J. Math. Kyoto Univ. (JMKYAZ)
14-1 (1974) 119-123

On representations of martingales

By

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(Communicated by Professor Yoshizawa, Feb. 6, 1973)

1. Introduction

The problem of which martingales can be transformed to Brownian motion by a random time change has been, under various conditions and methods considered by several authors ([1], [3], [4]).

In this paper we consider similar problems. In particular we drop the condition of nowhere or near nowhere constancy of paths of the random processes, which was used previously. Our method consists of a direct utilization of theorem 5.3 in [2], and a decomposition theorem for square integrable martingales, due to P. A. Meyer ([5], [6]). In section 3 we state and prove two representation theorems. Representations here are linked to an adjoining Brownian motion. In section 4 such an adjunction will be removed. Our approach seems to be simple and more transparent. We consider the one-dimensional case only.

2. Preliminaries

Let $(\Omega, \mathscr{A}, P)$ be a probability space. A random process $X=(X_t, \mathscr{A}_t)$ is a family of random variables X_t, $t\in[0, +\infty)$, defined on Ω, $\mathscr{A}_t$ is a family of an increasing sub-σ-algebra of $\mathscr{A}$, and where X_t is $\mathscr{A}_t$-measurable. All σ-algebras are assumed to be complete relative to P. A random process $(X_t, \mathscr{A}_t)$ is called martingale (square integrable martingale) if $E|X_t|<+\infty (EX_t^2<+\infty)$, $t\in[0, +\infty)$, $E\{X_t|\mathscr{A}_s\}=X_s$ whenever $t>s$. A random process is continuous if the paths $t\rightarrow X_t(\omega)$ are continuous for almost all $\omega\in\Omega$.

Let $\mathscr{A}_{t+}=\bigcap_{\epsilon>0}\mathscr{A}_{t+\epsilon}$. If $(X_t, \mathscr{A}_t)$ is a continuous martingale, then so is $(X_t, \mathscr{A}_{t+})$. In situations like this we may and do assume that:

$$\mathscr{A}_{t+} = \mathscr{A}_t,\ t \in [0, +\infty) \quad \text{i.e. the family of } \mathscr{A}_t\text{'s}$$

is right continuous. A process $(A_t, \mathscr{A}_t)$ is called an increasing process if the paths $t \to A_t$ are continuous and increasing.

Let $(X_t, \mathscr{A}_t)$ be a continuous, square integrable martingale. Then there is a unique increasing (natural) process $<X>_t$, $E<X>_t < +\infty$, and such that $X_t^2 - <X>_t$ is a martingale. This result is only a special case of a more general theorem due to P.A. Meyer ([5], [6]).

For convenience, we now state theorem 5.3 of [2].

Theorem 2.1 (Theorem 5.3 [2]).

Suppose $(X_t, \mathscr{A}_t)$ is a continuous square integrable martingale, and there exists a nonnegative random process $(\Phi_t, \mathscr{A}_t)$ measurable in (t, ω) relative to $\mathscr{B} \times \mathscr{A}$ where $\mathscr{B}$ is the Borel sets of $[0, +\infty)$, such that:

$$E\{(X_{t_2} - X_{t_1})^2 | \mathscr{A}_{t_1}\} = E\left\{\int_{t_1}^{t_2} \Phi_s ds | \mathscr{A}_{t_1}\right\}$$

whenever $t_2 > t_1$. If the set $\{(t, \omega): \phi(t, \omega) = 0\}$ has $dt \times dP$ measure zero, there exists a Brownian motion $(B_t, \mathscr{A}_t)$, such that:

$$X_t = X_0 + \int_0^t \Phi_s^{1/2} dB_s$$

Even without this additional hypothesis on the vanishing of Φ, this representation is valid, after a Brownian motion has been adjoined to $(X_t, \mathscr{A}_t)$ process.

Before proceeding to the next section, as a matter of notation, $I(A)$ will denote the indicator function of A; i.e. $I(A) = 1$ on A, $= 0$ off A. The value of $I(A)$ at a is denote by $I(A)(a)$. Also we write $a \wedge b$ for $\min(a, b)$. Random variables will be starred when viewed on the space obtained by adjoining to $(\Omega, \mathscr{A}, P)$ a space carrying a Brownian motion. Several statements below should be interpreted as holding almost everywhere P.

