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On potential densities of one-dimensional Lévy processes

By

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§1. Introduction

In this paper, we will study some behaviors near the origin of the derivatives of potential densities of some typical one-dimensional Lévy processes.

In the study of one-dimensional Lévy processes, their potential densities play an important role. For example, there is a close relation between the hitting probability for a single point and properties of potential densities: roughly, we can say that the positivity of hitting probability for a single point is equivalent to the existence of a bounded potential density and the regularity of a single point is equivalent to the existence of a bounded continuous density. These facts were well known and used in the study of stable processes (cf. Kac [3]) and have been established for general one-dimensional Lévy processes by Kesten [4] and Bretagnolle [1]. We note that Port and Stone [6] proved independently the existence of continuous densities (and hence, the regularity of a sinlge point) for asymmetric Cauchy processes.

Even in the case when a continuous potential density exists, its derivative behaves quite differently and it is our purpose of the present paper to study the behavior of derivatives near the origin for several one-dimensional Lévy processes. The behavior of derivatives reflects some aspects of the hitting of sample paths to a given point as is explained in Ikeda and Watanabe [2].

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§ 2. Results

Let $\{X_t, P_x\}_{t\geq 0, x\in R}$ be a one-dimensional Lévy process with exponent $\psi(\xi)$:

$$E_0(e^{i\xi X_t})=e^{t\psi(\xi)},$$

where

(2.1) $$\psi(\xi)=-ia\xi-(1/2)\sigma^2\xi^2+\int_R(e^{i\xi y}-1-i\xi y/(1+y^2))n(dy).$$

Its p-potential U_p $(p>0)$ is defined as follows:

(2.2) $$U_pf(x)=E_x\left[\int_0^\infty e^{-pt}f(X_t)dt\right]=\int_R U_p(dy)f(x+y),$$

where the measure $U_p(dy)$ is defined by

(2.3) $$U_p(B)=E_0\left[\int_0^\infty e^{-pt}I_B(X_t)dt\right].$$

If $U_p(dx)$ has the density $u_p(x)$, it is called the potential density.

Theorem 1. *Let X_t be a process with exponent given by* (2.1), *where $\sigma^2>0$ and the Lévy measure $n(dy)$ has a compact support. Then, the potential denisty $u_p(x)$ exists which is continuous in R and continuously differentiable in $R-\{0\}$. Further, $u_p'(0_+)=\lim_{x\downarrow 0}u_p'(x)$ and $u_p'(0_-)=\lim_{x\downarrow 0}u_p'(x)$ exist with finite values and*

(2.4) $$u_p'(0_-)-u_p'(0_+)=2/\sigma^2.$$

The following theorem was proved in [2]. We quote it here for reference.

Theorem 2. *Let X_t be a process with exponent ψ given by*

$$\psi(\xi) = -C_1|\xi|^{\alpha_1}(1 - i\beta_1 \tan(\pi\alpha_1/2)\cdot \operatorname{sgn}(\xi)) \\ - C_2|\xi|^{\alpha_2}(1 - i\beta_2 \tan(\pi\alpha_2/2)\cdot \operatorname{sgn}(\xi)), \tag{2.5}$$

where $\alpha_1 > 1$, $\alpha_1 > \alpha_2 > 0$, $\alpha_1 \neq 1$ *and* $-1 \leq \beta_i \leq 1$, $(i=1, 2)$. *i.e.* X *is an independent sum of stable processes with index* $\alpha_1 > 1$ *and* α_2. *Then, the continuous density* $u_p(x)$ *exists which is continuously differentiable in* $R - \{0\}$ *and*

$$u'_p(x) = -\frac{\Gamma(2-\alpha)\sin(\pi\alpha_{1/2})}{\pi(1+h_1^2)C_1}(\operatorname{sgn}(x) + \beta_1)|x|^{\alpha_1 - 2} \\ + (a_1 \operatorname{sgn}(x) + b_1)|x|^{2\alpha_1 - \alpha_2 - 2} \\ + (a_2 \operatorname{sgn}(x) + b_2)|x|^{3\alpha_1 - 2\alpha_2 - 2} + \cdots\cdots \\ + (a_n \operatorname{sgn}(x) + b_n)|x|^{\alpha_1 - 2 + n(\alpha_1 - \alpha_2)} + V_p(x), \tag{2.6}$$

where $h_1 = \beta_1 \tan(\pi\alpha_1/2)$ *and* $V_p(x)$ *is a bounded continuous function.* n *is the greatest integer such that* $\alpha_1 - 2 + n(\alpha_1 - \alpha_2) < 0$.

Theorem 3. *Let* X_t *be a asymmetric Cauchy process with exponent* ψ *given by*

$$\psi(\xi) = -ia\xi - C|\xi|(1 + i\cdot \operatorname{sgn}(\xi)h\cdot \log|\xi|), \tag{2.7}$$

where $C > 0$, $h = 2\beta/\pi$ *and* $-1 \leqq \beta \leqq 1$ $(\beta \neq 0)$. *If* $\beta = -1$, X_t *is a one-sided Cauchy process with positive jumps and if* $\beta = +1$, X *is a one-sided Cauchy process with negative jumps. Then a continuous density* $u_p(x)$ *exists which is continuously differentiable in* $R - \{0\}$ *and as* $|x| \downarrow 0$,

$$u'_p(x) = \frac{-\pi}{4\beta^2 C}(\operatorname{sgn}(x) + \beta)\frac{1}{|x|\left(\log\frac{1}{|x|}\right)^2} + O\left(\frac{1}{|x|\left(\log\frac{1}{|x|}\right)^3}\right) \tag{2.8}$$

Theorem 4. *Let* X_t *be a process with exponent* ψ *given by*

$$\psi(\xi) = -ia\xi - C_1|\xi|(1 + ih_1 \operatorname{sgn}(\xi)\log|\xi|) - C_2|\xi|^{\alpha}(1 + ih_2 \operatorname{sgn}(\xi)), \tag{2.9}$$

where $h_1 = 2\beta_1/\pi$, $h_2 = \beta_2 \tan(\pi\alpha/2)$, $-1 \leqq \beta_i \leqq 1$ $(i=1, 2)$, $\beta_1 \neq 0$, *and*

