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On the cut locus and the topology of Riemannian manifolds

By

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1. Introduction.

Let M be a connected complete Riemannian manifold with $\dim M \geqq 2$. Let p be a point in M and let $Q(p)$ (resp. $C(p)$) be the conjugate locus (resp. the cut locus) in the tangent space $T_p(M)$ to M at p. (For the precise definitions of $Q(p)$ and $C(p)$, see section 2.) We say that M satisfies condition (C) at p or the pair (M, p) satisfies condition (C) if $Q(p)$ and $C(p)$ do not have common points.

In this paper, we study the structure of the cut locus $C(p)$ and the topology of the Riemannian manifold M assuming that M satisfies condition (C) at a given point p.

A. D. Weinstein [8] showed that any compact manifold M with $\dim M \geqq 3$ always admits a Riemannian metric g which satisfies condition (C) at some point p in M. Therefore, for our purpose, we need some further assumptions on the Riemannian manifold. The principal tool in our study is the map $N_p: C(p) \to \mathbf{N} \cup \{+\infty\}$ defined by

$$N_p(v) = \#\{w \in C(p);\ \exp_p v = \exp_p w\}$$

for all $v \in C(p)$, where $\exp_p: T_p(M) \to M$ denotes the exponential map.

The main results are stated as follows.

Theorem A. *Assume that (M, p) satisfies condition (C). Then we have*

(1) *The set $N_p^{-1}(2) = \{v \in C(p);\ N_p(v) = 2\}$ is open and dense in*

$C(p)$;

(2) *Define a map* $f: N_p^{-1}(2) \to N_p^{-1}(2)$ *by* $f(v) \neq v$ *and* $\exp_p \cdot f(v) = \exp_p v$, *then* f *is a homeomorphism.*

Theorem B. *Assume that (i) M is compact, (ii) (M, p) satisfies condition (C) and (iii) $N_p \equiv 2$. Then we have*

(1) *The fundamental group of M is of order two;*

(2) *The universal covering space of M is homeomorphic to a sphere.*

Theorem C. *Assume that (i) (M, p) satisfies condition (C) and (ii) each geodesic emanating from p is a simple periodic curve with a common length. Then M is diffeomorphic to a real projective space.*

Theorem D. *Assume that (i) M is a 2-dimensional compact Riemannian manifold and (ii) (M, p) satisfies condition (C). Then M is not simply connected.*

Let H_p denote the group of isometries of M which fix a point p in M.

Theorem E. *Suppose that M is a 3-dimensional compact Riemannian manifold and that there is a point p in M such that* $\dim H_p \geqq 1$. *Further suppose that (M, p) satisfies condition (C). Then M is not simply connected.*

Combining Theorem D and Theorem E with Rauch's comparison theorem (cf. [5] p. 76 Theorem 4.1), we obtain

Theorem F. *Let k be a positive number. Suppose that M is a compact simply connected Riemannian manifold and that the sectional curvature of M is at most k. Further suppose that the following (i) or (ii) holds.*

(i) $\dim M = 2$.

(ii) $\dim M = 3$ *and there is a point p in M such that* $\dim H_p \geqq 1$.

Then the diameter of M is at least $\pi/\sqrt{k}$.

Remark 1. Given a point p in M, the Riemannian manifold M

is called a C_p-manifold with a common length $2l$ if it satisfies condition (ii) in Theorem C at the point p and if the common length is $2l$. Theorem C is a partial refinement of R. Bott [3] in which the cohomology groups of C_p-manifolds were studied by the application of the Morse theory. We will prove in section 4 that if M is not simply connected and if M is a C_p-manifold for some point p in M, then the pair (M, p) satisfies condition (C) (Proposition 4.1). Therefore this fact combined with Theorem C yields the following.

Theorem C′. *Assume that (i) M is not simply connected and (ii) M is a C_p-manifold for some point p in M. Then M is diffeomorphic to a real projective space.*

A theorem of L. W. Green (cf. [2] VIII. 9) states that if M is homeomorphic to the 2-dimensional real projective space and if M is a C_p-manifold for any point p in M, then M is isometric to the 2-dimensional real projective space with the standard metric.

Remark 2. Theorem D was first proved by S. B. Myers [7] in the real analytic case. We will give a different proof which is useful in the proof of Theorem E.

Remark 3. Let k be a positive number and let K_M be the sectional curvature of M. In case of even dimension, we know the following fact: (*) If M is a simply connected Riemannian manifold with $0 < K_M \leqq k$, then the diameter of M is at least $\pi/\sqrt{k}$. This follows immediately from the next theorem of W. Klingenberg [4].

Theorem. *If M is an even-dimensional compact simply connected manifold and if $0 < K_M \leqq k$, then we have $d(p, \tilde{C}(p)) \geqq \pi/\sqrt{k}$ for any point p in M, where d denotes the distance on M and $\tilde{C}(p)$ is the cut locus of p.* (For the precise definition of $\tilde{C}(p)$, see section 2.)

In case of odd dimension, the assertion of the above theorem is false in general. In fact, M. Berger [1] presented a 1-parameter family of counter examples $SU(2)\times\mathbf{R}/H_\alpha$ $(0<\alpha<\alpha_0)$ which are diffeomorphic to the 3-dimensional standard sphere. However Theorem F indicates that the weaker assertion (*) remains true even for the examples of M. Berger.

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2. General properties of the cut locus.

Let v be a non-zero tangent vector at a point p in M. We define $\mu(v)$ by

$$\mu(v)=\sup_{r>0}\{\|rv\|;\ d(p,\exp_p rv)=\|rv\|\},$$

where d denotes the distance function on M. Let $T_p(M)$ denote the tangent space to M at p.

Proposition 2.1. *The map* μ: $T_p(M)-\{0\}\to\mathbf{R}\cup\{+\infty\}$ *is continuous.* (cf. [5] p. 98 Theorem 7.3.)

We define the cut locus $C(p)$ of p in $T_p(M)$ by

$$C(p)=\{v\in T_p(M)-\{0\};\ \mu(v)=\|v\|\}$$

and the cut locus $\tilde{C}(p)$ of p by

$$\tilde{C}(p)=\exp_p(C(p)).$$

The point in $\tilde{C}(p)$ is said to be a cut point of p. The conjugate locus $Q(p)$ of p in $T_p(M)$ is defined by

$$Q(p)=\{v\in T_p(M);\ \exp_p \text{ is not of maximal rank at } v\}.$$

For each $v\in Q(p)$, the point $\exp_p v$ is said to be a conjugate point of p along the geodesic $\exp_p tv$ $(0\leqq t\leqq 1)$.

