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On the temporally global problem of the generalized Burgers' equation

By

Nobutoshi ITAYA

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E. Hopf discussed in details on "Burgers' equation" in his famous paper [5]. Since then, a great many papers on this equation and its related topics have appeared. The present author, however, thinks that there lies a very deep gap between the above equation and the system of equations for compressible viscous fluids. This paper aims at taking a step towards the latter, especially, from the view-point of treating temporally global problems.

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Contents

§1. Introduction and Notations.

We have shown in [7], [8] the existence of a temporally local solution of the system of fundamental equations for compressible viscous fluid:

$$
(1.1)\quad \begin{cases}
\dfrac{\partial v}{\partial t}=\dfrac{\mu}{\rho}\left(\triangle+\dfrac{1}{3}\nabla\cdot\mathrm{div}\right)v-(v\cdot\nabla)v-\dfrac{1}{\rho}\nabla p(\rho,\theta)+f,\\
\text{(v, velocity vector; ρ, density; θ, absolute temperature;}\\
\text{p, pressure; μ, viscosity coefficient; f, outer force),}\\
\dfrac{\partial\theta}{\partial t}=\dfrac{\kappa}{c_v\rho}\left[\Delta\theta+\Phi(\nabla v)+p\,\mathrm{div}\,v-c_v\rho v\cdot\nabla\theta\right],\\
\text{(c_v, specific heat at constant volume; κ, heat conductivity; $\Phi(\nabla v)$, dissipation function),}\\
\dfrac{\partial\rho}{\partial t}+\mathrm{div}\,\rho v=0,
\end{cases}
$$

where μ, κ, and c_v are assumed to be constants as functions in ρ and θ. The problem of the existence of a temporally global solution in (1.1) presents very great difficulties because of the complexity in non-linearity that the system (1.1) of differential equations contains in itself. In view of such circumstances at present, we attempt in this paper to discuss on the temporally global behavior of the solution of a much simplified model of the system (1.1), i.e.:

$$
\begin{cases}
(1.2)^1\quad \dfrac{\partial v}{\partial t}(x,t)=\dfrac{\mu}{\rho(x,t)}\dfrac{\partial^2}{\partial x^2}v(x,t)-v(x,t)\cdot\dfrac{\partial v}{\partial x}(x,t)\\
(1.2)^2\quad \dfrac{\partial\rho}{\partial t}(x,t)+\dfrac{\partial}{\partial x}(\rho v)=0,\qquad \left(\text{or }\dfrac{\partial\rho}{\partial t}+v\dfrac{\partial\rho}{\partial x}=-\rho\dfrac{\partial v}{\partial x}\right),
\end{cases}
$$

where v is a scalar function and $x\in R^1$. We name the above system (1.2) of differential equations the "generalized Burgers' equation" after the well-known Burgers' equation

$$
(1.3)\qquad \frac{\partial v}{\partial t}=\mu\frac{\partial^2}{\partial x^2}v-v\frac{\partial v}{\partial x}.
$$

As we shall see in the course of discussing on the subject under consideration, it seems that the study of the generalized Burgers' equation itself gives a great many suggestions on the treatments not only of the system (1.1) but also of non-linear problems in general.

Notations. In this paper we follow chiefly the notations used in [7], [8]. Generally speaking, the functions to be considered here are defined in R^1 or $R^1\times[0, T]$ $(0\leqq T<+\infty)$ and as many times continuously differentiable there as necessary.

$$
(1.4)\quad \begin{cases} |u(x)|^{(0)}\equiv \sup\limits_{x\in R^1}|u(x)|, \quad |u(x)|^{(\alpha)}\equiv \sup\limits_{\substack{x,x'\in R^1\\ x\neq x'}} \dfrac{|u(x)-u(x')|}{|x-x'|^{\alpha}}, \\ \hfill (0<\alpha<1), \\ |u(x)|^{(L)}\equiv \sup\limits_{x,x'\in R^1} \dfrac{|u(x)-u(x')|}{|x-x'|}. \end{cases}
$$

$$
(1.4)'\quad \begin{cases} \|u(x)\|^{(n)}\equiv \sum\limits_{i=0}^{n}\left|\dfrac{d^i}{dx^i}u(x)\right|^{(0)}, \\ \|u(x)\|^{(n+\alpha)}\equiv \|u(x)\|^{(n)}+\left|\dfrac{d^n}{dx^n}u(x)\right|^{(\alpha)}, \\ \|u(x)\|^{(n+L)}\equiv \|u(x)\|^{(n)}+\left|\dfrac{d^n}{dx^n}u(x)\right|^{(L)}, \qquad (n=0,\ 1,\ 2,\dots) \end{cases}
$$

$$
(1.5)\quad \begin{cases} |v(x,t)|_T^{(0)}\equiv \sup\limits_{(x,t)\in R^1\times[0,T]}|v(x,t)|, \\ |v(x,t)|_{x,T}^{(\alpha)}\equiv \sup\limits_{\substack{(x,t),(x',t)\in R^1\times[0,T]\\ x\neq x'}} \dfrac{|v(x,t)-v(x',t)|}{|x-x'|^{\alpha}} \\ |v(x,t)|_{t,T}^{(\alpha/2)}\equiv \sup\limits_{\substack{(x,t),(x,t')\in R^1\times[0,T]\\ t\neq t'}} \dfrac{|v(x,t)-v(x,t')|}{|t-t'|^{\alpha/2}}, \\ |v(x,t)|_T^{(\alpha)}\equiv |v(x,t)|_{x,T}^{(\alpha)}+|v(x,t)|_{t,T}^{(\alpha/2)}, \\ (|v(x,t)|_{x,T}^{(L)},\ \text{etc. are defined in an analogous way to} \\ |u(x)|^{(L)}). \end{cases}
$$

$$(1.5)' \quad \begin{cases} \|v(x, t)\|_T^{(n)} \equiv \sum_{2r+s=0}^{n} |D_t^r D_x^s v|_T^{(0)}, \qquad \left(D_t = \frac{\partial}{\partial t}, D_x = \frac{\partial}{\partial x}\right), \\ \|v\|_T^{(n+\alpha)} \equiv \|v\|_T^{(n)} + \sum_{2r+s=n} |D_t^r D_x^s v|_{x,T}^{(\alpha)} \\ \quad + \sum_{2r+s=\max[n-1,0]}^{n} |D_t^r D_x^s v|_{t,T}^{(\alpha/2)}, \\ <v>_T^{(n,\alpha)} \equiv \sum_{s=0}^{n} |D_x^s v|_T^{(0)} + \sum_{s=0}^{n-1} |D_x^s v|_{t,T}^{(\alpha/2)}, \\ \ll v \gg_T^{(n+\alpha)} \equiv <v>_T^{(n,\alpha)} + '<v>_T^{(n,\alpha)}, \\ '<v>_T^{(n,\alpha)} \equiv |D_x^n v|_T^{(\alpha)}, \qquad (n=0, 1, 2, \dots), \end{cases}$$

where r and s are non-negative integers.

$$(1.6) \quad \begin{cases} H^n \equiv \{u(x) : \|u\|^{(n)} < +\infty\}, \\ H^{u+\alpha(\text{or}+L)} \equiv \{u(x) : \|u\|^{(n+\alpha(\text{or}+L))} < +\infty\}, \\ H_T^{n+\alpha} \equiv \{v(x, t) : \|v\|_T^{(n+\alpha)} < +\infty\}, \\ \hat{H}_T^{n+\alpha} \equiv \{v(x, t) : \ll v \gg_T^{(n+\alpha)} < +\infty\}, \\ B_T^n \equiv \{v(x, t) : \sum_{r+s=0}^{n} |D_t^r D_x^s v|_T^{(0)} < +\infty\}, \\ B_T^{n+\alpha} \equiv \{v(x, t) : \sum_{r+s=0}^{n} |D_t^r D_x^s v|_T^{(0)} < + \sum_{r+s=n} |D_t^r D_x^s v|_T^{(0)} < +\infty\}. \end{cases}$$

If notations, not described above, appear hereafter, then they will be explained where they appear.

§2. A Fundamental Lemma.

We assume for (1.2) the following initial condition:

$$(2.1) \quad \begin{cases} v(x, 0) = v_0(x) \in H^{2+\alpha}, \\ \rho(x, 0) = \rho_0(x) \in H^1, \qquad (0 < \underline{\rho}_0 \text{ (constant) } \leqq \rho_0(x) \leqq \bar{\rho}_0 = |\rho_0|^{(0)}). \end{cases}$$

Let (v, ρ) be a solution in $H_T^{\frac{2+\alpha}{2}} \times B_T^{\frac{1}{2}}$ of (1.2) satisfying the initial condition (2.1), and $\bar{x}(\tau; x, t)$ be the solution curve of the characteristic equation for $(1.2)^2$ as a linear equation in ρ:

$$\text{(2.2)} \qquad \begin{cases} \dfrac{d}{d\tau}\bar{x}(\tau; x, t) = v(\bar{x}(\tau; x, t), \tau), & (0 \leqq \tau \leqq t \leqq T), \\ \bar{x}(t; x, t) = x. \end{cases}$$

We note that, since $\in H_T^{\frac{2+\alpha}{2}}$, the solution curve for (2.2) starting at an arbitrary point $(x, t) \in R^1 \times [0, T]$ is unique. From (2.2), we have

$$\text{(2.3)} \qquad \bar{x}_x(\tau; x, t) = \frac{\partial}{\partial x}\bar{x}(\tau; x, t) = e^{-\int_\tau^t v_x(\bar{x}(\tau'; x, t), \tau')d\tau'}$$

If $v(x, t) \in H_T^{\frac{2+\alpha}{2}}$ is given in $(1.2)^2$, then $\rho(x, t)$ is uniquely determined, being expressed by

$$\text{(2.4)} \qquad \begin{cases} \rho(x, t) = \rho_0(\bar{x}(0; x, t)) \cdot \bar{x}_x(0; x, t) \\ \qquad = \rho_0(\bar{x}(0; x, t)) e^{-\int_0^t v_x(\bar{x}(\tau; x, t,)\tau)d\tau}. \end{cases}$$

For simplicity's sake, we put

$$\text{(2.5)} \qquad \begin{cases} \bar{\rho}(\tau; x, t) \equiv \rho(\bar{x}(\tau; x, t), \tau), \\ \bar{v}(\tau; x, t) \equiv v(\bar{x}(\tau; x, t), \tau), \text{ etc.} \end{cases}$$

Directly from $(1.2)^1$, we have

$$\text{(2.6)} \qquad \bar{\rho}(\tau; x, t)\frac{d}{d\tau}\bar{v}(\tau; x, t) = \mu \overline{v_{xx}}(\tau; x, t), \qquad (0 \leqq \tau \leqq t \leqq T).$$

We remark here that

$$\text{(2.7)} \qquad \bar{\rho}(\tau; x, t)\bar{x}_x(\tau; x, t) = \rho_0(\bar{x}(0; \bar{x}(\tau; x, t), \tau)$$
$$\times e^{-\int_0^\tau \overline{v_x}(\tau'; \bar{x}(\tau; x, t), \tau)d\tau'} \times e^{-\int_0^t \overline{v_x}(\tau'; x, t)d\tau'}$$
$$= \rho_0(x(0; x, t))e^{-\int_0^t \overline{v_x}(\tau'; x, t)d\tau'} = \rho(x, t).$$

Therefore, by (2.6) and (2.7) we obtain a result that

$$(2.8)\qquad \begin{cases} \rho(x,t)\dfrac{d}{d\tau}\bar{v}(\tau;x,t)=\bar{\rho}(\tau;x,t)\bar{x}_x(\tau;x,t) \\ \times\dfrac{d}{d\tau}\bar{v}(\tau;x,t)=\mu\overline{v_{xx}}(\tau;x,t)\bar{x}_x(\tau;x,t). \end{cases}$$

Hence, by integrating both sides of (2.8) over $[0, t]$ we have:

Lemma 2.1. *If* (v, ρ) *is a solution of* (1.2) *in* $H_T^{2+\alpha}\times B_T^1$ *satisfying the initial condition* (2.1), *then the following equality holds, i.e.,*

$$(2.9)\qquad \mu\int_0^t\overline{v_{xx}}(\tau;x,t)\bar{x}_x(\tau;x,t)d\tau$$
$$=\rho(x,t)\{v(x,t)-v_0(x_0(x,t))\},$$

where

$$(2.10)\qquad x_0(x,t)\equiv\bar{x}(0;x,t).$$

§3. The Uniqueness of the Solution.

Now, let us direct ourselves towards the problem of uniqueness concerning the system (1.2) of differential equations. We assume that there exist two solutions (v, ρ_v) and (w, ρ_w) of (1.2) in $H_T^{2+\alpha}\times B_T^1$ satisfying one and the same initial condition (2.1). (N. B.: ρ is uniquely determined by $v\in H_T^{2+\alpha}$ under the condition (2.1).) Then, the following equalities hold:

$$(3.1)^1\qquad \begin{cases} v_t=\dfrac{\mu}{\rho_v}v_{xx}-vv_x, \quad (\rho_v)_t+(\rho_v v)_x=0, \\ v(x,0)=v_0, \quad \rho_v(x,0)=\rho_0, \end{cases}$$

$$(3.1)^2\qquad \begin{cases} w_t=\dfrac{\mu}{\rho_w}w_{xx}-ww_x, \quad (\rho_w)_t+(\rho_w w)_x=0, \\ w(x,0)=v_0, \quad \rho_w(x,0)=\rho_0, \qquad (0\leqq t\leqq T,\ x\in R^1). \end{cases}$$

The difference $v-w$ satisfies

$$(3.2)\qquad \begin{cases} (v-w)_t=\dfrac{\mu}{\rho_v}(v-w)_{xx}+\left\{\left(\dfrac{\mu}{\rho_v}-\dfrac{\mu}{\rho_w}\right)w_{xx}-v(v-w)_x\right. \\ \left.-(v-w)\cdot w_x\right\}, \qquad (v-w)(x,0)=0. \end{cases}$$

We note here that $v-w\times\in H_T^{2+\alpha}$ and that $\frac{\mu}{\rho_v}$ and $\frac{\mu}{\rho_w}\in H_T^{\alpha}$. Since it is known that the bounded solution of a linear parabolic equation is unique, by use of the fundamental solution $\Gamma(x, t; y, \tau; \frac{\mu}{\rho_v})$ of the linear parabolic equation

$$(3.3)\qquad \frac{\partial}{\partial t}W(x,t)=\frac{\mu}{\rho_v}\frac{\partial^2}{\partial x^2}W(x,t),$$

$v-w$ can be expressed as follows:

$$(3.4)\qquad \begin{cases} (v-w)(x,t)=\displaystyle\int_0^t d\tau\int_{R^1}\Gamma\left(x,t;y,\tau;\frac{\mu}{\rho_v}\right) \\ \times\left\{\left(\dfrac{\mu}{\rho_v}-\dfrac{\mu}{\rho_w}\right)(y,\tau)w_{yy}-v(v-w)_y-(v-w)\cdot w_y\right\}dy, \\ (N.B.: \|\{\cdots\}\|_T^{(\alpha)}<+\infty). \end{cases}$$

Lemma 3.1. *The norm* $\|v-w\|_{T_0}^{(1)}$ $(0\leqq T_0\leqq T)$ *is estimated from above in such a way that*

$$(3.5)\qquad \|v-w\|_{T_0}^{(0)}\leqq C_1\left(T_0, \left|\frac{\rho_v}{\mu}\right|_T^{(0)}+\left\|\frac{\mu}{\rho_v}\right\|_T^{(\alpha)}\right)|\tilde{N}(\xi,\tau)|_{T_0}^{(0)},$$

$$(\tilde{N}(\xi,\tau)=\{\cdots\} \text{ in } (3.4)),$$

where $C_1(T_0,\ldots)$ *is monotonically increasing in both arguments and decreases monotonically to* 0 *as* $T_0\searrow 0$.