$0<\alpha<1$ or $1<\alpha<2$. i.e. X_t is an independent sum of an asymmetric Cauchy process and a stable process with index $\alpha \neq 1$. Then, the continuous density $u_p(x)$ exists which is continuously differentiable in $R-\{0\}$ such that as $|x|\downarrow 0$,

$$(2.10) \qquad u_p'(x)=\frac{-\pi}{4\beta_1^2 C_1}(\operatorname{sgn}(x)+\beta_1)\frac{1}{|x|\left(\log\frac{1}{|x|}\right)^2} + O\left(\frac{1}{|x|\left(\log\frac{1}{|x|}\right)^3}\right) \qquad (0<\alpha<1)$$

and

$$(2.11) \qquad u_p'(x)=\frac{-\Gamma(2-\alpha)\sin\frac{\pi\alpha}{2}}{\pi(1+h_2^2)C_2}(\operatorname{sgn}(x)-\beta_2)|x|^{\alpha-2}+R(x),$$

where

$$R(x)=\begin{cases} O(|x|^{2\alpha-3}\log(1/|x|)) & (1<\alpha<3/2) \\ O((\log 1/|x|)^2) & (\alpha=3/2) \\ \text{a bounded continuous function} & (3/2<\alpha<2). \end{cases}$$

§3. Proof of theorem 1

3.1. The following two lemmas are well known or easily proved and their proofs are omitted.

Lemma 3.1. *If Lévy measure $n(dy)$ has a compact support (i.e. $^\exists M>0$, $\operatorname{supp}[n(dy)]\subset[-M, M]$), then*

$$\int_{-M}^{M}(1-\cos\xi y)n(dy) \quad \text{and} \quad \int_{-M}^{M}(\sin\xi y-\xi y/(1+y^2))n(dy)$$

are analytic functions of ξ. Particularly, they are bounded variation for real ξ in all bounded intervals.

Lemma 3.2. $\int_{-\infty}^{+\infty}(1-\cos\xi y)n(dy)$ *and* $\int_{-\infty}^{+\infty}(\sin\xi y-\xi y/(1+y^2))n(dy)$

are $o(\xi^2)$ *as* $|\xi|\uparrow\infty$. *These estimates are valid without the assumption that* $n(dy)$ *has a compact support.*

From the hypothesis of theorem 1, $u_p(x)$ can be written as follows.

$$(3.1)\qquad u_p(x)=\int_0^{\infty}e^{-pt}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-i\xi x}e^{t\psi(\xi)}d\xi\right]dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\{e^{-i\xi x}/(p-\psi(\xi))\}d\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\{f(\xi)\cos\xi x+g(\xi)\sin\xi x\}/(f^2(\xi)+g^2(\xi))d\xi,$$

where $f(\xi)=p+\sigma^2\xi^2/2+\int_{-M}^{M}(1-\cos\xi y)n(dy)$ and $g(\xi)=a\xi+\int_{-M}^{M}(\sin\xi y-\xi y/(1+y^2))n(dy)$.

3.2. *Continuity of* $u_p(x)$ *in* R. This is easily seen in view of (3.1) and lemma 3.2 and the bounded convergence theorem.

3.3. *Continuous differentiablity of* $u_p(x)$ *in* $R-\{0\}$. We can prove this by using the classical theorem of the change of orders of differentiation and integration. According to the theorem, we have only to check that the following condition is satisfied:

For every $\varepsilon>0$ and $\delta>0$, there exists $N=N(\varepsilon,\delta)>0$ such that for all $N'\geqq N$ and $|x|\geqq\delta$,

$$(3.2)\qquad \left|\int_{N'\geqq|\xi|\geqq N}\{\xi\sin\xi x\cdot f(\xi)+\xi\cos\xi x\cdot g(\xi)\}/(f^2(\xi)+g^2(\xi))d\xi\right|<\varepsilon.$$

Since $\xi g(\xi)/(f^2(\xi)+g^2(\xi))$ is a function of bounded variation of ξ by lemma 3.1, we have the following estimate by virtue of the second mean-value theorem:

For some P such that $N\leqq P\leqq N'$,

$$\left|\int_N^{N'}\frac{\xi g(\xi)\cos\xi x}{f^2(\xi)+g^2(\xi)}d\xi\right|\leqq\left|\frac{Ng(N)}{f^2(N)+g^2(N)}\right|\cdot\left|\int_N^P\cos\xi x d\xi\right|$$

$$+\left|\frac{N'g(N')}{f^2(N')+g^2(N')}\right|\cdot\left|\int_P^{N'}\cos\xi x d\xi\right|$$

Since $\left|\int_N^P\cos\xi x d\xi\right|$, $\left|\int_P^{N'}\cos\xi x d\xi\right|$ are at most $2/\delta$ if $|x|\geqq\delta$ and

$Ng(N)/(f^2(N)+g^2(N))$, $N'g(N')/(f^2(N')+g^2(N))\to 0$ as $N, N'\to\infty$ by lemma 3.2, (3.2) is proved for cosine term and we can prove similarly for sine term. Thus, $u_p(x)$ is continuously differentiable in $x\in R-\{0\}$ and we are allowed to write as follows.

$$(3.3)\qquad u_p'(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{-\xi f(\xi)\sin\xi x+\xi g(\xi)\cos\xi x}{f^2(\xi)+g^2(\xi)}d\xi \qquad (x\neq 0).$$

3.4. *Behavior of* $u_p'(x)$ *as* $|x|\downarrow 0$. For simplicity, we rewrite $u_p'(x)$ as follows. $u_p'(x)=I_1(x)+I_2(x)$, where $I_1(x)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\xi f(\xi)$ $\sin\xi x/(f^2(\xi)+g^2(\xi))d\xi$, $I_2(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\xi g(\xi)\cos\xi x/(f^2(\xi)+g^2(\xi))d\xi$.