Let $\mathscr{S}(p)$ denote the subset of $T_p(M)$ consisting of vectors w such that $d(p,\exp_p w)=\|w\|$. By Proposition 2.1, we can easily prove the following.

Proposition 2.2. (1) *$C(p)$ is the boundary of $\mathscr{S}(p)$.*

(2) *$\mathscr{S}(p)-C(p)$ is homeomorphic to the n-dimensional open ball, where $n=\dim M$.*

Since we have assumed that M is complete, we obtain

Proposition 2.3. *The map $\exp_p: \mathscr{S}(p)\to M$ is surjective.*

Proposition 2.4. *The map $\exp_p: \mathscr{S}(p)-C(p)\to M-\tilde{C}(p)$ is a diffeomorphism.* (cf. [5] p. 100 Theorem 7.4.)

Proposition 2.5. *Let q be a cut point of p such that $d(p, q)=d(p, \tilde{C}(p))$. Let $c_1(t)$ and $c_2(t)$ $(0\leqq t\leqq 1)$ be distinct geodesics from p to q. Suppose that the lengths of c_1 and c_2 are equal to $d(p, q)$ and that q is not a conjugate point of p along c_i $(i=1, 2)$. Then the curve $c(t)$ $(0\leqq t\leqq 1)$ defined by*

$$c(t)=c_1(2t) \qquad (0\leqq t\leqq 1/2)$$

$$c(t)=c_2(2-2t) \qquad (1/2\leqq t\leqq 1)$$

is smooth at $q=c(1/2)$.

Proof: Let $\dot{c}_i(t)$ be the tangent vector of the curve c_i at the point $c_i(t)$ $(i=1, 2)$. We clearly have $\|\dot{c}_1(t)\|=\|\dot{c}_2(t)\|=d(p, q)$. Suppose that $\dot{c}_1(1)\neq -\dot{c}_2(1)$. Then there is a tangent vector v at q such that both $g(\dot{c}_1(1), v)$ and $g(\dot{c}_2(1), v)$ are negative, where g denotes the Riemannian metric. Let $\gamma(\alpha)$ $(0\leqq\alpha\leqq\alpha_0)$ be a curve emanating from q with the initial tangent vector v. Since $\dot{c}_i(0)\notin Q(p)$, there is a curve $\gamma_i(\alpha)$ $(0\leqq \alpha\leqq\alpha_1\leqq\alpha_0)$ in $T_p(M)$ emanating from $\dot{c}_i(0)$ such that $\exp_p\cdot\gamma_i(\alpha)=\gamma(\alpha)$ $(i=1, 2)$. Let $c_{i,\alpha}(t)=\exp_p t\gamma_i(\alpha)$ and let $L_i(\alpha)$ denote the length of the curve $c_{i,\alpha}(t)$ $(0\leqq t\leqq 1)$. By the variation theory, we have

$$\frac{d}{d\alpha}L_i(0)=g\left(\frac{\dot{c}_i(1)}{\|\dot{c}_i(1)\|}, v\right)<0$$

(cf. [5] p. 80 Theorem 5.1). Hence there is a positive number α_2 such that $L_i(\alpha_2)<L_i(0)$ $(i=1, 2)$. Since c_1 and c_2 are distinct geodesics, we may assume that $c_{1,\alpha_2}(t)$ and $c_{2,\alpha_2}(t)$ $(0\leqq t\leqq 1)$ are distinct geodesics from p to $\gamma(\alpha_2)$. Moreover we may assume that $L_1(\alpha_2)\leqq L_2(\alpha_2)$. Then

we have $\mu(\gamma_2(\alpha_2)) \leqq L_2(\alpha_2)$, which implies that

$$d(p, \tilde{C}(p)) \leqq \mu(\gamma_2(\alpha_2)) \leqq L_2(\alpha_2) < L_2(0) = d(p, q).$$

It contradicts the choice of q.

3. The map N_p.

The following proposition is clear by the definition of N_p.

Proposition 3.1. *If (M, p) satisfies condition (C), then the map N_p: $C(p) \to \mathbf{N} \cup \{+\infty\}$ is upper semi-continuous.*

Proposition 3.2. *If (M, p) satisfies condition (C), then for any $v \in C(p)$ we have $2 \leqq N_p(v) < \infty$.* (cf. [5] p. 97 Theorem 7.1.)

Lemma 3.3. *Assume that (M, p) satisfies condition (C). Let u and v be two vectors in $C(p)$ such that* $\exp_p u = \exp_p v$. *Assume that N_p is locally constant around u. Then, for any neighborhood U of v in $C(p)$, there exists a neighborhood $U(u)$ of u in $C(p)$ such that* $\exp_p(U(u)) \subset \exp_p(U)$.

Proof: *Suppose that the conclusion is not true. Then we have a sequence* $\{u_i\}_{i=1,2,\dots}$ of vectors in $C(p)$ such that $\lim u_i = u$ and $\exp_p u_i \notin \exp_p(U)$ for any i. We have $N_p \equiv N_p(u)$ (which we denote by m) around u. Hence we can find vectors $u_i^j \in C(p)$ $(1 \leqq j \leqq m)$ having the properties that $u_i^1 = u_i$, $\exp_p u_i^j = \exp_p u_i$ and $u_i^j \neq u_i^k$ $(j \neq k)$. We may assume that the sequences $\{u_i^j\}_{i=1,2,\dots}$ are convergent. Let $\lim u_i^j = u^j$. By the choice of u_i^j, u^j $(j = 1, 2, \dots, m)$ are not contained in U. And condition (C) implies that $u^j \neq u^k$ $(j \neq k)$. Hence we have

$$\{w \in C(p);\ \exp_p w = \exp_p u\} \supset \{u^1, \dots, u^m, v\},$$

implying that $N_p(u) > m$. It is contradictory to the assumption.

Lemma 3.4. *Let u and v be the vectors as in Lemma 3.3. Assume that (M, p) satisfies condition (C) and that N_p is locally constant around u. Then we have neighborhoods $U(u)$ and $U(v)$ of u and v*

respectively in $C(p)$ such that the map $f_{vu}=(\exp_p|U(v))^{-1}\circ(\exp_p|U(u))$ is well defined and a homeomorphism of $U(u)$ onto $U(v)$.

In order to prove the above lemma, we need the following theorem in the dimension theory.

