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Deformation theoretic methods in the theory of algebraic transformation spaces*

By

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Introduction

The main object of this paper is to prove rough principal orbit theorems for actions of algebraic groups in characteristic $p>0$. The type of theorem which is intended is one which would give sufficient conditions so that in an action of a linear algebraic group G on a non-singular variety X over an algebraically closed field K, there would be a dense open subset $U\subset X$ on which connected components of stabilizers would be conjugate.

Our method is to apply the techniques of deformation theory to algebraic actions. We begin, in sections 1–3, with what is a treatment of the deformation theory of subgroups of an algebraic group. As we are concerned with a geometric application, we have found that neither the deformation theory of smooth groups, nor the theory of formal deformations is entirely adequate. A somewhat technical analysis of the deformations of finite (non-commutative) group-schemes has proven to be necessary. In later sections this analysis is applied to the stabilizer of the identity map of a space with an aciton. The arguments are analogous to arguments which proved effective in char-

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acteristic zero (see [3]).

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1. The fundamental construction.

We begin by considering certain special classes of finite group schemes. Throughout, it is assumed that all schemes are defined over a field of positive characteristic, $p>0$. If X is a scheme a **finite group scheme** over X is an affine group scheme over X, $G=\operatorname{Spec}\mathscr{A}$ where $\mathscr{A}$ is a locally free sheaf of commutative Hopf algebras of finite rank. The rank of $\mathscr{A}$ is called the order of G.

1.1 Definition. Let X be a scheme and let $G=\operatorname{Spec}\mathscr{A}$ be a finite group-scheme over X. Then G is said to be a non-singular group scheme of length r and exponent v, or more briefly a group scheme of type (v, r) over X if for each $x\in X$, there is a neighborhood U of x, and a locally free subsheaf of $\mathscr{A}\,|\,U$, ω_U, such that $\mathscr{A}=S_{\mathscr{O}_X|U}(\omega_U)/\mathscr{I}$ where $\mathscr{I}$ is the sheaf of ideals generated by p^v-th powers of elements of ω_U, ω_U is of rank r and ω_U generates the ideal defining the identi ty section in G.

Now if X is a scheme and G is a group scheme of type (v, r) over X, we consider a functor on the category of X-schemes which may be associated to G. Namely, if $p\colon Y\to X$ is an X-scheme, then $G\times_X Y$ is a group-scheme of type (v, r) over Y. Let s be an integer less than r and set $\operatorname{Inf}_{G/X}^{(v,s)}(Y)$ equal to the set of finite sub-group schemes of $G\times_X Y$ of type (v, s) over Y. This naturally is a contravariant functor on the category of X-schemes. The aim of the present section is to prove the following:

1.2 Theorem. *Let X be a scheme defined over a field, k, of characteristic $p>0$. Let G be a finite group scheme over X of type (v, r) and let s be a non-negative integer less than r.*

i) *There is an X scheme, denoted $\mathbf{Inf}_{G/X}^{(v,s)}$ with structure morphism p, such that $\mathrm{Hom}_X(Y, \mathbf{Inf}_{G/X}^{(v,s)}) = \mathrm{Inf}_{G/X}^{(v,s)}(Y)$, the set of finite subgroup-schemes of $G \times_X Y$ of type (v, s) over Y.*

ii) *Let $W = \mathbf{Inf}_{G/X}^{(v,s)}$. Then there is a subgroup scheme of $G \times_X W$ of type (v, s) over W, which we denote $H_{G/X}^{(v,s)}$ which is a universal subgroup-scheme of type (v, s) over W. That is, the isomorphism of functors in* (i) *assigns to a map $f \in \mathrm{Hom}_X(Y, W)$, the group scheme $f^* H_{G/X}^{(v,s)} = H_{G/X}^{(v,s)} \times_W Y$, where Y is a W scheme under f.*

iii) $\mathbf{Inf}_{G/X}^{(v,s)}$ *commutes with base extension. That is*

$$\mathbf{Inf}_{G\times Y/Y}^{(v,s)} = Y \times \mathbf{Inf}_{G/X}^{(v,s)} .$$

Proof. We begin by remarking that iii) is an immediate consequence of the definition of the functor which $\mathrm{Inf}_{G/X}^{(v,s)}$ represents. As for i) and ii) the proof is divided into a number of steps. The first is the following.

1.3 Proposition. *Let X be a scheme and let G be a finite group scheme of order n on X. Then there is an X scheme, $\boldsymbol{Gp}_{G/X}^r$, and a finite closed $\boldsymbol{Gp}_{G/X}^r$-subgroup scheme of $G \times_X \boldsymbol{Gp}_{G/X}^r$ of order r, denoted $H_{G/X}^r$ satisfying the following condition.*

Let $Gp_{G/X}^r(\cdot)$ denote the functor which assigns to each X scheme Y, the set of closed finite subgroup schemes of $G \times_X Y$ of order r. For any $f \in \mathrm{Hom}_X(Y, \boldsymbol{Gp}_{G/X}^r)$ let $\gamma_Y(f) = Y \times_Z H_{G/X}^r$ where $Z = \boldsymbol{Gp}_{G/X}^r$ and Y is regarded as a Z scheme with structure morphism, f. Then γ_Y is an isomorphism of functors. Moreover $\boldsymbol{Gp}_{G/X}^r$ is proper over X.

Proof. We begin with two lemmas:

1.3.1 Lemma. *Let X be a scheme, and let $\mathscr{A}$ be a locally free scheaf of commutative $\mathcal{O}_X$ algebras with unit. Let $p: \mathscr{A} \to \mathscr{B} \to \mathcal{O}$*

be a surjective morphism of locally free sheaves. Then there is a unique "maximal" closed subscheme of X, Z such that for any morphism $g: U \to X$ for which $g^*(\mathscr{B})$ is a sheaf of quotient algebras of $\mathscr{A}$, $g(U) \subset Z$. That is g factors through the injection $Z \hookrightarrow X$.

Proof. Let $\mathscr{I} = \ker(p)$. Then to say that $g^*(\mathscr{B})$ is a sheaf of quotient algebras is equivalent to the statement that $g^*(\mathscr{I})$ is a sheaf of ideals in $g^*(\mathscr{A})$. This is equivalent to saying that $g^*(\mathscr{I})\cdot g^*(\mathscr{A}) + g^*(\mathscr{I}) = g^*(\mathscr{I})$. Equivalently the rank of $g^*(\mathscr{A})/g^*(\mathscr{I})\cdot g^*(\mathscr{A}) + g^*(\mathscr{I})$ is equal to the rank of $g^*(\mathscr{A})/g^*(\mathscr{I}) = g^*(\mathscr{B})$. Let $n = \operatorname{rank} \mathscr{A}$, $r = \operatorname{rank} \mathscr{B}$, $s = \operatorname{rank}(\mathscr{I})$. Consider the sheaf $\mathscr{A}/(\mathscr{I}\cdot\mathscr{A} + \mathscr{I}) = \mathscr{Q}$. As the denominator contains $\mathscr{I}$ its rank at a point is at most r. Take Z equal to the maximal closed subset on which it is of rank r. (That is Z is the uppermost stratum in the flattening stratification of X associated to $\mathscr{Q}$.) Clearly Z is the subscheme in question.

1.3.2. Lemma *Let X be a scheme and let $\mathscr{A}^\vee$ be a finite locally free sheaf of $\mathcal{O}_X$-algebras (this time not necessarily commutative) with unit, and with an $\mathcal{O}_X$-involution, $s: \mathscr{A}^\vee \to \mathscr{A}^\vee$, Let $\mathscr{B}^\vee \hookrightarrow \mathscr{A}^\vee$ be a sub-bundle of $\mathscr{A}^\vee$. Then there is a unique "maximal" closed subscheme of X, denoted here, Z, such that if $g: Y \to X$ is any morphism for which $g^*(\mathscr{B}^\vee)$ is an s-stable sheaf of subalgebras of $g^*(\mathscr{A}^\vee)$ with unit, then g factors through the inclusion of Z in X.*

Proof. *Set* $\mathscr{Q} = \mathscr{B}^\vee\cdot\mathscr{B}^\vee + \mathscr{B}^\vee + s(\mathscr{B}^\vee) + \mathcal{O}_X$. Consider the points at which the rank of $\mathscr{A}^\vee/\mathscr{Q}$ equals the rank of $\mathscr{A}^\vee/\mathscr{B}^\vee$ (in the sense of the previous proof). We are done. We can now construct $\boldsymbol{Gp}^r_{G/X}$. First, suppose $G = \operatorname{Spec}\mathscr{A}$. Let T be the Grassman of r-quotient bundles of $\mathscr{A}$. Let $p: T \to X$ be the structure morphism. Let $\mathscr{B}$ be the universal quotient bundle of $p^*\mathscr{A}$ of rank r. Apply Lemma 1.3.1, and we arrive at a closed subscheme, $T_1 \hookrightarrow T$, a structure morphism $p_1: T_1 \to X$, and a universal quotient algebra $p_1^*\mathscr{A} \to \mathscr{B}_1 \to 0$. Let $\mu_{\mathscr{A}}$ be the comultiplication $\mu_{\mathscr{A}}: \mathscr{A} \to \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A}$, and let $s_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}$ be the antipode. Then $p_1^*(\mu_{\mathscr{A}})$ induces an algebra structure on $\mathscr{A}^\vee = \operatorname{Hom}_{\mathcal{O}_{T_1}}(p_1^*\mathscr{A}, \mathcal{O}_{T_1})$, while $p_1^*(s_{\mathscr{A}})$ induces an involution on $\mathscr{A}^\vee$. Let $\mathscr{B}^\vee = \operatorname{Hom}_{\mathcal{O}_{T_1}}(\mathscr{B}_1,$

$\mathcal{O}_{T_1}$). Then $\mathcal{B}^\vee$ is a sub-bundle of $\mathcal{A}^\vee$. Apply Lemma 1.3.2. We now have a scheme $Z \hookrightarrow T$, a structure morphism $p^1: Z \to X$ and a certain r-quotient bundle of $(p^1)^*(\mathcal{A})$, $\mathcal{B}_2$. We have arranged things so that Z is the unique maximal closed subscheme of T on which $\mathcal{B}$ is a quotient Hopf algebra of $p^*\mathcal{A}$ with antipode. Thus we may take $\boldsymbol{Gp}^r_{G/X}$ equal to Z and $H^r_{G/X}$ equal to $\mathrm{Spec}(\mathcal{B}_2)$. The proof of 1.3 is concluded.

We shall now proceed with a construction of $\mathbf{Inf}^{(v,s)}_{G/X}$ in such a way that we will not find proof of its properties necessary.