Considering oddness and evenness of $\sin\xi x$ and $\cos\xi x$ respectively, we consider the case $x>0$. As for $I_1(x)$, we have by the change of variable such as $\eta=\xi x$,

$$(3.4)\quad I_1(x)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\eta\sin\eta\left\{Px^2+\frac{\sigma^2\eta^2}{2}+x^2\int_{-M}^{M}\left(1-\cos\frac{\eta}{x}y\right)n(dy)\right\}d\eta}{\left\{Px^2+\frac{\sigma^2\eta^2}{2}+x^2\int_{-M}^{M}\left(1-\cos\frac{\eta}{x}y\right)n(dy)\right\}^2+\left\{a\eta x+x^2\int_{-M}^{M}\left(\sin\frac{\eta y}{x}-\frac{\eta y}{x(1+y^2)}\right)n(dy)\right\}^2}$$

Let $W(x,\eta)$ be the integrand of (3.4).

$$|W(x,\eta)|\leqq\left|\frac{\eta\sin\eta}{Px^2+\frac{\sigma^2\eta^2}{2}+x^2\int_{-M}^{M}\left(1-\cos\frac{\eta}{x}y\right)n(dy)}\right|\leqq\left|\frac{\eta\cdot\eta}{\frac{\sigma^2\eta^2}{2}}\right|=\frac{2}{\sigma^2}.$$

Therefore, $W(x,\eta)$ is integrable in η on each bounded interval $(-N, N)$ uniformly in x. Meanwhile, by the second mean-value theorem, we see that for every $\varepsilon>0$ there exists $N=N(\varepsilon)>0$ such that for all $N'>N$,

$$\left|-\frac{1}{2\pi}\int_{N\leqq|\eta|\leqq N'}W(x,\eta)d\eta\right|\leqq\varepsilon \qquad \text{uniformly in } |x|\leqq 1.$$

Thus, $$I_1(x) \longrightarrow -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\eta\sin\eta}{\frac{\sigma^2\eta^2}{2}}d\eta = -1/\sigma^2 \qquad (x\downarrow 0).$$

Now let us proceed to $I_2(x)$. We separate $I_2(x)$ into two terms and denote them by $\frac{1}{2\pi}J_1(x)$ and $\frac{1}{2\pi}J_2(x)$ erspectively:

$$(3.5)\quad I_2(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\xi\cos\xi x\left\{a\xi + \int_{-M}^{+M}\left(\xi y - \frac{\xi y}{1+y^2}\right)n(dy)\right\}}{f^2(\xi)+g^2(\xi)}d\xi$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\xi\cos\xi x\left(\int_{-M}^{M}(\sin\xi y - \xi y)n(dy)\right)}{f^2(\xi)+g^2(\xi)}d\xi$$

$$= \frac{1}{2\pi}J_1(x) + \frac{1}{2\pi}J_2(x)$$

Since the integrand of $J_1(x)$ is dominated by integrable function, we have only to consider $J_2(x)$.

$$(3.6)\qquad |J_2(x)| \leqq \int_{-\infty}^{+\infty}\frac{|\xi|\int_{-M}^{M}|\sin\xi y - \xi y|\,n(dy)}{\frac{\sigma^4\xi^4}{4}}d\xi$$

Let us consider $y>0$ part and change variable ξ by $\eta=\xi y$ in the right side of (3.6). Then,

$$\int_0^M y^2\left(\int_{-\infty}^{+\infty}\frac{\eta|\sin\eta - \eta|}{\frac{\sigma^4\eta^4}{4}}d\eta\right)n(dy) < +\infty.$$

As for $y>0$ part, the same is true. Therefore,

$$I_2(x) \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\xi g(\xi)}{f^2(\xi)+g^2(\xi)}d\xi \qquad (x\downarrow 0).$$

Hence, $$\lim_{|x|\downarrow 0} u_p'(x) = -\frac{1}{\sigma^2}\operatorname{sgn}(x) + A \qquad as \quad |x|\downarrow 0,$$

where $$A = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\xi g(\xi)}{f^2(\xi)+g^2(\xi)}d\xi.$$

§4. Proof of theorem 3

4.1. *Preliminary.* We provide here some simple results for the proof of theorem 3.

Lemma 4.1. *For an arbitrary positive integer N and $k \geqq 2$*

$$\sum_{n=N}^{\infty} \frac{1}{n\left(\log \frac{1}{x}+\log n\right)^{k}}=\frac{1}{k-1}\left(\log \frac{1}{x}\right)^{-(k-1)}\left\{1+O\left(\log \frac{1}{x}\right)^{-1}\right\}$$

as $x \downarrow 0$.

Proof. *Note the following:*

$$\int_{N}^{\infty} \frac{dt}{t\left(\log t+\log \frac{1}{x}\right)^{k}} \leqq \sum_{n=N}^{\infty} \frac{1}{n\left(\log n+\log \frac{1}{x}\right)^{k}} \tag{4.1}$$

$$\leqq \int_{N-1}^{\infty} \frac{dk}{t\left(\log t+\log \frac{1}{x}\right)^{k}}$$

$$\int_{N}^{\infty} \frac{dt}{t\left(\log t+\log \frac{1}{x}\right)^{k}}=\left[-\frac{1}{k-1}\left(\log t+\log \frac{1}{x}\right)^{k-1}\right]_{N}^{\infty} \tag{4.2}$$

Lemma 4.2. *Let n be a sufficiently large integer and $z \in [0, 1]$. Then,*

$$\log\left(\frac{2n+1+z}{2n+z} \cdot \frac{2n+1-z}{2n+2-z}\right)=\frac{1-2z}{(2n+z)(2n+1-z)}+O(1/n^{3}) \tag{4.3}$$

as $n \longrightarrow \infty$,

$$\log\left(\frac{z+2n+1}{z+2n}\right)-1/2n=\frac{-(2z+1)}{2(z+2n)^{2}}+O(1/n^{3}) \tag{4.4}$$

as $n \longrightarrow \infty$.