Theorem 3.5 (Brouwer's invariance theorem of domain). *Let Y and Y' be subsets of $\mathbf{R}^n$. Let f be a homeomorphism of Y onto Y'. If p is an inner point (resp. a boundary point) of Y, then $f(p)$ is also an inner point (resp. a boundary point) of Y'.*

Proof of Lemma 3.4: Let U' be a neighborhood of v in $T_p(M)$ such that the map $\exp_p|U'$ is a diffeomorphism onto some open set of M. Let $U=U'\cap C(p)$. Then, by Lemma 3.3, we obtain a neighborhood $U(u)$ of u in $C(p)$ such that $\exp_p(U(u))\subset\exp_p(U)$ and such that the map $\exp_p|U(u)$ is injective. It is clear that the map

$$f_{vu}=(\exp_p|U)^{-1}\circ(\exp_p|U(u))=(\exp_p|U')^{-1}\circ(\exp_p|U(u))$$

is well defined and a continuous injection. On the other hand, by Proposition 2.1, we know that $C(p)$ is a submanifold of $T_p(M)$ (in the C^0 sense). Especially $C(p)$ is a locally compact Hausdorff space, implying that f_{vu} is a homeomorphism. Hence we can apply Theorem 3.5 to the map $f_{vu}: U(u)\to f_{vu}(U(u))$ and we can conclude that f_{vu} is an open map. Therefore f_{vu} is a homeomorphism of $U(u)$ onto an open set of U. Put $U(v)=f_{vu}(U(u))$.

Lemma 3.6. *If (M, p) satisfies condition (C) and if N_p is locally constant around a vector $u\in C(p)$, then $N_p(u)=2$.*

Proof: By Proposition 3.2, we have a vector $v\in C(p)$ such that $\exp_p v=\exp_p u$ and $u\neq v$. By Lemma 3.4, we have neighborhoods $U(u)$ and $U(v)$ of u and v respectively in $C(p)$ such that

(i) the maps $\exp_p|\overline{U(u)}$ and $\exp_p|\overline{U(v)}$ are injective,

(ii) the map $f_{vu}=(\exp_p|U(v))^{-1}\circ(\exp_p|U(u)): U(u)\to U(v)$ is a homeomorphism,

(iii) there is a homeomorphism $h: B^{n-1}=\{x\in\mathbf{R}^{n-1};\ \|x\|<1\}$

$\rightarrow U(u)$,

where "——" denotes the closure and $\dim M = n$. Hence we can define a homeomorphism H from $B^{n-1} \times (0, 1)$ onto an open set of M as follows:

$$H(x, t) = \exp_p((1/2 + t)h(x)) \qquad (0 < t \leqq 1/2),$$

$$H(x, t) = \exp_p((3/2 - t)f_{vu} \circ h(x)) \qquad (1/2 \leqq t < 1).$$

If $N_p(u) \geqq 3$, there is a vector $w \in C(p) - \overline{U(u)} \cup \overline{U(v)}$ such that $\exp_p w = \exp_p u$. Since $\exp_p u$ is an inner point of $H(B^{n-1} \times (0, 1))$, the geodesic $\exp_p tw$ $(0 \leqq t < 1)$ must intersect the boundary of $H(B^{n-1} \times (0, 1))$ which is contained in $\{\exp_p ry;\ y \in \overline{U(u)} \cup \overline{U(v)}$ and $0 < r \leqq 1\}$. By Proposition 2.4, we see that $\{\exp_p tw;\ 0 \leqq t < 1\}$ and $\exp_p \overline{(U(u))}$ $(= \exp_p \overline{(U(v))})$ do not have common points. If $\{\exp_p tw;\ 0 \leqq t < 1\}$ and $\{\exp_p ry;$ $y \in \overline{U(u)} \cup \overline{U(v)}$ and $0 < r < 1\}$ have a common point, Proposition 2.4 implies that $w \in \overline{U(u)} \cup \overline{U(v)}$. It is contradictory to the choice of w.

By the lemmata and propositions above, we obtain Theorem A.

Proof of Theorem B: Let $\dim M = n$. Since M is compact, we can define a homeomorphism H: $V^n = \{v \in T_p(M);\ \|v\| \leqq 1\} \rightarrow \mathscr{S}(p)$ by

$$H(v) = \mu(v) \cdot v \qquad \text{for} \quad v \neq 0,$$

$$H(0) = 0.$$

Since $N_p^{-1}(2) = C(p)$, we can extend the map f defined in Theorem A to a map f: $T_p(M) \rightarrow T_p(M)$ in such a way that

$$f(rv) = r \cdot f(v), \quad \text{where} \quad v \in C(p) \quad \text{and} \quad r \geqq 0.$$

Let us identify $T_p(M)$ with $\mathbf{R}^n$ with respect to an orthonormal basis and define a map F: $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1};\ \sum_{i=1}^{n+1} x_i^2 = 1\} \rightarrow M$ by

$$F(x_1, \ldots, x_{n+1}) = \exp_p \cdot H(x_1, \ldots, x_n) \qquad \text{for} \quad x_{n+1} \geqq 0,$$

$$F(x_1, \ldots, x_{n+1}) = \exp_p \cdot f \cdot H(x_1, \ldots, x_n) \qquad \text{for} \quad x_{n+1} \leqq 0.$$

It is clear by Theorem 3.5 that F is a local homeomorphism. Hence

$F: S^n \to M$ is a double covering and the theorem follows.

4. C_p-manifolds.

Proposition 4.1. *Suppose that M is a C_p-manifold for some point p in M and that M is not simply connected. Then (M, p) satisfies condition (C).*

In fact, we can express $Q(p)$ as follows.

Proposition 4.2. *Suppose that M is a C_p-manifold with a common length $2l$ for some point p in M and that M is not simply connected. Then we have*

$$Q(p)=\{v \in T_p(M);\ \|v\|=2ml, \qquad m \in \mathbf{N}\}.$$

Lemma 4.3. *Let v be a non-zero tangent vector to M at p. Suppose that M is a C_p-manifold with a common length $2l$. Then we have*

$$\text{the rank of } \exp_p \text{ at } v = \text{the rank of } \exp_p \text{ at } \left(1+\frac{2lz}{\|v\|}\right)v$$

for any integer z with $1+\dfrac{2lz}{\|v\|} \neq 0$.

Proof: By our assumption, we have $\exp_p\left(1+\dfrac{2lz}{\|w\|}\right)w=\exp_p w$ for any $w \in T_p(M)-\{0\}$ and $z \in \mathbf{Z}$. Hence the assertion is clear.