Suppose that $\bar{G} = \mathrm{Spec}\,\bar{\mathcal{A}}$ is a finite group-scheme of type (v, m) over Z and suppose that $\bar{\mathcal{I}}_G$ is the sheaf of ideals in $\bar{\mathcal{A}}$ defining the identity section. Then, $\bar{G}$ is of order p^{vm}. Suppose that $\bar{H} = \mathrm{Spec}\,\bar{\mathcal{B}}$ is a closed subgroup scheme of $\bar{G}$ of order p^{vr}, with identity section defined by $\bar{\mathcal{I}}_H$. Then as $\bar{\mathcal{B}}$ is a quotient of $\bar{\mathcal{A}}$, for any $v \in \Gamma(U, \mathcal{I}_H)$ for all $U \subset Z$, $v^{p^v} = 0$. Since H is of order p^{vr}, it follows that $\bar{\mathcal{I}}_H/\bar{\mathcal{I}}^2_H$ is of rank at least r at each point of Z. A moment's thought reveals that $\bar{H}$ is of type (v, r) if and only if the rank of $\bar{\mathcal{I}}_H/\bar{\mathcal{I}}^2_H$ is precisely r.

Now let $G = \mathrm{Spec}\,\mathcal{A}$ be a finite group-scheme of type (v, n) over X with identily section defined by $\mathcal{I}_G$. Set $\alpha = p^{vn}$, $\beta = p^{vr}$ and suppose that $p: \boldsymbol{Gp}^\beta_{G/X} \to X$ is the structure morphism. Suppose that $H^\beta_{G/X} = \mathrm{Spec}\,\mathcal{B}$ with identity section defined by $\mathcal{I}_0$. Then $\mathcal{B}$ is a quotient of $p^*\mathcal{A}$ and by the argument above the set of points, x, at which $\mathcal{B}_x$ is of type (v, r) is just the set of points where $\mathcal{I}_0/\mathcal{I}_0^2$ is of rank r. As r is a lower bound for the rank of $\mathcal{I}_0/\mathcal{I}_0^2$, this is an open set $U \subset \boldsymbol{Gp}^\beta_{G/X}$. Clearly U together with $\mathrm{Spec}(\mathcal{B}|U)$ satisfy i) to iii) of 1.2.

2. The co-normal bundle and Hochschild cohomology.

This section is, for the most part, devoted to certain rather technical computations. We make use of Sweedler's summation notation (Sweedler [4]). We fix our notation for the remainder of this section.

Let A be a commutative Hopf algebra, finitely generated over a field, k (as usual of positive characteristic though this is a superfluous assumption in this section). Let $m_A: A \otimes A \to A$ be the multiplication, let $\mu_A\colon A \to A \otimes_k A$ be the co-multiplication, let $\varepsilon_A\colon A \to k$ be the augmentation, let $s_A\colon A \to A$ be the antipode and let $I_A = \ker \varepsilon_A$.

Let $B = A/J$ be a quotient Hopf algebra of A with corresponding data m_B, μ_B, ε_B, s_B and I_B. Let $p\colon A \to B$ be the projection. Throughout this section we make the following assumption.

2.1. Assumption. Let A, B be as above. Assume that $J \cdot I_A = J \cap I_A^2$. (For the astute reader who might wish to generalize we might add that in the case where k is not a field we would add the assumption that B and A be flat over k).

Consider J/J^2. Now, $\mu_A(J) \subset J \otimes_k A + A \otimes_k J$ and $\mu_A(J^2) \subset J^2 \otimes_k A + J \otimes_k J + A \otimes_k J^2 \subset J \otimes_k A + A \otimes_k J^2$. Thus μ_A induces $\lambda_B\colon J/J^2 \to B \otimes_k J/J^2$. Moreover λ_B satisfies $\lambda_B(bx) = \mu_B(b) \cdot \lambda_B(x)$ where $B \otimes_k J/J^2$ is a $B \otimes_k B$ module under $b_1 \otimes b_2 \cdot (a \otimes x) = b_1 a \otimes b_2 x$. Similary, by noting that $\mu_A(J^2) \subset J^2 \otimes A + A \otimes J$, we obtain $\rho_B\colon J/J^2 \to J/J^2 \otimes B$, such that $\rho_B(bx) = \mu_B(b) \cdot \rho_B(x)$. Moreover $id_B \otimes \rho_B \circ \lambda_B = \lambda_B \otimes id_B \circ \rho_B$. (We might observe here that these identies as well as the co-associativety of both λ_B and ρ_B depend on the flatness of B over k) Furthermore, by Sweedler [4], or by observing that either λ_B or ρ_B make J/J^2 into a homogeneous bundle over the affine group Spec(B), $J/J^2 \simeq B \otimes (J/J^2)^{\lambda_B} \simeq (J/J^2)^{\rho_B} \otimes B$ where $(J/J^2)^{\lambda_B}$ signifies the λ_B invariants of J/J^2 and $(J/J^2)^{\rho_B}$ signifies the ρ_B invariants. Moreover the isomorphisms are isomorphisms of homogeneous bundles or of B-Hopf modules in Sweedler's terminology. By assumption 2.1, $(J/J^2)^{\lambda_B} \simeq J/I_A^2 \cap J$. Moreover, the sequence

$$(2.2) \qquad 0 \longrightarrow J/I_A^2 \cap J \longrightarrow I_A/I_A^2 \longrightarrow I_B/I_B^2 \longrightarrow 0$$

is exact. Let $\mathfrak{l}_A$ be the Lie algebra of Spec(A) and let $\mathfrak{l}_B$ be that of Spec(B). Let $\mathfrak{n}_{B/A} = \mathrm{Hom}_k(J/I_A^2 \cap J, k)$. By the exactness of 2.2 we have:

$$(2.3) \qquad \mathfrak{n}_{B/A} \simeq \mathfrak{l}_A / \mathfrak{l}_B .$$

We shall see that 2.3 is actually a morphism of Spec(B) representations under the appropriate actions. Observe that we may set $\mathfrak{l}'_B =$

$(m_B\otimes id_{J/J^2})\circ(id_B\otimes s_B\otimes id_{J/J^2})\circ(id_B\otimes\tau')\circ id_B\otimes\rho_B\circ\lambda_B$ where $\tau'(x\otimes b)=b\otimes x$ for $b\in B$, $x\in J/J^2$. Then ℓ_B is just inner automorphism by Spec(B). Under ℓ'_B, both $(J/J^2)^{\lambda_B}$ and $(J/J^2)^{\rho_B}$ are stable. In particular ℓ'_B maps $(J/J^2)^{\lambda_B}$ into $B\otimes(J/J^2)^{\lambda_B}$ and the expression above for ℓ'_B shows that if $x\in(J/J^2)^{\lambda_B}$,

$$\ell'_B(x)=s_B\otimes id\circ\tau'\circ\rho_B(x) \tag{2.4}$$

Now, whenever $\alpha: V\to B\otimes_k V$ is a representation of Spec(B) on V, there is a contragredient representation defined as follows. Set $\alpha^0(f)(v)=id\otimes f\circ\alpha(v)$. Then α^0 maps V^* into $\mathrm{Hom}_k(V, B)$. Let η_V: $V^*\otimes B\to\mathrm{Hom}_k(V, B)$ be defined by $\eta_V(f\otimes b)(v)=f(v)\cdot b$. Then $\alpha^\vee=\eta_V^{-1}\circ\alpha^0$ maps V^* into $V^*\otimes_k B$ and it is a standard exercise to show that it is a representation. In particular, considering ℓ'_B on $(J/J^2)^{\lambda_B}$, we arrive at a map ℓ'^0_B: $((J/J^2)^{\lambda_B})^*\to\mathrm{Hom}_k((J/J^2)^{\lambda_B}, B)$ and a representation, $\ell'^\vee_B$: $((J/J^2)^{\lambda_B})^*\to((J/J^2)^{\lambda_B})^*\otimes B$. Observing that $\mathfrak{n}_{B/A}\simeq((J/J^2)^{\lambda_B})^*$ is now an isomorphism of B representations where Spec(B) operates on $\mathfrak{n}_{B/A}$ by inner automorphism, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{n}_{B/A} & \xrightarrow{\ \ell'^0_B\ } & \mathrm{Hom}_k((J/J^2)^{\lambda_B}, B) \\ & {\scriptstyle \ell'^\vee_B}\searrow \qquad \nearrow{\scriptstyle \eta} & \\ & \mathfrak{n}_{B/A}\otimes_k B & \end{array} \tag{2.5}$$

We would like, at this point, to utilize these facts to prove three formulae which, taken together, prove that certain maps which arise later in considering the tangent space to $\mathbf{Inf}_{G/k}^{(v,r)}$, are Hochschild cocycles and that certain others are Hochschild one boundaries. These results can essentially be found in [2] in a much more elegant formulation. However there, the group schemes under consideration are smooth. The smoothness hypothesis is replaced here by the fact that our group schemes are of type (v, r). We give a self-contained proof of our results.

Let $j_1: \mathfrak{n}_{B/A}\otimes_k B\to\mathrm{Hom}_B(J/J^2, B)$ be the map defined as follows.

If $x \in J/J^2$, the B-isomorphism $J/J^2 \simeq B \otimes (J/J^2)^{\lambda_B}$ implies that $x = \sum_{i=1}^{q} b_i \bar{x}_i$ with $\bar{x}_i \in (J/J^2)^{\lambda_B}$. Set $j_1(f \otimes b)(x) = \sum_{i=1}^{q} b_i s_B(b) f(\bar{x}_i)$. Then the isomorphism mentioned above implies that the definition is independent of the praticular expression for x chosen. Let ${}_{\mu}B \otimes_k B$ denote $B \otimes_k B$ viewed as a B-module via $b \cdot u = \mu_B(b) \cdot u$ for $u \in B \otimes_k B$. Define $j_2 : \mathfrak{n}_{B/A} \otimes_k B \otimes_k B \longrightarrow \mathrm{Hom}_B(J/J^2, {}_{\mu}B \otimes_k B)$ by $j_2(f \otimes c \otimes d)(x) = \sum_{i=1}^{q} \mu_B(b_i) \cdot s(d) \otimes s(c) f(\bar{x}_i)$ where x is as above. For convenience, set $\tau(c \otimes d) = d \otimes c$ for $c \otimes d \in B \otimes_k B$. Our aim is to prove the commutativity of thefollowing diagrams:

2.6.I

$$\begin{array}{ccc} \mathfrak{n}_{B/A} \otimes_k B & \xrightarrow{\ j_1\ } & \mathrm{Hom}_B(J/J^2, B) \\ \big\downarrow{\scriptstyle id_{\mathfrak{n}_{B/A}} \otimes \mu_B} & & \big\downarrow{\scriptstyle \mu_B^*} \\ \mathfrak{n}_{B/A} \otimes_k B \otimes_k B & \xrightarrow{\ j_2\ } & \mathrm{Hom}_B(J/J^2, {}_{\mu}B \otimes_k B) \end{array}$$

2.6.II

$$\begin{array}{ccc} \mathfrak{n}_{B/A} \otimes_k B & \xrightarrow{\ j_1\ } & \mathrm{Hom}_B(J/J^2, B) \\ \big\downarrow{\scriptstyle id_{\mathfrak{n}_{B/A}} \otimes id_B \otimes 1} & & \big\downarrow{\scriptstyle \lambda_B^*} \\ \mathfrak{n}_{B/A} \otimes_k B \otimes_k B & \xrightarrow{\ j_2\ } & \mathrm{Hom}_B(J/J^2, {}_{\mu}B \otimes_k B) \end{array}$$

and

2.6.III

$$\begin{array}{ccc} \mathfrak{n}_{B/A} \otimes_k B & \xrightarrow{\ j_1\ } & \mathrm{Hom}_B(J/J^2, B) \\ \big\downarrow{\scriptstyle i_{\check{B}} \otimes id_B} & & \big\downarrow{\scriptstyle \rho_B^*} \\ \mathfrak{n}_{B/A} \otimes_k B \otimes_k B & \xrightarrow{\ j_2\ } & \mathrm{Hom}_B(J/J^2, {}_{\mu}B \otimes_k B) \end{array}$$

where

$$(*) \qquad \mu_B^*(\delta)(x) = \mu_B \circ \delta(x)$$

$$(**) \qquad \lambda_B^*(\delta)(x) = (id_B \otimes \delta) \circ \lambda_B(x)$$

and

$$(***) \qquad \rho_B^*(\delta)(x) = \delta \otimes id_{J/J^2} \circ \rho_B(x).$$

In each case, due to the fact that $J/J^2 \simeq B \otimes (J/J^2)^{\lambda_B}$, B-maps from J/J^2 into any B module are uniquely determined by their values on $(J/J^2)^{\lambda_B}$. Our technique in each case will be to compare values on $(J/J^2)^{\lambda_B}$.