Lemma 4.3. *Let $z \in (0, 1/2]$. Then,*

$$\prod_{n=0}^{\infty}\left\{\frac{z+2n+1}{z+2n} \frac{2n+1-z}{2n+2-z}\right\}=\frac{1+\cos \pi z}{\sin \pi z}. \tag{4.5}$$

Proof. Since the above infinite product is absolutely convergent, we can change the order of multiplication to get

$$\prod_{n=0}^{\infty}\left\{\frac{z+2n+1}{z+2n}\frac{2n+1-z}{2n+2-z}\right\}=\frac{(1+z)^2(1-z)^2}{z(1-z)(2-z)(2+z)}$$

$$\frac{(3+z)^2(3-z)^2}{(3+z)(3-z)(4+z)(4-z)}\frac{(5+z)^2(5-z)^2}{(5+z)(5-z)(6+z)(6-z)}\cdots$$

$$=\frac{1}{z}\left[\left\{\prod_{n=0}^{\infty}\left(1-\frac{z^2}{(2n-1)^2}\right)\right\}^2\Big/\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right]\cdot\prod_{n=1}^{\infty}\frac{2n+1}{2n}\frac{2n-1}{2n}$$

$$=\frac{1+\cos\pi z}{\sin\pi z}.$$

Lemma 4.4.

$$\int_0^{1/2}\cos\pi z\log\left(\frac{1+\cos\pi z}{\sin\pi z}\right)dz=1/2. \tag{4.6}$$

4.2. *Continuity of $u_p(x)$ in R.* In this case $u_p(x)$ can be written as follows.

$$u_p(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\xi x}}{p-\psi(\xi)}d\xi=\frac{1}{\pi}\int_0^{\infty}\frac{(p+c\xi)\cos\xi x\,d\xi}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2} \tag{4.7}$$

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\xi(a+ch\log\xi)\sin\xi x\,d\xi}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2}$$

The first term of (4.7) is obviously continuous in $x\in R$ by the bounded convergence theorem. For the second term, we extract a non-absolutely integrable part and consider the following:

$$\frac{\xi(a+ch\log\xi)\sin\xi x}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2}=\frac{\sin\xi x}{ch\xi\log\xi} \tag{4.8}$$

$$+\sin\xi x\left\{\frac{\xi(a+ch\log\xi)}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2}-\frac{1}{ch\xi\log\xi}\right\}.$$

Non-absolutely integrable part $\frac{1}{\pi}\int_0^{\infty}\frac{\sin\xi x}{ch\xi\log\xi}d\xi$ is a continuous function of $x\neq 0$ by using the same method as in 3.3 (the second mean-value theorem). Moreover, it is seen that $\left|\frac{1}{\pi}\int_0^\infty \frac{\sin \xi x}{ch\xi \log \xi} d\xi\right| \to 0$ as $|x|\downarrow 0$ by the change of variable $\eta = \xi x$.

4.3. *Continuous differentiability of* $u_p(x)$, *when* $x \neq 0$. The proof is the same as that of 3.3 and $u_p'(x)$ is written as follows, when $x \neq 0$,

$$u_p'(x) = -\frac{1}{\pi}\int_0^\infty \frac{\xi(p+c\xi)\sin \xi x \, d\xi}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2} \tag{4.9}$$

$$-\frac{1}{\pi}\int_0^\infty \frac{\xi^2(a+ch\log\xi)\cos\xi x \, d\xi}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2}.$$

We denote the first and the second term of (4.9) by $K_1(x)$ and $K_2(x)$, respectively.

4.4. *Behavior of* $u_p'(x)$ *as* $|x|\downarrow 0$. We investigate $K_1(x)$ in 4.4.1 and $K_2(x)$ in 4.4.2.

4.4.1. We write $K_1(x)$ as follows, by extracting non-absolutely integrable part.

$$K_1(x) = -\frac{1}{\pi}\int_0^\infty \frac{\sin \xi x \, d\xi}{\left(c+\frac{a^2}{c}\right)+2ah\log\xi+ch^2\log^2\xi} \tag{4.10}$$

$$+\frac{1}{\pi}\int_0^\infty \sin\xi x\left[\frac{1}{\left(c+\frac{a^2}{c}\right)+2ah\log\xi+ch^2\log^2\xi}\right.$$

$$\left.-\frac{\xi(p+c\xi)}{(p+c\xi)^2+\xi^2(a+ch\log\xi)^2}\right]d\xi$$

The second term of (4.10) is a bounded continuous function of $x \in R$. Therefore, we have only to investigate the first term:

$$I_1(x) = -\frac{1}{\pi}\int_0^\infty \frac{\sin\xi x}{\left(c+\frac{a^2}{c}\right)+2ah\log\xi+ch^2\log^2\xi} d\xi. \tag{4.11}$$

By oddness of $K_1(x)$ and $I_1(x)$, we consider the case $x>0$. In (4.11), we divide the domain of integration into a half period of $\sin\xi x$ and change variable ξ by $\xi x=2n\pi x+\pi z$ in the intervals where $\sin\xi x$ is positive and by $\xi x=(2n+1)\pi+\pi z$ in the intervals where $\sin\xi x$ is negative. Then,

$$(4.12)\qquad I_1(x)=-\frac{1}{x}\sum_{n=0}^{\infty}\left[\int_0^1\log\left(\frac{z+2n+1}{z+2n}\right)P_n(z,z)\sin\pi z\,dz\right],\ \text{where}$$

$$P_n(z,x)=\frac{ch^2\left\{\log\left(\frac{2n+z}{x}\pi\right)\right.}{\left\{\left(c+\frac{a^2}{c}\right)+2ah\log\left(\frac{2n+z}{x}\pi\right)+ch^2\log^2\left(\frac{2n+z}{x}\pi\right)\right\}}$$

$$\frac{\left.+\log\left(\frac{2n+z+1}{x}\pi\right)\right\}}{\left\{\left(c+\frac{a^2}{c}\right)+2ah\log\left(\frac{2n+z+1}{x}\pi\right)+ch^2\log^2\left(\frac{2n+z+1}{x}\pi\right)\right\}}.$$