Lemma 4.4. *Let v and M be as in Lemma 4.3. Then there exists a positive number t_0 for which the following (1), (2) and (3) hold.*

(1) *If a geodesic $\exp_p tw$ $(0 \leqq t < \infty)$ passes the point $\exp_p t_0 v$, then v and w are linearly dependent.*

(2) *The point $\exp_p t_0 v$ is not the conjugate point of p along any geodesic.*

(3) *The geodesic $\exp_p tv$ $(0 \leqq t \leqq t_0)$ does not contain conjugate points of p along itself.*

Proof: Suppose that (1) is false for any $t_0 > 0$. Then there are tangent vectors $v_i \in T_p(M) - \mathbf{R}v$ and positive numbers t_i $(i=1, 2,\dots)$ such that $\exp_p v_i = \exp_p t_i v$ and $\lim t_i = 0$. Since any geodesic emanating from p is periodic with a length $2l$, we may assume that $\|v_i\| \leqq l$ $(i=1, 2,\dots)$. Hence we may assume that the sequence $\{v_i\}_{i=1,2,\dots}$ is convergent. Let $w = \lim v_i$. It is clear that $0 < \|w\| \leqq l$. On the other hand we have

$$\exp_p w = \lim \exp_p v_i = \lim \exp_p t_i v = p,$$

implying that the geodesic $\exp_p tw$ $(0 \leqq t \leqq \|w\|)$ is closed. Hence we obtain a closed geodesic emanating from p whose length is at most l. It contradicts the assumption. Hence there is a positive number t_1 such that (1) holds for any t_0 $(0 < t_0 \leqq t_1)$. We take $t_0 (\leqq t_1)$ small enough for which (3) holds. By Lemma 4.3, (2) also holds for the same t_0.

Proof of Proposition 4.2: It is clear that

$$Q(p) \supset \{v \in T_p(M);\ \|v\| = 2lm, \qquad m \in \mathbf{N}\}.$$

Suppose that there is a vector $v \in Q(p)$ such that $\|v\| \notin 2l\mathbf{N}$. By Lemma 4.3 we may assume that $\|v\| < 2l$. For this v we take the number t_0 as in Lemma 4.4 and we put $q = \exp_p t_0 v$. Let Ω denote the set of all curves in M joining p and q. Let $\lambda = \dim M -$ the rank of $\exp_p$ at v. Then it is clear that the index of the geodesic $\exp_p tv$ $(0 \leqq t \leqq t_0)$ is zero and that the indexes of the other geodesics in Ω are at least $\lambda (>0)$. Hence by the Morse theory we have

$$\pi_i(\Omega) = \{0\} \qquad (0 \leqq i < \lambda),$$

where π_i denotes the i-th homotopy group. (cf. [6] p. 95 Theorem 17.3.) Therefore we have $\pi_1(M) = \pi_0(\Omega) = \{0\}$. It contradicts the assumption.

Proposition 4.1 follows immediately from Proposition 4.2.

Proof of Theorem C: By our assumption (ii), any geodesic emanating from p is periodic with a common length, say $2l$. Hence we have $d(p, \tilde{C}(p)) \leqq l$. Let q be a cut point of p such that $d(p, q) =$

$d(p, \tilde{C}(p))$. Then by Proposition 3.2 and Proposition 2.5, we have a geodesic $c\colon [0, 1]\to M$ such that $c(0)=c(1)=p$, $c(1/2)=q$ and $L(c)=2\cdot d(p, q)$, where $L(c)$ denotes the length of c. We clearly have $2l\leqq L(c)$. Since $L(c)=2\cdot d(p, q)\leqq 2l$, it follows that $L(c)=2l$. Therefore we have $d(p, q)=d(p, \tilde{C}(p))=l$ and $d(p, q')=d(p, \tilde{C}(p))$ for any $q'\in\tilde{C}(p)$. Applying Proposition 3.2 and Proposition 2.5 again, we obtain $N_p\equiv 2$. Above lines also prove that $C(p)=\{v\in T_p(M);\ \|v\|=l\}$ and $f(v)=-v$ for any $v\in C(p)$. Hence the covering map defined in the proof of Theorem B is a local diffeomorphism and M is diffeomorphic to a real projective space.

5. 2-dimensional manifolds.

Throughout this section, we assume that M is a 2-dimensional compact Riemannian manifold and that (M, p) satisfies condition (C).

Lemma 5.1. *$N_p^{-1}(2)$ consists of a finite number of connected components.*

Proof: Assume that $N_p^{-1}(2)$ has an infinite number of connected components U_λ $(\lambda\in\Lambda)$ and take vectors $v_\lambda\in U_\lambda$. Since $C(p)$ is compact, $\{v_\lambda;\ \lambda\in\Lambda\}$ contains a convergent subsequence $\{v_i\}_{i=1,2,\dots}$. Let $u_i=f(v_i)$, where f is the map defined in Theorem A. Here we may assume that $\{u_i\}_{i=1,2,\dots}$ is also a convergent sequence. Let $\lim u_i=u$ and $\lim v_i=v$. Then, from condition (C), it follows that $u\neq v$. Since the map μ is continuous and since the sequences $\{v_i\}_{i=1,2,\dots}$ and $\{u_i\}_{i=1,2,\dots}$ are convergent, there exist simple curves c_i and c_i' in $\mathscr{S}(p)$ $(i=1, 2,\dots)$ such that

(a) $c_i(0)=v_i$, $c_i(1)=v_{i+1}$ and $c_i(t)\notin C(p)$ for $0<t<1$,

(b) $c_i'(0)=u_i$, $c_i'(1)=u_{i+1}$ and $c_i'(t)\notin C(p)$ for $0<t<1$,

(c) $\lim L(c_i)=\lim L(c_i')=0$.

We define closed curves $\gamma_i\colon [0, 1]\to M$ as follows:

$$\gamma_i(t)=\exp_p c_i(2t) \qquad (0\leqq t\leqq 1/2),$$

$$\gamma_i(t)=\exp_p c_i'(2-2t) \qquad (1/2\leqq t\leqq 1).$$

Then we have $\lim L(\gamma_i)=0$, implying that γ_i is simple for large i and that $\lim d(\gamma_i \cup \{\exp_p v\})=0$. Let V and V' be neighborhoods of v and u respectively in $T_p(M)$ such that $\exp_p|V$ and $\exp_p|V'$ are diffeomorphisms onto an open set U of M. Then there is an integer K such that γ_i $(i>K)$ are simple curves in U. By Jordan curve theorem, we see that $M-\gamma_i$ is composed of two connected components for large i. We denote the components by O_i and O_i' and suppose that $O_i' \ni p$. Then by the definition of γ_i it follows that O_i contains a point q_i which is in the image of the boundary points of U_i in $C(p)$. Hence there is a vector w_i in $C(p)-V \cup V'$ such that $\exp_p w_i=q_i$. Then the geodesic $\exp_p tw_i$ $(0<t<1)$ must intersect γ_i. By Proposition 2.4, we see that $\exp_p tw_i$ $(0<t<1)$ and $\gamma_i \cap \tilde{C}(p)$ have no common points. Therefore $\exp_p tw_i$ $(0<t<1)$ and $\gamma_i \cap (M-\tilde{C}(p))$ must have a common point. Then Proposition 2.4 implies that $w_i \in V \cup V'$, which is contradictory to the choice of w_i.