Proof of I: If $\bar{x} \in (J/J^2)^{\lambda_B}$,

$$\mu_B(j_1(f \otimes b))(\bar{x}) = \mu_B(f(\bar{x})s_B(b)) = f(\bar{x})(\mu_B \circ s_B)(b).$$

Moreover

$$\begin{aligned} & j_2(id_{\Pi_{B/A}} \otimes \mu_B(f \otimes b))(\bar{x}) \\ &= j_2(f \otimes \mu_B(b))(\bar{x}) \\ &= f(\bar{x})\tau \circ s_B \otimes s_B \circ \mu_B(b) \\ &= f(\bar{x})\mu_B \circ s_B(b). \end{aligned}$$

Thus I is proven.

Proof of II:

Recall that $\bar{x}$ is λ_B-invariant.

$$\begin{aligned} & \lambda_B^*(j_1(f \otimes b))(\bar{x}) = id_B \otimes (j_1(f \otimes b)) \circ \lambda_B(\bar{x}) \\ &= id_B \otimes (j_1(f \otimes b))(1_B \otimes \bar{x}) \\ &= 1_B \otimes f(\bar{x})s(b) \\ &= f(x)(1_B \otimes s(b)) = j_2(f \otimes b \otimes 1)(\bar{x}) \\ &= j_2(id_{\Pi_{B/A}} \otimes id_B \otimes 1_B(f \otimes b))(\bar{x}). \end{aligned}$$

That proves II.

Proof of III:

To prove III we recall identity 2.4 and diagram (2.5). We define γ: $\mathrm{Hom}_k((J/J^2)^{\lambda_B}, B) \otimes B \to \mathrm{Hom}_B(J/J^2, {}_\mu B \otimes_k B)$ by $[\gamma(F \otimes b)](\bar{x}) = s_B(b) \otimes s_B(F(\bar{x}))$ for $\bar{x} \in (J/J^2)^{\lambda_B}$. Assembling diagrams we obtain:

$$\text{III}^* \qquad \begin{array}{ccc}
\mathfrak{n}_{B/A}\otimes B & \xrightarrow{\quad j_1 \quad} & \operatorname{Hom}_B(J/J^2,\, B) \\
\Big\downarrow {\scriptstyle \ell'^{0}_{B}\otimes id_B} & & \Big\downarrow {\scriptstyle \rho^*_B} \\
\operatorname{Hom}_k((J/J^2)^{\lambda_B},\, B)\otimes B & \xrightarrow{\quad \gamma \quad} & \operatorname{Hom}_B(J/J^2,\, {}_{\mu}B\otimes B) \\
\Big\uparrow {\scriptstyle \eta\otimes id_B} & \nearrow {\scriptstyle j_2} & \\
\mathfrak{n}_{B/A}\otimes B\otimes B & &
\end{array}$$

As $\eta^{-1}\circ\ell'^{0}_{B}=\ell'^{\vee}_{B}$ the commutativity of III would follow from the commutativity of the upper rectangle and the lower triangle of III*. We begin with the lower triangle.
(Recall that $\bar{x}\in(J/J^2)^{\lambda_B}$.)

$$[\gamma\circ\eta\otimes id_B(f\otimes c\otimes d)](\bar{x})$$

$$=\gamma(\eta(f\otimes c)\otimes d)(\bar{x})=s_B(d)\otimes\{s_B\circ(\eta(f\otimes c))(\bar{x})\}$$

$$=s_B(d)\otimes s_B(f(\bar{x})c)=f(\bar{x})s_B(d)\otimes s_B(c)$$

$=j_2(f\otimes c\otimes d)(\bar{x})$. Thus the lower triangle commutes.

As for the upper rectangle, we proceed as follows:

$$[\gamma(\ell'^{0}_{B}\otimes id_B(f\otimes b))](\bar{x})$$

$$=\gamma(\ell'^{0}_{B}(f)\otimes b)(\bar{x})$$

$$=s_B(b)\otimes s_B(\ell'^{0}_{B}(f)(\bar{x}))$$

$$=s_B(b)\otimes s_B(id_B\otimes f\circ\ell'_B(\bar{x}))$$

(By the definition of ℓ'^{0}_{B})

$$=s_B(b)\otimes s_B(id_B\otimes f\circ s_B\otimes id_{J/J^2}\circ\tau'\circ\rho_B(\bar{x}))$$

(By identity (2.4))

Now as f maps $(J/J^2)^{\lambda_B}$ to k and the identification of $B\otimes_k k$ with B is implicit in the expression $id_B\otimes f$, we find that the expression in the right parentheses can be rewritten allowing us to continue:

$$=s_B(b)\otimes s_B(f\otimes s_B\cdot\rho_B(\bar{x}))$$

$$=s_B(b)\otimes(f\otimes id_{J/J^2}\circ\rho_B(\bar{x}))\,.$$

That is, in summary,

$$[\gamma(\ell'^{0}_{B}\otimes id_B(f\otimes b))](\bar{x})$$

$$=s_B(b)\otimes(f\otimes id_{J/J^2}\circ\rho_B(\bar{x}))$$

On the other hand,

$$\rho^*_B(j_1(f\otimes b))(\bar{x})=[j_1(f\otimes b)]\otimes id_B\circ\rho_B(x)$$

Now suppose $\rho_B(\bar{x})=\sum_{(x)}\bar{x}_{(0)}\otimes\bar{x}_{(1)}$ where $\bar{x}_{(0)}\in(J/J^2)^{\lambda_B}$, $\bar{x}_{(1)}\in B$. Then this latter is equal to

$$\sum_{(x)}s(b)f(\bar{x}_{(0)})\otimes\bar{x}_{(1)}=\sum_{(x)}s(b)\otimes f(\bar{x}_{(0)})\bar{x}_{(1)}=s(b)\otimes(f\otimes id_{J/J^2}\circ\rho_B(x))$$

Thus $\rho^*_B\circ j_1=\gamma\circ(\ell'^{0}_{B}\otimes id_B)$ as desired, and hence III is proven.

The astute reader will observe that the three identities we have just proven correspond to the terms of the Hochschild one co-boundary operation on the chains of Spec(B) with coefficients in $\mathfrak{n}_{B/A}$ under inner automorphism. We now proceed with calculations corresponding to the one boundaries.

The Hochschild one boundaries in $\mathfrak{n}_{B/A}\otimes B$ are just elements of the form $\ell'^{\vee}_{B}(f)-f\otimes 1_B$. Thus we must evaluate such elements under j_1. If $\bar{x}\in(J/J^2)^{\lambda_B}$,

$$j_1(\ell'^{\vee}_{B}(f)-f\otimes 1_B)(\bar{x})$$

$$=j_1(\ell'^{\vee}_{B}(f))(\bar{x})-f(\bar{x})\,.$$

Then, $$j_1(\ell'^{\vee}_{B}(f))(\bar{x})=s_B(f\otimes id_B\circ(id_{J/J^2}\otimes s_B)\circ\rho_B(\bar{x}))$$

$$=f\otimes id_B\circ\rho_B(\bar{x}).$$

Moreover, $id_B\otimes f\circ\lambda_B(\bar{x})=1_B\otimes f(\bar{x})$ and since this latter map agrees with $j_1(f\otimes 1_B)$ on $(J/J^2)^{\lambda_B}$ we have demonstrated the following

(2.7) $$j_1(\ell'^{\vee}_{B}(f)-f\otimes 1_B)=f\otimes id_B\circ\rho_B-id_B\otimes f\circ\lambda_B$$

Now suppose that Δ is in the Lie algebra of A. Then Δ is a derivation of A in A/I_A. Then Δ may be regarded as an element of $(I_A/I_A^2)^* = \mathrm{Hom}_k(I_A/I_A^2, k)$ and by restriction to $J/J \cap I_A^2$, it induces an element of $\mathfrak{n}_{B/A}$. The map thus defined is just the natural projection from $\mathscr{L}_A$ to $\mathfrak{n}_{B/A}$. Furthermore, Δ may also be used to define derivations of A in A. We concern ourselves with three different induced derivations. If $a \in A$ let $\mu_A(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$. Set $\Delta^l(a) = \sum_{(a)} a_{(1)}\Delta(a_{(2)})$ and $\Delta^r(a) = \sum_{(a)} \Delta(a_{(1)}) \cdot a_{(2)}$. Moreover, the inner action of Spec(A) on A is expressed by

$$j_A(a) = \sum_{(a)} a_{(1)} s(a_{(3)}) \otimes a_{(2)}. \quad \text{Set} \quad \Delta^i(a) = \Delta \otimes_1 \circ j_A(a).$$

Then
$$\Delta^i(a) = \sum_{(a)} \Delta(a_{(1)} s_A(a_{(3)})) \cdot a_{(2)}$$
$$= \sum_{(a)} \varepsilon_A(a_{(1)}) \Delta(s(a_{(3)}) a_{(2)} + \Delta(a_{(1)}) \varepsilon_A(s_A(a_{(3)})) a_{(2)}.$$

Now noting that $\varepsilon_A \circ s_A = \varepsilon_A$ and $\Delta \circ s_A = -\Delta$, this becomes:

$$\Delta^i(a) = \sum_{(a)} \Delta \circ s(a_{(2)}) \cdot a_{(1)} + \Delta(a_{(1)}) \cdot a_{(2)}$$
$$= \sum_{(a)} \Delta(a_{(1)}) \cdot a_2 - \Delta(a_{(2)}) \cdot a_{(1)}$$
$$= \Delta^r(a) - \Delta^l(a).$$

Now, if $\partial \colon A \to A$ is a derivation, ∂ induces a B-map, $\bar{\partial} : J/J^2 \to B$ by restriction to J. We wish to calculate the maps induced by Δ^r and Δ^l. We first observe that $\lambda_B(x)$ and $\rho_B(x)$ may be calculated by taking an element, $\bar{x}$, of J/J^2 lifting to a pre-image, $x \in J$ and considering the image of $\mu_A(x)$ in J/J^2. Let Δ^0 denote the image of Δ in $\mathfrak{n}_{B/A}$. The previous observation implies that