Proof. Calculating in the same way, for the case that the dimension of x is 1, as in [8], we have

$$\left|D_x^m\Gamma\left(x,t;\xi,\tau;\frac{\mu}{\rho_v}\right)\right|\leqq C_0^{(m)}\left(T_0, \left|\frac{\rho_v}{\mu}\right|_T^{(0)}+\left\|\frac{\mu}{\rho_v}\right\|_T^{(\alpha)}\right)$$

(3.6) $$\times(t-\tau)^{-\frac{1+m}{2}}\exp\left\{-d(m)\frac{|x-\xi|^2}{t-\tau}\right\},\qquad (m=0,\,1,\,2),$$

$$(T\geqq T_0\geqq t>\tau\geqq 0),$$

where $d(m)(m=0, 1, 2)$ have the form, respectively,

(3.7) $$\left\{ d(m)=\frac{A(m)}{\left|\frac{\mu}{\rho_v}\right|_T^{(0)}},\ (N.B.:\ (|\rho_v|_T^{(0)})^{-1}\leqq\rho_v^{-1}\leqq|\rho_v^{-1}|_T^{(0)},\ A(m)\text{'s are positive constants depending only on m}). \right.$$

Thus, for $m=0, 1$, it holds that

(3.8) $$|D_x^m(v-w)|_{T_0}^{(0)}\leqq C_0^{(m)}\left(T_0,\ \left|\frac{\rho_v}{\mu}\right|_T^{(0)}+\left\|\frac{\mu}{\rho_v}\right\|_T^{(\alpha)}\right)|\tilde{N}|_{T_0}^{(0)}$$

$$\times\left|\int_0^t d\tau\int_{R^1}(t-\tau)^{-\frac{1+m}{2}}e^{-d(m)\frac{|x-\xi|^2}{t-\tau}}d\xi\right|_{T_0}^{(0)}$$

$$=C_0^{(m)}(T_0,\dots)|\tilde{N}|_{T_0}^{(0)}\left(\frac{\pi}{d(m)}\right)^{\frac{1}{2}}\bullet\left|\int_0^t(t-\tau)^{-\frac{2}{m}}d\tau\right|_{T_0}^{(0)}$$

$$=C_0^{(m)}(T_0,\dots)|\tilde{N}|_{T_0}^{(0)}\left(\frac{\pi}{d(m)}\right)^{\frac{1}{2}}2^m T_0^{1-\frac{m}{2}}$$

$$='C_0^{(m)}\left(T_0,\ \left|\frac{\mu}{\rho_v}\right|_T^{(0)}+\left\|\frac{\mu}{\rho_v}\right\|_T^{(\alpha)}\right)|\tilde{N}|_T^{(0)},$$

(3.8)′ $$('C_0^{(m)}\equiv C_0^{(m)}\left(\frac{\pi}{d(m)}\right)^{\frac{1}{2}}\cdot 2^m\cdot T_0^{-1(m/2)}).$$

Therefore, we obtain

(3.5) $$\|v-w\|_{T_0}^{(1)}\leqq('C_0^{(0)}+'C_0^{(1)})|\tilde{N}|_{T_0}^{(0)}$$

$$=C_1\left(T_0,\ \left|\frac{\rho_v}{\mu}\right|_T^{(0)}+\left\|\frac{\mu}{\rho_v}\right\|_T^{(\alpha)}\cdot|\tilde{N}|_{T_0}^{(0)},\qquad (C_1\equiv'C_0^{(0)}+'C_1^{(0)}).\right.$$

Q. E. D.

Lemma 3.2. *The following inequality holds;*

(3.9) $$\left|\frac{1}{\rho_v}-\frac{1}{\rho_w}\right|_{T_0}^{(0)} \leqq C_2[T_0; v, w]\cdot\|v-w\|_{T_0}^{(1)},$$

where $C_2[T_0]\searrow 0$ *as* $T_0\searrow 0$.

Proof. Firstly, we have an equality

(3.10) $$\begin{cases} \dfrac{1}{\rho_v}-\dfrac{1}{\rho_w}=\dfrac{\rho_w-\rho_v}{\rho_v\cdot\rho_w}=\dfrac{1}{\rho_v\cdot\rho_w}[\{\rho_0(x_0(x, t); w)) \\ -\rho_0(x_0(x, t; v))\}\cdot e^{-\int_0^t \overline{w_x}(\tau; x, t)d\tau}+\rho_0(x_0(x, t; v)) \\ \times\{e^{-\int_0^t \overline{w_x}(\tau; x, t)d\tau}-e^{-\int_0^t \overline{v_x}(\tau; x, t)d\tau}\}]. \end{cases}$$

Therefore, we have the inequality

(3.11) $$\begin{cases} \left|\left(\dfrac{1}{\rho_v}-\dfrac{1}{\rho_w}\right)(x, t)\right| \leqq (\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}\cdot e^{T_0(|w_x|_T^{(0)}+|v_x|_T^{(0)})} \\ \times|x_0(x, t; w)-x_0(x, t; v)|+(\bar{\rho}_0)^{-1} \\ \times e^{2T_0(|w_x|_T^{(0)}+|v_x|_T^{(0)})}\cdot\left|\displaystyle\int_0^t(\overline{w_x}(\tau; x, t)-\overline{v_x}(\tau; x, t))d\tau\right|, \\ (0\leqq t\leqq T_0\leqq T,\ \rho_0'=\dfrac{d}{dx}\rho_0(x)). \end{cases}$$

Now, we note that

(3.12) $$\begin{cases} \dfrac{d}{d\tau}\bar{x}(\tau; x, t; v)-\bar{x}(\tau; x, t; w)=v(\bar{x}(\tau; x, t; v)) \\ -w(\bar{x}(\tau; x, t; w))=v(\bar{x}(\tau; x, t; v)-v(\bar{x}(\tau; x, t; w)) \\ +\{v(\bar{x}(\tau; x, t; w))-w(\bar{x}(\tau; x, t; w))\}. \end{cases}$$

Hence, it follows that

(3.13) $$|x_0(x, t; v)-x_0(x, t; w)|\leqq|v-w|_{T_0}^{(0)}(e^{T_0|v_x|_{T_0}^{(0)}}-1).$$

In the next place, we have an estimation that

$$(3.14)\quad \begin{cases} \left|\int_0^t \{\overline{w_x}(\tau;\, x,\, t) - \bar{v}_x(\tau;\, x,\, t)\} d\tau\right| \\ \leqq \left|\int_0^t \{w_x(\bar{x}(\tau;\, x,\, t;\, v),\, \tau) - v_x(\bar{x}(\tau;\, x,\, t;\, v),\, \tau)\} d\tau\right| \\ \quad + \left|\int_0^t \{w_x(\bar{x}(\tau;\, x,\, t;\, v),\, \tau) - v_x(\bar{x}(\tau;\, x,\, t;\, v),\, \tau)\} d\tau\right| \\ \leqq |w_{xx}|_T^{(0)} \cdot |v-w|_{T_0}^{(0)} (e^{T_0 |v_x|_T^{(0)}} - 1) + T_0 |(v-w)_x|_{T_0}^{(0)}. \end{cases}$$

Thus, by (3.11), (3.13), and (3.14), we have finally

$$(3.15)\quad \begin{cases} \left|\dfrac{1}{\rho_v} - \dfrac{1}{\rho_w}\right|_T^{(0)} \leqq [\{(\bar{\rho}_0)^{-2} |\rho_0'|^{(0)} e^{T_0(|v_x|_T^{(0)} + |w_x|_T^{(0)})} \\ \quad + (\bar{\rho}_0)^{-1} e^{2T_0(|v_x|_T^{(0)} + |w_x|_T^{(0)})} \cdot |w_{xx}|_T^{(0)}\} \\ \quad \times (e^{T_0 |v_x|_{T_0}^{(0)}} - 1) + (\bar{\rho}_0)^{-1} e^{\cdot 2T_0(|v_x|_T^{(0)} + |w_x|_T^{(0)})} T_0]_1 \\ \quad \times \|v-w\|_{T_0}^{(1)} = C_2[T_0;\, v,\, w] \|v-w\|_{T_0}^{(1)}, \end{cases}$$

where $C_2[T_0; \ldots]$ is equal to $[\ldots]_1$ in the right-hand side of the inequality in (3.15) and has the above-mentioned property.

Q.E.D.

Theorem 3.1. *If* (v, ρ) *and* $(w, \rho^*) \in H_T^{2+\alpha} \times B_T^1$ *satisfy* (1.2) *and* (2.1), *then* $(v, \rho) = (w, \rho^*)$, $(\rho = \rho^* = \rho_v)$.

Proof. We note here that

$$(3.16)\quad \begin{cases} |\tilde{N}(x, t)|_{T_0}^{(0)} \leqq \mu |w_{xx}|_T^{(0)} \cdot \left|\dfrac{1}{\rho_v} - \dfrac{1}{\rho_w}\right|_{T_0}^{(0)} \\ \quad + |v|_T^{(0)} |(v-w)_x|_{T_0}^{(0)} + |w_x|_T^{(0)} |v-w|_{T_0}^{(0)}. \end{cases}$$

Therefore, by the lemmas 3.1 and 3.2, we have

$$(3.17)\quad \begin{aligned} &\|v-w\|_{T_0}^{(1)} \leqq C_1(T_0) |\tilde{N}|_{T_0}^{(0)} \leqq C_1(T_0)\{\mu |w_{xx}|_T^{(0)} \|v-w\|_{T_0}^{(1)} \\ &\quad \times C_2[T_0;\, \ldots] + (|v|_T^{(0)} + |x_x|_T^{(0)})\} \|v-w\|_{T_0}^{(1)} \\ &= C_3[T_0;\, v,\, w] \|v-w\|_{T_0}^{(1)}, \end{aligned}$$

where C_3 is defined by

(3.17)′ $$C_3[T_0; \dots] \equiv C_1(T_0)C_2[T_0: \dots]\cdot\mu|w_{xx}|_T^{(0)} + C_1(T_0)(|v|_T^{(0)} + |w_x|_T^{(0)})$$

and $C_3[T_0]\searrow 0$ as $T_0\searrow 0$. Since it holds that, for a sufficiently small $T_1 \in (0, T]$,

(3.18) $$0 \in C_3[T_1; \dots] < 1,$$

we obtain

(3.19) $$\begin{cases} 0 \leqq (1 - C_3[T_1])\cdot \|v - w\|_{T_1}^{(1)} \leqq 0, \\ 0 < 1 - C_3[T_1] \leqq 1. \end{cases}$$

Hence, it follows that

(3.20) $$\|v - w\|_{T_1}^{(1)} = 0,$$

that is to say,

(3.20)′ $$v(x, t) = w(x, t), \qquad (0 \leqq t \leqq T_1 \leqq T).$$

According to the assumption of the theorem, we can continue this procedure again by starting at $t = T_1$. As a result we have

(3.21) $$\|v - w\|_{(T_1+T_1^*)\wedge T}^{(1)} = 0, \qquad (a \wedge b = \min[a, b]),$$

where T_1^* is a number such that (3.18) holds for C_3 as $(\bar{\rho}_0)^{-1}$ and $|\rho_0'|^{(0)}$ are replaced by $|(\rho_v)^{-1}|_T^{(0)} + |(\rho_w)^{-1}|_T^{(0)}$ and $|(\rho_v)_x|_T^{(0)} + |(\rho_w)_x|_T^{(0)}$, respectively. In this way, after a finite number of repetitions of this procedure, it is shown that the assertion of the theorem holds. (We remark that T_1^* can be taken uniformly with respect to the repetitions of the above-mentioned procedure.) Q.E.D.

§4. The Existence of a Temporally Local Solution.

In this section, we demonstrate the existence of a temporally local solution of (1.2) satisfying the initial condition (2.1), in a way analogous to that in (8). The method of demonstration is much simpler than in (8), so that we restrict ourselves to giving just an outline of the demonstration.

Let us take an arbitrary function $v(x, t)\in\hat{H}_T^{2+\alpha}$ satisfying (2.1). Then, it is obvious that $\rho_v\in B_T^1$, because it satisfies (2.1). Now we define a non-linear mapping G_T from

$$S_T=\{v: v\in\hat{H}_T^{2+\alpha},\ v(x, 0)=v_0(x)\in H^{2+\alpha}\} \tag{4.1}$$

into itself in the following way:

$$\tilde{v}(x, t)=(G_Tv)(x, t)=v_0(x)+\int_0^t d\tau\int_{R^1}\Gamma\Big(x, t; y, \tau; \frac{\mu}{\rho_v}\Big)\cdot\Big\{\frac{\mu}{\rho_v(y, \tau)}v_0''(y)-v(y, \tau)\frac{\partial}{\partial y}v(y, \tau)\Big\}dy, \tag{4.2}$$

$$(0\leqq t\leqq T;\ \text{as for } \Gamma,\ \text{cf. §3}).$$

We note that $S_T\neq\phi$, because $v(x, t)=v_0(x)\in S_T$. Referring ourselves to §5 of [8], we have

$$\begin{cases} <v>_T^{(2,\alpha)}\leqq C_4\Big(T, \Big|\Big|\Big|\frac{\mu}{\rho_v}\Big|\Big|\Big|_T\Big)\cdot\Big\|\frac{\mu}{\rho_v}v_0''-v\cdot v_x\Big\|_T^{(\alpha)}+\|v_0\|^{(2)}, \\ '<v>_T^{(2,\alpha)}\leqq C_5\Big(T, \Big|\Big|\Big|\frac{\mu}{\rho_v}\Big|\Big|\Big|_T\Big)\cdot\Big\|\frac{\mu}{\rho_v}v_0''-v\cdot v_x\Big\|_T^{(\alpha)}+|v_0''|^{(\alpha)}, \end{cases} \tag{4.3}$$

$$\Big(\Big|\Big|\Big|\frac{\mu}{\rho_v}\Big|\Big|\Big|_T=\Big|\frac{\rho_v}{\mu}\Big|_T^{(0)}+\Big\|\frac{\mu}{\rho_v}\Big\|_T^{(\alpha)};\ \text{we note that } |\rho^{-1}|_T^{(0)}\leqq\rho^{-1}\leqq(|\rho|_T^{(0)})^{-1}\Big), \tag{4.3$'$}$$

where C_4 and C_5 are monotonically increasing in each argument, $C_4(T, \cdot)\searrow 0$ as $T\searrow 0$, and $C_5(T, \cdot)\searrow a$ certain positive constant independent of $|||\mu/\rho_v|||_T$ as $T\searrow 0$. As for $|||\mu/\rho_v|||_T$, it holds that

$$\begin{aligned}\Big|\Big|\Big|\frac{\mu}{\rho_v}\Big|\Big|\Big|_T&=\Big|\frac{\rho_v}{\mu}\Big|_T^{(0)}+\Big|\frac{\mu}{\rho_v}\Big|_T^{(0)}+\Big|\frac{\mu}{\rho_v}\Big|_{x,T}^{(\alpha)}+\Big|\frac{\mu}{\rho_v}\Big|_{t,T}^{(\alpha/2)} \\ &\leqq\frac{\rho_0}{\mu}e^{T\cdot|v_x|_T^{(0)}}+\mu(\bar{\rho}_0)^{-1}e^{T|v_x|_T^{(0)}}+2\mu[\{(\bar{\rho}_0)^{-2}\cdot|\rho_0'|^{(0)} \\ &\quad+(\bar{\rho}_0)^{-1}e^{2T|v_x|_T^{(0)}}\cdot T|v_{xx}|_T^{(0)}\}+(\bar{\rho}_0)^{-1}e^{T|v_x|_T^{(0)}}]\end{aligned} \tag{4.4}$$

$$+2\mu[(\rho_0)^{-1}e^{T|v_x|_T^{(0)}}\cdot(1+|v_x|_T^{(0)})+|v|_T^{(0)}\{\bar\rho_0)^{-2}$$

$$\times|\rho_0'|^{(0)}+(\bar\rho_0)^{-1}e^{2T|v_x|_T^{(0)}}\cdot T|v_{xx}|_T^{(0)}\}],$$

since, for example, we have

$$(4.5)\quad \left|\frac{\mu}{\rho_v}(x,t)-\frac{\mu}{\rho_v}(x',t)\leqq\right|\frac{\mu}{\rho_v}(x,t)-\frac{\mu}{\rho_v}(x',t)\Big|^{\alpha}$$

$$\times\left(2\cdot\left|\frac{\mu}{\rho_v}\right|_T^{(0)}\right)^{1-\alpha}\leqq\mu\left(\left|\left(\frac{\mu}{\rho_v}\right)_x\right|_T^{(0)}|x-x'|\right)^{\alpha}\left(2\left|\frac{\mu}{\rho_v}\right|_T^{(0)}\right)^{1-\alpha}$$

$$\leqq2\mu\left\{\left|\left(\frac{\mu}{\rho_v}\right)_x\right|_T^{(0)}+\left|\frac{\mu}{\rho_v}\right|_T^{(0)}\right\}|x-x'|^{\alpha}.$$