We claim that, for an arbitrary natural number N,

$$(4.13)\qquad -\frac{1}{x}\sum_{n=N}^{\infty}\left[\int_0^1 P_n(z,x)\frac{1}{2n}\sin\pi z\,dz\right]$$

$$=\frac{-\pi}{4\beta^2cx\left(\log\frac{1}{x}\right)^2}+O\left(\frac{1}{x\left(\log\frac{1}{x}\right)^3}\right)\ \text{and}$$

$$(4.14)\qquad I_1(x)-\left\{-\frac{1}{x}\sum_{n=N}^{\infty}\left(\int_0^1 P_n(z,x)\frac{1}{2n}\sin\pi z\,dz\right)\right\}$$

$$=O\left(\frac{1}{x\left(\log\frac{1}{x}\right)^3}\right),\quad \textit{as } x\downarrow 0.$$

From (4.13) and (4.14),

$$(4.15)\qquad I_1(x)=\frac{-\pi}{4\beta^2c}\operatorname{sgn}(x)\frac{1}{|x|\left(\log\frac{1}{|x|}\right)^2}+O\left(\frac{1}{|x|\left(\log\frac{1}{|x|}\right)^3}\right),$$

$$\text{as } |x|\downarrow 0.$$

We now prove (4.13).

We put $Q_n(x)=(\log n+\log 1/x)$ $(n=1,2,\ldots)$ and rewrite $P_n(z,x)$ as follows.

$$(4.16)\qquad P_n(z, x)=\frac{2}{ch^2Q_n^3(x)}+\left(P_n(z, x)-\frac{2}{ch^3O_n^3(x)}\right)$$

The order of the second term of (4.16) is at most $1/Q_n^4(x)$ as $x\downarrow 0$, $n\to\infty$. Hence, using lemma 4.1,

$$-\frac{1}{x}\sum_{n=N}^{\infty}\left\{\int_0^1 P_n(z, x)\frac{1}{2n}\sin\pi z\, dz\right\}$$

$$=-\frac{1}{x}\sum_{n=N}^{\infty}\left\{\int_0^1\frac{2}{ch^2Q_n^3(x)}\frac{1}{2n}\sin\pi z\, dz\right\}$$

$$-\frac{1}{x}\sum_{n=N}^{\infty}\left\{\int_0^1\left[P_n(z, x)-\frac{2}{ch^2Q_n^3(x)}\right]\sin\pi z\, dz\right.$$

$$=\frac{-\pi}{4\beta^2c}\cdot\frac{1}{x\left(\log\frac{1}{x}\right)^2}+O\left(\frac{1}{x\left(\log\frac{1}{x}\right)^3}\right).$$

Now, we proceed to the proof of (4.14). Separating the first N terms and remaining terms and considering (4.4) and the fact that $P_n(z, x)$ is of order $1/Q_n^3(x)$,

$$(4.17)\qquad \left|I_1(x)-\left[-\frac{1}{x}\sum_{n=N}^{\infty}\left\{\int_0^1 P_n(z, x)\frac{1}{2n}\sin\pi z\, dz\right\}\right]\right|$$

$$\leqq\frac{1}{x}\left|\int_0^1\sum_{n=0}^{N-1}\log\left(\frac{z+2n+1}{z+2n}\right)P_n(z, x)\sin\pi z\, dz\right|$$

$$+\frac{1}{x}\left|\int_0^1\sum_{n=N}^{\infty}P_n(z, x)\sin\pi z\left(\frac{1}{2n}-\log\frac{z+2n+1}{z+2n}\right)dz\right|$$

$$=O\left(\frac{1}{x\left(\log\frac{1}{x}\right)^3}\right).$$

4.4.2. *Behavior of* $K_2(x)$ *as* $|x|\downarrow 0$. We write $K_2(x)$ as

$$(4.18)\qquad K_2(x)=-\frac{1}{\pi}\int_0^\infty\frac{(h\log\xi+\delta)\cos\xi x}{\theta+\omega\log\xi+ch^2\log^2\xi}d\xi$$

$$+\frac{1}{\pi}\int_0^\infty\left[\frac{h\log\xi+\delta}{\theta+\omega\log\xi+ch^2\log^2\xi}\right.$$

$$-\frac{\xi^2(a+ch\log\xi)}{(P+c\xi)^2+\xi^2(a+ch\log\xi)^2}\Big]\cos\xi x\,d\xi\,.$$

Here θ, ω, δ are determined so that the second term of (4.18) is integrable for all x. By a direct calculation, θ, ω, δ must be $(c^2+a^2)/c^3$, $ha(c^2+a^2)/c^2$, $a(c^2+a^2)/c^3$ respectively. By this choice of θ, ω, δ, the second term of (4.18) is a bounded continuous function. So, we have only to investigate

$$(4.19)\qquad I_2(x)=-\frac{1}{\pi}\int_0^\infty\frac{(h\log\xi+\delta)\cos\xi x}{\theta+\omega\log\xi+ch^2\log^2\xi}d\xi\,,$$

when $x\downarrow 0$, condisering evenness of the integrand. In (4.19), we devide the domain of integration by the period of $\cos\xi x$. Then,

$$(4.20)\qquad I_2(x)=\frac{-1}{\pi}\sum_{n=0}^{\infty}\int_{\frac{2n\pi}{x}}^{\frac{2n+2}{2}\pi}\frac{(h\log\xi+\delta)\cos\xi x}{\theta+\omega\log\xi+ch^2\log^2\xi}d\xi\,.$$

