such that $B(u)=f$, then for any $g\in L_0(\Gamma_\chi)^\perp$,

$$0=\int_\Delta f\bar{g}d\omega=<du,\ d(Ag)>.$$

Since by Lemma 6 $A[L_0(\Gamma_\chi)^\perp]=H_0(\Gamma_\chi^\perp)\cap N_0(\Gamma_{he})$, $u(\neq 0)$ belongs to

$[\overline{H_0(\Gamma_\chi^\perp)\cap N_0(\Gamma_{he})}]^\perp \cap H_0(\Gamma_\chi^\perp)$.

(2)$\Rightarrow$(1): Suppose $H_0(\Gamma_\chi^\perp)\cap N_0(\Gamma_{he})$ is not dense in $H_0(\Gamma_x)^\perp$. Then there exists a $u\in[\overline{H_0(\Gamma_\chi^\perp)\cap N_0(\Gamma_{he})}]^\perp\cap H_0(\Gamma_\chi^\perp)$, $u\neq 0$. Since $A(g)\in H_0(\Gamma_\chi^\perp)\cap N_0(\Gamma_{he})$ for any $g\in L_0(\Gamma_\chi)^\perp$,

$$0=<\mathrm{d}u,\ \mathrm{d}(Ag)>=\int_\Delta \hat{u}\bar{g}\mathrm{d}\omega\,,$$

and we get $\hat{u}\in L_0(\Gamma_\chi)\cap B[H_0(\Gamma_\chi^\perp)]$. (q.e.d.)

We write the class of harmonic Schottky differentials by Γ_s.

Let $$\Gamma_{es}=\Gamma_s\cap\Gamma_{he},\ \Gamma_{kd}={}^*\Gamma_{hse}\cap\Gamma_{he}\,.$$

We know the next orthogonal decompositions,

$$\Gamma_{he}=\Gamma_{ae}+\bar{\Gamma}_{ae}+\Gamma_{es}=\Gamma_{he}\cap{}^*\Gamma_{he}+\Gamma_{es}=\Gamma_{hm}+\Gamma_{kd}\,.$$

Here we have,

Corollary 3.1 (*M. Watanabe* [12]) *$H_0(\Gamma_{hm})\cap N_0(\Gamma_{he})$ is dense in $H_0(\Gamma_{hm})$.*

Proof. Let $\{u_n\}$ be a sequence of KD-functions such that $\{\hat{u}_n\}$ converges to $B(u)$ in the sense of L^2 where u is an HD-function. Then u_n converges to u uniformly on any compact subset, hence

$$\int_\gamma {}^*du=\lim_{n\to 0}\int_\gamma {}^*du_n=0$$

for any dividing cycle γ. This shows u is a KD-function and $B[H_0(\Gamma_{hm})]\cap L_0(\Gamma_{kd})=\{0\}$. By Proposition 3 we get $H_0(\Gamma_{hm})\cap N_0(\Gamma_{he})$ is dense in $H_0(\Gamma_{hm})$.

Corollary 3.2 *$H_0(\Gamma_{es})\cap N_0(\Gamma_{he})$, $H_0(\Gamma_{es}+\Gamma_{ae})\cap N_0(\Gamma_{he})$ and $H_0(\Gamma_{es}+\bar{\Gamma}_{ae})\cap N_0(\Gamma_{he})$ are dense in $H_0(\Gamma_{es})$, $H_0(\Gamma_{es}+\Gamma_{ae})$ and $H_0(\Gamma_{es}+\bar{\Gamma}_{ae})$ respectively.*

Proof. We can prove these by the similar argument as in Corollary 3.1.