(2.8) i) $1_B \otimes \Delta^0 \circ \lambda_B(x) = \overline{\Delta^l(x)}$

and

(2.8) ii) $\Delta^0 \otimes 1_B \circ \rho_B(x) = \overline{\Delta^r(x)}$.

where the upper bar denotes the residue class in the appropriate tensor

product.
Thus what we have shown by 2.7 is that

$$j_1(\ell^{\vee}_B(\Delta^0)-\Delta^0\otimes 1_B)=\overline{\Delta^r(x)}-\overline{\Delta^l(x)}=\overline{\Delta^i}\overline{(x)}$$

Now all of the calculations of this section may be summarized in the following:

2.9 Theorem. *Let $G=\mathrm{Spec}(A)$ be an affine group scheme of finite type over a field, and let $H=\mathrm{Spec}(B)$ be a closed subgroup scheme of G satisfying assumption 2.1. Let $B=A/J$, and let $\mathfrak{n}_{G/H}=\mathfrak{n}_{B/A}=\ell_G/\ell_H$ where ℓ_G is the Lie algebra of G and ℓ_H is the Lie algebra of H. Then there is a natural isomorphism j_1, from $\mathfrak{n}_{G/H}\otimes_k B$ to $\mathrm{Hom}_B(J/J^2, B)$ and:*

i) *Viewing $\mathfrak{n}_{G/H}\otimes_k B$ as the group of Hochschild one-cochains of H in $\mathfrak{n}_{G/H}$ under inner automorphism, $j_1(Z^1(H, \mathfrak{n}_{G/H}))=\{\delta\in \mathrm{Hom}_B(J/J^2, B)\colon \mu_B\circ\delta(x)=id_B\otimes\delta\circ\lambda_B(x)+\delta\otimes id_B\circ\rho_B(x).\}$*

ii) *Viewing $\mathfrak{n}_{G/H}\otimes B$ as above, $j_1(B^1(H, \mathfrak{n}_{G/H}))$ is precisely equal to $\zeta(\ell_G)$ were $\zeta\colon \ell_G\to \mathrm{Hom}_B(J/J^2, B)$ is the map defined by $\zeta(\Delta)=\bar{\Delta}^{i}$ where $\bar{\Delta}^{i}$ is the map from J/J^2 to B induced by the derivation of A in A associated to Δ by the action of A on itself by inner automorphism.*

Proof. i) is an immediate consequence of the commutativity of 2.6 I–III and ii) is just the calculation which immediately precedes the statement of the theorem.

3. Determination of the tangent space.

In this section we wish to give a rather explicit description of the tangent space to $\mathbf{Inf}_{G/X}^{(v,r)}$ in a certain special case.

Let $G=\mathrm{Spec}(A)$ be an affine group scheme over a field k and let $H=\mathrm{Spec}(B)$ be a closed subgroup-scheme of A. Let $k[t]$ be the ring of dual numbers, and let $T=\mathrm{Spec}(k[t])$. Let $E_{G/H}$ be the set of closed T-subgroup-schemes of $G\times_k T=G_T$ whose restriction to k-point of T is just H. Then if $\tilde{H}\in E_{G/A}$, and $\tilde{H}=\mathrm{Spec}(\tilde{B})$, let $\mathfrak{a}=\ker(\tilde{B}\to B)$.

Then $\mathfrak{a}$ is of square zero and so $\mathfrak{a}=\mathfrak{a}/\mathfrak{a}^2$ is a B-Hopf module in the language of Sweedler [4], or a homogeneous H-bundle in more geometric parlance. Let $E^0_{G/H}$ be the set of elements of $E_{G/H}$ such that $\mathfrak{a}\simeq B$ as a Hopf module. If $\tilde{H}_1$ and $\tilde{H}_2\in E^0_{G/H}$, then $\tilde{H}_1\times_T\tilde{H}_2\subset G_T\times_T G_T$. Set $\tilde{H}_1\cdot\tilde{H}_2=\Delta^{-1}(\tilde{H}_1\times_T\tilde{H}_2)$, where $\Delta: G_T\rightarrow G_T\times_T G_T$ is the diagonal. As Δ is a group-morphism. $\tilde{H}_1\cdot\tilde{H}_2$ is a T-subgroup of G_T, and as Δ commutes with interchange of factors, $\tilde{H}_1\cdot\tilde{H}_2=\tilde{H}_2\cdot\tilde{H}_1$. It is an exercise to show that $\tilde{H}_1\cdot\tilde{H}_2\in E^0_{G/H}$ and that this operation makes $E^0_{G/H}$ an abelian group. Moreover k operates as a monoid of endomorphisms of T. Just set $\zeta^0_x(t)=xt$ for $x\in k$. For any x let T_x denote T regarded as a T-scheme under the morphism opposite to ζ^0_x. Then if $\tilde{H}\in E^0_{G/H}$, $\tilde{H}\times_T T_x\hookrightarrow G_T\times_T T_x$. The latter is naturally isomorphic to G_T and so we arrive at a new element of $E^0_{G/H}$ which we will denote $x\cdot\tilde{H}$. By standard arguments this makes $E^0_{G/H}$ into a k-vector space.

Let $p: A\rightarrow B$ be the natural surjection and let $J=\ker p$. Then if $\mu_A: A\rightarrow A\otimes_k A$ is the co-multiplication μ_A induces Hopf-module actions $\rho_B: J/J^2\rightarrow J/J^2\otimes_k B$ and $\lambda_B: J/J^2\rightarrow B\otimes J/J^2$ as in the previous section. Moreover μ_B will denote the co-multiplication on B. We define $Z_{G/H}$ as follows:

$$(3.1.1)\qquad Z_{G/H}=\{\delta\in \mathrm{Hom}_B(J/J^2, B): \mu_B\circ\delta=id_B\otimes\delta\circ\lambda_B+\delta\otimes id_B\circ\rho_B\}.$$

3.1 Theorem. *Let $G=\mathrm{Spec}(A)$ be an affine group-scheme over a field k and let $H=\mathrm{Spec}(B)$ be a closed subgroup. Then there is a natural injection $\xi^0_{G/H}: E^0_{G/H}\rightarrow Z_{B/A}$. If, moreover, G is finite group scheme $\xi^0_{G/H}$ is surjective.*

Proof. We begin by considering a k-algebra R and a quotient algebra $S=R/J$. Let $p: R\rightarrow S$ be the projection. Let $k[t]$ be the ring of dual numbers and let $\pi: k[t]\rightarrow k$ be the natural map. Let $\bar{E}^0_{R/S}$ denote the set of quotient algebras of $R\otimes k[t]$, $\tilde{S}$ such that $\tilde{S}/t\tilde{S}\simeq S$ and $t\tilde{S}\simeq S$. Let $\tilde{R}=R\otimes_k k[t]$. For each element of $\bar{E}^0_{R/S}$ we have a commutative diagram:

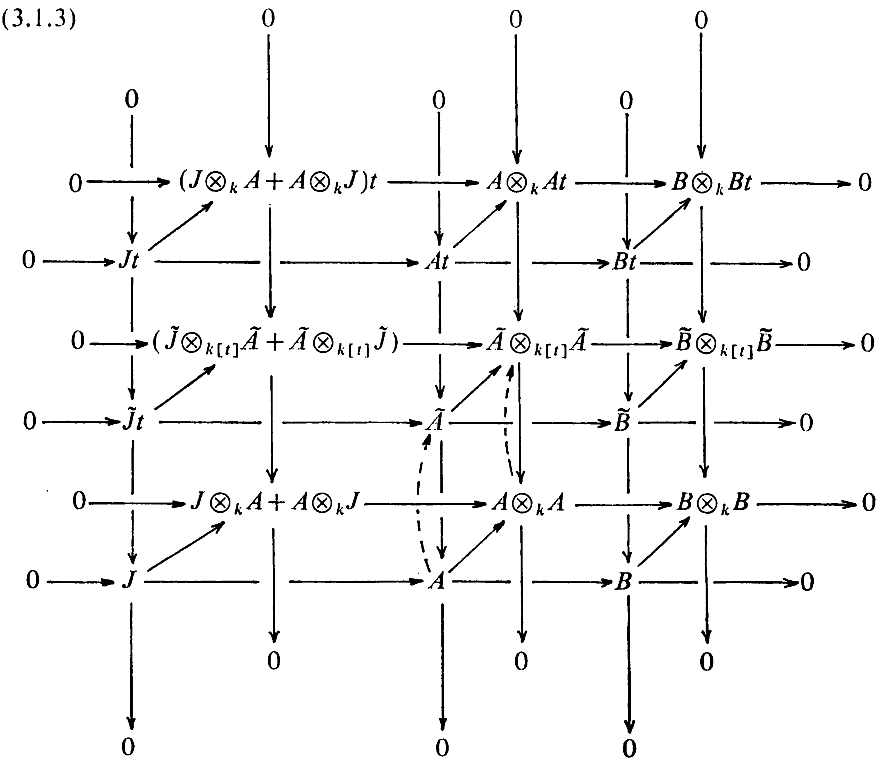
$$
(3.1.2)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Jt & \longrightarrow & Rt & \xrightarrow{p_0} & St & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & \tilde{J} & \longrightarrow & \tilde{R} & \xrightarrow{\tilde{p}} & \tilde{S} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_J} & & {\scriptstyle \sigma}\uparrow\downarrow{\scriptstyle \pi_R} & & \downarrow{\scriptstyle \pi_S} & & \\
0 & \longrightarrow & J & \xrightarrow{q} & R & \xrightarrow{p} & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the maps are the obvious ones, and $\sigma(r)=r\otimes 1$. For any $x\in J$ consider $\sigma\circ q(x)$. Then $\pi_S\circ\tilde{p}\circ\sigma\circ q(x)=p\circ\pi_R\circ\sigma\circ q(x)=p\circ q(x)=0$. Hence $\tilde{p}\circ\sigma\circ q(x)$ is in the image of u and hence this procedure defines a map, δ_0 from J to $S=St$. This is manifestly an R-map and so $\delta_0(J^2)=0$ and so it is clear that the procedeure defines an element, δ, of $\mathrm{Hom}_S(J/J^2, S)$. We will determine δ explicitly whence it will be clear that δ is uniquely determined by $\tilde{J}$. Suppose $\bar{x}\in J/J^2$ and x is a pre-image in J. Choose $x+ft\in\tilde{J}$ such that $\pi_J(x+ft)=x$. Then as $\sigma\cdot q(x)$ is just x and $x+ft\in\tilde{J}$, we find that $\tilde{p}(\sigma\circ q(x))=\tilde{p}(x)=\tilde{p}(x-(x+ft))=\tilde{p}(-ft)$. Now $-ft\in\ker\pi_R$ and so we may lift to Rt. Thus $p_0(-ft)$ is the element of St equal to $\tilde{p}\circ\sigma\circ q(x)$ in $\tilde{S}$. Thus $\delta(\bar{x})$ is just the image of $-f$ in S. Now we may reverse this procedure. Namely suppose $\delta\in\mathrm{Hom}_S(J/J^2, S)$ is gvien. Choose a set of generators for J say $\{x_i\colon i\in I\}$, and let $\bar{x}_i$ denote the residue calss of x_i in J/J^2. Then let $\delta(\bar{x}_i)=\tilde{f}_i\in S$ and choose f_i a pre-image of $\tilde{f}_i$ in R. Take $y_i=x_i-f_it$ and take $\tilde{J}_\delta$ equal to the ideal in $\tilde{R}$ generated by the y_i. It is an exercise to prove that $\tilde{J}_\delta$ is independent of the choices involved and that the diagram like (3.12) which it defines determines δ. Thus we have established a bijective correspondence between $\bar{E}^0_{R/S}$ and $\mathrm{Hom}_S(J/J^2. S)$.