Lemma 5.2. *Let* $N_p^{-1}(2)=\bigcup_{i=1}^{m} U_i$, *where* U_i *is the connected component of* $N_p^{-1}(2)$. *If* $m \geqq 2$, *i.e.,* $N_p^{-1}(2) \neq C(p)$, *then* $f(U_i)$ *is also a connected component of* $N_p^{-1}(2)$ *and* $f(U_i) \neq U_i$ *for all* i.

Proof: Since the map f defined in Theorem A is a homeomorphism, it is clear that $f(U_i)$ is a connected component of $N_p^{-1}(2)$ for each i. Suppose that $f(U_i)=U_i$ for some i. First we note that there is a homeomorphism $h\colon U_i \to (0, 1)$. Then the map $\alpha=h \circ f \circ h^{-1}\colon (0, 1) \to (0, 1)$ is a homeomorphism such that $\alpha^2=id$. Hence α is the identity map or a monotone decreasing map. In either case, α must have a fixed point, i.e., f has a fixed point, which is contradictory to its definition.

Proof of Theorem D: If $N_p^{-1}(2)=C(p)$, i.e., $N_p \equiv 2$, by Theorem B we see that the fundamental group of M is of order two. Therefore we have only to prove Theorem D in case that $N_p^{-1}(2) \neq C(p)$. Let $N_p^{-1}(2)=\bigcup_{i=1}^{m} U_i$ as in Lemma 5.2. Then we have a component U_j and vectors v and u such that:

(a) u and v belong to the different connected components of $C(p)-(f(U_j) \cup U_j)$,

(b) $\exp_p u = \exp_p v$.

(If not, we obtain an infinite sequence $\{U_{i_k}\}_{k=1,2\dots}$ of connected components of $N_p^{-1}(2)$ such that $U_{i_{k+1}}$ and $f(U_{i_{k+1}})$ are contained in the same connected component V_k of $C(p)-(f(U_{i_k}) \cup U_{i_k})$, where $V_k \subset V_{k-1}$. It is impossible by Lemma 5.1.) We consider the diagram

$$\begin{array}{ccc} \mathscr{S}(p) & \xrightarrow{\ h\ } & \mathscr{S}(p)' \\ \big\downarrow{\scriptstyle \exp_p} & & \big\downarrow{\scriptstyle \overline{\exp_p}} \\ M & \xrightarrow{\ \bar{h}\ } & M' \end{array}$$

defined as follows:

(c) $\mathscr{S}(p)'$ is the space obtained from $\mathscr{S}(p)$ by identifying U_j with $f(U_j)$ through f,

(d) $M' = M/(\tilde{C}(p) - \exp_p(U_j))$ [1],

(e) h and $\bar{h}$ are natural projections,

(f) $\overline{\exp_p}(x) = \bar{h} \circ \exp_p \circ h^{-1}(z)$ for $z \in \mathscr{S}(p)'$.

Then it is clear that the diagram is commutative and the maps are continuous. Let $c(t)$ $(0 \leqq t \leqq 1)$ be the closed curve in M defined by

$$c(t) = \exp_p 2tu \qquad (0 \leqq t \leqq 1/2),$$

$$c(t) = \exp_p(2-2t)v \qquad (1/2 \leqq t \leqq 1).$$

We fix an orientation of $C(p)$ and we endow U_i $(i=1, 2,\dots, m)$ with the orientation as its subsets. If $f|U_j$ is orientation-preserving, M' is homeomorphic to the 2-dimensional real projective space $P^2(\mathbf{R})$ and $\bar{h}(c)$ represents a generator of its fundamental group which is $\mathbf{Z}_2$. If $f|U_j$ is orientation-reversing, M' is homeomorphic to the space $S^1 \times S^1/S^1 \times \{\text{one point}\}$ and $\bar{h}(c)$ represents a generator of its fundamental group which is $\mathbf{Z}$. Hence c is not homotopic to a constant map.

6. Throughout this section, M will denote a 3-dimensional compact Riemannian manifold. We fix a point p in M and let H denote the

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1) For the pair (X, Y) of topological spaces such that $Y \subset X$, X/Y denotes the quotient space of X by the equivalence relation $\sim$ that $a \sim b$ if and only if $a=b$ or a, $b \in Y$.

identity component of H_p. First we study the case that $\dim H=1$. Let $\rho: H\to O(T_p(M))$ be the linear isotropy representation. Then $\rho(H)=SO(2)\subset O(3)=O(T_p(M))$, i.e., with respect to a suitable orthonormal basis (v_1, v_2, v_3) of $T_p(M)$, $\rho(H)$ can be expressed as:

$$\rho(H)=\{\sigma(\gamma);\ \gamma\in\mathbf{R}\},$$

where

$$\sigma(\gamma)=\begin{pmatrix}\cos\gamma & \sin\gamma & 0\\ -\sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We introduce a system of polar coordinates

$$\varphi: \mathbf{R}^2\times[0, \pi]\longrightarrow T_p(M)$$

as follows:

$$\varphi(r, \gamma, \alpha)=r\sin\alpha\cos\gamma\, v_1+r\sin\alpha\sin\gamma\, v_2+r\cos\alpha\, v_3.$$

Then it is clear that

$$h\cdot\exp_p\circ\varphi(r, \gamma, \alpha)=\exp_p\circ\rho(h)\circ\varphi(r, \gamma, \alpha)=\exp_p\circ\varphi(r, \gamma+\gamma', \alpha),$$

where $h\in H$ and $\rho(h)=\sigma(\gamma')$.

Lemma 6.1. *On the assumptions and notations above, we have*

$$\{v\in C(p);\ \exp_p v=\exp_p\mu(v_3)v_3\}=\{\mu(v_3)v_3,\ -\mu(-v_3)v_3\},$$

if (M, p) satisfies condition (C).