In (4.5) we have used the relation for a and $b\geqq0$ and for $\gamma\in[0,1]$

$$(4.6)\qquad a^{\gamma}\cdot b^{1-\gamma}\leqq\max[a,b]\leqq a+b.$$

We note also that

$$(4.7)\quad \begin{cases}\left(\frac{1}{\rho}\right)_x=-\frac{\rho_x}{\rho^2}=-\frac{\rho_0'}{\rho_0^2}+\{\rho_0\cdot\bar{x}_x(0;x,t)\}^{-1}\\ \qquad\times\int_0^t\overline{v_{xx}}(\tau;x,t)\bar{x}_x(\tau;x,t)d\tau,\\ \left(\frac{1}{\rho}\right)_t=-\frac{\rho_t}{\rho^2}=\frac{\rho v_x+v\rho_x}{\rho^2}=\frac{v_x}{\rho}+\frac{v\rho_x}{\rho^2}.\end{cases}$$

Now, we take an arbitrary constant M_0 such that

$$(4.8)\qquad \|v_0\|^{(2)}<M_0<+\infty,$$

and consider functions $C_i^*(T;M_0)(i=4,5)$ in T, such that

$$(4.9)\quad \begin{cases}C_i^*(T;M_0)\equiv C_i(T,A(T,M_0))\{|v_0''|^{(\alpha)}A(T,M_0)+M_0^2\},\\ \qquad(i=4,5),\end{cases}$$

where

$$(4.10)\quad A(T,M_0)\equiv\left\{\frac{\bar{\bar\rho}_0}{\mu}+\mu(5+2M_0)(\bar\rho_0)^{-1}\right\}e^{M_0T}+2\mu(1+M_0)$$

$$\times\{(\bar\rho_0)^{-2}\cdot|\rho_0'|^{(0)}+(\bar\rho_0)^{-1}M_0\cdot e^{2M_0T}T\}.$$

By the property of C_4^*, for a sufficiently small $T_1 \in (0, T]$,

$$C_4^*(T_1;\ M_0) + \|v_0\|^{(2)} \leqq M_0. \tag{4.11}$$

Thus, if we choose T from the beginning in such a way that $T = T_1$, and if we define $\hat{S}_T$ by

$$\hat{S}_T = \{v:\ <v>_T^{(2,\alpha)} \leqq M_0,\ v \in S_T\} (\subset S_T), \tag{4.12}$$

then it is obvious that

$$G_T \hat{S}_T \subset \hat{S}_T \subset S_T. \tag{4.13}$$

For an arbitrary $v \in \hat{S}_T$,, we have

$$<v>_T^{(2,\alpha)} \leqq C_5^*(T;\ M_0) + |v_0''|^{(\alpha)} < +\infty. \tag{4.14}$$

Furthermore, defining S_T^* by

$$\begin{aligned} S_T^* \equiv \{v:\ &<v>_T^{(2,\alpha)} \leqq M_0,\ '<v>_T^{(2,\alpha)} \leqq C_5^*(T;\ M_0) + |v_0''|^{(\alpha)}, \\ &v \in S_T\} (\subset \hat{S}_T \subset S_T \subset \hat{H}_T^{2+\alpha}), \qquad (S_T^* \neq \phi), \end{aligned} \tag{4.15}$$

we have likewise an inclusion relation

$$G_T S_T^* \subset S_T^*. \tag{4.16}$$

As is easily seen, S_T^* is a convex set. Next, if we consider S_T^* as a subset of a Fréchet space $\dot{H}_T^{2,\alpha\beta} (\beta \in (0, 1))$ defined by a countable system of seminorms

$$<v>_{N,T}^{(2,\alpha\beta)}, \qquad (N = 1, 2, \ldots), \tag{4.17}$$

where the suffix 'N, T' indicates that the supremum is considered on $\{(x, t):\ x \in R^1,\ t \in [0, T],\ |x| \leqq N + M_0(T-t)\}$ instead of $R^1 \times [0, T]$, then it is shown that S_T^* is a compact subset of $\dot{H}_T^{2,\alpha\beta}$. Finally, it is demonstrated in the same way as in §5 of [8] that G_T is a continuous operator from S_T^* as a subset of $\dot{H}_T^{2,\alpha\beta}$ into itself. Thus, we can apply Tikhonov's fixed point theorem to the result obtained above. Hence, we have the proposition that there exists at least one v satisfying

(4.18) $$\hat{v}(x, t)=(G_T v)(x, t)=v(x, t)\in S_T^*\subset \hat{H}_T^{2+\alpha}.$$

In virtue of Theorem 3.1 and the fact that $\rho_v\in B_T^1$, we have:

Theorem 4.1. *For some* $T\in(0, +\infty)$, *there exists a unique solution of* (1.2) *in* $H_T^{2+\alpha}\times B_T^1$, *satisfying the initial condition* (2.1).

Proof. We have only to remark that, for $v\in\hat{H}_T^{2+\alpha}$, $\tilde{v}=G_T\cdot v\in H_T^{2+\alpha}$.
Q.E.D.

§5. Preliminaries for the Temporally Glodal Problem.

The main purpose of this section is to demonstrate some preparatory lemmas necessary for obtaining a priori estimates for v_x, etc. The well-known following lemma plays an important role here (cf. [4], [11], etc.).

Lemma 5.1. (i) *If a continuous function* $u(x, t)$ *defined in* $R^1\times[0, T]$ *is bounded in modulus by* $A\cdot\exp\{Bx^2\}$ *for certain non-negative constants A and B, and, moreover, if it satisfies regularly the equation*

(5.1) $$\begin{cases} \dfrac{\partial u}{\partial t}=a(x, t)\dfrac{\partial^2 u}{\partial x^2}+b(x, t)\dfrac{\partial u}{\partial x}+c(x, t)u+f(x, t), \\ \qquad\qquad (0<t\leqq T), \end{cases}$$

where $a(x, t)$, $b(x, t)$, *and* $f(x, t)$ *are continuous in* $R^1\times[0, T]$ *and satisfy*

(5.1)′ $$\begin{cases} 0\leqq a(x, t)\leqq |a|_T^{(0)}<+\infty,\ |b|_T^{(0)}<+\infty,\ |c|_T^{(0)}<+\infty, \\ |f|_T^{(0)}<+\infty, \end{cases}$$

then it satisfies (5.1) *uniquely in the class of functions having the above-mentioned property.* (ii). *Especially, if* $B=0$, *then, for* $u(x, t)$ *satisfying* (5.1), *it holds that*

(5.2) $$|u|_T^{(0)}\leqq\{|u(x, 0)|^{(0)}+T\cdot|f|_T^{(0)}\}\cdot e^{T\cdot|c|_T^{(0)}}.$$

Directly, by the above lemma, we have:

Lemma 5.2. *If* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *satisfies* (1.2) *and* (2.1), *then it holds that*

$$|v|_T^{(0)}\leqq|v_0|^{(0)}. \tag{5.3}$$

Proof. Put in (5.1)

$$a(x, t)=\frac{\mu}{\rho},\ b(x, t)=-v,\ c(x, t)=0. \qquad \text{Q.E.D.}$$

If $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ satisfies (1.2) and (2.1), then, by the lemmas 2.1 and 5.2, we have

$$\begin{aligned}\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}&=\left|\frac{\mu}{\rho}\right|_T^{(0)}+\left|\frac{\mu}{\rho}\right|_{x,T}^{(\alpha)}+\left|\frac{\mu}{\rho}\right|_{t,T}^{(\alpha/2)}\leqq(\bar\rho_0)^{-1}\\ &\times|\bar x_x(0;x,t)|_T^{(0)}+2^{1-\alpha}\mu\{(\bar\rho_0)^{-1}|\bar x_x(0;x,t)|_T^{(0)}\\ &+(\bar\rho_0)^{-2}|\rho_0'|^{(0)}+\frac{1}{\mu}|v-v_0(x_0(x,t))|_T^{(0)}\}+2^{1-\alpha}\mu[(\bar\rho_0)^{-1}\\ &\times|\bar x_x(0;x,t)|_T^{(0)}(1+|v_x|_T^{(0)})+|v|_T^{(0)}\{(\bar\rho_0)^{-2}|\rho_0'|^{(0)}\\ &+\frac{1}{\mu}|v-v_0(x_0(x,t))|_T^{(0)}\}]\leqq(5+2|v_x|_T^{(0)})\mu(\bar\rho_0)^{-1}\\ &\times\exp\left\{\left|\int_0^t\bar v_x(\tau;x,t)d\tau\right|_T^{(0)}\right\}+4|v_0|^{(0)}(1+|v_0|^{(0)})\\ &+2\mu(\bar\rho_0)^{-2}|\rho_0'|^{(0)}(1+|v_0|^{(0)}).\end{aligned} \tag{5.4}$$

From (5.4), we know that, in order to have a priori estimates for $\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}$, we have to obtain beforehand those for $|v_x|_T^{(0)}$. Hereafter in §5 and §6, we shall endeavor to have an a priori estimate for $|v_x|_T^{(0)}$.

Now, we perform, under the same assumption as in Lemma 2.5, a co-ordinates transformation such that

$$x_0=x_0(x, t)\equiv\bar x(0; x, t), \qquad t_0=t_0(x, t)\equiv t, \tag{5.5}$$

where we remark that $\bar x(\tau; x, t)$ is the solution curve of the characteristic equation for $(1.2)^2$ as a linear equation in ρ. Since $v\in H_T^{2+\alpha}$, this transformation is a one-to-one mapping from $R^1 \times [0, T]$ onto itself. We call (x_0, t_0) the characteristic co-ordinates for (x, t). It is obvious that x and t are inversely expressed by

$$x = x(x_0, t_0) \equiv \bar{x}(t_0; x_0, 0), \qquad t = t(x_0, t_0) \equiv t_0 . \tag{5.5)'$$

If we define $\hat{v}(x_0, t_0)$ by

$$\hat{v}(x_0, t_0) = v(x(x_0, t_0), t = t_0), \tag{5.6}$$

then we see that v is expressed by use of $\hat{v}$ in the form

$$v(x, t) = \hat{v}(x_0(x, t), t_0 = t). \tag{5.6)'$$

From the relation

$$\begin{pmatrix} \frac{\partial x_0}{\partial x} & \frac{\partial x_0}{\partial t} \\ \frac{\partial t_0}{\partial x} = 0 & \frac{\partial t_0}{\partial t} = 1 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial t_0} = \hat{v}(x_0, t_0) \\ \frac{\partial t}{\partial x_0} = 0 & \frac{\partial t}{\partial t_0} = 1 \end{pmatrix} \tag{5.7}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where we remark that

$$\begin{cases} \frac{\partial x}{\partial t_0} = \frac{\partial}{\partial t_0} \bar{x}(t_0; x_0, 0) = v(\bar{x}(t_0; x_0, 0), t_0) \\ \quad = v(\bar{x}(x_0, t_0), t_0) = \hat{v}(x_0, t_0), \end{cases}$$

it follows that

$$\frac{\partial x_0}{\partial x} = \left(\frac{\partial x}{\partial x_0}\right)^{-1}, \quad \frac{\partial x_0}{\partial t} = -v \cdot \left(\frac{\partial x}{\partial x_0}\right)^{-1}. \tag{5.8}$$

Lemma 5.3.

$$\frac{\partial x_0}{\partial x} = \exp\left\{-\int_0^t \bar{v}_x(\tau; x, t) d\tau\right\} = \left(\frac{\partial x}{\partial x_0}\right)^{-1} \tag{5.9}$$

$$=\left\{1+\int_0^{t=t_0}\hat{v}_{x_0}(x_0,\tau)d\tau\right\}^{-1}.$$

Proof.

$$\frac{d}{d\tau}\bar{x}(\tau; x_0, 0)=v(\bar{x}(\tau; x_0, 0), \tau)=\hat{v}(x_0, \tau).$$

Hence, we have

(5.10) $$\frac{d}{d\tau}\bar{x}_{x_0}(\tau; x_0, 0)=\hat{v}_{x_0}(x_0, \tau),$$

where, as for the differentiability of $\hat{v}$ in x_0, we refer to (5.6) and the theorems on inverse functions in general. Thus, the following equality is obtained.

(5.11) $$\begin{cases}\dfrac{\partial x}{\partial x_0}=\bar{x}_{x_0}=1+\displaystyle\int_0^{t_0}\hat{v}_{x_0}(x_0,\tau)d\tau, \qquad (N.B.: x_{x_0}(0; x_0, \\ 0)=1).\end{cases}$$

Q.E.D.

For simplicity, we put

(5.12) $$\omega(x_0, t_0)=\int_0^{t_0}\hat{v}_{x_0}(x_0,\tau)d\tau.$$

By theassumption that $v\in H_T^{2+\alpha}$,

(5.13) $$0<e^{-T\cdot|v_x|_T^{(0)}}\leqq\left(\frac{\partial x}{\partial x_0}\right)^{-1}=\frac{1}{1+\omega(x_0,t_0)}\leqq e^{+T\cdot|v_x|_T^{(0)}}.$$

Lemma 5.4. *Under the same assumption as in Lemma 5.2, it holds that*

(5.14) $$\hat{v}(x_0, t_0)\in H_T^{2+\alpha}.$$

Proof. By (5.9) and the assumption that $v\in H_T^{2+\alpha}$, we can easily show that $(x_0)_{xx}$ and $x_{x_0x_0}$ exist and are expressed as follows:

(5.15) $$(x_0)_{xx}=-\bar{x}_w(0; x, t)\cdot\int_0^t\overline{v_{xx}}(\tau; x, t)\bar{x}_x(\tau; x, t)d\tau,$$

$$x_{x_0x_0}=\frac{\partial}{\partial x_0}\left\{\frac{\partial x_0}{\partial x}(x(x_0, t_0), t=t_0)\right\}^{-1}=+\left(\frac{\partial x_0}{\partial x}\right)^{-2}$$

$$\times\frac{\partial^2 x_0}{\partial x^2}\frac{\partial x}{\partial x_0}=-\left(\frac{\partial x_0}{\partial x}\right)^{-3}\frac{\partial^2 x_0}{\partial x^2}=-\{\bar{x}_x(0; x, t)\}^{-2}$$

$$\times\int_0^t \overline{v_{xx}}(\tau; x, t)\bar{x}_x(\tau; x, t)d\tau.$$

Furthermore, $\hat{v}_{x_0}$, $\hat{v}_{x_0x_0}$, and $\hat{v}_{t_0}$ exist, being expressed in the forms

(5.16) $$\hat{v}_{x_0}(x_0, t_0)=\frac{\partial}{\partial x_0}v(x(x_0, t_0), t=t_0)=v_x\frac{\partial x}{\partial x_0},$$

$$\hat{v}_{x_0x_0}(x_0, t_0)=\frac{\partial}{\partial x_0}\cdot v_x(x(x_0, t_0), t=t_0)\frac{\partial x}{\partial x_0}$$

$$=v_{xx}\cdot\left(\frac{\partial x}{\partial x_0}\right)^2+v_x\cdot x_{x_0x_0},$$

$$\hat{v}_{t_0}(x_0, t_0)=\frac{\partial}{\partial t_0}v(x(x_0, t_0), t=t_0)=v_x\cdot\frac{\partial x}{\partial t_0}+v_t.$$

By making use of (5.7), (5.8), (5.9), (5.15), (5.16), and the assumption that $\in H_T^{2+\alpha}$, it is easily shown that $\hat{v}\in H_T^{2+\alpha}$. Q.E.D.

