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Dependence of local homeomorphisms and local C^r-structures

By

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Introduction

Let $\Sigma(X)$ be the set of all C^r-structures on a topological manifold X. The study of the diffeomorphism classes of $\Sigma(X)$ has been an important subject in differential topology. We, however, consider $\Sigma(X)$ itself paying attention to its dependence relation ($\subset$) defined below. We give some results which are chiefly reduced to a local theory of homeomorphisms of $\mathbf{R}^n$. We begin by the following problems.

Problem I G. *For given C^r-structures $\mathscr{D}$, $\mathscr{D}'\in\Sigma(X)$, can we find a third $\mathscr{D}''\in\Sigma(X)$ such that $\mathscr{D}\subset\mathscr{D}''$, $\mathscr{D}'\subset\mathscr{D}''$?*

Problem II G. *For given C^r-structures $\mathscr{D}$, $\mathscr{D}'\in\Sigma(X)$, can we find a third $\mathscr{D}''\in\Sigma(X)$ such that $\mathscr{D}''\subset\mathscr{D}$, $\mathscr{D}''\subset\mathscr{D}'$?*

These problems are quite raw and more suitable presentations will be found according to the stages of our study. First, we localize the problems.

By a *local C^r-structures* on $\mathbf{R}^n$ we mean the germ at 0 of a C^r-structure of a neighbourhood of $0\in\mathbf{R}^n$ (we shall give a more detailed definition in Section 1). By a *local homeomorphism*(1) of $\mathbf{R}^n$ we mean the germ at 0 of that homeomorphism between neighbourhoods of 0

(1) We use this term following Sternberg, who investigated local homeomorphisms in connection with the theory of flow and found normal forms of conjugate classes of local diffeomorphisms.

which leaves 0 fixed. All the local homeomorphisms of $\mathbf{R}^n$ form a group ${}_n\mathbf{C}$ with respect to the operation ($\circ$) induced by the composition of maps. If two local C^r-structures $\mathscr{D}$ and $\mathscr{D}'$ are given, take admissible charts $f(t)$ and $g(t)$ of their representative C^r-structures in neighbourhoods of 0. The germ $f\circ g^{-1}$ of $f\circ g^{-1}(t)$ is a local homeomorphism of $\mathbf{R}^n$. We say that $\mathscr{D}$ is dependent on $\mathscr{D}'$ and write $\mathscr{D}\subset\mathscr{D}'$ when $f\circ g^{-1}$ is of class C^r. This defines an order in the space of local C^r-structures. Then we obtain the local forms, **(I L)** and **(II L)**, of the problems (I G) and (II G) in an obvious way. Global dependence is defined by the pointwise dependence of admissible charts (see Section 4).

(I L) and (II L) are easily reduced to problems about the subsemigroup[2] ${}_n\mathbf{E}_r\subset{}_n\mathbf{C}$ that consists of the germs of class C^r. Church's smoothing lemma (2.1) gives a sufficient condition for (II L) to be answered in the affirmative. Next, we show by the examples (2.4), (2.5) and (2.6) neither (I L) nor (II L) is unconditionally answered in the affirmative. These examples reveal somewhat complicated aspects of ${}_n\mathbf{C}$ relative to ${}_n\mathbf{E}$. If we restrict ourselves to the 1-dimensional case we obtain a sharp positive result (as a consequence of Theorem 3.1):

Theorem 0.1. *If $n=1$ and if (I L) is answered in the affirmative for a pair $(\mathscr{D}, \mathscr{D}')$ then (II L) is also.*

In substance this assertion gives the simple expression ${}_1\mathbf{E}_r^{-1}\circ{}_1\mathbf{E}_r$ of the subgroup of ${}_1\mathbf{C}$ generated by ${}_1\mathbf{E}_r$ (Corollary 3.7). A difficulty in the proof lies in the removal of the singularities of a function which is smooth almost everywhere. We remark that the converse of the above theorem is false for every $\mathbf{R}^n$ (Example 2.5).

All of these local results yield corresponding answers to (I G) and (II G). The positive results for (II G) are best explained by a reduction theory of structures (4.1 and 4.4). We can also treat subsemigroups of the homeomorphism group of a C^r manifold (see (4.6), (4.7) and

(2) A subsemigroup of a group is a subset closed with respect to multiplication.

(4.8)). In the last section we remark on the diffeomorphism class of a C^∞-structure in some dependence relation with another. By the way, transforming Schoenflies theorem, we show that Milnor's group ${}_nA$ of exotic spheres is a quotient group of a subgroup of ${}_n\mathbf{C}$ (Theorem 5.3.)

The essential part, Section 2 and 3, of this paper depends upon real analysis. Especially we use a basic knowledge of measure theory and the simplest case of Sard's theorem(3). We refer to results in ordinary differential topology only in the additional section 5.

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1. Preliminary consideration

First we clarify the relationship between local homeomorphisms and local C^r-structures. We shall often express a property of a germ by one of its representative.

Let us fix an affine coordinate system $t=(t^1, t^2,\dots, t^n)$ of the Euclidian n-space $\mathbf{R}^n$. t assins naturally a C^r-structure on $\mathbf{R}^n$. Then it determines the following subsemigroups of the group ${}_n\mathbf{C}$ of local homeomorphisms at $0 \in \mathbf{R}^n$:

$${}_n\mathbf{D}_r = \{\text{the elements of } {}_n\mathbf{C} \text{ of class } C^r \text{ except at } 0\},$$

$${}_n\mathbf{E}_r = \{\text{the elements of } {}_n\mathbf{D}_r \text{ of class } C^r\},$$

$${}_n\mathbf{J}_r = \{\text{the elements of } {}_n\mathbf{E}_r\ r\text{-flat at } 0\}.$$

Here a (local) homeomorphism of class C^r does not mean the C^r-differentiability of its inverse. r-flatness means vanishing of all the partial derivatives of order up to r.

Let ${}_n\mathbf{D}_r^*$ (resp. ${}_n\mathbf{E}_r^*$, ${}_n\mathbf{J}_r^*$) denotes the subsemigroup of ${}_n\mathbf{D}_r$ (resp. ${}_n\mathbf{E}_r$, ${}_n\mathbf{J}_r$) consisting of all the elements whose Jacobian do not vanish except at 0. Let ${}_n\mathbf{E}_r^{**}$ be the subsemigroup of ${}_n\mathbf{E}_r^*$ consisting of all the elements whose Jacobian do not vanish at 0. When we do not need to restrict the dimension of the Euclidean space, the left

(3) According to Sard [8], this case is due to M. Morse.

subscripts n of ${}_n\mathbf{C}$, ${}_n\mathbf{D}_r$ etc. are omitted. r is always assumed to be a natural number or ∞. $\mathbf{D}_r^*$ and $\mathbf{E}_r^{**}$ are subgroups of $\mathbf{C}$ and satisfy

$$\mathbf{D}_r^{*-1}=\mathbf{D}_r^*, \qquad \mathbf{E}_r^{**-1}=\mathbf{E}_r^{**}.$$

It is also easy to see that

$$\mathbf{E}_r\circ\mathbf{E}_r^{-1}=\mathbf{J}_r\circ\mathbf{J}_r^{-1},\ \ \mathbf{E}_r^*\circ\mathbf{E}_r^{*-1}=\mathbf{J}_r^*\circ\mathbf{J}_r^{*-1},$$

$$\mathbf{E}_r^{-1}\circ\mathbf{E}_r=\mathbf{J}_r^{-1}\mathbf{J}\circ_r,\ \ \mathbf{E}_r^{*-1}\circ\mathbf{E}_r^*=\mathbf{J}_r^{*-1}\circ\mathbf{J}_r^*.$$