Proof: Suppose that there is a vector $v\in C(p)-\mathbf{R}v_3$ such that $\exp_p v=\exp_p\mu(v_3)v_3$. Then we have

$$\exp_p\rho(H)v=H\cdot\exp_p v=H\cdot\exp_p\mu(v_3)v_3=\exp_p\mu(v_3)v_3.$$

This implies that v is contained in the conjugate locus $Q(p)$ of p, which is contradictory to condition (C). Therefore we have

$$\{v \in C(p);\ \exp_p v = \exp_p \mu(v_3)v_3\} \subset \{\mu(v_3)v_3,\ -\mu(-v_3)v_3\}\,.$$

On the other hand, we have

$$2 \leqq N_p(\mu(v_3)v_3) = \#\{v \in C(p);\ \exp_p v = \exp_p \mu(v_3)v_3\}$$

by Proposition 3.2. Hence we have the lemma.

By Lemma 6.1, we see that $N_p(\mu(v_3)v_3) = N_p(-\mu(-v_3)v_3) = 2$ if (M, p) satisfies condition (C).

Let $N_p^{-1}(2) = \bigcup_{i \in I} U_i$, where U_i $(i \in I)$ are the connected components of $N_p^{-1}(2)$.

Lemma 6.2. *On the assumptions and notations above, we have* $\#(I) < \infty$ *if* (M, p) *satisfies condition* (C).

Proof: Suppose that $\#(I) = \infty$. For each $i \in I$ we take a vector v_i in U_i. Since $C(p)$ is compact, there is a convergent subsequence $\{v_j\}_{j=1,2\ldots}$ in $\{v_i;\ i \in I\}$. We may assume that $\{f(v_j)\}_{j=1,2,\ldots}$ is also a convergent sequence, where f is the map defined in Theorem A. Let $\lim v_j = v$ and $\lim f(v_j) = v'$. By condition (C) we have $v \neq v'$. Since N_p is upper semi-continuous and larger than 1, we have $N_p(v) = N_p(v') \geqq 3$, implying that $v,\ v' \notin \mathbf{R}v_3$. Therefore there are open sets U and U' in $\mathbf{R}^2 \times [0, \pi]$ and neighborhoods $U(v)$ and $U(v')$ of v and v' respectively in $T_p(M)$ such that:

(a) $\psi = \varphi | U$ is a diffeomorphism of U onto $U(v)$,

(b) $\psi' = \varphi | U'$ is a diffeomorphism of U' onto $U(v')$,

(c) $\exp_p | U(v)$ and $\exp_p | U(v')$ are diffeomorphisms onto an open set V of M.

Let

$$F = \psi'^{-1} \circ (\exp_p | U(v'))^{-1} (\exp_p | U(v)) \circ \psi :\ U \longrightarrow U',$$

$$F(r, \gamma, \alpha) = (r', \gamma', \alpha')\,,$$

and let

$$\psi^{-1}(w) = (r(w), \gamma(w), \alpha(w)) \qquad \text{for } w \in U(v)\,,$$

$$\psi'^{-1}(w) = (r'(w), \gamma'(w), \alpha'(w)) \qquad \text{for } w \in U(v')\,.$$

Since $\rho(H)\cdot U_i = U_i$ for any $i \in I$, we can assume that $v_j \in U(v)$, $\gamma(v_j) = \gamma(v)$ and $\alpha(v_j) < \alpha(v_{j+1})$. Since $r(v) = r'((\exp_p|U(v'))^{-1} \circ (\exp_p|U(v))(v))$ and since $\partial r'/\partial r \neq 1$ at $\psi^{-1}(v)$ by the variation theory, we see that the equation

$$r - r' = 0$$

can be solved for r as a function of γ and α around $\psi^{-1}(v)$. Let $r(\gamma, \alpha)$ denote the solution and let

$$K = \{(r, \gamma, \alpha) \in U;\ r = r(\gamma, \alpha)\}\ .$$

Here we can assume that $U - K$ is composed of two connected components which we denote by V_1 and V_2. Let $V_i' = F(V_i)$ $(i = 1, 2)$ and $K' = F(K)$. By the definition of $\mathscr{S}(p)$, we have:

$$r \leqq \mu(\psi(r, \gamma, \alpha)) \leqq r(\gamma, \alpha) \text{ if } (r, \gamma, \alpha) \in U \text{ and } \psi(r, \gamma, \alpha) \in \mathscr{S}(p)\ .$$

Therefore we may assume that $\psi(V_1 \cup K) \supset \mathscr{S}(p) \cap U(v)$ and $\psi'(V_2' \cup K') \supset \mathscr{S}(p) \cap U(v')$. Since $\exp_p v_j = \exp_p f(v_j)$ where $v_j \in \mathscr{S}(p) \cap U(v)$ and $f(v_j) \in \mathscr{S}(p) \cap U(v')$, it is clear that $v_j \in \psi(K)$ and $f(v_j) \in \psi'(K')$. Hence there are positive numbers r_0 and γ_0 for which we can define a family of (continuous) embeddings

$$E_j\colon (-r_0, r_0) \times (-\gamma_0, \gamma_0) \times (0, 1) \longrightarrow M \qquad (j = 1, 2, \dots)$$

as follows:

(d) $R_j(x, y, z) = (1 - x)\cdot r(\gamma(v) + y,\ z\cdot\alpha(v_j) + (1 - z)\cdot\alpha(v_{j+1}))$
for $(x, y, z) \in [0, r_0) \times (-\gamma_0, \gamma_0) \times (0, 1)$,

(e) $\Phi_j(x, y, z) = (R_j(x, y, z),\ \gamma(v) + y,\ z\cdot\alpha(v_j) + (1 - z)\cdot\alpha(v_{j+1}))$
for $(x, y, z) \in [0, r_0) \times (-\gamma_0, \gamma_0) \times (0, 1)$,

(f) $(R_j(y, z), \Gamma_j(y, z), A_j(y, z)) = F(\Phi_j(0, y, z))$
for $(y, z) \in (-\gamma_0, \gamma_0) \times (0, 1)$,

(g) $E_j(x, y, z) = \exp_p \circ \psi \circ \Phi_j(x, y, z)$
for $(x, y, z) \in [0, r_0) \times (-\gamma_0, \gamma_0) \times (0, 1)$,

(h) $E_j(x, y, z) = \exp_p \circ \psi'((1 + x)\cdot R_j(y, z), \Gamma_j(y, z), A_j(y, z))$
for $(x, y, z) \in (-r_0, 0] \times (-\gamma_0, \gamma_0) \times (0, 1)$.