Let us describe (1.2) in the characteristic co-ordinates under the above-mentioned assumption. In a way analogous to (5.16), we have

(5.17) $$v_x(x, t)=\frac{\partial}{\partial x}\hat{v}(x_0(x, t), t_0=t)=\hat{v}_{x_0}(x_0, t_0)\frac{\partial x_0}{\partial x}$$

$$=\hat{v}_{x_0}\{1+\omega(x_0, t_0)\}^{-1},\ v_{xx}(x, t)=\frac{\partial}{\partial x}\left(\hat{v}_{x_0}\frac{\partial x_0}{\partial x}\right)$$

$$=\hat{v}_{x_0x_0}\left(\frac{\partial x_0}{\partial x}\right)^2+\hat{v}_{x_0}\frac{\partial^2 x_0}{\partial x^2}.$$

By Lemma 2.1 and (5.15),

(5.18) $$(x_0)_{xx}=-\rho_0(x_0)\{1+\omega(x_0, t_0)\}^{-2}\frac{\hat{v}(x_0, t_0)-v_0(x_0)}{\mu}$$

$$x_{x_0x_0}=(1+\omega)_{x_0}=\frac{1+\omega}{\mu}\rho_0(\hat{v}-v_0).$$

Finally, we have

$$(5.19)\qquad \hat{v}_{t_0}=v_t+vv_x=\frac{\mu}{\rho}v_{xx}=\frac{\mu(1+\omega)}{\rho_0(x_0)}\Big\{\hat{v}_{x_0x_0}\cdot(1+\omega)^{-2}$$

$$+\hat{v}_{x_0}\cdot\frac{-\rho_0}{\mu}(1+\omega)^{-2}(\hat{v}-v_0)\Big\}=\frac{\mu}{\rho_0(x_0)}\frac{1}{1+\omega(x_0,t_0)}$$

$$\times\hat{v}_{x_0x_0}-\frac{\hat{v}-v_0}{1+\omega}\hat{v}_{x_0},\ \rho(x,t)=\rho_0(x_0(x,t))\Big(\frac{\partial x}{\partial x_0}\Big)^{-1}$$

$$=\rho_0(x_0)(1+\omega)^{-1}=\hat{\rho}(x_0,t_0),$$

where $\hat{\rho}$ is defined analogously to $\hat{v}$. By the second relation in (5.18), it holds that

$$(5.20)\qquad \int_a^{x_0}\frac{\omega_{x_0}}{1+\omega}dx_0=\int_a^{x_0}\frac{\rho_0}{\mu}(\hat{v}-v_0)dx_0,$$

where a is an arbitrary fixed constant. Thus, we have

$$(5.20)'\qquad \log(1+\omega(x_0,t_0))-\log(1+\omega(a,t_0))$$

$$=\int_a^{x_0}\frac{\rho_0}{\mu}(\hat{v}-v_0)dx_0.$$

Let us define $\Psi(x_0,t_0)$ by

$$(5.21)\qquad \Psi(x_0,t_0)=\int_a^{x_0}\frac{\rho_0}{\mu}\cdot\hat{v}(x_0,t_0)dx_0.$$

Then, Ψ satisfies

$$(5.22)\qquad \frac{\partial}{\partial t_0}\Psi(x_0,t_0)=\int_a^{x_0}\frac{\rho_0}{\mu}\hat{v}_{t_0}(x_0,t_0)dx_0=\int_a^{x_0}\{(1+\omega)^{-1}$$

$$\times\hat{v}_{x_0x_0}-(1+\omega)^{-2}(1+\omega)_{x_0}\cdot\hat{v}_{x_0}\}dx_0=\frac{\hat{v}_{x_0}(x_0,t_0)}{1+\omega(x_0,t_0)}$$

$$-\frac{\hat{v}_{x_0}(a,t_0)}{1+\omega(a,t_0)}-\int_a^{x_0}\Big\{\frac{\partial}{\partial x_0}(1+\omega)^{-1}\Big\}\hat{v}_{x_0}dx_0$$

$$-\int_a^{x_0}(1+\omega)^{-2}(1+\omega)_{x_0}\cdot\hat{v}_{x_0}dx_0=\frac{\hat{v}_{x_0}(x_0,t_0)}{1+\omega(x_0,t_0)}$$

$$-\frac{\hat{v}_{x_0}(a, t_0)}{1+\omega(a, t_0)}=\frac{1}{1+\omega(x_0, t_0)}\left(\frac{\mu}{\rho_0}\Psi_{x_0}\right)_{x_0}$$

$$-\frac{\hat{v}_{x_0}(a, t_0)}{1+\omega(a, t_0)}.$$

Hence, we have

$$\frac{\partial}{\partial t_0}\{\Psi(x_0, t_0)+\log(1+\omega(a, t_0))\} \tag{5.23}$$

$$=\frac{1}{1+\omega(x_0, t_0)}\frac{\partial}{\partial x_0}\left[\frac{\mu}{\rho_0}\frac{\partial}{\partial x_0}\{\Psi(x_0, t_0)+\log(1+\omega(a, t_0))\}\right].$$

The function $\Psi(x_0, t_0)+\log(1+\omega(a, t_0))$ takes at $t_0=0$ the value

$$\Psi(x_0, 0)+\log(1+\omega(a, 0))=\Psi(x_0, 0)=\int_a^{x_0}\frac{\rho_0}{\mu}v_0 dx_0. \tag{5.24}$$

Lemma 5.5. *If* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *satisfies* (1.2) *and* (2.1) *and, moreover, if* $v_0\in L^1(R^1)$ *and* $\rho_0\in H^{1+\alpha}$, *then we have for* $(1+\omega)^{-1}=(x_0)_x$ *the following a priori estimates*:

$$\exp\left\{-\frac{1}{\mu}\|\rho_0\cdot v_0\|_{L^1(R^1)}\right\}\leqq\frac{1}{1+\omega(x_0, t_0)} \tag{5.25}$$

$$\leqq\exp\left\{\frac{1}{\mu}\|\rho_0\cdot v_0\|_{L^1(R^1)}\right\}.$$

Proof. By Lemma 5.3, it holds that

$$\begin{cases} |\log(1+\omega(x_0, t_0))|\leqq T|v_x|_T^{(0)}, \\ (\bar{\bar{\rho}}_0)^{-1}e^{-T|v_x|_T^{(0)}}\leqq\dfrac{1}{\rho_0(x_0)}\dfrac{1}{1+\omega(x_0, t_0)} \\ \quad\leqq(\bar{\rho}_0)^{-1}e^{T|v_x|_T^{(0)}}. \end{cases} \tag{5.26}$$

Furthermore, by the condition for ρ_0 and by the fact that $\hat{v}\in H_T^{2+\alpha}$, we have

(5.26)′ $$\frac{1}{1+\omega}\frac{\mu}{\rho_0} \text{ and } \frac{\mu}{1+\omega}\frac{\rho_0'}{\rho_0^2}\in H_T^{\alpha}.$$

Next, we have, by Lemma 5.2 and (5.16),

(5.27) $$|\Psi(x_0, t_0)+\log(1+\omega(a, t_0))|\leqq\frac{\bar{\bar{\rho}}_0}{\mu}|v_0|^{(0)}|x-a| + T|v_x|_T^{(0)}\leqq\left(\frac{\bar{\bar{\rho}}_0}{\mu}|v_0|^{(0)}+T|v_x|_T^{(0)}\right)e^{2+|a|}\cdot e^{2x^2}.$$

Therefore, the first assertion of Lemma 5.1 guarantees that $\Psi(x_0, t_0)+\log(1+\omega(a, t_0))$ is to be expressed by utilizing the fundamental solution of the linear parabolic equation

(5.28) $$V_{t_0}=\frac{\mu}{(1+\omega)\rho_0}V_{x_0x_0}+\frac{\mu}{1+\omega}\frac{\rho_0'}{\rho_0^2}V_{x_0}$$

in the following way:

(5.29) $$\Psi(x_0, t_0)+\log(1+\omega(a, t_0))=\int_{R^1}\Gamma^*(x_0, t_0; \xi, 0)\times\Psi(\xi, 0)d\xi=\int_{R^1}\Gamma^*(x_0, t_0; \xi, 0)\times\left\{\int_a^{\xi}\frac{\rho_0(\xi')}{\mu}v_0(\xi')d\xi'\right\}d\xi,$$

where Γ^* is the fundamental solution for (5.28). By the well-known property of Γ^*

(5.30) $$\int_{R^1}\Gamma^*(x_0, t_0; \xi, 0)d\xi=1,$$

we have

(5.31) $$|\Psi(x_0, t_0)+\log(1+\omega(a, t_0))|^{(0)}\leqq\left|\int_a^{x_0}\frac{\rho_0}{\mu}v_0dx_0\right|^{(0)}\leqq\frac{1}{\mu}\cdot\|\rho_0\cdot v_0\|_{L^1(R^1)}.$$

Finally, by (5.30) it holds that

(5.32) $$|\log(1+\omega(x_0, t_0))| = \Big| \Psi(x_0, t_0) + \log(1+\omega(a, t_0)) - \int_a^{x_0} \frac{\rho_0}{\mu} v_0 dx_0 \Big| \leqq \frac{1}{\mu} \cdot \|\rho_0 \cdot v_0\|_{L^1(R^1)}.$$

As a result, we have (5.25). Q.E.D.

§6. A Priori Estimates for $|\hat{v}_{x_0}|_T^{(0)}$ and $|v_x|_T^{(0)}$.

Lemma 6.1. *Under the initial condition* (2.1) *and an additional one*

(6.1) $$\begin{cases} v_0, v_0', v_0'' \in L^1(R^1), \\ \rho_0 \in H^{2+\alpha}, \end{cases}$$

$|\hat{v}_{x_0}|_T^{(0)}$ *is bounded by a constant depending only on the quantities appearing in* (2.1) *and* (6.1) (*but independent of* T).

Proof. The procedure of the demonstration is divided into three steps.

((*1st step*)). Firstly, we note that (5.25) holds by Lemma 5.5. Now, we define $V_\lambda(x_0, t_0)$ by

(6.2) $$V_\lambda(x_0, t_0) \equiv \hat{v}_{x_0}(x_0, t_0)^2 + \lambda \hat{v}(x_0, t_0)^2,$$

where λ is a constant to be determined at a later time. The fact that $\rho_0 \in H^{2+\alpha}$ and $\hat{v} \in H_T^{2+\alpha}$ guarantees us that $D_{t_0}^r D_{x_0}^s \hat{v}$ $(2r+s=3$ or 4; r and s, non-negative integers) exist in $R^1 \times (0, T]$, being continuous there, since, by the fact that

(6.3) $$\frac{\mu}{\rho_0} \frac{1}{1+\omega}, \frac{\hat{v}-v_0}{1+\omega} \in H_T^{2+\alpha}, \text{ (cf. Lemma 2.1)},$$

the well-known theorems on the differentiability of the solution which are based upon the a priori interior estimates can be applied to this case, (cf. [4], etc.). We note here that $\hat{v}_{x_0}$ satisfies the equation

(6.4) $$(\hat{v}_{x_0})_{t_0} = \frac{\mu}{\rho_0} \frac{1}{1+\omega} (\hat{v}_{x_0})_{x_0 x_0} + \Big\{ \Big(\frac{\mu}{\rho_0} \frac{1}{1+\omega} \Big)_{x_0} - \frac{\hat{v}-v_0}{1+\omega} \Big\} (\hat{v}_{x_0})_{x_0} - \Big(\frac{\hat{v}-v_0}{1+\omega} \Big)_{x_0} (\hat{v}_{x_0}).$$

Let $\mathscr{L}$ be defined by

$$\mathscr{L} \equiv -\frac{\mu}{\rho_0(x_0)} \frac{1}{1+\omega(x_0, t_0)} \frac{\partial^2}{\partial x_0^2} + \frac{\hat{v}(x_0, t_0)-v_0}{1+\omega} \frac{\partial}{\partial x_0} + \frac{\partial}{\partial t_0}. \tag{6.5}$$

Then, we have

$$\begin{aligned}\mathscr{L}V_\lambda(x_0, t_0) &= -\frac{\mu}{\rho_0(1+\omega)} \frac{\partial^2}{\partial x_0^2}(\hat{v}_{x_0}^2+\lambda\hat{v}^2) \\ &+\frac{\hat{v}-v_0}{1+\omega}\frac{\partial}{\partial x_0}(\hat{v}_{x_0}^2+\lambda\hat{v}^2)+\frac{\partial}{\partial t_0}(\hat{v}_{x_0}^2+\lambda\hat{v}^2)=2\hat{v}_{x_0} \\ &\times\Big[(\hat{v}_{x_0})_{t_0}-\frac{\mu}{\rho_0(1+\omega)}(\hat{v}_{x_0})_{x_0x_0}+\Big\{\frac{\hat{v}-v_0}{1+\omega}-\Big(\frac{\mu}{\rho_0} \\ &\times\frac{1}{1+\omega}\Big)_{x_0}\Big\}(\hat{v}_{x_0})_{x_0}+\Big(\frac{\hat{v}-v_0}{1+\omega}\Big)_{x_0}(\hat{v}_{x_0})\Big]+2\lambda\hat{v}\mathscr{L}\hat{v} \\ &-\Big[\frac{2\mu}{\rho_0(1+\omega)}(\hat{v}_{x_0x_0})^2-\frac{2\lambda\mu}{\rho_0(1+\omega)}(\hat{v}_{x_0})^2 \\ &+\Big(\frac{2\mu}{\rho_0(1+\omega)}\Big)_{x_0}\cdot\hat{v}_{x_0}\hat{v}_{x_0x_0}-2\Big(\frac{\hat{v}-v_0}{1+\omega}\Big)_{x_0}(\hat{v}_{x_0})^2\Big]' \\ &=-[\cdots]'.\end{aligned} \tag{6.6}$$

Now, we see that

$$\begin{cases}\Big(\dfrac{\mu}{\rho_0}\dfrac{1}{1+\omega}\Big)_{x_0}=-\dfrac{\mu}{1+\omega}\Big(\dfrac{\rho_0'}{\rho_0^2}+\dfrac{\hat{v}-v_0}{\mu}\Big), \\ \Big(\dfrac{\hat{v}-v_0}{1+\omega}\Big)_{x_0}=\dfrac{\hat{v}_{x_0}-v_0'}{1+\omega}-\dfrac{\rho_0(\hat{v}-v_0)^2}{\mu(1+\omega)},\end{cases} \tag{6.7}$$

and that, for an arbitrary positive number ε,

$$\hat{v}_{x_0}\cdot\hat{v}_{x_0x_0} \leqq \varepsilon(\hat{v}_{x_0x_0})^2+\frac{1}{4\varepsilon}(\hat{v}_{x_0})^2. \tag{6.8}$$

Hence, it follows that

(6.9) $$\mathscr{L}V_\lambda(x_0, t_0) \leqq -\frac{2\mu}{\bar{\bar{\rho}}_0(1+\omega)}(\hat{v}_{x_0x_0})^2 - \frac{2\mu\lambda}{\bar{\bar{\rho}}_0(1+\omega)}(\hat{v}_{x_0})^2 + \frac{2}{1+\omega}Q_0{}^{\dagger)}\left\{\varepsilon(\hat{v}_{x_0x_0})^2 + \frac{1}{4\varepsilon}(\hat{v}_{x_0})^2\right\} - \frac{2}{1+\omega}\left\{\hat{v}_{x_0} - v_0' - \frac{\rho_0}{\mu}(\hat{v}-v_0)^2\right\}(\hat{v}_{x_0})^2 = \frac{2}{1+\omega}\left\{-\frac{\mu}{\bar{\bar{\rho}}_0} + \varepsilon Q_0\right\}^2(\hat{v}_{x_0x_0})^2 + \frac{2}{1+\omega}\left\{\frac{1}{4\varepsilon}Q_0 - \frac{\lambda\mu}{\rho_0} - \hat{v}_{x_0} + v_0' + \frac{\rho_0}{\mu}(\hat{v}-v_0)^2\right\}(\hat{v}_{x_0})^2 \leqq \frac{2}{1+\omega}\left\{-\frac{\mu}{\bar{\bar{\rho}}_0} + \varepsilon Q_0\right\}(\hat{v}_{x_0x_0})^2 + \frac{2}{1+\omega}\left\{\frac{1}{4\varepsilon}Q_0 - \frac{\lambda\mu}{\bar{\bar{\rho}}_0} + |\hat{v}_{x_0}|_T^{(0)} + |v_0'|^{(0)} + \frac{4\bar{\rho}_0(|v_0|^{(0)})^2}{\mu}\right\}(\hat{v}_{x_0})^2.$$

We choose $\varepsilon=\varepsilon_0$ in such a way that

(6.10) $$\varepsilon_0 \equiv \begin{cases} \dfrac{\mu}{Q_0\bar{\bar{\rho}}_0}, & (\text{if } |\rho_0'| + |v_0|^{(0)} \neq 0), \\ 1, & (\text{if } |\rho_0'| + |v_0|^{(0)} = 0). \end{cases}$$

For such a fixed number $\varepsilon_0(>0)$, it holds that

(6.11) $$\mathscr{L}V_\lambda(x_0, t_0) \leqq \frac{2}{1+\omega}\left\{\frac{1}{4\varepsilon_0}Q_0 - \frac{\lambda\mu}{\bar{\bar{\rho}}_0} + |\hat{v}_{x_0}|_T^{(0)} + |v_0'|^{(0)} + \frac{4\bar{\rho}_0(|v_0|^{(0)})^2}{\mu}\right\}(\hat{v}_{x_0})^2.$$

Thus, if we take $\lambda=\lambda_0$ in such a way that

(6.12) $$\lambda_0 = \frac{\bar{\bar{\rho}}_0}{\mu}\left\{\frac{1}{4\varepsilon_0}Q_0 + |\hat{v}_{x_0}|_T^{(0)} + |v_0'|^{(0)} + \frac{4\bar{\bar{\rho}}_0}{\mu} \times (|v_0|^{(0)})^2\right\},$$

then we have an inequality

† $Q_0 \equiv \mu|\rho_0'|^{(0)}(\bar{\rho}_0)^{-2} + 2|v_0|^{(0)}$.