Now take an element $f\in\mathbf{C}$ and its representative $f(t)$ defined on a neighbourhood U of 0. Regarding $(U,(t\circ f)(t))$ as a chart we can define another C^r-structure on U. The set $\mathscr{E}_r(f)={}_n\mathscr{E}_r(f)$ of the germs of all the C^r functions at 0 in the new sense is determined by the germ f and is independent of the choice of the representative $(U, f(t))$. It is clear that $\mathscr{E}_r(f)\subset\mathscr{E}_r(g)$ (resp. $\mathscr{E}_r(f)=\mathscr{E}_r(g)$) if and only if $f\in\mathbf{E}_r\circ g$ (resp. $f\in\mathbf{E}_r^{**}\circ g$). Thus a local C^r-structure defined in the introduction can be naturally identified with a subring $\mathscr{E}_r(f)$ of the ring of germs of continuous functions at $0\in R^n$. Dependence of local C^r-structures just corresponds to inclusion of subrings. The left coset space $\mathbf{E}_r^{**}\backslash\mathbf{C}$ can be also identified with the space of local C^r-structures. We remark that $\mathbf{E}_r^{**}$ is not a normal subgroup of $\mathbf{C}$, which we see in the following:

Example 1.1. Put

$$\alpha(x)=2+\sin\left[(\pi/2)-\pi\log|x|\right],$$

$$\beta(x)=\int_0^x\alpha(\xi)\,d\xi\,.$$

The germs f, g at 0 of the maps

$$f(t)=(\beta(t^1),\ t^2,\ t^3,\dots,\ t^n)\,,$$

$$g(t)=(et^1,\ t^2,\ t^3,\dots,\ t^n)$$

belong to $\mathbf{D}_r^*$, $\mathbf{E}_r^{**}$ respectively. Suppose that $\mathbf{E}_r^{**}$ is a normal subgroup of $\mathbf{D}_r^*$. $h=f^{-1}\circ g\circ f$ belongs to $\mathbf{E}_r^{**}$. The first component $h^1(t)$ of $h(t)$ depends only upon t^1 and then we have

$$h^1_{t^1}(0, 0,\dots, 0)>0$$

Putting

$$k(x)=h^1(x, 0, 0,\dots, 0)$$

we have

$$\beta\circ k(x)=e\,\beta(x) \qquad (0\leqq x<\delta)\,,$$

$$k(x)\in C^r[0, \delta), \quad k(x)>0 \qquad (0<x<\delta)$$

for a small positive δ.
(If A is a subset of $\mathbf{R}^n$, by $C^r(A)$ we mean the set of functions obtained by restriction to A of C^r-functions on an open set including A.) By differentiation we have

$$\alpha\circ k(x)\times k'(x)=e\,\alpha(x) \qquad (0<x<\delta)\,.$$

Since

$$\underline{\lim}_{x\downarrow 0}\,\alpha(x)=1,\ \overline{\lim}_{x\downarrow 0}\,\alpha(x)=3,\ \lim_{x\downarrow 0} k(x)=+0$$

we deduce

$$k'(0)=\lim_{x\downarrow 0} k'(x)=e\,.$$

Then

$$k(x)=ex+\mathrm{o}(x)\,.$$

Putting $x=e^{-2m}$ we have

$$\alpha\{e^{-2m+1}[1+\mathrm{o}(1)]\}\times k'(e^{-2m})=e\,\alpha(e^{-2m})\,.$$

On the other hand

$$\lim_{m\to\infty}\alpha\{e^{-2m+1}[1+\mathrm{o}(1)]\}\times k'(e^{-2m})=e\,,$$

$$\lim_{m\to\infty} e\,\alpha(e^{-2m})=3e\,.$$

Thus our first assumption lead to a contradiction and $\mathbf{E}_r^{**}$ is not a normal subgroup of $\mathbf{D}_r^*$.

Remark. It is easier to find f in $\mathbf{E}_r$ than in $\mathbf{D}_r^*$.

(I L) and (II L) are translated into the following problems about local homeomorphisms.

Problem I L′. *Let f and g be elements of $\boldsymbol{C}$. Do hold the following mutually equivalent statements* (i)~(iv)?

(i) $f\circ g^{-1}\in \boldsymbol{E}_r\circ \boldsymbol{E}_r^{-1}$.

(ii) $\boldsymbol{E}_r^{-1}\circ f\cap \boldsymbol{E}_r^{-1}\circ g\neq\phi$.

(iii) $f,\ g\in \boldsymbol{E}_r\circ\varphi$ *for some* $\varphi\in\boldsymbol{C}$.

(iv) $\mathscr{E}_r(f),\ \mathscr{E}_r(g)\subset\mathscr{E}_r(\varphi)$ *for some* $\varphi\in\boldsymbol{C}$.

Problem II L′. *Let f and g be elements of $\boldsymbol{C}$. Do hold the following mutually equivalent statements* (i)~(iv)?

(i) $f\circ g^{-1}\in \boldsymbol{E}_r^{-1}\circ \boldsymbol{E}_r$.

(ii) $\boldsymbol{E}_r\circ f\cap \boldsymbol{E}_r\circ g\neq\phi$.

(iii) $f,\ g\in \boldsymbol{E}_r^{-1}\circ\psi$ *for some* $\psi\in\boldsymbol{C}$.

(iv) $\mathscr{E}_r(f),\ \mathscr{E}_r(g)\supset\mathscr{E}_r(\psi)$ *for some* $\psi\in\boldsymbol{C}$.

We show a simple example to illustrate the situation meant by these problems.

Example 1.2. We put

$$f(t)=t\,,\qquad g(t)=\begin{cases} t & (t\geqq 0)\\ 2t & (t<0)\,,\end{cases}$$

$$\alpha(t)=\begin{cases}\exp(-1/t) & (t>0)\\ 0 & (t=0)\\ -2\exp(1/t) & (t<0),\end{cases}\qquad \beta(t)=\begin{cases}\exp(-1/t) & (t>0)\\ 0 & (t=0)\\ -\exp(2/t) & (t<0),\end{cases}$$

$$\varphi(t)=\begin{cases}-1/\log t & (t>0)\\ 0 & (t=0)\\ 1/\log(-t) & (t<0),\end{cases}\qquad \psi(t)=\begin{cases}\exp(-1/t) & (t>0)\\ 0 & (t=0)\\ -\exp(1/t). & (t<0).\end{cases}$$

Then we have

$$f=\psi\circ\varphi\in{}_1E_\infty\circ\varphi\,,\qquad f=\psi^{-1}\circ\psi\in{}_1E_\infty^{-1}\circ\psi\,,$$

$$g=\alpha\circ\varphi\in{}_1E_\infty\circ\varphi\,,\qquad g=\beta^{-1}\circ\psi\in{}_1E_\infty^{-1}\circ\psi$$

and hence

$${}_1\mathscr{E}_\infty(\psi)\subset\{{}_1\mathscr{E}_\infty(f)\cap{}_1\mathscr{E}_\infty(g)\}\subset\{{}_1\mathscr{E}_\infty(f)\cup{}_1\mathscr{E}_\infty(g)\}\subset{}_1\mathscr{E}_\infty(\varphi).$$

Thus the problems (I L) and (II L) are answered in the affirmative for the germs f and g.