Now suppose that $\delta\in\mathrm{Hom}_S(J/J^2, S)$ corresponds to $\tilde{p}\colon\tilde{R}\to\tilde{S}$ as in diagram (3.1.2). There is a surjective map $\tilde{p}\otimes\tilde{p}\colon\tilde{R}\otimes_{k[t]}\tilde{R}\to\tilde{S}\otimes_{k[t]}\tilde{S}$ and so $\tilde{S}\otimes_{k[t]}\tilde{S}$ determines an element of $\mathrm{Hom}_{S\otimes_k S}(J_2/J_2^2, S\otimes_k S)$, where $J_2=\ker p\otimes p=J\otimes R+R\otimes J$. We should like to calculate the corresponding map. We note that $J_2/J_2^2 = J/J^2 \otimes R/J \oplus R/J \otimes J/J^2$. The explicit determination of δ makes it self-evident that the element of $\mathrm{Hom}_{S\otimes_k S}(J_2/J_2^2, S\otimes S)$ corresponding to $\tilde{S}\otimes_{k[t]}\tilde{S}$ is just $\delta\otimes id_S \coprod id_S\otimes\delta$.

We now return to the case under consideration in the theorem. We begin by defining $\xi^0_{G/H}$. Namely if $\tilde{H}\in E^0_{G/H}$, $\tilde{H}$ determines a diagram like (3.1.2) with R replaced by A and S by B. We can thus determine an element in $\mathrm{Hom}_B(J/J^2, B)$ by the procedure defined above. This is the element $\xi^0_{G/H}(\tilde{H})$. All that remains to be proven is that $\xi^0_{G/H}(\tilde{H})$ satisfies the identity of definition 3.1.1. To this end we observe that if $\tilde{H}=\mathrm{Spec}(\tilde{B})$, $\tilde{B}\otimes\tilde{B}$ determines an element of $\bar{E}^0_{A\otimes A/B\otimes B}$. Moreover as $\tilde{B}$ is the affine ring of a group scheme it has a co-multiplication, $\tilde{\mu}_B\colon \tilde{B}\to\tilde{B}\otimes_{k[t]}\tilde{B}$. Let $\tilde{\mu}_A$ denote the co-multiplication on $\tilde{A}$. Then as all of our morphisms are group-scheme morphisms, we obtain a commutative diagram:

(3.1.3)



The restriction of μ_A to J induces a morphism $\mu_0: J/J^2 \to \frac{J\otimes A+A\otimes J}{(J\otimes A+A\otimes J)^2}$ $=J/J^2\otimes B\oplus B\otimes J/J^2$. As we have observed in the previous section the induced morphism is just $\rho_B \amalg \lambda_B$. Now, the commutativity of diagram 3.1.3 implies that if we start at J, pass to $J\otimes A+A\otimes J$ and apply the procedure defining the map corresponding to the extension, the result would be the same as if we had first applied the procedure to the rear face of the diagram and then passed into $B\otimes B$ via the restriction of $\tilde{\mu}_B$ to $B\otimes B\cdot t$. If $\xi^0_{G/H}(\tilde{H})$ is δ, then the corresponding map associated to the front face of the diagram is $\delta\otimes id_B \amalg id_B\otimes\delta$. Consequently we arrive at the following diagram:

(3.1.4)
$$\begin{array}{ccc} J/J^2 & \xrightarrow{\rho_B\amalg\lambda_B} & J/J^2\otimes B\amalg B\otimes J/J^2 \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta\otimes id_B\amalg id_B\otimes\delta} \\ B & \xrightarrow{\mu_B} & B\otimes B \end{array}$$

The commutativity of this diagram is expressed equationally, as

$$\mu_B\circ\delta=\delta\otimes id_B\circ\rho_B+id_B\otimes\delta\circ\lambda_B\,.$$

This is just the identity of 3.1.1 and so $\xi^0_{G/H}(\tilde{H})\in Z_{G/H}$.

We shall now prove the surjectivity of $\xi^0_{G/H}$ in the case when G is a finite group scheme. First we prove that if $\delta\in Z_{G/H}$ and $\tilde{J}_\delta$ is the ideal in $\tilde{A}$ corresponding to δ, then $\tilde{\mu}_A(\tilde{J}_\delta)\subset\tilde{J}_\delta\otimes_{k[t]}\tilde{A}+\tilde{A}\otimes_{k[t]}\tilde{J}_\delta$. Suppose that $x\in J$ and $\bar{x}$ is its residue class in J/J^2. Let f be an element in A which maps to $-\delta(x)$ in B. Then $x-ft\in\tilde{J}_\delta$ and conversely every element of $\tilde{J}_\delta$ may be expressed in this way. Consider $\tilde{\mu}_A(x-ft)=\mu_A(x)-\mu_A(f)t$. Now the residue class of $\mu_A(x)$ in $(J\otimes A+A\otimes J)/(J\otimes A+A\otimes J)^2$ is just $(\rho_B(x),\lambda_B(x))$, and the identity of 3.1.1 shows that $(\delta\otimes id_B, id_B\otimes\delta)((\rho_B(x),\lambda_B(x))$ is just $-\mu_B(\bar{f})$ where $\bar{f}$ is the residue class of f in B. It follows that $\mu_A(x)-\mu_A(f)t\in\tilde{J}_\delta\otimes\tilde{A}+\tilde{A}\otimes\tilde{J}_\delta$. Thus what we have shown is that the elements of $Z_{G/H}$ correspond precisely to the ideals $\tilde{J}$ of the requisite type in A such that $\tilde{\mu}_A(\tilde{J})\subset\tilde{J}\otimes\tilde{A}+\tilde{A}\otimes\tilde{J}$. These are the ideals defining closed subschemes of G which are sub-monoids. Now the surjectivity of $\xi^0_{G/H}$ follows from this Lemma:

3.1.5 Lemma. *Let R be a commutative Artinian algebra over a field, k, and let $G=\mathrm{Spec}(A)$ be a finite group-scheme over R. Let $H=\mathrm{Spec}(B)$ be a faithfully flat closed subscheme of G. Then H is a closed subgroup-scheme of G if and only if H is a closed submonoid of G.*

Proof. Let the co-multiplication, antipode and augmentation on A be μ_A, s_A and ε_A respectively and let $I_A=\ker\varepsilon_A$. Let $p: A\to B$ be the map defined by the embedding $H\hookrightarrow G$ and let $J=\ker p$. Then the statement that H induces a sub-monoid functor of G is equivalent to the statement that $\mu_A(J)\subset J\otimes A+A\otimes J$ and hence μ_A induces a co-associative co-multiplication $\mu_B: B\to B\otimes B$. Let $\bar{h}: B\to T$ be an R-algebra morphism from B into a finite R algebra and let $h=\bar{h}\circ p$. Then $h\in G(\mathrm{Spec}(T))=G(T)$ and so it induces an automorphism, left translation, λ_h: $T\otimes_R A\to T\otimes_R A$, by $\lambda_h(t\otimes a)=\sum th(a_{(2)})\otimes a_{(1)}$. The fact that h vanishes on J implies the commutativity of the diagram:

$$\begin{array}{ccc} T\otimes_R A & \xrightarrow{\lambda_h} & T\otimes_R A \\ \downarrow{\scriptstyle id_T\otimes p} & & \downarrow{\scriptstyle id_T\otimes p} \\ T\otimes_R B & \xrightarrow{\lambda_{\bar{h}}} & T\otimes_R B \end{array}.$$

It follows that $\lambda_h(T\otimes_R J)\subset T\otimes_R J$. Moreover the vertices of the diagram are finite dimensional vector spaces over k and the arrows k-morphisms. Hence $\lambda_h(T\otimes_R J)=T\otimes_R J$ and hence $\lambda_{\bar{h}}$ is an automorphism. Observe that if $j(b)=1\otimes b\in T\otimes B$, then $id_T\otimes\bar{h}'\circ\lambda_{\bar{h}}\circ j=\bar{h}'\cdot\bar{h}$ where $\bar{h}'\cdot\bar{h}$ means the product of $\bar{h}'$ and $\bar{h}$ as elements of $H(\mathrm{Spec}(T))$. It follows that the map from $H(\mathrm{Spec}(T))$ to $H(\mathrm{Spec}(T))$ induced by multiplication by $\bar{h}$ is an automorphism of sets and hence $H(\mathrm{Spec}(T))$ is a group. In particular, as B is finite over R, we may set $T=B$ and $\bar{h}=id_B$. We first find a map $f: B\to B$ such that $f\cdot id_B=id_B$. That is f satisfies $\sum f(b_{(1)})\cdot b_{(2)}=b$. Moreover there is a map s_B, such that $\sum_{(b)} s_B(b_{(1)})\cdot b_{(2)}=f(b)$. The inclusion $H(H)\subset G(H)$ and the fact that $H(H)$ is a subgroup of $G(H)$ imply that $f\circ p$ must satisfy $f\circ p(a)=\varepsilon_A(a)\cdot 1_B$. It follows that f factors through R, and hence determines $\varepsilon_B: B\to R$ such that $\varepsilon_B\circ p=\varepsilon_A$. Consequently ε_B is an augmentation and s_B an antipode and the lemma is proven.

We now will apply these results to our main object of study. For the remainder of this section assume that k is an algebraically closed field of positive characteristic, $p>0$. Let G be a linear algebraic group over k. If X is a scheme and $H=\operatorname{Spec}\mathscr{A}$ is an affiine group scheme over X with identity section defined by the sheaf of ideals, $\mathscr{I}_H$, then H_ν will always denote $\operatorname{Spec}\mathscr{A}/\mathscr{I}_{H,\nu}$ where $\mathscr{I}_{H,\nu}$ is the ideal generated, locally, by p^νth powers of sections of $\mathscr{I}_H$. In particular, if G is of dimension n, G_ν is always a finite group scheme of type (ν, n) over k. If H is an infinitesimal closed subgroup of G, and $\mathscr{I}_H$ is the ideal defining the identity in H, then, if $x\in\mathscr{I}_H$ implies $x^{p^\nu}=0$, $H\subset G_\nu$. Thus, if X is a k-scheme and H is a closed finite subgroup scheme of $G\times_k X$ which is of type (ν, r) it follows that H is actually a subscheme of $G_\nu\times_k X$. This motivates the next definition.