(6.13) $$\mathscr{L} V_{\lambda_0}(x_0, t_0) \leqq 0.$$

For an arbitrary positive number N, we define $\bar{N}_T$ and $S(\bar{N}_T)$ by

(6.14) $$\begin{cases} \bar{N}_T \equiv [-N, N] \times [0, T], \\ S(\bar{N}_T) \equiv \{N\} \times [0, T] \subset \{-N\} \times [0, T]. \end{cases}$$

By (6.13) and the maximum principle, it holds that

(6.15) $$\begin{aligned} &\max_{(x_0, t_0) \in \bar{N}_T} V_{\lambda_0}(x_0, t_0) \leqq \max [\max_{x_0 \in [-N, N]} V_{\lambda_0}(x_0, 0), \\ &\max_{(x_0, t_0) \in S(\bar{N}_T)} V_{\lambda_0}(x_0, t_0)] \leqq \max [\max_{x_0 \in [-N, N]} v_0'(x_0)^2 \\ &+ \lambda_0 \max_{x_0 \in [-N, N]} v_0(x_0)^2, \max_{(x_0, t_0) \in S(\bar{N}_T)} \hat{v}_{x_0}(x_0, t_0)^2 \\ &+ \lambda_0 \max_{(x_0, t_0) \in S(\bar{N}_T)} \hat{v}(x_0, t_0)^2] \leqq \max [(|v_0'|^{(0)})^2 \\ &+ \lambda_0 (|v_0|^{(0)})^2, \max_{(x_0, t_0) \in S(\bar{N}_T)} \hat{v}_{x_0}(x_0, t_0)^2 \\ &+ \lambda_0 (|v_0|^{(0)})^2] \leqq |(v_0'|^{(0)})^2 + \lambda_0 (|v_0|^{(0)})^2 \\ &+ \{\max_{(x_0, t_0) \in S(\bar{N}_T)} \hat{v}_{x_0}(x_0, t_0)\}^2, \qquad \text{(cf. Lemma 5.2)}. \end{aligned}$$

Thus, we obtain the result that

(6.16) $$\begin{aligned} &\max_{(x_0, t_0) \in \bar{N}_T} \hat{v}_{x_0}(x_0, t_0)^2 = \{\max_{(x_0, t_0) \in \bar{N}_T} |v_{x_0}(x_0, t_0)|\}^2 \\ &\leqq \max_{(x_0, t_0) \in \bar{N}_T} V_{\lambda_0}(x_0, t_0) \leqq (|v_0'|^{(0)})^2 + \lambda_0 (|v_0|^{(0)})^2 \\ &\quad + \{\max_{(x_0, t_0) \in S(\bar{N}_T)} |\hat{v}_{x_0}(x_0, t_0)|\}^2. \end{aligned}$$

((*2nd step*)). According to the assumption of the lemma, $\hat{v}_{x_0 x_0}$ satisfies in $0 < t \leqq T$ the equation

(6.17) $$\begin{aligned} (\hat{v}_{x_0 x_0})_{t_0} &= \left\{ \frac{\mu}{\rho_0} \frac{1}{1+\omega} \hat{v}_{x_0 x_0} - \frac{\hat{v} - v_0}{1+\omega} \hat{v}_{x_0 x_0 x_0} \right\} \\ &= \frac{\mu}{\rho_0} \frac{1}{1+\omega} (\hat{v}_{x_0 x_0})_{x_0 x_0} + \left\{ 2 \left(\frac{\mu}{\rho_0} \frac{1}{1+\omega} \right)_{x_0} - \frac{\hat{v} - v_0}{1+\omega} \right\}_1 \end{aligned}$$

$$\times(\hat{v}_{x_0x_0})_{x_0}+\left\{\left(\frac{\mu}{\rho_0}\frac{1}{1+\omega}\right)_{x_0x_0}-2\left(\frac{\hat{v}-v_0}{1+\omega}\right)_{x_0}\right\}_{\text{II}}\hat{v}_{x_0x_0}$$

$$-\left(\frac{\hat{v}-v_0}{1+\omega}\right)_{x_0x_0}\cdot\hat{v}_{x_0}=\frac{\mu}{\rho_0}\frac{1}{1+\omega}(\hat{v}_{x_0x_0})_{x_0x_0}$$

$$+\{\cdots\}_{\text{I}}(\hat{v}_{x_0x_0})_{x_0}+\left[\{\cdots\}_{\text{II}}-\frac{\hat{v}_{x_0}}{1+\omega}\right]\hat{v}_{x_0x_0}$$

$$+\frac{\hat{v}_{x_0}}{1+\omega}\left\{\frac{\rho_0}{\mu}(\hat{v}-v_0)(\hat{v}_{x_0}-v_0')-\frac{\rho_0'}{\mu}(\hat{v}-v_0)^2\right.$$

$$\left.+(\hat{v}-v_0)^3+v_0''\right\}_{\text{III}}.$$

We put

$$F(x_0, t_0)\equiv\frac{\hat{v}_{x_0}}{1+\omega}\{\cdots\}_{\text{III}}. \tag{6.18}$$

Since $\hat{v}\in H_T^{2+\alpha}$, $|\hat{v}_{x_0x_0}|_T^{(0)}$ is finite. Therefore, by Lemma 5.1, which asserts the uniqueness of the solution, $\hat{v}_{x_0x_0}$ is expressed by making use of the fundamental solution $\hat{\Gamma}(x_0, t_0; \xi, \tau)$ of the linear parabolic equation

$$W_{t_0}=\frac{\mu}{\rho_0(1+\omega)}W_{x_0x_0}+\{\cdots\}_{\text{I}}W_{x_0}+\{\cdots\}_{\text{II}}W, \tag{6.19}$$

$$\left(N.B.:\ \frac{\mu}{\rho_0(1+\omega)}\in H_1^{\alpha},\ |\{\cdots\}_i|_T^{(0)}+|\{\cdots\}_i|_{x,T}^{(\alpha)}<+\infty\right.$$

$$\left.(i=\text{I or II})\right),$$

in the following way:

$$\hat{v}_{x_0x_0}(x_0, t_0)=\int_{R^1}\hat{\Gamma}(x_0, t_0; \xi, 0)v_0''(\xi)d\xi \tag{6.20}$$

$$+\int_0^{t_0}d\tau\int_{R^1}\hat{\Gamma}(x_0, t_0; \xi, \tau)F(\xi, \tau)d\xi.$$

Analogously, $\hat{v}_{x_0}$ is expressed by use of the fundamental solution $\tilde{\Gamma}(x_0, t_0; \xi, \tau)$ of the linear equation

(6.21) $$V_{t_0}=\frac{\mu}{\rho_0(1+\omega)}V_{x_0x_0}+\left\{\left(\frac{\mu}{\rho_0}\frac{1}{1+\omega}\right)_{x_0}-\frac{\hat{v}-v_0}{1+\omega}\right\}V_{x_0}+\left\{-\left(\frac{\hat{v}-v_0}{1+\omega}\right)_{x_0}\right\}\cdot V$$

in the form

(6.22) $$\hat{v}_{x_0}(x_0, t_0)=\int_{R^1}\tilde{\Gamma}(x_0, t_0; \xi, 0)v_0'(\xi)d\xi.$$

As is well known, $\tilde{\Gamma}$ is estimated as follows:

(6.23) $$0<\tilde{\Gamma}\leqq A_1(t_0-\tau)^{-\frac{1}{2}}\exp\left\{-A_2\frac{|x_0-\xi|^2}{t_0-\tau}\right\}\times\exp\left\{t_0\left|\left(\frac{\hat{v}-v_0}{1+\omega}\right)_{x_0}\right|_T^{(0)}\right\}, \qquad (A_1, A_2>0).$$

Therefore, since $v_0'\in L^1(R^1)$, we have

(6.24) $$\begin{cases} \hat{v}_{x_0}(\cdot t_0)\in L^1(R^1), & (t_0\in[0, T]); \\ \hat{v}_{x_0}(\cdot, t_0) \text{ is continuous on } [0, T] \text{ in the topology of } L^1(R^1). \end{cases}$$

Hence, the same assertion also holds for $F(x_0, t_0)$. Thus, we have the results that

(6.25)
(i) the proposition (6.24) as "$\hat{v}_{x_0}$" is replaced by "$\hat{v}_{x_0x_0}$" holds, and

(ii) for an arbitrary positive number ε, there exists a positive number $\bar{N}(\varepsilon)$ uniformly in $t_0\in[0, T]$ such that

$$\int_{R^1-[-N,N]}|\hat{v}_{x_0x_0}(x_0, t_0)|dx_0<\varepsilon, \qquad (N\geqq\bar{N}(\varepsilon)).$$

If we note that $\tilde{\Gamma}$ is estimated in a way analogous to (6.23), i.e.,

(6.26) $$0<\hat{\Gamma}\leqq A_1'\frac{1}{(t_0-\tau)^{\frac{1}{2}}}\exp\left\{-A_2'\frac{|x_0-\xi|^2}{t_0-\tau}\right\}$$

$$\times \exp\{t_0|\{\cdots\}_{\text{II}}|_T^{(0)}\}, \qquad (A_1', A_2'>0),$$

then the assertion (6.25) (ii) is ascertained in such a way that

$$\text{(6.27)} \quad \int_{R^1-[-N,N]} |\hat{v}_{x_0x_0}(x_0, t_0)|dx_0 \leqq \int_{R^1-[-N,N]} dx_0$$

$$\times \int_{R^1} \hat{\Gamma}(x_0, t_0; \xi, 0)|v_0''(\xi)|d\xi$$

$$+\int_{R^1-[-N,N]} dx_0 \int_0^{t_0} d\tau \int_{R^1} \hat{\Gamma}(x_0, t_0; \xi, \tau)|F(\xi, \tau)|d\xi$$

$$\leqq \int_{R^1} |v_0''(\xi)|d\xi \int_{R^1-[-N,N]} \hat{\Gamma}(x_0, t_0; \xi, \tau)dx_0$$

$$+\int_0^{t_0} d\tau \int_{R^1} |F(\xi, \tau)|d\xi \int_{R^1-[-N,N]} \hat{\Gamma}(x_0, t_0; \xi, \tau)dx_0$$

$$\leqq A_1' \exp\{T|\{\cdots\}_{\text{II}}|_T^{(0)}\}\Big[\int_{[-\frac{N}{2}, \frac{N}{2}]} |v_0''(\xi)|d\xi$$

$$\times \int_{R^1-[-N,N]} t_0^{-\frac{1}{2}} \cdot \exp\Big\{-A_2' \frac{|x_0-\xi|^2}{t_0} dx_0$$

$$+\int_{R^1-[-\frac{N}{2}, \frac{N}{2}]} |v_0''(\xi)|d\xi \int_{R^1} t_0^{-\frac{1}{2}} \exp\Big\{-A_2' \frac{|x_0-\xi|^2}{t_0}\Big\}dx_0$$

$$+\int_0^{t_0} d\tau \int_{[-\frac{N}{2}, \frac{N}{2}]} |F(\xi, \tau)|d\xi \int_{R^1-[-N,N]} (t_0-\tau)^{-\frac{1}{2}}$$

$$\times \exp\Big\{-A_2' \frac{|x_0-\xi|^2}{t_0-\tau}\Big\}dx_0 + \int_0^{t_0} d\tau \int_{R^1-[-\frac{N}{2}, \frac{N}{2}]} |F(\xi, \tau)|d\xi$$

$$\times \int_{R^1} (t_0-\tau)^{-\frac{1}{2}} \exp\Big\{-A_2' \frac{|x_0-\xi|^2}{t_0-\tau}\Big\}dx_0\Big]$$

$$\leqq A_1' \exp[T|\{\cdots\}_{\text{II}}|_T^{(0)}]\Big[\Big\{\int_{-\frac{N}{2}}^{\frac{N}{2}} |v_0''(\xi)|d\xi$$

$$\times (A_2')^{-\frac{1}{2}} \int_{R^1-[-\sqrt{A_2'}(N+\xi)/\sqrt{t_0}, \sqrt{A_2'}(N-\xi)/\sqrt{t_0}]} e^{-y^2}dy$$

$$+\left(\frac{\pi}{A_2'}\right)^{\frac{1}{2}}\int_{R^1-[-\frac{N}{2},\frac{N}{2}]}|v_0''(\xi)|d\xi\Big\}$$

$$+\Big\{\int_0^{t_0}d\tau\int_{-\frac{N}{2}}^{\frac{N}{2}}|F(\xi,\tau)|d\xi\cdot(A_2')^{-\frac{1}{2}}$$

$$\times\int_{R^1-[-\sqrt{A_2'}(N+\xi)/\sqrt{t_0-\tau},\sqrt{A_2'}(N-\xi)/\sqrt{t_0-\tau}]}e^{-y^2}dy$$

$$+\int_0^{t_0}d\tau\cdot\left(\frac{\pi}{A_2'}\right)^{\frac{1}{2}}\int_{R^1-[-\frac{1}{2},\frac{1}{2}]}|F(\xi,\tau)|d\xi\Big\}\Big]$$

$$\leqq A_1'\dots\Big[\|v_0''\|_{L^1(R^1)}\cdot(A_2')^{-\frac{1}{2}}\cdot\int_{R^1-[-\frac{N}{2}(\frac{A_2'}{T})^{\frac{1}{2}},\frac{N}{2}(\frac{A_2'}{T})^{\frac{1}{2}}]}$$

$$e^{-y^2}dy+\left(\frac{\pi}{A_2'}\right)^{\frac{1}{2}}\cdot\int_{R^1-[-\frac{1}{2},\frac{1}{2}]}|v_0''(\xi)|d\xi$$

$$+(A_2')^{-\frac{1}{2}}\int_0^T\|F(\cdot,\tau)\|_{L^1(R^1)}d\tau$$

$$\times\int_{R^1-[-\frac{N}{2}(\frac{A_2'}{T})^{\frac{1}{2}},\frac{N}{2}(\frac{A_2'}{T})^{\frac{1}{2}}]}e^{-y^2}dy$$

$$+\left(\frac{\pi}{A_2'}\right)^{\frac{1}{2}}\cdot\int_0^T d\tau\int_{R^1-[-\frac{1}{2},\frac{1}{2}]}|F(\xi,\tau)|d\xi.$$