Now we make some notational agreements. We put $\|t\|=\sqrt{\sum_{i=1}^n t^it^i}$ for $t=(t^1, t^2,\dots, t^n)\in\mathbf{R}^n$. Let $f(t)$ be a C^r-function on a domain $\Omega\subset\boldsymbol{R}^n$, A a subset of Ω, $\nu=(\nu_1, \nu_2,\dots, \nu_n)$ an element of $\{0, 1, 2,\dots, r\}^n$ $|\nu|=\nu_1+\nu_2+\cdots+\nu_n$. We put

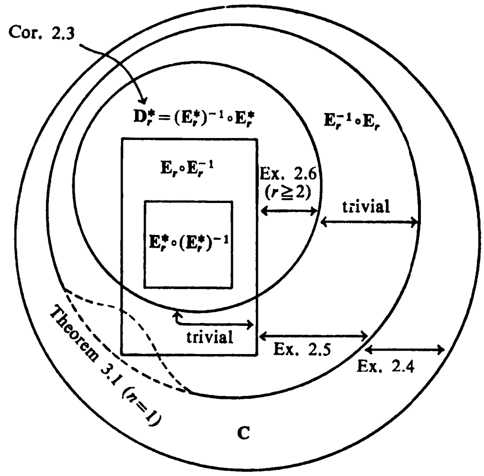
$$\|f(t)\|_A^i=\sup_{0\leqq|\nu|\leqq i}\ \sup_{t\in A}\left|\frac{\partial^{|\nu|}f(t)}{(\partial t^1)^{\nu_1}(\partial t^2)^{\nu_2}\cdots(\partial t^n)^{\nu_n}}\right|\qquad(0\leqq i\leqq r)\,.$$

Next, let I be an interval of $\mathbf{R}$. We define $M^r(\mathrm{I})$ to be the set of all properly monotone increasing function of $C^r(\mathrm{I})$. When we mention of the measure $\mu[A]$ of a subset $A\subset\mathbf{R}^n$, it is the ordinary Lebesgue measure.

2. Dependence of local homeomorphisms

In this section we show a sufficient condition for the assertions of (II L′) in Section 1 and a few examples of f and g for which (I L′)

or (II L′) are answered in the negative. We illustrate these situation in advance.



Here discs mean subgroups of C.

Lemma 2.1 (*Church's smoothing lemma*). *Let M and N be connected paracompact C^r manifold without boundary, $A = \{x_i\}$ be a discrete subset of N and $\{(U_i, (t_i^1, t_i^2, \dots, t_i^n))\}$ coordinate neighbourhoods centered at x_i. If $f(t)\colon M \to N$ is continuous and proper on M and C^r on $M - f^{-1}(A)$, then there exists a C^r homeomorphism $g(t)$ of N satisfying the following*:

(i) *$g \circ f(t)$ is C^r on M.*

(ii) *$g(t)$ can be expressed in the form*

$$(t_i^1 \cdot \sigma_i(\|t_i\|),\ t_i^2 \cdot \sigma_i(\|t_i\|), \dots,\ t_i^n \cdot \sigma_i(\|t_i\|))$$

such that $\sigma_i(x)$ is flat at 0 and $\sigma_i(x) = 1$ for $x \geqq k_i\,(>0)$ where $\{t\colon \|t_i\| \leqq k_i\} \subset U_i$.

(iii) *$g(t)$ is a C^r diffeomorphism on $N - A$ and the identity map on $N - \cup U_i$. If N is a $C^s\,(s \geqq r)$ manifold, g can be chosen to be C^s.*

Remrak 2.2. The condition (ii) has a special meaning. The homeomorphism $g(t)$ induces a new C^r-structure (pull back by $g(t)$) on N.

It is easy to see that (ii) means this structure is diffeomorphic to original one. See (5.3) and (5.5).

Corollary 2.3. *If $r \leqq s$ it holds that $D_r^* = (E_s^*)^{-1} \circ E_r^* = (E_r^*)^{-1} \circ E_s^*$. Since $(E_r^*)^{-1} \circ E_r^* \subset E_r^{-1} \circ E_r$, (II L′) is answered in the affirmative f, $g \in C$ satisfying $f \circ g^{-1} \in D_r^*$.*

Example 2.4. Here we give $f, g \in \mathbf{C}$ such that (II L′) is answered in the negative i.e. $f \circ g^{-1} \notin \mathbf{E}_r^{-1} \circ \mathbf{E}_r$.
Take a function $\alpha(x)$ defined on $[-1, 1]$ with the following properties.

(a) $\alpha(x) \in M^0[-1, 1]$.

(b) $\alpha(x)$ is not differentiable on a dense subset A of $[-1, 1]$.

(c) $\alpha(0) = 0$.

We can construct such a function summing monotone increasing functions whose graphs are open polygons. The germ f of $f(t) = (\alpha(t^1), t^2, t^3, \dots, t^n)$ at 0 belongs to $\mathbf{C}$. If $\mathbf{E}_r^{-1} \circ \mathbf{E}_r = \mathbf{C}$ we may assume that $f = h^{-1} \circ k$ for some $h, k \in \mathbf{E}_r$. For sufficiently small positive δ we have $h \circ f(t) = k(t)$ $(t \in I^n = [-\delta, \delta]^n)$ and $h^i(t), k(t) \in C^r(I^n)$. We put

$$l^i(x) = H^i(x, t_0^1, t_0^2, \dots, t_0^n).$$

If $(l^i)' \circ \alpha(a) \neq 0$ $(a \in A \cap I)$, there exists the differentiable inverse function $(l^i)^{-1}(x)$ defined in a neighbourhood of $l^i \circ \alpha(a)$ such that

$$\alpha(x) = (l^i)^{-1} \circ k^i(x, t_0^1, t_0^2, \dots, t_0^n).$$

This contradicts to the assumption that $\alpha(x)$ is not differentiable at a. Thus we have proved $(l^i)' \circ \alpha(x) = 0$ on $A \cap I$. Since A is dense and $(l^i)' \circ \alpha(x)$ is continuous,

$$(l^i)' \circ \alpha(x) = (\partial h^i / \partial t^1) f(t) = 0$$

and $h \circ f$ is constant on $I \times t_0^2 \times t_0^3 \times \cdots \times t_0^n$, a contradiction. Thus $f \notin \mathbf{E}_r^{-1} \circ \mathbf{E}_r$.

Example 2.5. If $r \geqq 1$ we have $\mathbf{E}_r^{-1} \circ \mathbf{E}_r \not\subset \mathbf{E}_1 \circ \mathbf{E}_1^{-1}$ and hence there is a pair $(f, g) \subset \mathbf{C}$ such that (II L′) is answered in the affirmative and (I L′) in the negative.

If $r \geqq 2$, this example is weaker than the next simple example (2.6). Thus the reader can neglect this.

Let A be a subset of $I=(-1, 1)$ such that:

(a) A is a closed subset of I.

(b) A has no interior point.

(c) A has positive measure.

(d) 0 is a density point of A, i.e. $\displaystyle\lim_{a \uparrow 0, b \downarrow 0} \frac{\mu[A \cap (a, b)]}{b-a} = 1$.

(e) A is symmetric with respect to 0.