3.2 Definition. Let G be a linear algebraic group over an algebraically closed field k of characteristic $p>0$. Then let $\mathbf{Inf}^{(\nu,r)}_{G/k}$ equal $\mathbf{Inf}^{(\nu,r)}_{G_\nu/k}$ and let $H^{(\nu,r)}_{G/k}=H^{(\nu,r)}_{G_\nu/k}$.

As k is algebraically closed we may identify the closed points of a k-scheme of finite type with its k-points, and for the remainder of this section we shall adhere to this convention. Thus if H is an infinitesimal subgroup of G of type (ν, r) we may think of it as a k-point of $\mathbf{Inf}^{(\nu,r)}_{G/k}$. We shall, however, write $x(H)$ for the point corresponding to H, and $H^{(x)}$ for the infinitesimal subgroup of G corresponding to a closed point of $\mathbf{Inf}^{(\nu,r)}_{G/k}$.

3.3 Theorem. *Let k be an algebraically closed field of characteristic $p>0$ and let G be a linear algebraic group over k. Let H be an infinitesimal subgroup of G of type (ν, r). Let ℓ_G be the Lie algebra of G, let ℓ_H be the Lie algebra of H and let $\mathfrak{n}_{G/H}$ denote ℓ_G/ℓ_H with the H-representation arising from the inner action of H on ℓ_G. Let $x=x(H)$ be the point of $\mathbf{Inf}^{(\nu,r)}_{G/k}$ corresponding to H, and let T_x denote the tangent space to $\mathbf{Inf}^{(\nu,r)}_{G/k}$ at the point x corresponding to H. Then, there is a natural isomorphism, $\xi_{G/H}$, from T_x to*

$Z^1(H, \mathfrak{n}_{G/H})$, *the Hochschild one-cycles of* H *with coefficients in* $\mathfrak{n}_{G/H}$.

Proof. Let T be the spectrum of the dual numbesrs. Then by the functorial description of $\mathbf{Inf}_{G/k}^{(v,r)}$, $\mathbf{Inf}_{G/k}^{(v,r)}(T)$ is just the set of finite group-schemes of type (v, r) over T which are closed T-subgroups of $G\times_k T$. A moments consideration will show that the points of $\mathbf{Inf}_{G/k}^{(v,r)}(T)$ lying over H, correspond to closed subgroups of type (v, r) in $G\times_k T$ whose restriction to the closed point of T is just H. Let one such be $\tilde{H}$. Let $H=\mathrm{Spec}(B)$, let $G_v=\mathrm{Spec}(A_v)$, let $\tilde{H}=\mathrm{Spec}(\tilde{B})$, let $k[t]$ be the dual numbers and let $A_v\otimes_k k[t]=\tilde{A}$. Let $J=\ker(A\to B)$. The fact that H and $\tilde{H}$ are of type (v, r) have the following consequences:

3.3 i) B and J satisfy assumption 2.1.

3.3 ii) $\ker(\tilde{B}\to B)=Bt$.

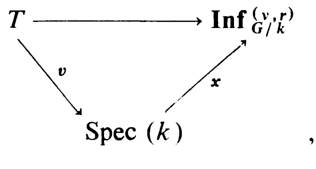
As a consequence of 3.3 ii) $T_x=E^0_{G/H}$. Theorem 3.2 yields an isomorphism $\xi^0_{G/H}\colon T_x\to Z_{G/H}$. Since 3.3 i) is true we may apply Theorem 2.9. We may take $\xi_{G/H}=j_1^{-1}\circ\xi^0_{G/H}$. The theorem is proven.

Before passing to the next theorem we observe that G operates on $\mathbf{Inf}_{G/k}^{(v,r)}$ on the right by inner automorphisms. We have described the tangent space to $\mathbf{Inf}_{G/k}^{(v,r)}$ at $x(H)$. Now we will describe the tangent space to the orbit of $x(H)$ under G.

3.4 Theorem. *Let* $H\subset G$ *be as above. That is,* H *is an infinitesimal subgroup of* G *of type* (v, r). *Let* $x(H)=x$ *be the point of* $\mathbf{Inf}_{G/k}^{(v,r)}$ *corresponding to* H *and let* $G(x)$ *denote the orbit of* x *under inner automorphism. Let* T^0_x *denote the tangent space to* $G(x)$ *at* x. *Then* $T^0_x\subset T_x$, *the tangent space to* $\mathbf{Inf}_{G/k}^{(v,r)}$, *and* $\xi_{G/H}(T^0_x)$ *is just* $B^1(H, \mathfrak{n}_{G/H})$, *the Hochschild one boundaries of* H *with coefficients in* $\mathfrak{n}_{G/H}$.

Proof. To describe the tangent space to $G(x)$ we begin with a functorial description of $G(x)$. In general, let X be a scheme with a G-action $X\times_k G\to X$. Then $G(T)$ acts on $X(T)$ on the right for any k-scheme T. In any case if the structure morphism of T is $v\colon T\to \mathrm{Spec}(k)$ and x is thought of as a map from $\mathrm{Spec}(k)$ to X, then $G(x)(T)$

is just the image of $x\circ v$ in $X(T)$ under the action. If $T=\mathrm{Spec}(k[t])$ where $k[t]$ is the dual numbers, then, $G(x)(T)$ is the tangent space to $G(x)$. If x and the action of G on x can be described explicitly, then, it is quite feasible to describe the tangent space to $G(x)$ at x. Let us turn to the case at hand. Namely $x=x(H)\in \mathbf{Inf}_{G/k}^{(v,r)}$. First we evaluate $G(T)$ where T is the spectrum of the dual numbers. A point of G in T is a pair (σ, Δ_σ) where $\sigma\in G(k)$ and Δ_σ is a derivation of A in k at σ, i.e. $\Delta_\sigma(ab)=\sigma(a)\Delta_\sigma(b)+\sigma(b)\Delta_\sigma(a)$. Then $(\sigma, \Delta_\sigma)(a)=\sigma(a)+\Delta_\sigma(a).t$. We now describe the action of $G(T)$ on $\mathbf{Inf}_{G/k}^{(v,r)}(T)$. Inner automorphism on the right is described by a left coaction ℓ'_A: $A\to A\otimes A$, where $\ell'_A(a)=\sum_{(a)} a_{(1)}s(a_{(3)})\otimes a_{(2)}$. Since G_v is a characteristic subgroup of G, ℓ'_A reduces to a co-action, $\ell'_A\colon A_v\to A\otimes A_v$. Let $\tilde{A}=A\otimes_k k[t]$. Now, ℓ'_A induces $\tilde{\ell'}_{A_v}\colon \tilde{A}_v\to \tilde{A}\otimes_{k[t]}\tilde{A}_v=A\otimes_k\tilde{A}_v$. Let $\tilde{\sigma}=(\sigma, \Delta_\sigma)$ be a T-point of G. Define $\varphi_{\tilde{\sigma}}$ by $\varphi_{\tilde{\sigma}}=m_{A_v}\circ(\tilde{\sigma}\otimes_k id_{\tilde{A}_v})\circ\tilde{\ell'}_{A_v}$. Then $\varphi_{\tilde{\sigma}}$ is an automorphism of $\tilde{A}_v$ and $\varphi_{\tilde{\sigma}}\circ\varphi_{\tilde{\tau}}=\varphi_{\tilde{\sigma}\tilde{\tau}}$. If $\tilde{B}$ is the coordinate ring of an infinitesimal subgroup of $\mathrm{Spec}(\tilde{A}_v)$ of type (v, r), and if $p:\tilde{A}_v\to\tilde{B}$ is the natural projection, then $p\circ\varphi_{\tilde{\sigma}}$ defines a Hopf algebra morphism from $\tilde{A}_v$ to $\tilde{B}$ whose kernel is $\varphi_{\tilde{\sigma}}^{-1}(\ker p)$. Then, this is a point of $\mathbf{Inf}_{G/k}^{(v,r)}(T)$ distinct from the point y corresponding to $\tilde{B}$; it is, in fact, $y\cdot\tilde{\sigma}$. Now the structure morphism from T to $\mathrm{Spec}(k)$ corresponds to the injection $k\hookrightarrow k[t]$ and as the group scheme over T corresponding to $f\colon T\to \mathbf{Inf}_{G/k}^{(v,r)}$ is gotten by pulling back the universal subgroup scheme, we can easily determine the T-point of $\mathbf{Inf}_{Gk}^{(v,/r)}$ corresponding to $x(H)=x$. Namely we have the diagram



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where $v: T\to\mathrm{Spec}(k)$ is the structure morphism. Then $x.v$ corresponds to $(x\circ v)^*(H_{G/k}^{(v,r)})=v^*(x^*(H_{G/k}^{(v,r)}))$. That is, the point $x\circ v$ corresponds to $k[t]\otimes_k B$. Write B_t for $k[t]\otimes_k B$. The points of $G(x(H))(T)$ are subgroup schemes of A_v arrived at as follows. For $\tilde{\sigma}\in G(T)$ consider the morphism $p\circ\varphi_{\tilde{\sigma}} : \tilde{A}_v \to B_t$.

This determines a closed $k[t]$ subgroup scheme of $\tilde{A}_v$ of type (v, r) which we denote ${}^{\tilde{\sigma}}B_t$, and we write p^σ for $p\circ\varphi_{\tilde{\sigma}}$. (Observe that $\ker p^\sigma \neq \ker p$. That is why we must distinguish ${}^{\tilde{\sigma}}B_t$ from B_t). To see which of these points lie above $H = \mathrm{Spec}(B)$ we observe that $\varphi_{\tilde{\sigma}}$ actually may be written as $(\varphi_\sigma, \Delta'_\sigma)$ where $\varphi_{\tilde{\sigma}}(a) = \varphi_\sigma(a) + \Delta'_\sigma(a)\cdot t$ for $a \in A_v \subset \tilde{A}_v$. Here Δ'_σ is defined as in section 2. Then Δ'_σ satisfies $\Delta'_\sigma(ab) = \varphi_\sigma(a)\cdot\Delta'_\sigma(b) + \varphi_\sigma(b)\cdot\Delta'_\sigma(a)$, and φ_σ is an automorphism of A_v defined by $\varphi_\sigma = \sigma\otimes id_{A_v}\circ\iota'_{A_v}$, while $\Delta'_\sigma = \Delta_\sigma\otimes id_{A_v}\circ\iota'_{A_v}$. Writing $\varphi_{\tilde{\sigma}}$ in this form we see that $\varphi_{\tilde{\sigma}}(a + a't) = \varphi_\sigma(a) + (\varphi_\sigma(a') + \Delta'_\sigma(a))t$. It is then clear that ${}^{\tilde{\sigma}}B_t$ lies above B only if φ_σ induces an automorphism of $\mathrm{Spec}(B)$ as a subgroup scheme of $\mathrm{Spec}(A)$. That is, $\sigma \in N_G(H)$ where $H = \mathrm{Spec}(B)$, and Δ_σ is in the tangent space to G at σ (not the tangent space to $N_G(H)$ at σ). The determination of ${}^{\tilde{\sigma}}B_t$ is synonymous with the determination of ${}^{\tilde{\sigma}}\tilde{J} = \ker(\tilde{A}_v \to {}^{\tilde{\sigma}}B_t)$. Now $x + yt \in {}^{\tilde{\sigma}}\tilde{J}$ if and only if $p\circ\varphi_{\tilde{\sigma}}(x + yt) = 0$, that is if $p(\varphi_\sigma(x) + (\varphi_\sigma(y) + \Delta'_\sigma(x))t) = 0$. Then $x + yt \in {}^{\tilde{\sigma}}\tilde{J}$ if and only if $\varphi_\sigma(x) \in J$ and $\varphi_\sigma(y) + \Delta'_\sigma(x) \in J$. But φ_σ preserves J and so this is true if and only if $x \in J$ and $y + \varphi_\sigma^{-1}\Delta'_\sigma(x) \in J$. Now observe that $\varphi_\sigma^{-1}\circ\Delta'_\sigma$ is a derivation of A_v in itself. (It is not a φ_σ-semi-derivation as was Δ'_σ). We shall describe it in detail later, but for the moment we shall merely re-lable it D_σ. Thus, the element of $\mathrm{Hom}_B(J/J^2, B)$, $\delta_{\tilde{\sigma}}$, associated to ${}^{\tilde{\sigma}}B_t$ may be desrcibed as follows. We know that $x + yt \in {}^{\tilde{\sigma}}\tilde{J}$ if and only if y is congruent to $-D_\sigma(x)$ modulo J. Thus $\delta_\sigma(x)$, which is the residue class of $-y$ modulo J is just the residue class of $D_\sigma(x)$. Hence $\delta_{\tilde{\sigma}}$ is just the map from J/J^2 to B induced by D_σ.