Now, we note that directly from (6.22) and (6.23) follows the inequality

$$(6.27)'\qquad\begin{cases}\displaystyle\int_{R^1-[-N,N]}|\hat{v}_{x_0}(x_0,t_0)|dx_0<\varepsilon(\text{arbitrary number }>0),\\(N\geqq N_0'(\varepsilon),\ \text{uniformly in } t_0\in[0,T]),\end{cases}$$

and that a similar relation holds for $F(\xi,\tau)$. Furthermore, it is to be noted that

$$(6.28)\qquad \hat{v}_{x_0}(x_0,t_0)-\hat{v}_{x_0}(x_0,t_0)=\int_{x_0}^{x_0'}\hat{v}_{x_0x_0}(x_0,t_0)dx_0,$$

and, therefore, that

$$(6.28)'\qquad |\hat{v}_{x_0}(x_0,t_0)-\hat{v}_{x_0}(x_0',t_0)|$$

$$\leqq \int_{x_0'}^{x_0} \hat{v}_{x_0 x_0}(x_0, t_0) dx_0, \qquad (x_0 \geqq x_0').$$

Thus, by (6.25) (ii), which is ascertained by (6.27) and (6.27)′, for an arbitrary number $\varepsilon>0$, there exists a positive number $N_0''(\varepsilon)$ independent of $t_0 \in [0, T]$ such that

(6.29) $$\left\{ \begin{array}{l} |\hat{v}_{x_0}(x_0, t_0) - \hat{v}_{x_0}(x_0', t_0)| < \varepsilon, \text{ (if } x_0 \text{ and } x_0' \geqq N_0''(\varepsilon) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{or } -N_0''(\varepsilon)). \end{array} \right.$$

Hence, from the fact that $\hat{v}_{x_0}(\cdot, t_0) \in L^1(R^1)(t_0 \in [0, T])$, we have a result that

(6.30) $$\hat{v}_{x_0}(x_0, t_0) \to 0 \text{ (as } |x_0| \to +\infty, \text{ uniformly in } t_0 \in [0, T]).$$

Accordingly, from (6.16) it follows that for an arbitrary number $\varepsilon>0$, if $N \geqq N_0''(\varepsilon)$, then

(6.31) $$\{ \max_{(x_0, t_0) \in N|_T} \hat{v}_{x_0}(x_0, t_0)\}^2 \leqq (|v_0'|^{(0)})^2 + \lambda_0(|v_0|^{(0)})^2 + \varepsilon^2.$$

Therefore, finally, we have

(6.32) $$|\hat{v}_{x_0}|_T^{(0)} \leqq \{(|v_0'|^{(0)})^2 + \lambda_0(|v_0|^{(0)})^2\}^{\frac{1}{2}}.$$

((3*rd step*)). By (6.12) and (6.32), we have

(6.33) $$\begin{aligned} (|\hat{v}_{x_0}|_T^{(0)})^2 &\leqq (|v_0'|^{(0)})^2 + \lambda_0(|v_0|^{(0)})^2 = (|v_0'|^{(0)})^2 \\ &+ (|v_0|^{(0)})^2 \cdot \frac{\bar{\bar{\rho}}_0}{\mu} \cdot \left\{ \frac{Q_0}{4\varepsilon_0} + |\hat{v}_{x_0}|_T^{(0)} + |v_0'|^{(0)} \right. \\ &\left. + \frac{4\varepsilon_0(|v_0|^{(0)})^2}{\mu} \right\} = (|v_0'|^{(0)})^2 + \frac{\bar{\bar{\rho}}_0}{\mu}(|v_0|^{(0)})^2 \\ &\times \left\{ \frac{Q_0}{4\varepsilon_0} + |v_0'|^{(0)} + \frac{4\bar{\bar{\rho}}_0(|v_0|^{(0)})^2}{\mu} \right\} \\ &+ \frac{\bar{\bar{\rho}}_0}{\mu}(|v_0|^{(0)})^2 \cdot |\hat{v}_{x_0}|_T^{(0)} = a_0 + b_0 |\hat{v}_{x_0}|_T^{(0)}, \end{aligned}$$

where

(6.34)
$$\begin{cases} a_0 \equiv (|v_0'|^{(0)})^2 + (|v_0|^{(0)})^2 \dfrac{\bar{\bar{\rho}}_0}{\mu}\left\{\dfrac{Q_0}{4\varepsilon_0} + |v_0'|^{(0)} \right. \\ \left. \quad + \dfrac{4\bar{\rho}_0(|v_0|^{(0)})^2}{\mu}\right\}, \ b_0 \equiv \dfrac{\rho_0}{\mu}(|v_0|^{(0)})^2. \end{cases}$$

Thus it holds that

(6.35)
$$|\hat{v}_{x_0}|_T^{(0)} \leqq \frac{b_0 + (b_0^2 + 4a_0)^{\frac{1}{2}}}{2}.$$

This completes the proof of the lemma. Q.E.D.

Lemma 6.2. *Under the initial condition* (2.1)–(6.1), *as for the solution* (v, ρ) *of* (1.2) *in* $H_T^{2+\alpha} \times B_T^1$, $|v_x|_T^{(0)}$ *is a priori bounded by a constant depending only on the quantities appearing in* (2.1) *and* (6.1) *but independent of T.*

Proof. By (5.17), (5.25), and (6.35), it holds that

(6.36)
$$|v_x(x, t)|_T^{(0)} \leqq |\hat{v}_{x_0}(x_0, t_0)|_T^{(0)} \cdot \left|\frac{1}{1+\omega(x_0, t_0)}\right|_T^{(0)}$$
$$\leqq \frac{b_0 + (b_0^2 + 4a_0)^{\frac{1}{2}}}{2} \exp\left\{\frac{\bar{\bar{\rho}}_0}{\mu}\|v_0\|_{L^1(R^1)}\right\},$$

which completes the assertion of the lemma. Q.E.D.

Thus, we have an a priori estimate for $\left\|\dfrac{\mu}{\rho}\right\|_T^{(\alpha)}$.

Lemma 6.3. *For the function* ρ *in* $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ *satisfying* (1.2)–(2.1)–(6.1), *the following inequalities hold*:

(6.37)
$$\left|\frac{\rho}{\mu}\right|_T^{(0)} \leqq \frac{\bar{\rho}_0}{\mu} \exp\left\{\frac{\bar{\bar{\rho}}_0}{\mu}\|v_0\|_{L^1(R^1)}\right\},$$
$$\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)} \leqq 5\frac{\mu}{\bar{\rho}_0} \exp\left\{\frac{\bar{\bar{\rho}}_0}{\mu}\|v_0\|_{L^1(R^1)}\right\} + 2\mu\left[\frac{1}{\bar{\rho}_0}\right.$$
$$\times \frac{b_0 + (b_0^2 + 4a_0)^{\frac{1}{2}}}{2} + (|v_0|^{(0)} + 1)\left\{\frac{|\rho_0'|^{(0)}}{(\bar{\rho}_0)^2}\right\}$$

$$\times \exp\left\{\frac{\bar{\bar{\rho}}_0}{\mu}\|v_0\|_{L^1(R^1)}\right\}+\frac{2|v_0|^{(0)}}{\bar{\rho}_0}\Bigg].$$

Proof. See (4.5), (4.6), (4.7), and the lemmas 5.4, 5.5, and 6.1. Especially, remark that

$$\frac{v_x}{\rho}=\frac{\hat{v}_{x0}}{\rho_0}. \tag{6.38}$$

Q.E.D.

The inequalities of (6.37) show that $\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}$ is also independent of T.

§7. A Priori Estimates for $\ll v \gg_T^{(2+\alpha)}$.

As for $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ satisfying (1.2), (2.1), and (6.1), it has been shown in §6 that $|v_x|_T^{(0)}$ is a priori bounded by a constant depending only the quantities appearing in (2.1) and (6.1). Based on this fact, here, we estimate $\|v\|_T^{(1+\alpha)}$ in an a priori way from above, utilising the fundamental solution $\Gamma\left(x, t; y, \tau; \frac{\mu}{\rho}\right)$ of the linear equation (3.3). Since $|v|_T^{(0)}$ and $|v_x|_T^{(0)}$ are already estimated, we need only estimate $|v|_{t,T}^{(\alpha/2)}$, $|v_x|_{x,T}^{(\alpha)}$, and $|v_x|_{t,T}^{(\alpha/2)}$.

In the same way as in [8], we have:

Lemma 7.1. *For* $t\geqq t'>\tau$,

$$\left|\Gamma\left(x, t; y, \tau; \frac{\mu}{\rho}\right)-\Gamma\left(x, t'; y, \tau; \frac{\mu}{\rho}\right)\right| \tag{7.1}$$

$$\leqq C_6^{(0)}\left(T; \left|\frac{\mu}{\rho}\right|_T^{(0)}+\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\right)(t-t')(t'-\tau)^{-\frac{3}{2}}$$

$$\times\exp\left\{-\bar{d}\,\frac{|x-y|^2}{t-\tau}\right\}, \qquad (\bar{d}=v_0 d(0),\ 0<v_0<1),$$

$$\left|\frac{\partial}{\partial x}\Gamma\left(x, t; y, \tau; \frac{\mu}{\rho}\right)-\frac{\partial}{\partial x}\Gamma\left(x, t'; y, \tau; \frac{\mu}{\rho}\right)\right|$$

$$\leqq C_6^{(1)}\left(T; \left|\frac{\rho}{\mu}\right|_T^{(0)}+\left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\right)\{(t-t')(t'-\tau)^{-2}+(t-t')^{\frac{1+\alpha}{2}}$$

$$\times (t'-\tau)^{-\frac{3}{2}}\} \exp\left\{-\bar{d}\frac{|x-y|^2}{t-\tau}\right\},$$

where $C_6^{(i)}(T; A)(i=0, 1)$ *increase monotonically as each argument increases.*

Lemma 7.2. *It holds that*

$$|v|_{t,T}^{(\alpha/2)} \leqq C_7\left(T; \left|\frac{\rho}{\mu}\right|_T^{(0)} + \left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\right)\cdot |\hat{N}|_T^{(0)}, \tag{7.2}$$

$$\left(\hat{N} \equiv \frac{\mu}{\rho_v} v_0'' - v\cdot v_x;\ C_7(T;\ \cdot)\searrow 0 \text{ as } T\searrow 0\right),$$

where it is to be remarked that $|\hat{N}|_T^{(0)}$ *is finite.*

Proof. By (3.6), Lemma 7.1, and the expression of $v(x, t)$

$$v(x, t) = \int_0^t d\tau \int_{R^1} \Gamma\left(x, t; y, \tau; \frac{\mu}{\rho}\right)\hat{N}(y, \tau)dy + v_0(x), \tag{7.3}$$

we have, for $t \geqq t' > 0$,

$$\begin{aligned} |v(x, t) - v(x, t')| &\leqq \left|\int_0^{t'} d\tau \int_{R^1} \left\{\Gamma\left(x, t; y, \tau; \frac{\mu}{\rho}\right)\right.\right. \\ &\quad \left.\left. - \Gamma\left(x, t'; y, \tau; \frac{\mu}{\rho}\right)\right\}\hat{N}(y, \tau)dy\right| + \left|\int_{t'}^{t} d\tau \int_{R^1} \Gamma\left(x, t;\right.\right. \\ &\quad \left.\left. y, \tau; \frac{\mu}{\rho}\right)\hat{N}(y, \tau)dy\right| \leqq \int_0^{t'} d\tau \int_{R^1} |\Gamma(x, t; \ldots) \\ &\quad - \Gamma(x, t'; \ldots)|^{\frac{1}{1}}\{\Gamma(x, t; \ldots) + \Gamma(x, t'; \ldots)\}^{1-\frac{\alpha}{2}} \\ &\quad \times |\hat{N}(y, \tau)|dy + (t-t')|\hat{N}|_T^{(0)} \\ &\quad \left(N.B.: \int_{R^1} \Gamma\left(x, t; y, \tau; \frac{\mu}{\rho}\right)dy = 1\right) \\ &\leqq \left[\left\{C_6^{(0)}\left(T; \left\|\frac{\mu}{\rho}\right\|_T\right)\right\}^{\frac{\alpha}{2}} \cdot \left\{2C_6^{(0)}\left(T; \left\|\frac{\mu}{\rho}\right\|_T\right)\right\}^{1-\frac{\alpha}{2}} (t-t')^{\frac{\alpha}{2}} \right. \\ &\quad \times \int_0^{t'} d\tau \int_{R^1} (t'-\tau)^{-\frac{1+\alpha}{2}} e^{-\bar{d}|x-y|^2/(t-\tau)} dy \end{aligned} \tag{7.4}$$

$$+(t-t')^{\frac{\alpha}{1}}T^{1-\frac{\alpha}{2}}\Big]\cdot|\hat{N}|_T^{(0)}\leqq(t-t')^{\frac{\alpha}{2}}\Big[\Big\{C_6^{(0)}\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)\Big\}^{\frac{\alpha}{2}}$$

$$\times\frac{2}{1-\alpha}\cdot\Big\{2C_0^{(0)}\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)\Big\}^{1-\frac{\alpha}{2}}\cdot T^{1-\frac{\alpha}{2}}(\pi/\bar{d})^{1/2}$$

$$+T^{1-\frac{\alpha}{2}}\Big]_1\cdot|\hat{N}|_T^{(0)}$$

$$=(t-t')^{\frac{\alpha}{2}}C_7\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)|\hat{N}|_T^{(0)},$$

where

(7.5) $$C_7=[\cdots]_1,\qquad \Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\equiv\Big|\frac{\rho}{\mu}\Big|_T^{(0)}+\Big\|\frac{\mu}{\rho}\Big\|_T^{(\alpha)}.$$

Q.E.D.

Lemma 7.3.

(7.6) $$|v_x|_{x,T}^{(\alpha)}\leqq(|v_0''|^{(0)})^\alpha(2|v_0'|^{(0)})^{1-\alpha}+C_8\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)|\hat{N}|_T^{(0)}.$$

Proof. Remarking (4.6), we have

(7.7) $$|v_x(x,t)-v_x(x',t)|\leqq|v_0'(x)-v_0'(x')|$$

$$+\int_0^t d\tau\int_{R^1}\Big|\frac{\partial^2}{\partial x^2}\Gamma(x'',t;y,\tau)(x-x')\Big|^\alpha\Big\{\Big|\frac{\partial}{\partial x}\Gamma(x,t;$$

$$y,\tau)\Big|+\Big|\frac{\partial}{\partial x}\Gamma(x',t;y,\tau)\Big|\Big\}^{1-\alpha}\cdot|\hat{N}|_T^{(0)}dy$$

$$\leqq|x-x'|^\alpha[(|v_0''|^{(0)})^\alpha(2|v_0'|^{(0)})^{1-\alpha}+\{C_0^{(2)}\}^\alpha\{2C_0^{(1)}\}^{1-\alpha}$$

$$\times\int_0^t(t-\tau)^{-1-(\alpha/2)}d\tau\int_{R^1}\{e^{-d(2)\frac{|x''-y|^2}{t-\tau}}$$

$$+e^{-d(1)\frac{|x-y|^2}{t-\tau}}+e^{-d(1)\frac{|x'-y|^2}{t-\tau}}\}dy\cdot|\hat{N}|_T^{(0)}$$

$$\leqq|x-x'|^\alpha\Big[(|v_0''|^{(0)})^\alpha(2|v_0'|^{(0)})^{1-\alpha}+\Big\{(C_0^{(2)})^\alpha(2C_0^{(1)})^{1-\alpha}$$

$$\times\frac{2}{1-\alpha}T^{\frac{1-\alpha}{2}}\cdot\pi^{1/2}\cdot(d(2)^{-1/2}+2d(1)^{-1/2})\Big\}_A\cdot|\hat{N}|_T^{(0)}\Big]$$

$$=|x-x'|^\alpha\cdot\Big[\cdots+C_8\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)|\hat{N}|_T^{(0)}\Big],$$

where

$$C_8\equiv\{\ldots\}_A, \tag{7.8}$$

and $C_8(T;\,a)\searrow 0$ as $T\searrow 0$, and C_8 decreases monotonically as $a\searrow 0$.