We can realize such a set using a generalized Cantor set. Let I_i $(i=1, 2, 3, \dots)$ be the connected components of $I-A$ and p_i, $2r_i$ be the center and the width of I_i. Take a C^∞-function $\sigma(x)$ such that $\sigma(x)=-1$ $(x \leqq -1)$, $\sigma(x)=1$ $(x \geqq 1)$ and $\sigma(x)|_{[-1,1]} \in M^\infty[-1, 1]$. $\rho(x)$ is defined by $\rho(x)=\sigma \circ \sigma(x)$. If we put

$$c_i = \max\{\|\rho(x)\|_R^i, \|\sigma(x)\|_R^i\}$$

and

$$\alpha(x) = \sum_{i=1}^{\infty} 2^{-i} \frac{r_i^r}{c_i} \left[\rho\left(\frac{x-p_i}{r_i}\right) + \operatorname{sign} p_i\right],$$

$$\beta(x) = \sum_{i=1}^{\infty} 2^{-i} \frac{r_i^i}{c_i} \left[\sigma\left(\frac{x-p_i}{r_i}\right) + \operatorname{sign} p_i\right],$$

then $\alpha(x)$, $\beta(x) \in M^\infty(I)$. It is a consequence of the condition (b) that $\alpha(x)$ and $\beta(x)$ are properly monotone increasing. To see the smoothness, confer the proof of (3.2) given later. Obviously we have

$$\alpha(p_i \pm r_i) = \beta(p_i \pm r_i).$$

If $x \in \alpha(I_i) = \beta(I_i)$ we have

$$\alpha'\circ\alpha^{-1}(x)=2^{-i}\frac{r_i^{i-1}}{c_i}\sigma'\circ\sigma^{-1}\left\{[x-\alpha(p_i-r_i)]\,2^i\frac{c_i}{r_i^i}-1\right\}$$

$$\times\sigma'\circ\rho^{-1}\left\{[x-\alpha(p_i-r_i)]2^i\frac{c_i}{r^i}-1\right\},$$

$$\beta'\circ\beta^{-1}(x)=2^{-i}\frac{r_i^{i-1}}{c_i}\sigma'\circ\sigma^{-1}\left\{[x-\alpha(p_i-r_i)]\,2^i\frac{c_i}{r^i}-1\right\}.$$

Thus we obtain

$$\frac{\alpha'\circ\alpha^{-1}(x)}{\beta'\circ\beta^{-1}(x)}=\sigma'\circ\rho^{-1}\left\{[x-\alpha(p_i-r_i)]\,2^i\frac{c_i}{r^i}-1\right\}$$

$$\longrightarrow 0\ (x\downarrow\alpha(p_i-r_i)=\beta(p_i-r_i)).$$

Similarly

$$\frac{\alpha'\circ\alpha^{-1}(x)}{\beta'\circ\beta^{-1}(x)}\longrightarrow 0\quad(x\uparrow\alpha(p_i+r_i)=\beta(p_i+r_i)).$$

We put

$$f^{-1}(t)=(\alpha(t^1),\ \alpha(t^2),\dots,\ \alpha(t^n)),$$

$$g^{-1}(t)=(\beta(t^1),\ \beta(t^2),\dots,\ \beta(t^n))$$

and then $f\circ g^{-1}\in{}_n\mathbf{E}_r^{-1}\circ{}_n\mathbf{E}_r$. Assume that there exist $h,\ k\in{}_n\mathbf{E}_1$ such that

$$f\circ g^{-1}=h\circ k^{-1}$$

then

$$f\circ g^{-1}\circ k=h.$$

Comparing each component on the t^1-axis we obtain

$$\alpha^{-1}\circ\beta\circ k^i(t^1,0,0,\dots,0)=h^i(t^1,0,0,\dots,0)$$

for $i=1,2,3,\dots,n$ and $|t^1|\leqq\delta$. We may assume that

$$h^1(\delta,0,0,\dots,0)>0$$

without loss of generality. Putting

$$l(x)=k^1(x, 0, 0,\dots, 0)$$

we have

$$\frac{\alpha'\circ\alpha^{-1}\circ(\beta\circ l)(x)}{\beta'\circ\beta^{-1}\circ(\beta\circ l)(x)}=\frac{l'(x)}{(\partial h^1/\partial t^1)(x, 0, 0,\dots, 0)}\,.$$

The numerater and the denominator on the right are continuous on $J=[0, \delta]$. The left expression tends to 0 when $l(x)$ approaches $p_i \pm r$ from the inside of I_i. Thus

$$l'(x)=0$$

provided

$$l(x)=p_i \pm r_i \in K\,,$$

where K is a compact interval such that $K \Subset l(J)$ and $\mu[A \cap K]>0$. Then

$$B=\{p_i \pm r_i\} \cap K$$

is included in the set of the critical values of l. Being closed, the latter includes the set

$$\bar{B}=A \cap K$$

and thus has positive measure. This contradicts the theorem of Sard on critical value and proves

$$f\circ g^{-1} \notin {}_n\mathbf{E}_1 \circ {}_n\mathbf{E}_1^{-1}, \qquad q.e.d.$$

Example 2.6. We give an element $f \in \mathbf{D}_\infty^* = \mathbf{E}_r^{*-1}\circ\mathbf{E}_r^*$ that does not belong to $\mathbf{E}_r\circ\mathbf{E}_r^{-1}$ for $r \geqq 2$.

Put

$$\rho(x)=\int_0^x\left(1+\xi\sin\frac{1}{\xi^3}\right)d\xi\,,$$

$$f^i(t)=\sqrt{\rho(\|t\|^2)}\cdot t^i \qquad (i=1, 2, 3,\dots, n).$$

Then the germ f of the map

$$f(t)=(f^1(t), f^2(t),\dots, f^n(t))$$

at 0 belongs to $\mathbf{D}_\infty^*$, because

$$\frac{\partial(f^1, f^2,\dots, f^n)}{\partial(t^1, t^2,\dots, t^n)}=\frac{\det(t^i t^j+\delta_{ij}\|t\|^2)+0(\|t\|^{2n+2})}{[\sqrt{\rho(\|t\|^2)}]^n}$$

$$=\frac{2\|t\|^{2n}+0(\|t\|^{2n+2})}{[\sqrt{\rho(\|t\|^2}]^n}$$

But $f\notin \mathbf{E}_r\circ\mathbf{E}_r^{-1}$ if $r\geqq 2$. To see this let us assume that

$$f\in\mathbf{E}_r\circ\mathbf{E}_r^{-1}=\mathbf{J}\circ{}_r\mathbf{J}_r^{-1}$$

or $f\circ\varphi\in\mathbf{J}_r$ for some $\varphi\in\mathbf{J}_r$. Then $\sqrt{\rho[\|\varphi(t)\|^2]}\cdot\varphi^i(t)$ and $\rho[\|\varphi(t)\|^2]\cdot\|\varphi(t)\|^2$ are r-flat at 0. Putting

$$\sigma(x)=\|\varphi(x, 0, 0,\dots, 0)^2\|,$$

$$\tau(x)=\rho[\sigma(x)]\cdot\sigma(x),$$

we have

$$\tau''=\sigma\cdot(\sigma')^2\cdot\rho''(\sigma)+2(\sigma')^2\cdot\rho'(\sigma)$$

$$+\sigma\cdot\sigma''\cdot\rho'(\sigma)+\sigma''\cdot\rho(\sigma).$$

Since $\sigma(x)$ and $\tau(x)$ are r-flat $(r\geqq 2)$, $\tau''(x)$, $\sigma(x)$, $\sigma'(x)$, $\sigma''(x)$ and $\rho[\sigma(x)]$ approach 0 and $\rho'[\sigma(x)]$ is bounded when x tends to 0. Then $\sigma\cdot(\sigma')^2\cdot\rho''(\sigma)$, and consequently

$$\frac{\sigma'}{\sigma}\sqrt{\left|\cos\frac{1}{\sigma^3}\right|}$$

approaches 0. On the other hand, putting

$$\Phi(x)=\int_x^1\frac{1}{s}\sqrt{\left|\cos\frac{1}{s^3}\right|}\,ds$$

we have

$$\lim_{x\to 0} \Phi[\sigma(x)] = \lim_{x\downarrow 0} \int_x^1 \frac{1}{s} \sqrt{\left|\cos \frac{1}{s^3}\right|}\, ds = \infty .$$

The previous observation means

$$\lim_{x\to 0} \frac{d}{dx} \Phi[\sigma(x)] = 0 .$$

These two limit equations are not compatible. Thus we have proved

$$f \in \mathbf{D}_\infty^*,\ f \notin \mathbf{E}_r \circ \mathbf{E}_r^{-1} \qquad (r \geqq 2).$$

3. The case of $\mathbf{R}^1$

In the previous section we have seen that the assertions of the problems (I L′) and (II L′) suppose some condition on f and g. Corollary 2.3 gives such a condition. If we restrict ourselves to one dimensional case, we obtain a sharper result which yields Theorem 0.1 in Introduction.