Moreover, we divide each term of (4.20) into four parts each of which is 1/4-period of $\cos\xi x$. In each 1/4-peroid of $\cos\xi x$, we change the variable by $\xi x=\pi x+2n\pi$, $\xi x=\pi(1/2-z)+(2n+1/2)\pi$, $\xi x=\pi z+(2n+1)\pi$, $\xi x=\pi(1/2-z)+(2n+3/2)\pi$ respectively. Then,

$$(4.21)\qquad I_2(x)=-\frac{1}{x}\sum_{n=0}^{\infty}\int_0^{1/2}\cos\pi z\left\{\log\left(\frac{2n+1+z}{2n+z}\right)A_n(z,x)\right.$$
$$\left.-\log\left(\frac{2n+2-z}{2n+1-z}\right)A_n(1-z,x)\right\}dz$$

where, $$A_n(z,x)=\frac{ch^3\log\left(\frac{2n+z}{x}\pi\right)\log\left(\frac{2n+1+z}{x}\pi\right)}{\left\{\theta+\omega\log\left(\frac{2n+z}{x}\pi\right)+ch^2\log^2\left(\frac{2n+z}{x}\pi\right)\right\}}$$
$$\frac{+ch^2\left\{\log\left(\frac{2n+z}{x}\pi\right)+\log\left(\frac{2n+1+z}{x}\pi\right)\right\}+\left(\delta\omega-\theta h\right)}{\left\{\theta+\omega\log\left(\frac{2n+1+z}{x}\pi\right)+ch^2\log^2\left(\frac{2n+1+z}{x}\pi\right)\right\}}$$

Now, we extract $\dfrac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}$ or $\dfrac{1}{ch\left(\log\frac{1}{x}\right)^2}$ from $A_n(z,x)$ and $A_n(1-z,x)$, when $n\neq 0$ or $=0$ respectively. Then,

$$(4.22)\quad A_n(z, x)=\frac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}+\left\{A_n(z, x)-\frac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}\right\}$$

$$(n=1, 2,\dots),$$

$$(4.23)\quad A_0(z, x)=\frac{1}{ch\left(\log\frac{1}{x}\right)^2}+\left\{A_0(z, x)-\frac{1}{ch\left(\log\frac{1}{x}\right)^2}\right\}.$$

By a direct calculation, it is seen that the order of the second term of (4.22) is $\frac{1}{\left(\log\frac{1}{x}+\log n\right)^4}$, since the order $\frac{1}{\left(\log\frac{1}{x}+\log n\right)^3}$ vanishes. Quite similarly the second term of (4.23) is $1/\left(\log\frac{1}{x}\right)^4$. As for $A_n(1-z, x)$, the same is true. By (4.3) of lemma 4.2 and the fact that $\log\left(\frac{2n+1+z}{2n+z}\right)$ as well as $\log\left(\frac{2n+2-z}{2n+1-z}\right)$ is $O(1/n)$, we see easily the following:

$$(4.24)\quad I_2(x)=-\frac{1}{x}\int_0^{1/2}\cos\pi z\sum_{n=0}^{\infty}\log\left(\frac{2n+1+z}{2n+z}\,\frac{2n+1-z}{2n+2-z}\right)\frac{1}{ch\left(\log\frac{1}{x}\right)^2}dz$$

$$-\frac{1}{x}\int_0^{1/2}\cos\pi z\left[\log\left(\frac{1+z}{z}\right)\left\{A_0(z, x)-\frac{1}{ch\left(\log\frac{1}{x}\right)^2}\right\}\right.$$

$$\left.-\log\left(\frac{2-z}{1-z}\right)\left\{A_0(z, x)-\frac{1}{ch\left(\log\frac{1}{x}\right)^2}\right\}\right]dz$$

$$-\frac{1}{x}\int_0^{1/2}\cos\pi z\sum_{n=1}^{\infty}\left[\log\left(\frac{2n+1+z}{2n+z}\right)\right.$$

$$\left\{\frac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}-\frac{1}{ch\left(\log\frac{1}{x}\right)^2}\right.$$

$$\left.+\left(A_n(z, x)-\frac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}\right)\right\}$$

$$-\log\left(\frac{2n+2-z}{2n+1-z}\right)\left\{\frac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}-\frac{1}{ch\left(\log\frac{1}{x}\right)^2}\right.$$

$$\left.+\left(A_n(1-z,x)-\frac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}\right)\right\}\Bigg]dz.$$

The first term of (4.24) is equal to $\dfrac{-1}{2chx\left(\log\frac{1}{x}\right)^2}=\dfrac{-\pi}{4c\beta x\left(\log\frac{1}{x}\right)^2}$ by lemma 4.4. The second term of (4.24) is obviously $O\left(\dfrac{1}{x\left(\log\frac{1}{x}\right)^3}\right)$ as $x\downarrow 0$. The third term of (4.24) is also of order $O\left(\dfrac{1}{x\left(\log\frac{1}{x}\right)^3}\right)$, noting (4.3) of lemma 4.2, lemma 4.1 and the fact that $\dfrac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}-\dfrac{1}{ch\left(\log\frac{1}{x}\right)^2}$ is of order $\dfrac{1}{\left(\log\frac{1}{x}\right)^3}$ for fixed n and $A_n(z,x)-\dfrac{1}{ch\left(\log\frac{1}{x}+\log n\right)^2}$ is of order $\dfrac{1}{\left(\log\frac{1}{x}+\log n\right)^4}$ for fixed z.

§5. Proof of theorem 4

5.1. *Continuity in R and continuous differentiability in $R-\{0\}$ of $u_p(x)$.* In this case, $u_p(x)$ can be written as follows:

$$(5.1)\qquad u_p(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\xi x}}{p-\psi(\xi)}\,d\xi$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{(p+c_1\xi+C_2\xi^\alpha)\cos\xi x-(c_1h_1\xi\log\xi+c_2h_2\xi^\alpha+a\xi)\sin\xi x}{(p+c_1\xi+c_2\xi^\alpha)^2+(c_1h\xi\log\xi+c_2h_2\xi^\alpha+a\xi)^2}d\xi$$

When $1<\alpha<2$, continuity of $u_p(x)$ in R is obvious from the bounded convergence theorem. When $0<\alpha<1$, continuity of $u_p(x)$ in R is proved as in 4.2. The continuous differentiability in $R-\{0\}$ of $u_p(x)$ is proved as in 3.3 and for $x\neq 0$, $u_p'(x)$ can be written as follows.

$$(5.2)\qquad u_p'(x)=-\frac{1}{\pi}\int_0^\infty \frac{\xi(p+c_1\xi+c_2\xi^\alpha)\sin\xi x}{(p+c_1\xi+c_2\xi^\alpha)^2} \frac{+\xi(c_1h_1\xi\log\xi+c_2h_2\xi^\alpha+a\xi)\cos\xi x}{+(c_1h_1\xi\log\xi+c_2h_2\xi^\alpha+a\xi)^2}d\xi$$

For simplicity, we put the denominator of the integrand of (5.2) as $F(\xi)$;

$$(5.3)\qquad F(\xi)=(p+c_1\xi+c_2\xi^\alpha)^2+(c_1h_1\xi\log\xi+c_2h_2\xi^\alpha+a\xi)^2$$

The behavior of $u_p'(x)$ as $|x|\downarrow 0$ is different according as $0<\alpha<1$ or $1<\alpha<2$.