Let us now consider which derivations, D_σ, arise in this context. We recall the construction of D_σ. We began with a $\sigma \in G(k)$ and a Δ_σ such that $\Delta_\sigma(ab) = \sigma(a)\Delta_\sigma(b) + \sigma(b)\Delta_\sigma(a)$. We then induced a φ_σ-semi-derivation on A_v, by composing $\Delta_\sigma\circ id_A$ with the left inner coaction on A (because G is acting from the right), to obtain $\Delta'_{\tilde{\sigma}}$, and composed $\varphi_\sigma^{-1} = \varphi_{\sigma^{-1}}$ with Δ'_σ to obtain D_σ. All of the operations in question can be performed on A rather than A_v as $\mathrm{Spec}(A_v)$ is characteristic in G, and so we shall perform our calculations in this way. What we propose to show is that there is an element $\Delta_1 \in \ell_G$, the Lie algebra of G, such that $D_\sigma = \Delta_1^{\epsilon}$ and conversely that each Δ_1^{ϵ} corresponds to some D_σ where Δ_1^{ϵ} signifies what it did in the previous section.

Proving the second assertion amounts to nothing more than the observation that $\Delta_1^{\epsilon} = D_\sigma$ for $\tilde{\sigma} = (\varepsilon_A, \Delta_1)$. Hence we must only show that $D_\sigma = \Delta_1^{\epsilon}$ for some Δ_1. Now $D_\sigma = \varphi_{\sigma^{-1}} \Delta_\sigma^{\epsilon}$. We prove that $\varphi_{\sigma^{-1}} \Delta_\sigma^{\epsilon} = (\Delta_\sigma \circ \lambda_{\sigma^{-1}})^{\epsilon}$ where $\lambda_{\sigma^{-1}}$ is left translation by σ^{-1}; i.e. $\lambda_{\sigma^{-1}}(a) = \sum_{(a)} a_{(1)} \sigma \circ s_A(a_{(2)})$. By definition $\sigma^{-1} = \sigma \circ s_A$, and $\varphi_{\sigma^{-1}} \circ \Delta_\sigma^{\epsilon}(a)$

$$= \sum_{(a)} \Delta_\sigma(a_{(1)} s_A(a_{(5)})) \cdot \sigma \circ s_A(a_{(2)} s_A(a_{(4)})) \cdot a_{(3)}$$

$$= \sum_{(a)} \Delta_\sigma(a_{(1)} \sigma \circ s_A(a_{(2)}) s_A(a_{(5)}) \sigma \circ s_A(s_A(a_{(4)}))) a_{(3)}$$

$$= \sum_{(a)} \Delta_\sigma(\lambda_{\sigma^{-1}}(a_{(1)}) \cdot \lambda_{\sigma^{-1}}(s_A(a_{(3)}))) \cdot a_{(2)}$$

$$= \sum_{(a)} \Delta_\sigma \circ \lambda_{\sigma^{-1}}(a_{(1)} s_A(a_{(3)})) \cdot a_{(2)} .$$

Thus, the identity is proven. Observe that $\Delta_\sigma \circ \lambda_{\sigma^{-1}} \in \ell_G$. Summarizing, we have shown that if $\tilde{A}_v \to \tilde{B} \to 0$ is a point of the tangent space to the orbit of B under inner automorphism by G, then the corresponding element δ of $\mathrm{Hom}_B(J/J^2, B)$ is induced by a derivation of the form Δ^{ϵ} for $\Delta \in \ell_G$. By Theorem 2.9 this shows that the tangent space to the orbit of B is precisely $j_1^{-1}(B^1(H, \mathfrak{n}_{G/H}))$. Theorem 3.4 is proven.

3.5 Corollary. *Let G be a linear algebraic group over an albegraically closed field of positive characteristic, $p>0$ and let $H_v \subset G$ be a closed infinitesimal subgroup of type (v, r), Then if $H^1(H, \mathfrak{n}_{G/H}) = 0$, the closure of the orbit of H_v under inner automorphism is a component of $\mathbf{Inf}_{G/k}^{(v,r)}$*

Proof. To say that $H^1(H, \mathfrak{n}_{G/H}) = 0$ is the same as saying that $Z^1(H, \mathfrak{n}_{G/H}) = B^1(H, \mathfrak{n}_{G/H})$. Thus, if $x = x(H_v)$ is the point of $\mathbf{Inf}_{G/k}^{(v,r)}$ corresponding to H_v, the tangent space to x in $\mathbf{Inf}_{G/k}^{(v,r)}$ is equal to the tangent space to x in $G(x)$. The corollary is hence proven.

4. Reductive normal subgroups.

The purpose of this section is to prove the following:

4.1 Theorem. *Let G be a linear algebraic group over an algebraically closed field k, of positive characteristic. Let $H \subset G$ be a closed connected reductive normal subgroup of G and let K be a connected affine group of automorphisms of G. Then K leaves H stable.*

Proof. Let $H = \mathrm{Spec}(B)$, $G = \mathrm{Spec}(A)$, $K = \mathrm{Spec}(C)$ and let $J = \ker(A \to B)$. Moreover I_A, I_B, A, B, G, H will denote what they did in previous sections.

The idea of the proof is to show that the tangent space to the orbit of H_ν in $\mathbf{Inf}_{G/k}^{(\nu,r)}$ under the action of K consists of cycles from $Z^1(H, \mathfrak{n}_{G/H})$, and to demonstrate that $H^1(H, \mathfrak{n}_{G/H}) = 0$. This will imply the result.

Let $u : A \to C \otimes_k A$ be the coaction corresponding to the action of K on G. Let ∂ be an element of the Lie algebra of K. Then $\partial \otimes id_A \circ u$ is a derivation of A in itself, which we denote $\tilde{\partial}$. Then $\tilde{\partial}$ induces a B-map, $\tilde{\partial}^* : J/J^2 \to B$. The fact that K operates as automorphisms on G implies that $\mu_B \circ \tilde{\partial}^* = id_B \otimes \tilde{\partial}^* \circ \lambda_B + \tilde{\partial}^* \otimes id_B \circ \rho_B$. Consequently $\tilde{\partial}^*$ determines an element of $Z^1(H, \mathfrak{n}_{G/H})$. Now we may reason exactly as we did in the proof of 3.4, and we would find that the elements of $Z^1(H_\nu, \mathfrak{n}_{G/H})$ which are in the tangent space to the orbit of H_ν under the action of K are defined by just the same procedure as is $\tilde{\partial}^*$. We would, hence, find that the tangent space to the orbit of H under K lies in the image of $Z^1(H, \mathfrak{n}_{G/H})$ in $Z^1(H_\nu, \mathfrak{n}_{G/H})$ under the natural map arising from the inclusion $H_\nu \hookrightarrow H$.

However, H is reductive, and the fact that H is normal implies that H acts on $\mathfrak{n}_{G/H}$ trivially. Consequently, $B^1(H, \mathfrak{n}_{G/H}) = 0$, and $Z^1(H, \mathfrak{n}_{G/H}) = \mathfrak{n}_{G/H} \otimes_k \mathrm{Hom}(H, G_a) = 0$. Thus the tangent space to the orbit of H_ν under K in $\mathbf{Inf}_{G/k}^{(\nu,r)}$ is zero for each ν. This implies that the orbit is just a point for each ν. Now, the fact that H_ν is K-stable is expressed as follows. Let $I_{A,\nu}$ be the ideal generated by p^ν-th powers of elements in I_A, as usual. Then the stability of H_ν means that $u(J + I_{A,\nu}) \subset C \otimes_k (J + I_{A,\nu})$. Thus $u(J) = u(\bigcap_{\nu \geq 0} J + I_{A,\nu}) \subset \bigcap_{\nu \geq 0} (C \otimes J + C \otimes I_{A,\nu})$. As tensoring over a field is quite flat, the intersection on the right is just $C\otimes J$. Thus $u(J)\subset C\otimes J$ and hence H is K-stable. The theorem is proven.

5. Transformation spaces.

Let X be a non-singular variety of finite type over an algebraically closed field, k. Let G be a linear algebraic group over k and assume that $\alpha: G\times_k X\rightarrow X$ is an action of G on X.

5.1 Definition. Let $\alpha: G\times_k X\rightarrow X$ be an action of G on X. Then α will be called a fully separable action if the following conditions hold:

i) For any point $x\in X$, the stabilizer of x is reduced.

ii) If G_x is the stabilizer of a point, x, and G_x^0 is its connected component, then the fixed point set of G_x^0 is reduced.

Let X be a non-singular quasi-compact algebraic variety of finite type over a field k and assume that $\alpha: G\times_k X\rightarrow X$ is a fully separable action of G on X. Consider the stabilizer of the identity map of X into itself. Namely take G_Δ equal to the inverse image of the diagonal in $X\times_k X$ under the map Φ: $(g, x)\wedge\!\!\rightarrow(\alpha(g, x), x)$. Then G_Δ is a group-scheme over X and the fibre over a point, x, is just the stabilizer of x. Assume that the orbits of G in X are equideminsional, of dimension s. Let $n=\dim G$ and let $r=n-s$. Then G_Δ has fibres of dimension r over G and moreover by full separability, G_Δ has non-singular fibres. Let $\mathscr{A}_\Delta$ be the sheaf of rings of functions on G_Δ over $\mathcal{O}_X$, and let $\mathscr{I}_\Delta$ be the sheaf of ideals defining the identity section in G_Δ. Let $\mathscr{I}_\Delta^{\{p^\nu\}}$ denote the ideal generated by p^ν-th powers of sections in $\mathscr{I}_\Delta$. Let $G_{\Delta,\nu}=\operatorname{Spec}\mathscr{A}/\mathscr{I}_\Delta^{\{p^\nu\}}$. Then $G_{\Delta,\nu}$ is clearly a group-scheme of type (ν, r) over X and hence it determines a morphism, $\eta_{X,G}^{(\nu,r)}: X\rightarrow \mathbf{Inf}_{G/k}^{(\nu,r)}$.