Theorem 3.1. *For $r, s \geqq 1$ we have the following*:

(i) $${}_1\boldsymbol{E}_r \circ {}_1\boldsymbol{E}_r^{-1} \subset {}_1\boldsymbol{E}_s^{-1} \circ \boldsymbol{E}_r .$$

(ii) *If*

$${}_1\boldsymbol{E}_r^{-1} \circ f \cap {}_1\boldsymbol{E}_r^{-1} \circ g \neq \phi$$

then

$${}_1\boldsymbol{E}_r \circ f \cap {}_1\boldsymbol{E}_s \circ g \neq \phi .$$

(iii) *If $f, g \in {}_1\boldsymbol{E}_r \circ \varphi$ for some $\varphi \in {}_1\boldsymbol{C}$ then $f \in {}_1\boldsymbol{E}_r^{-1} \circ \psi$, $g \in {}_1\boldsymbol{E}_s^{-1} \circ \psi$ for some $\psi \in {}_1\boldsymbol{C}$.*

(iv) *If ${}_1\mathscr{E}_r(f)$, ${}_1\mathscr{E}_r(g) \subset {}_1\mathscr{E}_r(\varphi)$ for some $\varphi \in {}_1\boldsymbol{C}$ then ${}_1\mathscr{E}_r(\psi) \subset {}_1\mathscr{E}_r(f)$, ${}_1\mathscr{E}_s(\psi) \subset {}_1\mathscr{E}_s(g)$ for some $\psi \in \boldsymbol{C}_1$.*

The assertions (i)~(iv) are mutually equivalent. We arrange a few lemmata before proof.

Lemma 3.2. *Let Ω be an open subset of $\boldsymbol{R}^n$, $f(x)\in C^0(\Omega)$ and let $f(x)\geqq 0$. Then there exists a function $\varphi(x)\in C^\infty(\Omega)$ such that*

$$0<\varphi(x)<f(x) \qquad \textit{if } f(x)>0,$$

$$\varphi(x) \textit{ is } \infty\textit{-flat at } x_0 \qquad \textit{if } f(x_0)=0.$$

Proof. We may assume that

$$f(x)\leqq M<\infty$$

without loss of generality. Put

$$P=\{x: x\in\Omega, f(x)>0\}$$

and take a countable covering V_i of P, where V_i is an open ball of center p_i and radious $r_i<1$ and the closure $\overline{V}_i$ is contained in P. Let $\rho(x)$ be a C^∞-function on $\boldsymbol{R}^n$ such that

$$\rho(x)>0 \ (\|x\|<1), \qquad \rho(x)=0 \ (\|x\|\geqq 1).$$

Putting

$$c_i=\|\rho(x)\|^i_{\mathbf{R}^n}, \qquad m_i=\inf_{x\in V_i} f(x),$$

we show that

$$\varphi(x)=\sum_{i=1}^\infty 2^{-i}m_i\frac{r_i^i}{c_i}\rho\left(\frac{x-p_i}{r_i}\right)$$

has the required properties. Since

$$\left|\left[\rho\left(\frac{x-p_i}{r_i}\right)\right]^{(\nu)}\right|=\frac{1}{r_i^{|\nu|}}\left|\rho^{(\nu)}\left(\frac{x-p_i}{r_i}\right)\right|\leqq\frac{c_{|\nu|}}{r_i^{|\nu|}},$$

we have

$$\left\|2^{-i}m_i\frac{r_i^i}{c_i}\rho\left(\frac{x-p_i}{r_i}\right)\right\|^i_{\mathbf{R}^n}<2^{-i}M.$$

Then $\varphi(x)$ is in $C^\infty(\mathbf{R}^n)$ and ∞-flat on $\Omega-P$.
By the inequalities

$$0<m_i\leqq f(x) \quad (x\in V_i)\,,$$

$$0\leqq\frac{1}{c_i}\rho\left(\frac{x-p_i}{r_i}\right)\leqq 1\,,$$

we have

$$0<\varphi(x)\leqq\sum_{i=1}^{\infty}2^{-i}f(x)r_i^i<f(x) \quad (x\in p)\,.$$

This completes the proof.

If $r\geqq 1$, the set of critical points of a function of $M^r(I)$ is a closed subset without inner point. We can see that the converse is also true putting $f(x)=x$ in the next corollary.

Corollary 3.3. *Let I be an interval of $\boldsymbol{R}$ and Z a closed subset of O without inner point. For a function $f(x)\in M^r(I)$ $(r\geqq 1)$ we put $J=f(I)$. Then we can find a function $F(x)\in M^\infty(J)$ such that the set of zero points of $(F'\circ f)(x)$ is Z. (Consequently the set of critical points of $(F\circ f)(x)$ is the union of one of $f(x)$ and Z.)*

Proof. Since f is a homeomorphism, $f(Z)$ is also a closed subset without inner point. Put

$$\sigma(x)=(\text{the distance of } x \text{ and } f(Z))\,,$$

then $\sigma(x)\in C^0(J)$, $\sigma(x)\geqq 0$ and the set of zero points of $\sigma(x)$ is $f(Z)$. By (3.2) there is a function $\tau(x)$ such that $\tau(x)\in C^\infty(J)$, $\tau(x)>0$ on $J-f(Z)$ and $\tau(x)=0$ on $f(Z)$. It is clear that any primitive function of $\tau(x)$ has the required property, *q.e.d.*

Lemma 3.4. *Let Ω be an open set in $\boldsymbol{R}^n$ and A its closed null subset. If a homeomorphism $f(x)$ of Ω into $\boldsymbol{R}^n$ is C^1 diffeomorphism on $\Omega-A$ then $f^{-1}(x)$ is absolutely continuous on $f(\Omega)$. If $g(x)$ and and $h(x)$ are C^1 homeomorphisms of Ω into $\boldsymbol{R}^n$ such that the critical* *set, Z, of $g(x)$ includes that, W, of $h(x)$, then $g\circ h^{-1}(x)$ is absolutely continuous.*

Proof. Let S be a null set in $f(\Omega)$. $f^{-1}(S)\cap(\Omega-A)$ and $f^{-1}(S)\cap A$ are obviously null sets. This means that $f^{-1}(x)$ is absolutely continuous. $h\circ g^{-1}(x)$ is a diffeomorphism on $g(\Omega-Z-W)=g(\Omega-Z)$. This set is the critical value of $g(x)$ and hence a null set by the Sard's theorem. It is also closed. Thus, we have only to apply the former part to $h\circ g^{-1}(x)$, *q.e.d.*

Lemma 3.5. *Let I be an interval of $\boldsymbol{R}$ and A be its closed subset. Let $f(x)$ be a continuous function that takes a fixed value on A. If $f(x)$ is differentiable on $I-A$ and if $f'(x)$, defined on $I-A$, has a continuous extension $g(x)$ over I which vanishes almost everywhere on A, then $f(x)$ is of class C^1 and $f'(x)=g(x)$.*