5.2. *Behavior of* $u_p'(x)$ *as* $|x|\downarrow 0$ *when* $1<\alpha<2$. We rewrite $u_p'(x)$ as follows:

$$(5.4)\quad u_p'(x)=-\frac{1}{\pi}\left\{\int_0^\infty \frac{c_2\xi^{\alpha+1}\sin\xi x}{F(\xi)}d\xi+\int_0^\infty \frac{c_2h_2\xi^{\alpha+1}\cos\xi x}{F(\xi)}d\xi\right\}$$
$$-\frac{1}{\pi}\int_0^\infty \frac{c_1\xi^2\sin\xi x}{F(\xi)}d\xi-\frac{1}{\pi}\int_0^\infty \frac{(c_1h_1\xi^2\log\xi+a\xi^2)\cos\xi x}{F(\xi)}d\xi$$

The third term of (5.4) is a bounded continuous function by the bounded convergence theorem. When $3/2<\alpha<2$, the second term is also a bounded continuous function by the same theorem.

5.2.1. *The case* $3/2<\alpha<2$. In this case, we have only to investigate the first term of (5.4). By evenness and oddness of $\cos\xi x$ and $\sin\xi x$ respectively, we consider the case $x>0$. Let us rewrite the first term of (5.4) as follows:

$$(5.5)\qquad -\frac{1}{\pi}\left\{\int_0^\infty \frac{c_2\xi^{\alpha+1}\sin\xi x}{F(\xi)}d\xi+\int_0^\infty \frac{c_2h_2\xi^{\alpha+1}\cos\xi x}{F(\xi)}d\xi\right\}$$
$$=-\frac{1}{\pi}\left\{\frac{1}{c_2(1+h_2^2)}\int_0^\infty \frac{\sin\xi x}{\xi^{\alpha-1}}d\xi+\frac{h_2}{c_2(1+h_2^2)}\int_0^\infty \frac{\cos\xi x}{\xi^{\alpha-1}}d\xi\right\}$$
$$+\frac{1}{\pi}\left\{\frac{1}{c_2(1+h_2^2)}\int_0^\infty \frac{\sin\xi x}{\xi^{\alpha-1}}d\xi-\int_0^\infty \frac{c_2\xi^{\alpha+1}\sin\xi x}{F(\xi)}d\xi\right\}$$
$$+\frac{1}{\pi}\left\{\frac{h_2}{c_2(1+h_2^2)}\int_0^\infty \frac{\cos\xi x}{\xi^{\alpha-1}}d\xi-\int_0^\infty \frac{c_2h_2\xi^{\alpha+1}\cos\xi x}{F(\xi)}d\xi\right\}.$$

Since $3/2<\alpha<2$, the second and the third term of (5.5) are bounded continuous functions by the bounded convergence theorem. The first term of (5.5) can be calculated by the following formulas:

$$\int_0^\infty \frac{\sin \xi x}{\xi^a} d\xi = \frac{\pi x^{a-1}}{2\Gamma(a)\sin\frac{\pi a}{2}}, \quad (0<a<2,\ 0<x)$$

$$\int_0^\infty \frac{\cos \xi x}{\xi^a} d\xi = \frac{\pi x^{a-1}}{2\Gamma(a)\cos\frac{\pi a}{2}}, \quad (0<a<1,\ 0<x).$$

Rherefore,

$$u_p'(x) = -\frac{\Gamma(2-\alpha)\sin\frac{\pi a}{2}}{c_2\pi(1+h_2^2)}\{\mathrm{sgn}\,(x)-\beta_2\}\,|x|^{\alpha-2} + \text{(a bounded continuous function)}.$$

5.2.2. *The case* $1<\alpha<3/2$. We rewrite $u_p'(x)$ as follows:

$$(5.6) \qquad u_p'(x) = -\frac{1}{c_2\pi(1+h_2^2)}\left\{\int_0^\infty \frac{\sin\xi x}{\xi^{\alpha-1}}d\xi + \int_0^\infty \frac{h_2\cos\xi x}{\xi^{\alpha-1}}d\xi\right\}$$

$$+\frac{1}{\pi}\int_0^\infty\left\{\frac{1}{c_2(1+h_2^2)\xi^{\alpha-1}} - \frac{\xi p + c_1\xi + c_2\xi^\alpha}{F(\xi)}\right\}\sin\xi x\, d\xi$$

$$+\frac{1}{\pi}\int_0^\infty\left\{\frac{h_2}{c_2(1+h_2^2)\xi^{\alpha-1}} - \frac{\xi(c_1h_1\xi\log\xi + c_2h_2\xi^\alpha + a\xi)}{F(\xi)}\right\} \cos\xi x\, dx.$$

For simplicity, we denote the second and the third term of (5.6) by $\frac{1}{\pi}\int_0^\infty R(\xi)\sin\xi x\, d\xi$ and $\frac{1}{\pi}\int_0^\infty T(\xi)\cos\xi x\, d\xi$ respectively. The behavior of the first term of (5.6) is the same sa in 5.2.1. As for the second term of (5.6), we use the theorem 1 of [5]. Then,

$$\frac{1}{\pi}\int_0^\infty R(\xi)\sin\xi x\, d\xi \sim \frac{1}{\pi x}S(2\alpha-2)\,R\left(\frac{1}{\pi}\right)$$

$$=O\left(x^{2\alpha-3}\log\frac{1}{x}\right), \text{ where } S(m)=\frac{\pi}{2\Gamma(m)\sin\frac{\pi m}{2}}.$$