Q.E.D.

Lemma 7.4. *For* $|v_x|_{t,T}^{(\alpha/2)}$, *we have the following a priori estimate:*

$$|v_x|_{t,T}^{(\alpha/2)}\leqq C_9\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)\cdot|\hat{N}|_T^{(0)}, \tag{7.9}$$

where C_9 *has the same property as* C_8.

Proof. 1°). For $t>t'>0$ and $t>2t'$ (i.e., $t'<t-t'$, $t<2(t-t')$), by the relation

$$|v_x(x,t)-v_0'(x)|=\Big|\int_0^t d\tau\int_{R^1}\Gamma_x(x,t;\,y,\tau)\hat{N}(y,\tau)\,dy\Big|\leqq C_0^{(1)}\Big(T;\Big\|\Big|\frac{\mu}{\rho}\Big\|\Big|_T\Big)\int_0^t d\tau\int_{R^1}(t-\tau)^{-1}e^{-d(1)\frac{|x-y|^2}{t-\tau}}\times|\hat{N}(y,\tau)|_T^{(0)}dy\leqq C_0^{(1)}2(\pi/d(1))^{1/2}\cdot t^{1/2}\cdot|\hat{N}|_T^{(0)}, \tag{7.10}$$

we have

$$\begin{aligned}|v_x(x,t)-v_x(x,t')|&\leqq|v_x(x,t)-v_0'(x)|+|v_x(x,t')-v_0'(x)|\leqq C_0^{(1)}2(\pi/d(1))^{1/2}\{t^{1/2}+t'^{1/2}\}|\hat{N}|_T^{(0)}\\&\leqq\cdots\{\sqrt{2}(t-t')^{1/2}+(t-t')^{1/2}\}|\hat{N}|_T^{(0)}\\&\leqq(t-t')^{\alpha/2}[C_0^{(1)}2(\pi/d(1))^{1/2}(\sqrt{2}+1)T^{(1-\alpha)/2}]_1|\hat{N}|_T^{(0)}\\&=(t-t')^{\alpha/2}C_{9,1}\cdot|\hat{N}|_T^{(0)},\qquad(C_{9,1}=[\cdots]_1)\,.\end{aligned} \tag{7.11}$$

2°). For $0<t-t'\leqq t'$ (i.e., $t<2t'<2t$), we utilize the following expression of $v_x(x, t)-v_x(x, t')$:

(7.12) $$v_x(x, t)-v_x(x, t')=\{v_x(x, t)-v_0'(x)\}-\{v_x(x, t') -v_0'(x)\}=\int_{2t'-t}^{t}d\tau\int_{R^1}\Gamma_x(x, t; y, \tau)\hat{N}(y, \tau)dy -\int_{2t'-t}^{t'}d\tau\int_{R^1}\Gamma_x(x, t'; y, \tau)\hat{N}(y, \tau)dy +\int_0^{2t'-t}d\tau\int_{R^1}\{\Gamma_x(x, t; y, \tau)-\Gamma_x(x, t'; y, \tau)\} \times\hat{N}(y, \tau)dy=J_1+J_2+J_3.$$

As for J_1, we obtain

(7.13) $$|J_1|\leqq C_0^{(1)}\left(T; \left|\left|\left|\frac{\mu}{\rho}\right|\right|\right|_T\right)|\hat{N}|_T^{(0)}\cdot\int_{2t'-t}^{t}d\tau\int_{R^1}(t-\tau)^{-1} \times e^{-d(1)\frac{|x-y|^2}{t-\tau}}dy=C_0^{(1)}|\hat{N}|_T^{(0)}(\pi/d(1))^{1/2}\cdot 2^{3/2}(t-t')^{1/2} \leqq(t-t')^{\alpha/2}\bar{C}_{9,1}\left(T; \left|\left|\left|\frac{\mu}{\rho}\right|\right|\right|_T\right)|\hat{N}|_T^{(0)},$$

where

(7.14) $$\bar{C}_{9,1}=2^{3/2}T^{(1-\alpha)/2}\cdot(\pi/d(1))^{1/2}\cdot C_0^{(1)}.$$

In the same way, we have

(7.15) $$|J_2|\leqq(t-t')^{\alpha/2}\cdot\bar{C}_{9,2}\left(T; \left|\left|\left|\frac{\mu}{\rho}\right|\right|\right|_T\right)\cdot|\hat{N}|_T^{(0)}.$$

It remains to estimate J_3. By Lemma 7.1 and the formula

(7.16) $$(a+b)^\gamma\leqq a^\gamma+b^\gamma, \ (a, \ b\geqq 0, \text{ and } 0\leqq\gamma\leqq 1),$$

we have

(7.17) $$|J_3|\leqq(C_6^{(1)})^\gamma\cdot(C_0^{(1)})^{1-\gamma}|\hat{H}|_T^{(0)}\cdot\int_0^{2t'-t}d\tau\int_{R^1}\{(t-t')$$

$$\times (t'-\tau)^{-2}+(t-t')^{-\frac{1+\alpha}{2}}(t'-\tau)^{-\frac{2}{3}}\}^{\gamma}\cdot e^{-\bar{d}\frac{|x-y|^2}{t-\tau}}$$

$$\times \{(t-\tau)^{-1}e^{-d(1)\frac{|x-y|^2}{t-\tau}}+(t-\tau)^{-1}e^{-d(1)\frac{|x-y|^2}{t'-\tau}}\}^{1-\gamma}$$

$$dy \leqq 2\cdot(C_0^{(1)}+C_6^{(1)})|\hat{N}|_T^{(0)}\int_0^{2t'-t}d\tau\int_{R^1}\{(t-t')^{\gamma}$$

$$\times (t'-\tau)^{-1-\gamma}+(t-t')^{\gamma(1+\alpha)/2}(t'-\tau)^{-1-(\gamma/2)}$$

$$\times e^{-\bar{d}\frac{|x-y|^2}{t-\tau}}dy, \qquad (\bar{d}=v_0\cdot d(0) \text{ for some } v_0\in(0,1)),$$

where it has been used that the inequality $\bar{d}<d(1)$ holds, since it is known concerning the $A(m)$'s in (3.7) that $A(0)\geqq A(1)>A(2)$ (e.g., cf. [8]). Now, we put

$$\gamma=\frac{\alpha}{1+\alpha}, \qquad \alpha(\in(0,1)), \tag{7.18}$$

and as a result we have

$$\gamma\in\left(\frac{\alpha}{2},\frac{1}{2}\right). \tag{7.18$'$}$$

Hence, it follows that

$$|J_3|\leqq 2(C_0^{(1)}+C_6^{(1)})\cdot|\hat{N}|_T^{(0)}(t-t')^{\alpha/2}\int_0^{2t'-t}\{T^{\gamma-\frac{\alpha}{2}} \tag{7.19}$$

$$\times (t-\tau)^{1/2}(t'-\tau)^{-1-\gamma}+(t-\tau)^{1/2}(t'-\tau)^{-1-\frac{\gamma}{2}}d\tau$$

$$\times\int_{R^1}e^{-\bar{d}\frac{|x-y|^2}{t-\tau}}\cdot(t-\tau)^{-1/2}dy.$$

If we remark that

$$(t-\tau)^{1/2}(t'-\tau)^{-1-\beta}=\left(\frac{t-t'+t'-\tau}{t'-\tau}\right)^{\frac{1}{2}}(t'-\tau)^{-\frac{1}{2}-\beta} \tag{7.20}$$

$$\leqq\{1+(t-t')^{1/2}(t'-\tau)^{-1/2}\}(t'-\tau)^{-\frac{1}{2}-\beta}$$

$$=(t'-\tau)^{-\frac{1}{2}-\beta}+(t-t')^{1/2}(t'-\tau)^{-1-\beta},$$

where $\beta=\gamma$ or $\gamma/2$, then we have

$$(7.21) \qquad \int_0^{2t'-t}(t-\tau)^{1/2}\cdot(t'-\tau)^{-1-\beta}d\tau \leqq \left(\frac{1}{2}-\beta\right)^{-1}\cdot T^{\frac{1}{2}-\beta}$$

$$+\beta^{-1}(t-t')^{\frac{1}{2}-\beta} \leqq \left\{\left(\frac{1}{2}-\beta\right)^{-1}+\beta^{-1}\right\}T^{\frac{1}{2}-\beta}.$$

Therefore,

$$(7.22) \qquad |J_3| \leqq (t-t')^{\alpha/2}|\hat{N}|_T^{(0)}\cdot\Big[2(C_0^{(1)}+C_6^{(1)})(\pi/\bar{d})^{1/2}$$

$$\times\left\{\left(\frac{1}{2}-\gamma\right)^{-1}+{}^{-}\gamma\right\}T^{\frac{1}{2}-\gamma}+\left\{\left(\frac{1-\gamma}{2}\right)^{-1}+\left(\frac{\gamma}{2}\right)^{-1}\right\}$$

$$\times T^{\frac{1-\gamma}{2}}\Big]_{\mathrm{II}}=(t-t')^{\alpha/2}C_{9,3}|\hat{N}|_T^{(0)}$$

$$\left(C_{9,3}=[\cdots]_{\mathrm{II}}, \quad \gamma=\frac{\alpha}{1+\alpha}\right).$$

Thus, if we define

$$(7.23) \qquad C_9=C_{9,1}+\bar{C}_{9,1}+\bar{C}_{9,2}+\bar{C}_{9,3},$$

then we obtain (7.9). Q.E.D.

From the lemmas 5.2, 6.2, 6.3, 7.2, 7.3, and 7.4 follows:

Lemma 7.5.

$$(7.24) \qquad \|v_x\|_T^{(1+\alpha)}C_{10}[T;\, v_0,\, \rho_0],$$

where $C_{10}[T; \ldots\}$ *increases monotonically as* T *increases.*

Thus, we have

$$(7.25) \qquad \|\hat{N}\|_T^{(\alpha)} \leqq C_{11}[T;\, v_0,\, \rho_0],$$

where C_{11} has the same property as C_{10}.

By making use of the expression (7.3) and the inequality (7.25), we can estimate the other quantities constructing the norm of v in $\hat{H}_T^{2+\alpha}$ (cf. [8]), i.e.,

$$(7.26)\qquad |v_{xx}|_T^{(0)} \leqq C_{12}\left(T; \left\|\left|\frac{\mu}{\rho_v}\right|\right\|_T\right)\cdot \|\hat{N}\|_T^{(\alpha)} + |v_0''|^{(0)}$$

$$\leqq |v_0''|^{(0)} + C_{12}\left(T; \left\|\left|\frac{\mu}{\rho_v}\right|\right\|_T\right)\cdot C_{11}[T; v_0, \rho_0],$$

$$|v_{xx}|_T^{(\alpha)} \leqq C'_{12}\left(T; \left\|\left|\frac{\mu}{\rho_v}\right|\right\|_T\right)\cdot \|\hat{N}\|_T^{(\alpha)} + |v_0''|^{(\alpha)}$$

$$\leqq |v_0''|^{(\alpha)} + C'_{12}\left(T; \left\|\left|\frac{\mu}{\rho_v}\right|\right\|_T\right)\cdot C_{11}[T; v_0, \rho_0],$$

where each of C_{12} and C'_{12} increases monotonically as each of the arguments increases, and $C_{12}\searrow 0$ as $T\searrow 0$. From the discussions made thus far follows:

Lemma 7.6. *For the initial condition* (2.1)–(6.1), *if there exists a solution* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *of* (1.2), *then* $\|v\|_T^{(2+\alpha)} + [\rho]_T^{(1)}$ *has a priori bounds in* T, *where*

$$(7.27)\qquad [\rho]_T^{(n)} = \sum_{r+s=0}^{n} |D_t^r D_x^s \rho|_T^{(0)}, \qquad (r \text{ and } s, \text{ non-negative integers}).$$

Proof. We have only to note that

$$(7.28)\qquad \|v\|_T^{(2+\alpha)} \leqq \ll v \gg_T^{(2+\alpha)} + \|v_t\|_T^{(\alpha)} = <v>_T^{(2,\alpha)} + '<v>_T^{(2,\alpha)}$$

$$+ \left\|\frac{\mu}{\rho} v_{xx} - v\cdot v_x\right\|_T^{(\alpha)} \leqq <v>_T^{(2,\alpha)} + '<v>_T^{(2,\alpha)}$$

$$+ \left\|\frac{\mu}{\rho}\right\|_T^{(\alpha)}\cdot \|v_{xx}\|_T^{(\alpha)} + \|v\cdot v_x\|_T^{(\alpha)}$$

The discussions having been made since §5 guarantee that each term of the right-hand side of (7.28) has a priori bounds in T. Q.E.D.

§8. Main Theorems.

(I). Basing ourselves upon Lemma 7.6, we show here that there exists a temporally global regular solution, unique in a certain sense, of (1.2). For this purpose, we prepare the following three lemmas.

Lemma 8.1. *For the parabolic equation*

$$
(8.1)\quad \begin{cases} \dfrac{\partial u}{\partial t}(x, t) = a(x, t)\dfrac{\partial^2}{\partial x^2}u(x, t) + b(x, t)\dfrac{\partial}{\partial x}u(x, t) \\ \quad + c(x, t)u, \qquad (0 < t \leqq T), \\ u(x, 0) = u_0(x) \in H^0, \end{cases}
$$

if we require that

$$
(8.2)\quad \begin{cases} a \in H_T^\alpha, \quad 0 < a_0 \leqq a(x, t) \leqq |a|_T^{(0)} < +\infty \qquad (a_0, \text{ constant}), \\ |b|_T^{(0)} + |b|_{x,T}^{(\alpha)} < +\infty, \quad |c|_T^{(0)} + |c|_{x,T}^{(\alpha)} < +\infty, \end{cases}
$$

then we can construct a fundamental sclution $\Gamma(x, t; \xi, \tau; a, b, c)$ of (8.1). [*By Lemma* 5.1, *the fundamental solution is unique, so far as bounded solutions, regular in* $R^1 \times (0, T]$, *are concerned.*]

Proof. E. g., see [4]. Q.E.D.

Lemma 8.2. *For the inhomogeneous parabolic equation*

$$
(8.3)\quad \begin{cases} u_t = a(x, t)u_{xx} + b(x, t)\cdot u_x + c(x, t)u + f(x, t), \\ [0 < t \leqq T;\ a,\ b, \text{ and } c \text{ satisfy } (8.2);\ |f(x, t)| \leqq A \\ \quad \times \exp\{Bx^2\}(A,\ B \geqq 0,\ t \in [0, T])], \\ u(x, 0) = u_0(x) \in H_0, \end{cases}
$$

if we require that

$$
(8.4)\quad \begin{cases} f(x, t) \text{ is locally Hölder-continuousin } x \text{ with the exponent } \alpha \\ \text{in } R^1 \times [0, T], \text{ uniformly with respect to } t, \end{cases}
$$

then the function

(8.5)
$$U(x, t)^{\dagger)}=\int_{R^1}\Gamma(x, t; \xi, 0; a, b, c)u_0(\xi)d\xi + \int_0^t d\tau\int_{R^1}\Gamma(x, t; \xi, \tau; a, b, c)f(\xi, \tau)d\xi$$

is the unique bounded regular solution of (8.3).

Proof. E.g., see [4]. Q.E.D.