Proof. We may assume that I is a compact interval $[a, b]$. If we put

$$s(x)=f(x)-\int_a^x g(\xi)d\xi\,,$$

$s(x)$ is a continuous function with differential coefficient 0 on $I-$A. We have only to prove that $s(x)$ is a constant function on I. Let $I_1, I_2, I_3,\dots$ be the connected component of $I-A$. Then $f(x)$ takes a same value on the boundary of I_n, so that

$$\int_{I_n} f'(\xi)d\xi=0\,.$$

If $x_1, x_2\in A$, $[x_1, x_2]$ is a disjoint sum B of some of $I_1, I_2, I_3,\dots$ and a subset C of A. Then we have

$$\int_a^{x_2} g(\xi)d\xi-\int_a^{x_1} g(\xi)d\xi=\int_{B-C} f'(\xi)+\int_C g(\xi)d\xi=0\,.$$

Thus each of

$$f(x),\quad \int_a^x g(\xi)d\xi$$

and consequently $s(x)$ takes a fixed value on A. Since $s(x)$ is constant on I_n, so is on I, *q.e.d.*

The proof of Theorem 3.1. For arbitrary elements $f, g \in {}_1\mathbf{E}_r$ we have only to find suitable elements $h \in {}_1\mathbf{E}_r$ and $k \in {}_1\mathbf{E}_\infty$ such that

$$h \circ f = k \circ g$$

We may assume that f and g are orientation preserving. Let $f(t)$, $g(t)$ be representatives of f and g respectively, which are defined on a finite interval $I=(-a, a)$ and belong to $M^r(I)$. The set Z_0 of critical points of $f(t)$ is a closed set without inner point. Applying (3.3), we can find a function $F(t) \in M^\infty(J)$ $(J=g(I))$ such that $F(0)=0$ and the set of critical points of $(F\circ g)(t)$ is the union of Z_0 and one for $g(t)$. Since the germ $F\circ g$ of $(F\circ g)(t)$ belongs to ${}_1\mathbf{E}_r \circ g$ we can replace g by $F\circ g$. Thus we may assume that the set of zero points of $g'(t)$ includes one for $f'(t)$, Z_0. Similarly we assume that Z_0 contains 0. (3.2) assures the existence of $\varphi(t) \in C^\infty(J)$ such that

$$0<\varphi(t)<f'\circ g^{-1}(t) \qquad \text{if} \quad f'\circ g^{-1}(t)>0\,,$$

$$\varphi(t)\text{: flat at } t_0 \qquad \text{if} \quad f'\circ g^{-1}(t_0)=0\,.$$

Now take the C^∞-function $\theta_0(t)$ on $\mathbf{R}$ defined by

$$\theta_0(t)=\exp(-1/|t|) \quad (t\neq 0), \qquad \theta_0(0)=0\,.$$

Then

$$\theta(t)=\operatorname{sign} t\cdot\theta_0(t)$$

defines a function of $M^\infty(\mathbf{R})$. It is flat at 0 and has a positive differential coefficient anywhere else.

We put

$$\psi(t)=\int_0^t \theta_0\circ\varphi(\tau)d\tau\,,$$

$$h(t)=\theta\circ\psi\circ g\circ f^{-1}(t)\,.$$

Since the set of zero points of $\psi'(t)$, or equivalently of $f'\circ g^{-1}(t)$, is homeomorphic to that of $f'(t)$ and has no inner point, we have

$$\psi(t)\in M^{\infty}(J)\,, \qquad h(t)\in M^0(K)\,,$$

where $K=f(I)$. Further we shall show that $h(t)\in M^r(K)$. It is obvious that $h(t)$ is of class C^r outside the set Z of zero points of $f'\circ f^{-1}(t)$. By mathematical induction on v we can see that any higher derivative $h^{(v)}(t)$ $(1\leqq v\leqq r)$ is a linear combination of terms of the following type on $K-Z$ excepting possible multiplication by sign t:

$$(*)\qquad \frac{\theta\circ\psi\circ g\circ f^{-1}(t)}{[\psi\circ g\circ f^{-1}(t)]^{\alpha}}\times\frac{[\theta\circ\varphi\circ g\circ f^{-1}(t)]^{\delta}}{[\varphi\circ g\circ f^{-1}(t)]^{\beta}[f'\circ f^{-1}(t)]^{\gamma}}$$

$$\times\prod_{l=1}^{L}[\varphi^{(l)}\circ g\circ f^{-1}(t)]^{a_l}\times\prod_{m=1}^{M}[g^{(m)}\circ f^{-1}(t)]^{b_m}\times\prod_{n=2}^{N}[f^{(n)}\circ f^{-1}(t)]^{c_n}$$

where

$$\alpha,\ \beta,\ \gamma,\ \delta,\ a_l,\ b_m,\ c_n,\ L,\ M,\ N$$

are all non-negative integers and $M,\ N\leqq r,\ \delta>0$.
Since

$$0<\varphi(t)<f'\circ g^{-1}(t)$$

and

$$\lim_{t\to 0}\theta(t)/t^s=0$$

every factor of $(*)$ is bounded when t tends to a point of Z from the inside of $K-Z$. Especially the second factor tends to zero. Then $h^{(v)}(t)$ defined on $K-Z$ has a continuous extension $\widetilde{h_v}(t)$ which vanishes on Z. We have assumed that the set Z_0 of zero points of $f'(t)$ is included in that of $g'(t)$. Then by (3.4), we see that $g\circ f^{-1}(t)$ is absolutely continuous on K. Then $h(t)$ is also absolutely continuous and its derivative on $K-Z$ has a continuous extension $\widetilde{h_1}(t)$. Since Z is a null set as well as W, $\widetilde{h_1}(t)$ is the derivative of $h(t)$. We proceed by induction on v. Assume that $h(t)\in C^{v-1}(K)$ and $h^{(v-1)}(t)=\widetilde{h_{v-1}}(t)$

$(2 \leqq \nu \leqq r)$. Applying (3.5) to the triple $\widetilde{h_{\nu-1}}(t)$, K, Z, we have

$$h^{(\nu)}(t)=\widetilde{h'_{\nu-1}}(t)=\widetilde{h_\nu}(t).$$

Thus we have proved that

$$h(t) \in M^0(K) \cap C^r(K)=M^r(K).$$

Putting

$$k(t)=\theta \circ \psi(t) \in M^\infty(J),$$

we have

$$h \circ f(t)=k \circ g(t).$$

Taking the germ at 0

$$h \circ f=k \circ g, \qquad h \in {}_1\mathbf{E}_r,\ k \in {}_1\mathbf{E}_\infty .$$

This completes the proof of the theorem.

Let us deduce some results from this theorem. To do this we prove an algebraic lemma. (We use only (iv)).

Lemma 3.6. *Let G be a group and A, B, C and D be its subsets satisfying*

$$A \supset D, \qquad B \supset C,$$

$$AB^{-1} \subset C^{-1}D \qquad (\text{i.e. } BA^{-1} \subset D^{-1}C).$$

(i) *If C and D are subsemigroups of G, then $C^{-1}D \cap D^{-1}C$ is a subgroup of G.*

(ii) *If A and B are subsemigroups of G, then $A^{-1}B \cap B^{-1}A$ is a subgroup of G.*

(iii) *If $A \cap B \ni 1$ (identity of G) and if A and C (or B and D) are subsemigroups, then*

$$A^{-1}B \cap B^{-1}A=C^{-1}D \cap D^{-1}C$$

is a subgroup of G and

$$A^{-1}B=D^{-1}C \qquad (i.e.\ B^{-1}A=C^{-1}D).$$

(iv) *If $A=B\ni 1$ and if $A=B$ and one of C, D are subsemigroups, then*

$$A^{-1}A=C^{-1}D=D^{-1}C$$

is a subgroup of G.