Corollary 3.3 *Suppose Γ_{he}-normal compactification R^* has the ideal boundary $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_1 \cap \Delta_2 = \phi$ such that Δ_1 consists of connected components $\{\Delta_e\}$ each of which has a positive harmonic measure, and Δ_2 is of harmonic measure zero, then every HD-function u which is constant ω a.e. on each Δ_e is a harmonic measure if and only if $H_0(\Gamma_{kd}) \cap N_0(\Gamma_{he})$ is dense in $H_0(\Gamma_{kd})$.*

Proof. It is sufficient that we show this for real u. We have a decomposition $u = u \vee 0 + u \wedge 0$. Since boundary function of $u \in HD$ is constant ω a.e. on each Δ_e, boundary function of HD-function $u \vee 0$ (resp. $u \wedge 0$) is also constant ω a.e. on each Δ_e. Now we consider a harmonic function v_e which takes a positive constant ω a.e. on only one Δ_e and vanishes ω a.e. on the other components. v_e is obviously a harmonic measure if Δ_e is isolated. In general v_e can be approximated in the sense of L^2 by decreasing harmonic measures. Since the number of $\{\Delta_e\}$ each of whose element has a positive harmonic measure is countable, by monotone convergence theorem we know that $u \vee 0$(resp. $u \wedge 0$) can be approximated in the sense of L^2 by a linear combination of those harmonic functions $\{v_e\}$ and so $\hat{u} - u(a_0)$ belongs to $L_0(\Gamma_{hm})$. Moreover by orthogonal decomposition we may assume that du belongs to Γ_{kd}, From Proposition 3 we get the conclusion.

At last we note that the *set of all KD-functions each of whose function has a Γ_{he}-generalized normal derivative is not always dense in the space of all KD-functions.* Because in Example 2 we know easily $-\log|z| \in L_0(\Gamma_{hm})$, so by Proposition 3 Example 2 gives this example. In fact for arbitrary small $\varepsilon > 0$ there are real numbers $\{r_i\}_{1 \leq i \leq n}$ such that $r_i \notin K$ (K: the generalized Cantor set in Example 2) and $\frac{1}{3} - \delta = r_1 < r_2 < r_n = \frac{1}{2} + \delta$, $\delta > 0$, $\log r_{i+1} - \log r_i < \varepsilon$. Let $E_i = \{(r, \theta) \in E;\ r_i < r < r_{i+1}\}_{1 \leq i \leq n-1}$, $E_0 = \left\{z;\ |z| = \frac{1}{4}\right\}$, f_i be a harmonic measure such that $f_i = -\log r_i$ on E_i, $=0$ on the other connected components of boundary. We set $f = \sum_{i=0}^{n-1} f_i \in H_0(\Gamma_{hm})$.

Then it is clearly $\int_\Delta |-\log|z| - f|^2 d\omega < \varepsilon^2$, and $-\log|z| \in L_0(\Gamma_{hm})$.

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References

[1] Ahlfors, L., and Sario, L.: Riemann Surfaces. Princeton (1960)

[2] Constantinescu, C., and Cornea, A.: Ideal Ränder Riemannscher Flächen. Springer-Verlag (1963).

[3] Doob, J. L.: Boundary properties of functions with finite Dirichlet integrals. Ann. Inst. Fourier, 12, 573–621 (1962)

[4] Ikegami, T.: A mapping of Martin boundary into Kuramochi boundary by means of poles. Osaka J. Math., 5, 285–297 (1968).

[5] Kusunoki, Y.: Characterization of canonical differentials. J. Math. Kyoto Univ., 5, 197–207 (1966).

[6] ——— : On some boundary properties of harmonic Dirichlet functions. Proc. Japan Acad. 46, 277–282 (1970).

[7] Kusunoki, Y., and Mori, S.: Some remarks on boundary values of harmonic functions with finite Dirichlet integrals. J. Math. Kyoto Univ., 7, 315–324 (1968).

[8] Maeda, F.-Y.: Normal derivatives on an ideal boundary. J. Sci. Hiroshima Univ. Ser. A-I, 28, 113–131 (1964).

[9] Mizohata, S.: Henbibunhôteishikiron. Iwanami, Tokyo (1965).

[10] Tanaka, H.: On function-theoretic separative conditions on compactifications of hyperbolic Riemann surfaces. Hiroshima Math. J., 2, 33–65 (1972).

[11] Watanabe, M.: On a boundary property of principal functions. Pacific J. Math., 31, 537–545 (1969).

[12] ——— : Generalized normal derivatives of principal functions. J. Math. Kyoto Univ., 12, 33–54 (1972).