Let $0 < \gamma_1 < \gamma_0$ and let $\{r_k\}_{k=1,2,\dots}$ be a sequence of positive numbers such that $r_0 > r_1 > r_2 > \dots$ and $\lim r_k = 0$. Since the map μ is continuous, there is a subsequence $\{v_{j_k}\}_{k=1,2,\dots}$ of $\{v_j\}_{j=1,2,\dots}$ such that

$$E_{j_k}(-r_k \times [-\gamma_1, \gamma_1] \times (0, 1)) \subset \exp_p((\mathscr{S}(p) - C(p)) \cap U(v'))$$

$$E_{j_k}(r_k \times [-\gamma_1, \gamma_1] \times (0, 1)) \subset \exp_p((\mathscr{S}(p) - C(p)) \cap U(v)).$$

Let $D_k^\gamma = E_{j_k}((-r_k, r_k) \times \gamma \times (0, 1))$ and let $R_k = E_{j_k}((-r_k, r_k) \times (-\gamma_1, \gamma_1) \times (0, 1))$. Then the boundary of R_k is contained in $\exp_p((\mathscr{S}(p) - C(p)) \cup N_p^{-1}(2)) \cup D_k^{-\gamma_1} \cup D_k^{\gamma_1}$. Since we assumed that v_{j_k} and v_{j_k+1} belong to the different connected components U_{j_k} and U_{j_k+1} of $N_p^{-1}(2)$ respectively, we have a vector $w_k \in U(v) \cap C(p)$ which is a boundary point of U_{j_k} and which is such that $\alpha(v_{j_k}) < \alpha(w_k) < \alpha(v_{j_k+1})$ and $\gamma(w_k) = \gamma(v)$. This means that $q_k = \exp_p w_k \in D_k^0$ is an inner point of R_k. Since $N_p(w_k) \geqq 3$, we have a vector w_k' in $C(p) - U(v) \cup U(v')$ such that $\exp_p w_k' = q_k$. Let q_k' be the first point on the geodesic $\exp_p t w_k'$ $(0 \leqq t < 1)$ which is in the boundary of R_k. Then Proposition 2.4 implies that $q_k' \in D_k^{-\gamma_1} \cup D_k^{\gamma_1}$. Let $P(\gamma) = \exp_p(\sigma(\gamma)v)$. Since $\lim d(D_k^\gamma) = 0$ and since $\lim d(D_k^\gamma, P(\gamma)) = 0$, we obtain

$$\begin{aligned}
\|v\| &= d(p, P(0)) = \lim d(p, D_k^0) = \lim d(p, q_k) \\
&= \lim d(p, q_k') + \lim d(q_k', q_k) \\
&\geqq \lim d(p, D_k^{-\gamma_1} \cup D_k^{\gamma_1}) + \lim d(D_k^{-\gamma_1} \cup D_k^{\gamma_1}, D_k^0) \\
&= d(p, \{P(-\gamma_1), P(\gamma_1)\}) + d(\{P(-\gamma_1), P(\gamma_1)\}, P(0)) \\
&= \|v\| + d(\{P(-\gamma_1), P(\gamma_1)\}, P(0)) > \|v\|.
\end{aligned}$$

It is a contradiction.

Proof of Theorem E: Let H be the identity component of H_p and $\rho: H \to SO(T_p(M))$ be the linear isotropy representation as above. Since $SO(3, \mathbf{R})$ does not have 2-dimensional subgroups, $\dim \rho(H) = 1$ or 3. In case that $\dim \rho(H) = 3$, we have $\rho(H) = SO(T_p(M))$, implying that N_p is constant. By Theorem A, we see that $N_p \equiv 2$. Hence the theorem follows from theorem B. Therefore we suppose that $\dim \rho(H) = 1$. As in the beginning of this section, $\rho(H)$ can be expressed as:

$\rho(H)=\{\sigma(\gamma); \gamma \in \mathbf{R}\}$ with respect to a suitable orthonormal basis (v_1, v_2, v_3) of $T_p(M)$. For each vector $v \in C(p)$ we define $\alpha(v)$ by

$$g(v, v_3)=\|v\| \cdot \cos \alpha(v) \quad \text{and} \quad 0 \leqq \alpha(v) \leqq \pi,$$

where g is the Riemannian metric on M. By Lemma 6.2, we see that $N_p^{-1}(2)$ is composed of a finite number of connected components which we denote by U_i ($i=1, 2,\dots, m$). There is a sequence $0=\alpha_0<\alpha_1<\alpha_2<\cdots<\alpha_m=\pi$ such that:

$$U_1 =\{v \in C(p);\ \alpha_0 \leqq \alpha(v)<\alpha_1\}$$

$$U_i =\{v \in C(p);\ \alpha_{i-1}<\alpha(v)<\alpha_i\} \qquad (0<i<m)$$

$$U_m=\{v \in C(p);\ \alpha_{m-1}<\alpha(v) \leqq \alpha_m\} .$$

This is clear from the fact that $\rho(H) \cdot U_i=U_i$ and $\alpha(v)=\alpha(\rho(H)v)$ and from Lemma 6.1. If $m=1$, we see that $C(p)=N_p^{-1}(2)$, which implies that M is not simply connected as in the first part of this proof. Hence we suppose that $m>1$. Let $\tilde{\alpha}: \rho(H)\backslash C(p) \to [0, \pi]$ be the map defined by $\tilde{\alpha}(\rho(H)v)=\alpha(v)$. Then it is clear that $\tilde{\alpha}$ is a homeomorphism. We have $f \circ h=h \circ f$ for any $h \in \rho(H)$, where f is the map defined in Theorem A. Hence the map

$$\tilde{\alpha} \circ f \circ \tilde{\alpha}^{-1}:\ [\alpha_0, \alpha_1) \cup (\alpha_1, \alpha_2) \cup \cdots \cup (\alpha_{m-1}, \alpha_m] \to [\alpha_0, \alpha_1) \cup \cdots \cup (\alpha_{m-1}, \alpha_m]$$

is well defined and a homeomorphism. To simplify the notation, we write (α_0, α_1) and (α_{m-1}, α_m) for $[\alpha_0, \alpha_1)$ and $(\alpha_{m-1}, \alpha_m]$ respectively. We distinguish three cases.