3. Representation theorems

Theorem 3.1

If $(X_t, \mathscr{A}_t)$ *is a continuous square integrable martingale satisfying* $X(0)=0$, *then* $X_t^*=B(<X>_t^*)$. *B is a Brownian motion.*

PROOF. $X_t^2-<X>_t$ is a martingale by P. A. Meyer decomposition. $<X>_t$ is continuous by continuity of X_t. Let $T(t)=\inf\{s: <X>_s>t\}$ where by convention $\inf\phi=+\infty$. $T(t)$ is a stopping time, and so is $T(t)\wedge r$ for each positive number r. Consequently, $Y(t)=X(T(t)\wedge r)$ is a continuous square integrable martingale. Applying the decomposition theorem on $Y(t)$ we obtain:

$$<Y>_t=<X>_{T(t)\wedge r}$$

But $<X>_{T(t)\wedge r}=t\wedge<X>_r$, so that

$$\frac{d<Y>_t}{dt}=I(\{s: s<<x>_r\})(t)$$

This $I(\{s: s<<X>_r\})(t)$, relative to $Y(t)$ process, satisfies the conditions required on Φ_t in theorem 2.1. Hence

$$Y^*(t)=\int_0^t I^*(\{s: s<<X>_r\})^{1/2}(\tau)dB_\tau$$

$$=B(t\wedge<X>_r^*)$$

where B is a Brownian motion. Or $X^*(T^*(t)\wedge r)=B(t\wedge<X>_r^*)$. From which follows that $X^*(t\wedge r)=B(<X>_t^*\wedge<X>_r^*)$. Letting $r\to\infty$, $X^*(t)=B(<X>_t^*)$

Theorem 3.2

If $(X_t, \mathscr{A}_t)$ *is a continuous, unbounded martingale satisfying* $X(0)=0$, *then* $X_t^*=B(A_t^*)$. *B is a Brownian motion, and* A_t^* *is an increasing process.*

Remark. A theorem similar to this was proved in [3], under further condition that X_t is nowhere constant.

PROOF. For an integer N, let $T_N=\inf\{t: |X_t|>N\}$ then T_N is finite, and $Y_N(t)=X(t\wedge T_N)$ satisfies the hypothesis of theorem 3.1. Hence

$$Y_N^*(t)=B(<Y_N>_t^*).$$

Since $<Y_N>_t=<Y_{N'}>_t$ on $[0, T_N]$ whenever $N'>N$, $\lim_{N\to\infty} <Y_N>_t=A_t$ exists and is increasing process. Clearly $X^*(t)=B(A_t^*)$

4. Remarks.

To simplify and clarify our notation let us denote $I(\{s: s<<X>_r\})(t)$ by $I_r(t)$. Using (2 pp. 449)

$$B_r(t)=\int_0^t I_r^{*-1/2}(\tau)dY_r^*(\tau)+\int_0^t(1-I_r^{*-1/2}(\tau)I_r^{*1/2}(\tau))d\hat{B}^*(\tau)$$

where $Y_r(t)=X(T(t)\wedge r)$, $I_r^{-1/2}(t)=0$ whenever $I_r^{1/2}(t)=0$, and $\hat{B}$ is an independent Brownian motion adjoined to $Y_r(t)$ process is a Brownian motion. This $\hat{B}$ is fixed throughout.

Defining $r_\infty=\inf\{s: <X>_t$ is constant on $[s, +\infty)\}$, it is easy to see that $\lim_{r\to\infty} B_r(t)=B(t)$

$$=\begin{cases} X^*(T^*(t)), & t<<X>_{r_\infty} \\ X^*(r_\infty^*)+\hat{B}^*(t)-\hat{B}^*(<X>_{r_\infty}), & t\geq <X>_{r_\infty} \end{cases}$$

Being limit of Brownian motions, B itself is a Brownian motion. The full Brownian motion in theorem 3.1 can be take to be this B. $X(T(t))$, $t<<X>_{r_\infty}$ viewed on $(\Omega, \mathscr{A}, P)$ is a Brownian motion stopped at $<X>_{r_\infty}$, i.e. $X(T(t))$ and $B(t)$, $t<<X>_{r_\infty}$ have the same distribution. Letting $B_s(t)=X(T(t))$, $t<<X>_{r_\infty}$,

$$X(t)=B_s(<X>_t),\ t\geq 0$$

is a representation of X on $(\Omega, \mathscr{A}, P)$ in terms of B_s, a stopped Brownian motion at $<X>_{r_\infty}$.

The above considerations, applied to theorem 3.2, will result, in view of sample paths unboudeness, $X(t)=B(A_t)$ where B is a full Brownian motion on $(\Omega, \mathscr{A}, P)$.

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