Proof. (i) Since

$$(C^{-1}D)^2\subset C^{-1}AB^{-1}D\subset C^{-1}C^{-1}DD\subset C^{-1}D,$$

$$(D^{-1}C)^2\subset D^{-1}BA^{-1}C\subset D^{-1}D^{-1}CC\subset D^{-1}C,$$

we have

$$(C^{-1}D\cap D^{-1}C)^2\subset C^{-1}D\cap D^{-1}C.$$

While it is obvious that

$$(C^{-1}D\cap D^{-1}C)^{-1}=C^{-1}D\cap D^{-1}C.$$

Then $C^{-1}D\cap D^{-1}C$ is a subgroup of G.

(ii) Since $A\,B^{-1}\subset C^{-1}D\subset B^{-1}A$, we have only to put $C=B$, $D=A$ in (i).

(iii) Since

$$B^{-1}A=1\cdot B^{-1}A\subset A\ B^{-1}A\subset C^{-1}DA\subset C^{-1}A\ AA=C^{-1}A$$

$$=C^{-1}A\cdot 1\subset C^{-1}C^{-1}D\subset C^{-1}D\subset B^{-1}A,$$

we have $B^{-1}A=C^{-1}D$ and then

$$(C^{-1}D)^2\subset C^{-1}C^{-1}D\ D\subset C^{-1}A\ A\subset C^{-1}A\subset B^{-1}A=C^{-1}D,$$

$$(D^{-1}C)^2=(C^{-1}D)^{-2}\subset D^{-1}C.$$

Again it is obvious that

$$(C^{-1}D\cap D^{-1}C)^{-1}=C^{-1}D\cap D^{-1}C.$$

Hence

$$A^{-1}B\cap B^{-1}A=C^{-1}D\cap D^{-1}C$$

is a subgroup of G.

(iv) Immediate from (iii).

Combining (2.4), (2.5), (3.1) and (iv) of (3.6), we have the following:

Corollary 3.7. *If* $1\leqq r\leqq s\leqq\infty$, ${}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_r$ *is the subgroup of* ${}_1\boldsymbol{C}$ *generated by* ${}_1\boldsymbol{E}_r$ *and*

$$ {}_1\boldsymbol{E}_r\circ{}_1\boldsymbol{E}_r^{-1}\subsetneqq{}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_r\subsetneqq{}_1\boldsymbol{C},$$

$$ {}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_r={}_1\boldsymbol{E}_s^{-1}\circ{}_1\boldsymbol{E}_r={}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_s.$$

Corollary 3.8. *If* $r,\ s\geqq 1$ *and* $f_i\in{}_1\boldsymbol{C}$ $(i=1,2,\dots,m)$ *belong to the same left coset of* ${}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_r\backslash{}_1\boldsymbol{C}$, *then*

(i) $\quad {}_1\boldsymbol{E}_s\circ f_1\cap\left(\bigcap_{i=2}^{m}{}_1\boldsymbol{E}_r\circ f_i\right)\neq\phi$

or, what amounts to the same,

(ii) $\quad {}_1\mathscr{E}_s(\psi)\subset{}_1\mathscr{E}_s(f_1),\qquad {}_1\mathscr{E}_r(\psi)\cap\bigcap_{i=2}^{m}{}_1\mathscr{E}_r(f_i)$

for some $\psi\in\boldsymbol{C}$.

The proof is formal and we omit it. I wonder whether this corollary can be generalized to the case of enumerably infinite number of f_i.

Corollary 3.9. *In* (3.1) (i) ~ (iii), (3.7) *and* (3.8) (i), *we may replace* $({}_1\boldsymbol{E}_r,{}_1\boldsymbol{E}_s)$ *by* $({}_1\boldsymbol{E}_r^*,{}_1\boldsymbol{E}_s^*)$ *or* $({}_1\boldsymbol{E}_r^{\mathfrak{a}},{}_1\boldsymbol{E}_s^{\mathfrak{a}})$, *where* ${}_1\boldsymbol{E}_r^{\mathfrak{a}}$ *denotes the subsemigroup of* ${}_1\boldsymbol{E}_r$ *consisting of elements whose critical points are at most enumerable. Hence*

$$ {}_1\boldsymbol{E}_r^{**}\subsetneqq{}_1\boldsymbol{E}_r^{*-1}\circ{}_1\boldsymbol{E}_r^*={}_1\boldsymbol{D}_r^*\subsetneqq({}_1\boldsymbol{E}_r^{\mathfrak{a}})^{-1}\circ{}_1\boldsymbol{E}_r^{\mathfrak{a}}\subsetneqq{}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_r$$

are proper subgroups of $_1\boldsymbol{C}$.

This is a consequence of the fact that the sets of critical points of $h(t)$ and $k(t)$ in the proof of the theorem are at most homeomorphic to the union of those of the original $f(t)$ and $g(t)$ and 0.

4. Global versions

Here we treat the dependence of C^r-structures on a topological manifold and the dependence of homeomorphisms of a C^r manifold. In the local theory these are mutually equivalent. (compare (I L) and (II L) with (I L′) and (II L′)) but not in the global theory. The problems (I G) and (II G) in Introduction are concerned with the former. The negative examples (2.4), (2.5) and (2.6) immediately yield the corresponding negative examples for (I G) and (II G). The positive results (2.3) and (3.1) have also global versions. We will explain them as a reduction theory of some structures. There arises no topological obstruction as far as we are concerned with the global versions of the local results we have obtained. The dependence of homeomorphism is defined by subsemigroups of the homeomorphism group. We show global versions of (2.3) and (3.1) also in this sense. Hereafter we assume that a manifold X is a connected paracompact topological one without boundary.

We review the definition of structures on X in our terminology. For a homeomorphism $f(t)\colon U \to V$ of open sets in $\mathbf{R}^n$, we define $\tilde{f}_{t_0}$ to be the germ of the homeomorphism

$$\tilde{f}_{t_0}(t) = f(t+t_0) - f(t_0)$$

at 0. $\tilde{f}_t$ is an element of $_n\boldsymbol{C}$ for $t \in U$. If $(U, \varphi(x))$ is a chart on X, we define $\bar{\varphi}_{x_0}$ to be the germ of the homeomorphism

$$\bar{\varphi}_{x_0}(x) = \varphi(x) - \varphi(x_0)$$

at x_0. Now let $\mathbf{H}$ be a subsemigroup of $_n\mathbf{C}$ $(n = \dim X)$ which contains the identity and satisfies the following condition:

(†) If $f(t)\colon U \to V$ is a homeomorphism of open sets in $\mathbf{R}^n$ and if $\tilde{f}_{t_0} \in \mathbf{H}$, then $\tilde{f}_t \in \mathbf{H}$ for all t near t_0.

If two charts $(U, \varphi(x))$, $(V, \psi(x))$ and $x \in U \cap V$ satisfy $\bar{\varphi}_x \in \mathbf{H} \circ \bar{\psi}_x$, we say that $\varphi(x)$ is **H**-*dependent* on $\psi(x)$ at x. Then by (†), the set

$$\{x_0 \in U \cap V: \varphi(x) \text{ is } \mathbf{H}\text{-dependent on } \psi(x) \text{ at } x_0\}$$