(1) The case where $f(U_i)=U_i$ for some i. By Lemma 6.1, we see that $i \neq 1, m$. Hence it implies that the map $\exp_p: U_i \to \exp_p(U_i)$ is a covering of order two. Let $c(t)$ ($0 \leqq t \leqq \pi$) be a curve in $C(p)$ such that $\alpha(c(t))=t$. By Lemma 6.1, we see that $\tilde{c}(t)=\exp_p c(t)$ ($0 \leqq t \leqq \pi$) is a closed curve in $\tilde{C}(p)$. First suppose that the map $\tilde{\alpha} \circ f \circ \tilde{\alpha}^{-1}$: $(\alpha_{i-1}, \alpha_i) \to (\alpha_{i-1}, \alpha_i)$ is monotone increasing. Since $(\tilde{\alpha} \circ f \circ \tilde{\alpha}^{-1})^2=id$, $\tilde{\alpha} \circ f \circ \tilde{\alpha}^{-1}$ must be the identity map. It implies that the space $\tilde{C}(p)/(\tilde{C}(p)-\exp_p(U_i))$ is homeomorphic to the space $S^1 \times S^1/S^1 \times \{\text{one point}\}$. It

is clear that the image of the curve $\tilde{c}$ in $\tilde{C}(p)/(\tilde{C}(p)-\exp_p(U_i))$ represents a generator of its fundamental group which is $\mathbf{Z}$. Hence $\pi_1(M)=\pi_1(\tilde{C}(p))\neq\{0\}$. Now suppose that the map $\tilde{\alpha}\circ f\circ\tilde{\alpha}^{-1}$: $(\alpha_{i-1}, \alpha_i)\to(\alpha_{i-1}, \alpha_i)$ is monotone decreasing. Then it is clear that $\tilde{C}(p)/(\tilde{C}(p)-\exp_p(U_i))$ is homeomorphic to $P^2(\mathbf{R})$ and the image of the curve $\tilde{c}$ in $\tilde{C}(p)/(\tilde{C}(p)-\exp_p(U_i))$ represents a generator of its fundamental group which is $\mathbf{Z}_2$. Hence $\pi_1(M)=\pi_1(\tilde{C}(p))\neq\{0\}$.

(2) The case where $f(U_i)=U_{i+1}$ and $\exp_p^{-1}(\exp_p(\bar{U}_i\cap\bar{U}_{i+1}))\cap C(p)=\bar{U}_i\cap\bar{U}_{i+1}$ for some i. It is clear that the map $\tilde{\alpha}\circ f\circ\tilde{\alpha}^{-1}$: $(\alpha_{i-1}, \alpha_i)\to(\alpha_i, \alpha_{i+1})$ is monotone decreasing. Let v be a vector in $\bar{U}_i\cap\bar{U}_{i+1}$. Let $\gamma_0=\inf_{\gamma>0}\{\gamma; \exp_p\sigma(\gamma)v=\exp_p v\}$. By condition (C), we see that $\gamma_0>0$. Since $\rho(H)$ acts on $\bar{U}_i\cap\bar{U}_{i+1}$ as a rotation, for any $w\in\bar{U}_i\cap\bar{U}_{i+1}$ we see that $\exp_p\sigma(\gamma)w=\exp_p w$ if and only if $\gamma\in\gamma_0\mathbf{Z}$. Hence $2\pi/\gamma_0$ is an integer which we denote by n. And the map $\exp_p$: $\bar{U}_i\cap\bar{U}_{i+1}\to\exp_p(\bar{U}_i\cap\bar{U}_{i+1})$ is a covering of order n. By the choice of i, we have $n\geqq 3$. Let $\tilde{c}(t)$ $(0\leqq t\leqq 1)$ be the curve defined by $\tilde{c}(t)=\exp_p\sigma(t\gamma_0)v$. Here it is clear that the space $\tilde{C}(p)/\exp_p(\bigcup_{j\neq i,i+1}\bar{U}_j)$ is homeomorphic to the space $V^2\cup_\xi S^1$, where $V^2=\{x\in\mathbf{R}^2;\ \|x\|\leqq 1\}$, ξ: $\partial V^2=S^1\to S^1$ is a covering map of order n and $V^2\cup_\xi S^1$ is given by identifying ∂V^2 with S^1 through ξ. Moreover the image of the curve $\tilde{c}$ in $V^2\cup_\xi S^1$ represents a generator of its fundamental group which is $\mathbf{Z}_n$. Hence $\pi_1(M)=\pi_1(\tilde{C}(p))\neq\{0\}$.

(3) The other case. We consider the orbit spaces $\rho(H)\backslash\mathscr{S}(p)$ and $H\backslash M$ and the commutative diagram

$$\begin{array}{ccc} \mathscr{S}(p) & \xrightarrow{\pi} & \rho(H)\backslash\mathscr{S}(p) \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \overline{\exp_p}} \\ M & \xrightarrow{\bar{\pi}} & H\backslash M \end{array}$$

defined as follows:

(a) π and $\bar{\pi}$ are the natural projections,

(b) $\overline{\exp_p}(\rho(H)v)=H\cdot\exp_p v$.

Since $\rho(H)\backslash N_p^{-1}(2)$ is composed of a finite number of connected components $\rho(H)\backslash U_i$ $(i=1, 2,\dots, m)$, we have a component U_j and vectors

u and v in $C(p)$ such that

(c) $\rho(H)u$ and $\rho(H)v$ belong to the different connected components of $\rho(H)\backslash C(p)-\rho(H)\backslash U_j \cup \rho(H)\backslash f(U_j)$,

(d) $\exp_p u=\exp_p v$.

Let $c(t)$ $(0\leqq t\leqq 1)$ be the curve defined by

$$c(t)=\exp_p 2tu \qquad (0\leqq t\leqq 1/2),$$

$$c(t)=\exp_p(2-2t)v \qquad (1/2\leqq t\leqq 1).$$

Then we can prove that $\bar{\pi}(c)$ is not homotopic to a constant map, applying the method in the proof of Theorem D to the diagram

$$\begin{array}{ccc} \rho(H)\backslash\mathscr{S}(p) & \xrightarrow{\ k\ } & (\rho(H)\backslash\mathscr{S}(p))' \\ \Big\downarrow{\scriptstyle \overline{\exp_p}} & & \Big\downarrow{\scriptstyle \overline{\overline{\exp_p}}} \\ H\backslash M & \xrightarrow{\ \bar{k}\ } & (H\backslash M)/(H\backslash\tilde{C}(p)-H\backslash\exp_p(U_j)) \end{array}$$

defined as follows:

(e) $(\rho(H)\backslash\mathscr{S}(p))'$ is the space obtained from $\rho(H)\backslash\mathscr{S}(p)$ by identifying $\rho(H)w$ with $\rho(H)\cdot f(w)$ for $w\in U_j$,

(f) k and $\bar{k}$ are the natural projections,

(g) $\overline{\overline{\exp}}_p x=\bar{k}\circ\overline{\exp}_p\circ k^{-1}(x)$ for $x\in(\rho(H)\backslash\mathscr{S}(p))'$.

Hence the curve c in M is not homotopic to a constant map, implying that M is not simply connected.

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