As for the third term of (5.6), we rewrite it by the integration by part so that the theorem 1 of [5] can be applied:

$$\frac{1}{\pi}\int_{\varepsilon}^{M} T(\xi)\cos\xi x\,d\xi=\frac{1}{\pi}\left\{\left[T(\xi)\frac{\sin\xi x}{x}\right]_{\xi=\varepsilon}^{\xi=M}-\int_{\varepsilon}^{M}\frac{dT(\xi)}{d\xi}\frac{\sin\xi x}{x}d\xi\right\}.$$

Letting $M\to\infty$ and $\varepsilon\to 0$ for fixed $x>0$,

$$(5.7)\qquad \frac{1}{\pi}\int_0^{\infty} T(\xi)\cos\xi x\,d\xi=\frac{-1}{\pi x}\int_0^{\infty}\frac{dT(\xi)}{d\xi}\sin\xi x\,d\xi\,.$$

Since $\frac{dT(\xi)}{d\xi}$ has the index $1-2\alpha$ as $\xi\to\infty$, the theorem 1 of [5] can be applied if $\alpha\neq 3/2$. Then,

$$\frac{1}{\pi}\int_0^{\infty} T(\xi)\cos\xi x\,d\xi=\frac{-1}{\pi x}\int_0^{\infty}\frac{dT(\xi)}{d\xi}\sin\xi x\,d\xi\sim$$

$$\frac{-1}{\pi x}\cdot\frac{1}{x}S(2\alpha-1)\frac{dT(\xi)}{d\xi}\bigg|_{\xi=\frac{1}{x}}=O\left(x^{2\alpha-3}\log\frac{1}{x}\right).$$

When $\alpha=3/2$, we must use the theorem 3 of [5] for a function with index -2. Then,

$$(5.8)\quad \frac{1}{\pi}\int_0^{\infty} T(\xi)\cos\xi x\,d\xi=\frac{-1}{\pi x}\int_0^{\infty}\frac{dT(\xi)}{d\xi}\sin\xi x d\xi\sim\frac{-1}{\pi x}x\int_0^{1/x}\xi\frac{dT(\xi)}{d\xi}d\xi\,.$$

Describing the highest power of ξ in the numerator and denominator of $\xi\frac{dT(\xi)}{d\xi}$,

$$(5.9)\qquad \xi\frac{dT(\xi)}{d\xi}=k\frac{\xi^6\log\xi+\cdots}{\xi^7+\cdots+h}\,.$$

Here k and h are constants $\neq 0$.

Therefore, the behavior of $\int_0^{1/x}\xi\frac{dT(\xi)}{d\xi}d\xi$ as $x\downarrow 0$ is determined by $k\int_1^{1/x}\frac{\log\xi}{\xi}d\xi$, since $\xi\frac{dT(\xi)}{d\xi}$ has no singularity except powers of $\log\xi$ from (5.9). Now, the proof of theorem 4 for $\alpha=3/2$ is accomplished by nothing

$$\int_1^{1/x}\frac{\log\xi}{\xi}d\xi=\frac{\left(\log\frac{1}{x}\right)^2}{2}\,.$$

5.3. *Behavior of* $u'_p(x)$ *as* $|x|\downarrow 0$ *when* $0<\alpha<1$. We rewrite $u'_p(x)$ as follows:

$$(5.10)\qquad u'_p(x)=-\frac{1}{\pi}\left\{\int_0^\infty \frac{\sin\xi x}{\left(c_1+\frac{a^2}{c_1}\right)+2ah_1\log\xi+c_1h_1^2\log^2\xi}d\xi\right.$$

$$\left.+\int_0^\infty \frac{(h_1\log\xi+\delta)\cos\xi x}{\theta+\omega\log\xi+c_1h_1^2\log^2\xi}d\xi\right\}$$

$$+\frac{1}{\pi}\int_0^\infty G(\xi)\sin\xi x d\xi+1/\pi\int_0^\infty H(\xi)\cos\xi x\, d\xi,\quad \text{where}$$

$$(5.11)\qquad G(\xi)=\frac{1}{\left(c_1+\frac{a^2}{c_1}\right)+2ah_1\log\xi+c_1h_1^2\log\xi}-\frac{\xi(p+c_1\xi+c_2\xi^\alpha)}{F(\xi)}\quad \text{and}$$

$$(5.12)\qquad H(\xi)=\frac{h_1\log\xi+\delta}{\theta+\omega\log\xi+c_1h_1^2\log^2\xi}-\frac{\xi(c_1h_1\xi\log\xi+c_2h_2\xi^\alpha+a\xi)}{F(\xi)}$$

The first term of (5.10) has been already investigated in 4.4. As for the second term of (5.10), we can apply the theorem 1 of [5], since $G(\xi)$ has the index $\alpha-1$. Then,

$$(5.13)\qquad \frac{1}{\pi}\int_0^\infty G(\xi)\sin\xi x\, d\xi\sim\frac{1}{\pi}x^{-1}S(1-\alpha)\,G\left(\frac{1}{x}\right).$$

From (5.13), $\frac{1}{\pi}\int_0^\infty G(\xi)\sin\xi x\, d\xi$ has a lower order of infinity than $O\left(\frac{1}{x\left(\log\frac{1}{x}\right)^3}\right)$ as $x\downarrow 0$. Therefore, it has no effect on the assertion of theorem 3. The third term fo (5.10) has also no effect, since $dH(\xi)/d\xi$ has the index $\alpha-2$ and the theorem 1 of [5] can be applied and

$$\frac{1}{\pi}\int_0^\infty H(\xi)\cos\xi x\, d\xi=\frac{-1}{\pi x}\int_0^\infty\frac{dH(\xi)}{d\xi}\sin\xi x\, d\xi\sim$$

$$\frac{1}{\pi x}x^{-1}S(2-\alpha)\frac{dH(\xi)}{d\xi}\bigg|_{\xi=\frac{1}{x}}\cdot$$

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