In the next place, by virtue of the above-mentioned two lemmas, we demonstrate:

Lemma 8.3. *If* (v, ρ) *satisfies regularly in* $R^1\times[0, T]$, *the initial condition is* (2.1)–(6.1), *and, moreover,* v *and* v_x *are bounded there, then,* $(v, \rho)\in H_T^{2+\alpha}\times B_T^1$ *(more particularly speaking,* $\rho\in B_T^{1+\alpha}\subset B_T^1$*), where we define* $v_t(x, 0)$ *and* $v_t(x, T)$ *by the right derivative at* $(x, 0)$ *and the left one at* (x, T) *of* $v(x, t)$ *in* t, *respectively.*

Proof. By the assumption that v and v_x are bounded in $R^1\times[0, T]$, it holds that

(8.6)
$$\begin{cases} 0<\bar{\rho}_0\cdot e^{-T|v_x|_T^{(0)}}\leqq\rho(x, t)\leqq\bar{\bar{\rho}}_0\cdot e^{+T|v_x|_T^{(0)}}<+\infty, \\ 0<\mu(\bar{\rho}_0)^{-1}e^{-T|v_x|_T^{(0)}}\leqq\dfrac{\mu}{\rho(x, t)}\leqq\mu(\bar{\rho}_0)^{-1} \\ \quad\times e^{+T|v_x|_T^{(0)}}<+\infty. \end{cases}$$

By lemma 2.1, we have, e.g.,

(8.7)
$$\left|\frac{\mu}{\rho(x, t)}-\frac{\mu}{\rho(x', t)}\right|\leqq 2\mu|x-x'|^{\alpha}\cdot\left\{\left|\left(\frac{1}{\rho}\right)_x\right|_T^{(0)} + \left|\frac{1}{\rho}\right|_T^{(0)}\right. \leqq|x-x'|^{\alpha}2\mu\{(\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}+2\mu^{-1}|v|_T^{(0)} \left. +(\bar{\rho}_0)^{-1}e^{T\cdot|v_x|_T^{(0)}}\right\}, \quad (\text{cf. } 4.7).$$

Thus, it follows from the boundedness of v and v_x that

† If $u_0\in H^{2+\alpha}$, then U satisfies (8.3) regularly in $R^1\times[0, T]$.

$$\frac{\mu}{\rho} \in H_T^\alpha. \tag{8.8}$$

By the assumption, v_{xx} is continuous in $R^1 \times [0, T]$. Therefore, $-v \cdot v_x$ is locally Hölder-continuous in x with an arbitrary exponent $\beta \in (0, 1)$, uniformuly with respect to t. According to Lemma 8.2 (N. B.: (0, T]$\subset$[0, T]) $v(x, t)$ is expressed in the form

$$\begin{aligned} v(x, t) &= \int_{R^1} \Gamma\left(x, t; \xi, 0; \frac{\mu}{\rho_v}\right) v_0(\xi) d\xi \\ &\quad + \int_0^t d\tau \int_{R^1} \Gamma\left(x, t; \xi, \tau; \frac{\mu}{\rho_v}\right)(-v \cdot v_\xi) d\xi \\ &= v_0 + \int_0^t d\tau \int_{R^1} \Gamma\left(x, t; \xi, \tau; \frac{\mu}{\rho_v}\right)\left\{\frac{\mu}{\rho_v} v_0''(\xi) - v \cdot v_\xi\right\} d\xi, \end{aligned} \tag{8.9}$$

where the fact has been used that $v - v_0$ satisfies

$$\begin{cases} (v - v_0)_t = \dfrac{\mu}{\rho_v}(v - v_0)_{xx} + \left(\dfrac{\mu}{\rho_v} v_0'' - v \cdot v_x\right), \\ (v - v_0)(x, 0) = 0, \end{cases} \tag{8.9$'$}$$

and we note that $\hat{N}(\xi, \tau)$ has the same property as $-v \cdot v_\xi$. In the same way as we did in §7, it is shown that

$$\|v\|_T^{(1+\alpha)} < +\infty. \tag{8.10}$$

Thus, we have the result that

$$v \in H_T^{2+\alpha}, \text{ and } \rho \in B_T^{1+\alpha} \subset B_T^1. \tag{8.11}$$

Q.E.D.

From Theorem 3.1, the a priori estimates obtained in §7, and Lemma 8.3 follows a theorem on the existence of a temporally global regular solution of (1.2).

Theorem 8.1. *For the initial condition* (2.1)–(6.1), *there exists a unique regular solution* (v, ρ) *of* (1.2) *in* $R^1 \times [0, +\infty)$ *such that, for an arbitrary* $T \in (0, +\infty)$, v *and* v_x *are bounded in* $R^1 \times [0, T]$. *Furthermore, it holds that*

(8.12) $$|v(x, t)| \leqq |v_0|^{(0)}, \quad 0 < \underline{\rho}_0 \cdot e^{-\|\rho_0 v_0\|_{L^1(R^1)}} \leqq \rho(x, t)$$
$$\leqq \bar{\bar{\rho}}_0 e^{\|\rho_0 v_0\|_{L^1(R^1)}},$$
$$|v_x(x, t)| \leqq K_1\left(\|v_0\|_{L^1(R^1)}, \|v_0\|^{(1)}, \|\rho_0\|^{(1)}, \frac{1}{\underline{\rho}_0}\right) < +\infty,$$

where $K_1 \nearrow$ as each argument$\nearrow$. Thus, v and v_x are bounded in $R^1 \times [0, +\infty)$ under the above-mentioned conditions.

The following theorems are variations of the above theorem.

Theorem 8.2. *Besides* (2.1) *and* (6.1), *if a condition*

(8.13) $$\rho_0 \in H^{3+\alpha}, \quad v_0''' \in H^\alpha \cap L^1(R^1)$$

is added, then, in addition to the assertion of Theorem 8.1, *we have*

(8.14) $$|v_{xx}(x, t)| \leqq K_2(\|v_0\|_{L^1(R^1)}, \|v_0\|^{(2)}, \|\rho_0\|^{(2)}, (\rho_0)^{-1})$$
$$< +\infty, \qquad (K_2 \nearrow \text{ as each argument} \nearrow).$$

Proof. The assertion is ascertained almost in the same way as in Theorem 8.1. In the course of demonstration, we have only to consider the function W_λ defined by

(8.15) $$W_\lambda(x_0, t_0) = (\hat{v}_{x_0 x_0})^2 + \lambda(\hat{v}_{x_0})^2$$

instead of V_λ in (6.2). Q.E.D.

Theorem 8.3. *For the initial conditions* (2·1) *and, instead of* (6.1),

(8.16) $$\begin{cases} v_0 - A, \quad (v_0 - A)', \quad (v_0 - \mathrm{A})'' \in L^1(R^1) \text{ for some constant} \\ A \in (-\infty, +\infty), \quad \rho_0 \in H^{2+\alpha}, \end{cases}$$

there exists a unique temporally global regular solution (v, ρ) of (1.2) *such that, for an arbitrary $T \in (0, +\infty)$, v and v_x are bounded in $R^1 \times [0, T]$. Moreover, we have*

(8.17) $$|v(x, t)| \leqq |v_0|^{(0)}, \quad 0 < \underline{\rho}_0 e^{-\|\rho_0 v_0\|_{L^1(R^1)}} \leqq \rho(x, t)$$

$$\leqq \bar{\bar{\rho}}_0 \cdot e^{+\|\rho_0 v_0\|_{L^1(R^1)}}, \quad |v_x(x, t)| \leqq K_1(\|v_0\|^{(1)},$$

$$\|v_0 - A\|_{L^1(R^1)}, \quad \|\rho_0\|^{(1)}, \quad (\rho_0)^{-1}) < +\infty,$$

where $K_1 \nearrow$ *as each argument* $\nearrow$.

Proof. We can rewrite (5.20)′ in the following way:

$$\log(1+\omega(x_0, t_0)) = \log(1+\omega(a, t_0)) + \int_a^{x_0} \frac{\rho_0}{\mu}(\hat{v} - A)dx_0 + \int_a^{x_0} \frac{\rho_0}{\mu}(A - v_0)dx_0. \tag{8.18}$$

It is easily to be seen that the function Ψ^* defined by

$$\Psi^*(x_0, t_0) \equiv \log(1+\omega(a, t_0)) + \int_a^{x_0} \frac{\rho_0}{\mu}(\hat{v} - A)dx_0 \tag{8.19}$$

satisfies the equation

$$\begin{cases} \Psi^*_{t_0} = \dfrac{\mu}{1+\omega}\left(\dfrac{\Psi^*_{x_0}}{\rho_0}\right)_{x_0}, \\ \Psi^*(x_0, 0) = \displaystyle\int_a^{x^0} \frac{\rho_0}{\mu}(v_0 - A)dx_0. \end{cases} \tag{8.20}$$

Seeing that $\hat{v} - A (\in H_T^{2+\alpha})$ satisfies

$$\begin{cases} (\hat{v} - A)_{t_0} = \dfrac{\mu}{\rho_0(1+\omega)}(\hat{v} - A)_{x_0 x_0} - \dfrac{\hat{v} - v_0}{1+\omega}(\hat{v} - A)_{x_0}, \\ (\hat{v} - A)(x_0, 0) = v_0 - A \in H^{2+\alpha} \cap L^1(R^1), \end{cases} \tag{8.21}$$

we have the boundness of $\|\rho_0(\hat{v} - A)(\cdot, t_0)\|_{L^1(R^1)}$ on $[0, T]$. Thus, by Lemma 5.1 it holds that

$$|\Psi^*(x_0, t_0)| \leqq \left\| \frac{\rho_0}{\mu}(v_0 - A) \right\|_{L^1(R^1)}. \tag{8.22}$$

Hereafter, the proecedure of the proof is analogous to that of Theorem 8.1. Q.E.D.

(II). In (I), we have demonstrated some theorems on the existence of a temporally global regular solution of (1.2) under the initial condition (2.1)–(6.1). There occurs a question whether we have not any such solution of (1.2) under other initial conditions. Here, we shall discuss somewhat on it.

Theorem 8.4. *For the initial condition*

$$(8.23)\quad \begin{cases} v(x,0)=v_0(x)\in H^{2+\alpha}, \quad v_0'\geqq 0, \\ \rho(x,0)=\rho_0(x)\in H^{1+\alpha}, \qquad (0<\bar{\rho}_0\leqq \rho_0(x)\leqq \bar{\bar{\rho}}=|\rho_0|^{(0)} \\ \qquad\qquad\qquad\qquad\qquad\qquad < +\infty), \end{cases}$$

there exists a unique temporally global regular solution of (1.2) *such that, for an arbitrary* $T\in(0,+\infty)$, v *and* v_x *are bounded in* $R^1\times[0,T]$. *Moreover, it holds that*

$$(8.24)\quad \begin{cases} |v(x,t)|\leqq |v_0|^{(0)}, \quad 0\leqq v_x(x,t)\leqq |v_0'|^{(0)}, \\ 0<\bar{\rho}_0 e^{-|v|_0'^{(0)}\cdot T}\leqq \rho(x,t)\leqq |\rho_0|^{(0)}. \end{cases}$$

Proof. First we estimate $\ll v \gg_T^{(2+\alpha)}$ a priori. By the assumption and Lemma 8.3, (v,ρ) satisfying (2.1) and (8.23) in $R^1\times[0,T]$ belongs to $H_T^{2+\alpha}\times B_T^1$, and, moreover, v_{xxx} and v_{xt} exist in $R^1\times(0,T]$, being continuous there. Thus, v_x satisfies the equation

$$(8.25)\quad \begin{cases} (v_x)_t=\dfrac{\mu}{\rho}(v_{xx})_{xx}+\left\{\left(\dfrac{\mu}{\rho}\right)_x-v\right\}\cdot(v_x)_x-v_x(v_x), \\ \qquad\qquad\qquad\qquad (0<t\leqq T), \\ v_x(x,0)=v_0'(x), \end{cases}$$

where, by Lemma 2.1,

$$(8.26)\quad \left(\frac{\mu}{\rho}\right)_x-v=-\mu\{\rho_0(x_0(x,t))\}^2\cdot\rho_0'(x_0(x,t))+v_0(x_0(x,t))$$

It is obvious that

$$(8.27)\quad \frac{\mu}{\rho}\in H_T^\alpha, \quad \left|\left(\frac{\mu}{\rho}\right)_x-v\right|_T^{(0)}+\left|\left(\frac{\mu}{\rho}\right)_x-v\right|_{x,T}^{(\alpha)}<+\infty,$$

$$|-v_x|_T^{(0)}+|-v_x|_{x,T}^{(\alpha)}<+\infty,$$

Accordingly, we can construct the fundamental solution $\Gamma'(x, t; y, \tau)$ of the linear equation

(8.28) $$V_t=\frac{\mu}{\rho}V_{xx}+\left\{\left(\frac{\mu}{\rho}\right)_x-v\right\}\cdot V_x-v_x\cdot V.$$

We remark here that Γ' is unique so far as bounded solutions, regular in $R^1\times(0, T)$, are concerned. By making use of Γ', v_x is expressed in the form

(8.29) $$v_x(x, t)=\int_{R^1}\Gamma'(x, t; y, 0)v_0'(y)dy.$$

Hence, by the non-negativity (or, strictly, positivity) of Γ' and by the non-negativity of v_0',

(8.30) $$v_x(x, t)\geqq 0, \qquad (0\leqq t\leqq T).$$

We note that, if $v_0'\geqq 0$ and $v_0'\not\equiv 0$, then $v_x>0$. Moreover, clearly we have

(8.31) $$|v(x, t)|\leqq|v_0|^{(0)}.$$

Furthermore, by the theorem of comparison, $v_x(x, t)$ is bounded from above by the unique bounded regular solution $w(x, t)$ of the linear equation

(8.32) $$\begin{cases} w_t=\frac{\mu}{\rho}w_{xx}+\left\{\left(\frac{\mu}{\rho}\right)_x-v\right\}\cdot w_x, & (0<t\leqq T), \\ w(x, 0)=v_0'(x) \quad (\geqq 0), & (\text{N. B.: } -v_x\cdot v_x\leqq 0), \end{cases}$$

i.e., we have

(8.33) $$0\leqq v_x(x, t)\leqq w(x, t)\leqq|v_0'|^{(0)}.$$

The procedure following this is analogous to those made in §5~§8(I). Thus, we have the following a priori estimate for $\langle\!\langle v\rangle\!\rangle_T^{(2+\alpha)}$:

(8.34) $$\langle\!\langle v\rangle\!\rangle_T^{(2+\alpha)}\leqq K_3[T; v_0, \rho_0]<+\infty, \qquad (K_3\nearrow \text{as} T\nearrow).$$

Therefore, the assertion of the theorem is completed. Q. E. D.

In (8.23), if $v_0' \leqq 0$ and $\neqq 0$, then $-v_x$ satisfies

$$(8.35) \qquad (-v_x)_t = \frac{\mu}{\rho}(-v_x)_{xx} + \left\{\left(\frac{\mu}{\rho}\right)_x - v\right\}\cdot(-v_x)_x + (-v_x)^2,$$

which suggests that v_x may blow up in the course of a finite time. This problem presents us a great interest and is worth while to be solved. We give an example in which v and v_x are not generally bounded in x and, in some case, blow up in a finite time. If we define v and ρ by

$$(8.36) \qquad \begin{cases} v(x, t) = \dfrac{ax}{1+at} & \text{(hence, } v_x = \dfrac{a}{1+at} \text{ and } v_0(x) = ax), \\ \rho(x, t) = \rho_0^* \dfrac{1}{1+at} & \text{(hence, } \rho_0(x) = \rho_0^*), \quad (t \geqq 0), \end{cases}$$

where a is a constant and ρ_0^* is a positive constant, then (v, ρ) satisfies (1.2). The behaviours of v, v_x, and ρ in (8.36) vary according as $a \geqq 0$ or $a < 0$, as is easily to be seen.

Finally, we add that, in the one-dimentional problem of (1.1), the uniqueness of (v, θ, ρ) in $H_T^{2+\alpha} \times H_T^{2+\alpha} \times B_T^1$ under the initial condition

$$(8.37) \qquad (v_0, \theta_0, \rho_0) \in H^{2+\alpha} \times H^{2+\alpha} \times H^{1+\alpha}$$

is proved on the basis of a lemma similar to Lemma 2.1 and yet more complicated. The details of the proof will be given on another occasion.

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