5.2 Definition. Let, X, G be as above. Then the map $\eta_{X,G}^{(\nu,r)}$ is called the equivariant cotangent map of order ν associated to α.

For the remainder of this section we assume that the field k is algebraically closed. For simplicity, we shall write $S(\nu)$ for $\mathbf{Inf}_{G/k}^{(\nu,r)}$ and η^ν for $\eta_{X,G}^{(\nu,r)}$. We return to our discussion of X. Let $Z_\nu=X\times_{S(\nu)}X$.

Then Z_ν is closed in $X \times_k X$ and since the maps $\eta^\nu: X \to S(\nu)$ are a directed system, the Z_ν are a decreasing sequence of closed subsets of $X \times_k X$. As X is quasi-compact and of finite type over k, the sequence $\{Z_\nu\}_{\nu>0}$ eventually stabilizes. That is there is an integer, ν, such that $\nu \geq N \Rightarrow Z_\nu = Z_{\nu+1}$.

We shall analyze the significance of this integer. First we observe that since k is algebraically closed, we may think of points as k-points. Hence a point in $S(\nu)$ corresponds to a subgroup of G, say K_ν of type (ν, r). A point $x \in X$ is in the fibre over K_ν if and only if the stabilizer of x, truncated at exponent ν is equal to K_ν. Hence the fibre over K_ν is X^{K_ν}, the fixed point set of K_ν. We observe that at the fibre over x in Z_ν under either projection may now be described explicitly. Namely let G_x be the stabilizer of x, and let K_ν be its truncation at exponent ν for each ν. Then the fibre over x in Z_ν is just X^{K_ν} base extended by the residue class field of x. This latter, being k, may be forgotten. Now the stability of Z_ν for $\nu \geq N$ implies that $X^{K_\nu} = X^{K_{\nu+1}} = X^{K_{\nu+2}} \ldots$ etc. Let $K = G_x^0$, the connected component of G_x. It is clear that if $X^{K_\nu} = X^{K_{\nu+1}} \ldots$ etc., then $X^{K_\nu} = X^K$. That is for $\nu \geq N$, $X^{K_\nu} = X^K$. We have proven the following:

5.3 Lemma. *Let k be algebraically closed. Let X be a non-singular, quasi-compact algebraic variety over k, and let $\alpha: G \times_k X \to X$ be a G-action with equidiemensional non-singular stabilizers, all of dimension r. Let $\eta^\nu: X \to \mathbf{Inf}_{G/k}^{(\nu,r)}$ be the equivariant cotangent map of exponent ν associated to α. For any x, let $K_{(x)}$ be the connected component of the stabilizer of x. Then there is an integer $N>0$ such that for $\nu > N$, $(\eta^\nu)^{-1}(\eta^\nu(x)) = X^K(x)$.*

We shall now consider the maps η^ν more in detail. Assume that $\alpha: G \times_k X \to X$ is fully separable. In particular for any $x \in X$, η^ν induces a map $d\eta^\nu$, from $T_X(x)$, the Zariski tangent space of X at x, to the Zariski tangent space of $\mathbf{Inf}_{G/k}^{(\nu,r)}$ at $\eta_\nu(x)$. We should like to analyze the map in detail. This can be accomplished by examining the map functorially.

Let H be the connected component of the stabilizer of x in G. Then H is a subgroup of G of dimension r. Thus $\eta^\nu(x)$ is the point of $\mathbf{Inf}_{G/k}^{(v,r)}$ corresponding to H_v, the unique infinitesimal closed subgroup scheme of H of type (v, r). Let $k[t]$ be the ring of dual numbers. Then a point of $T_X(x)$ is just a point, $\delta_X \in X(k[t])$, lying over x. In fact, $\eta^v(\delta_X)$ is just precisely the stabilizer of δ_X. The stabilizer of δ_X is a subgroup-scheme of $G \times_k \mathrm{Spec}(k[t])$. Since the action of G on X is fully separable, the stabilizer of δ_X is smooth and of dimension r over $k[t]$. We shall write $\tilde{H}$ for the connected component of the stabilizer of δ_X. Now $\tilde{H}$ contains a unique group-scheme of type (v, r) over $k[t]$, $\tilde{H}_v$. This in turn determines a cycle in $Z^1(H_v, \mathfrak{n}_{G/H})$. Then, $d\eta^v(\delta_X)$ is just the cycle determined by $\tilde{H}_v$. However we might proceed to determine $d\eta^v(\delta_X)$ in a slightly different fashion. Namely, let A_G be the ring of functions on G, A_H the ring of functions on H, and J the ideal defining A_H. Then $\tilde{H} = \mathrm{Spec}(\tilde{A}_H)$ where $\tilde{A}_H = A_G \otimes k[t]/\tilde{J}$ and $\tilde{J}/t\tilde{J} \cong J$. Then just as in section 3, $\tilde{H}$ determines a map $\beta: J/J^2 \to A_H$, which determines a cycle $\tilde{\delta} \in Z^1(H, \mathfrak{n}_{G/H})$. Rather than strive for completeness we may just cite the well known theory of deformations of smooth groups, or alternatively remark that the arguments of section 3 remain valid. It is clear from our constructions that $\tilde{H}_v$ is determined by the image of $\tilde{\delta}$ in $Z^1(H, \mathfrak{n}_{G/H})$ under the natural map determined by the inclusion $H_v \hookrightarrow H$. Thus we have proven the following.

5.4 Lemma. *Let k be algebraically closed and let G be linear algebraic over k. Let X be non-singular, connected quasi-compact and of finitely generated type over k. Let $\alpha: G \times_k X \to X$ be a fully separable action with stabilizers all of dimension r. Let $\eta^v: X \to$ $\mathbf{Inf}_{G/k}^{(v,r)}$ be the equivariant cotangent map of exponent v associated to α, and let $d\eta_x^v$ denote the associated map of Zariski tangent spaces. Let $x \in X$ and let H be the connected component of the stabilizer of x in G. Let H_v denote the subgroup of H of type (v, r) and let $\mathfrak{n}_{G/H}$ be the quotient of the Lie algebra of G by that of H. Then, $d\eta^v(T_X(x))$ is contained in the image of $Z^1(H, \mathfrak{n}_{G/H})$ in $Z^1(H_v, \mathfrak{n}_{G/H})$.*

We shall now apply these lemmas to prove our main theorem.

5.5 Theorem. *Let k be an algebraically closed field of charcteristic*

$p>0$. Let G be a linear algebraic group over k and let X be a non-singular connected quasi-compact variety of finite type over k. Let $\alpha: G\times_k X\to X$ be a fully separable regular action. Assume that the stabilizers of points in X are of fixed dimension. Suppose that for some closed point $x\in X$, $H^1(H, \mathfrak{n}_{G/H})=0$, where H is the connected component of the stabilizer of x, and $\mathfrak{n}_{G/H}$ is the quotient of the Lie algebra of G by that of H. Then there is an open set in X, U, such that if u is a closed point of U, $(G_u)^0$, the connected component of the stabilizer of u, is conjugate to H.

Proof. The technique of our proof will be to analyze the equivariant cotangent maps and to show that for sufficiently large exponent, the image of X is contained in the closure of the orbit of the corresponding truncation of H.

We begin by choosing N large enough to meet the conditions of Lemma 5.3. Let the stabilizers of points be of dimension r. For $\nu>N$, we consider the map $\eta^\nu=\eta_{X,G}^{(\nu,r)}: X\to \mathbf{Inf}_{G/k}^{(\nu,r)}$, and the corresponding map of tangent spaces. As the action of G on X is fully separable, the fibres of η^ν are reduced for $\nu>N$. Moreover for $\nu>N$, the fibre over $\eta^\nu(u)$ is the fixed point set of $(G_u)^0$. Consider a point x satisfying the hypotheses of the theorem. Let $H=(G_x)^0$ and H_ν denote the truncation of H at exponent ν were $\nu>N$. Then $X^H=(\eta^\nu)^{-1}(\eta^\nu(x))$ is a reduced subvariety of X and so contains a dense open non-singular subset. Let $V\subset X^H$ denote this subset and let v be a point of V. We consider $d\eta_v^\nu$, the map of tangent spaces at v. Then $d\eta_v^\nu(T_X(v))$ is in the image of $Z^1(H, \mathfrak{n}_{G/H})$ in $Z^1(H_\nu, \mathfrak{n}_{G/H})$. As $H^1(H, \mathfrak{n}_{G/H})=0$, $Z^1(H, \mathfrak{n}_{G/H})=B^1(H, \mathfrak{n}_{G/H})$ and so $d\eta_v^\nu(T_X(v))\subset B^1(H_\nu, \mathfrak{n}_{G/H})$. This latter is the tangent space to the orbit of H under inner conjugation. Let $Z=\mathbf{Inf}_{G/k}^{(\nu,r)}$, and let W be the orbit of H_ν under inner action by G in $\mathbf{Inf}_{G/k}^{(\nu,r)}$. Now let us review the facts which we have at our disposal:

i) $\eta^\nu(X^H)\subset \overline{W}$,

ii) $d\eta_v^\nu(T_X(v))\subset T_W(\eta^\nu(x))$,

iii) The fibres of η^{v} are reduced,

iv) v is non-singular in the fibre over $\eta^{v}(v)$.

We leave it to the reader to write down the ring theoretic statements corresponding to these facts. He will be forced to conclude that $\eta^{v}(X)\subset\overline{W}$, the closure of W. As $\eta^{v}(v)\in W$, then $\eta^{v^{-1}}(W)$ contains a dense open subset of X. The theorem is proven.

We conclude this paper with the remark that all of the conditions of theorem 5.5 are somewhat mysterious. First of all, regular actions which fail to meet one or both of the conditions required for full separability are easily constructed. Secondly, the condition $H^{1}(H, \mathfrak{n}_{G/H})=0$ is a rather enigmatic one. The astute reader will notice that this cohomology group is closely related to $H^{1}(G/H, \Theta_{G/H})$ where G/H is the homogeneous space and $\Theta_{G/H}$ is the sheaf of holomorphic tangent vector fields on G/H. The cohomology is the ordinary Zariski cohomology. Hence it would appear that $H^{1}(H, \mathfrak{n}_{G/H})$ is closely related to the infinitesimal deformations of the homogeneous space, G/H, in the sense of Kodaira-Spencer. Any attempt to analyze the condition in a way that would relate it either to group-theoretic considerations or to geometric aspects of actions would require some analysis of this relationship. The author intends to initiate such a study in a forthcoming paper and he also suggests that it be examined as a suitiable topic for further research.

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