is open. If $\mathbf{G}$ is a subgroup of ${}_n\mathbf{C}$, $\mathbf{G}$-dependence is an equivalence relation and hence we call it $\mathbf{G}$-equivalence. Let $\mathfrak{B} = \{(U_\lambda, \varphi_\lambda(x))\}_{\lambda \in \Lambda}$ be a collection of charts covering X. If $\varphi_\lambda(x)$ and $\varphi_{\lambda'}(x)$ are $\mathbf{C}$-equivalent for all $\lambda, \lambda' \in \Lambda$ and all $x \in U_\lambda \cap U_{\lambda'}$, we call $\mathfrak{B}$ a **G**-*basis*. If the join of two $\mathbf{G}$-basis $\mathfrak{B}, \mathfrak{B}'$ is again a $\mathbf{G}$-basis, we say $\mathfrak{B}$ and $\mathfrak{B}'$ are ($\mathbf{G}$-) equivalent. An equivalence class of a $\mathbf{G}$-basis is called a **G**-*structure* on X. A $\mathbf{G}$-structure can be identified with its unique maximal $\mathbf{G}$-basis. If a chart belongs to the maximal basis, we call it an *admissible chart* of the $\mathbf{G}$-structure. If $\mathbf{G} \subset \mathbf{G}'$, a basis of a $\mathbf{G}$-structure $\mathscr{D}$ is a $\mathbf{G}$-basis of a uniquely determined $\mathbf{G}'$-structure $\mathscr{D}'$ and then we say $\mathscr{D}$ *generates* $\mathscr{D}'$. Let Y be an open set of X and $\mathfrak{B} = \{(U_\lambda, \varphi_\lambda(x))\}_{\lambda \in \Lambda}$ be a basis of a $\mathbf{G}$-structure $\mathscr{D}$ on X. Then the restriction $\mathfrak{B} | Y = \{(U_\lambda \cap Y, (\varphi_\lambda | U_\lambda \cap Y)(x))\}_{\lambda \in \Lambda}$ is a $\mathbf{G}$-basis on Y and $(Y, \mathscr{D} | Y) = (Y, \mathscr{D})$ becomes a manifold with $\mathbf{G}$-structure.

Consider a subgroup $\mathbf{G}$ and a subsemigroup $\mathbf{H}$ of ${}_n\mathbf{C}$ satisfying $\mathbf{G} = \mathbf{H} \cap \mathbf{H}^{-1}$. If any admissible chart $(U, \varphi(x))$ of a $\mathbf{G}$-structure $\mathscr{D}$ is $\mathbf{H}$-dependent on any admissible chart $(V, \psi(x))$ of another $\mathbf{G}$-structure $\mathscr{D}'$ on $U \cap V$, we say $\mathscr{D}$ is **H**-*dependent* on $\mathscr{D}'$. This is an order relation. $\mathbf{E}_r$-dependence of C^r-structures (i.e. $\mathbf{E}_r^{**}$-structures) corresponds to the "dependence" of local C^r-structures.

The group $\mathbf{D}_r^*$ satisfies (†), so we consider $\mathbf{D}_r^*$-structures. Let $\mathfrak{B}$ be a basis of a $\mathbf{D}_r^*$-structures $\mathscr{D}$. We define $S_\mathfrak{B}$ to be the closure of the set of all x where all charts of $\mathfrak{B}$ containing x are not $\mathbf{E}_r^{**}$-equivalent. If $\mathfrak{B}$ is locally finite, $S_\mathfrak{B}$ is discrete and $\mathfrak{B} | X - S_\mathfrak{B}$ is a $\mathbf{E}_r^{**}$-basis of a C^r-structure on $X - S_\mathfrak{B}$.

Proposition 4.1. *Any ${}_n\mathbf{D}_r^*$-structure $\mathscr{D}$ is generated by a C^r-structure $\mathscr{D}'$. If a locally finite basis $B = \{(U_\lambda, \varphi_\lambda(x))\}_{\lambda \in \Lambda}$ of $\mathscr{D}$ is given, we can choose $\mathscr{D}'$ such that:*

(i) *$\mathscr{D}'|U_\lambda$ is E_r-dependent on $\varphi_\lambda(x)$ for any $\lambda \in \Lambda$ in a obvious sense.*

(ii) *$\mathfrak{B}|_{X-S_{\mathfrak{B}}}$ is a basis of $\mathscr{D}'|_{X-S_{\mathfrak{B}}}$.*

The proof is easy by Church's lemma (2.1).

Corollary 4.2. *Let $\mathscr{D}$ and $\mathscr{D}'$ be two C^r-structures differing only on a discrete subset A. Then there is a third C^r-structure $\mathscr{D}''$ such that*

(i) *$\mathscr{D}''$ is $\boldsymbol{E}_r$-dependent both on $\mathscr{D}$ and $\mathscr{D}'$.*

(ii) *$\mathscr{D}|X-A=\mathscr{D}'|X-A=\mathscr{D}''|X-A$.*

(iii) *$(X, \mathscr{D}'')$ is diffeomorphic to $(X, \mathscr{D})$.*

(iii) follows from the special form of the map $g(t)$ in (2.1) (cf. (2.2)). If $n=1, 2, 3, 5, 6, 12$, $(X, \mathscr{D})$, $(X, \mathscr{D}')$ and $(X, \mathscr{D}'')$ can be chosen to be diffeomorphic to each other by (5.6). This corollary gives a positive answer to the problem (II G) (cf. (2.3)).

Next, we give the global version of (3.1).

Lemma 4.3. (*an extension of a smooth monotone function*) *If*

$a<b<c<c'<b'<a'$,

$f(x) \in C^r([a, c] \cup [c', a'])$,

$f(x)|[a, c] \in M^r[a, c], \qquad f(x)|[c', a'] \in M^r[c', a']$,

$f(c)<f(c')$

then there exist a function $g(x) \in M^r[a, a']$ such that

$$g(x)=f(x) \qquad (a \leqq x \leqq b,\ b' \leqq x \leqq a'),$$
$$g'(x)>0 \qquad (c \leqq x \leqq c').$$

The proof is omitted.

Theorem 4.4. *Let* $\dim X=1$. *Any $({}_1\boldsymbol{E}_r^{-1} \circ {}_1\boldsymbol{E}_r)$-structure $\mathscr{D}$ on X*

comes from a C^r-structure $\mathscr{D}'$. Moreover, if a locally finite bases $\{(U_\lambda, \varphi_\lambda(x))\}_{\lambda\in\Lambda}$ of $\mathscr{D}$ is given, we can choose $\mathscr{D}'$ such that any admissible chart $(V, \psi(x))$ is ${}_1\boldsymbol{E}_r$-dependent on $(U_\lambda, \varphi_\lambda(x))$ on $U_\lambda \cap V$.

Proof. By our assumption we can choose a collection $\{(W_i, \theta_i(x))\}_{i\in N}$ of charts satisfying the following conditions ((g) is possible by (3.8)).

(a) N is at most countable.

(b) $W_i \subset W_j$ implies $i=j$.

(c) $W_i \cap W_j$ is connected.

(d) Three members from $\{W_i\}_{i\in N}$ do not intersect.

(e) $X = \bigcup_{i\in N} W_i$.

(f) $\theta_i(w_i)$ is an interval of R.

(g) $\theta_i(x)$ is C^r-dependent upon $\varphi_\lambda(x)$ for any λ satisying $x\in U_\lambda$.

Take a point $p=p_{ij}\in W_i\cap W_j$ for each non empty $W_i\cap W_j$. From (g)

$$\bar{\theta}_{ip}\in {}_1\mathbf{E}_r\circ\bar{\varphi}_\lambda \subset {}_1\mathbf{E}_r\circ {}_1\mathbf{E}_r^{-1}\circ\bar{\theta}_{jp}\subset {}_1\mathbf{E}_r^{-1}\circ {}_1\mathbf{E}_r\circ\bar{\theta}_{jp}.$$

We have used (3.1) to show the last inclusion. Then we obtain

$$f_i\circ\theta_{ip}(x) = f_j\circ\theta_{jp}(x) \qquad (x\in W),$$

$$\varepsilon_i f_i(t)\in M^r[\theta_{ip}(W)]\,, \qquad \varepsilon_j f_j(t)\in M^r[\theta_{jp}(W)] \ (\varepsilon_i,\ \varepsilon_j=\pm 1)\,,$$

$$f_i(0)=f_j(0)=0$$

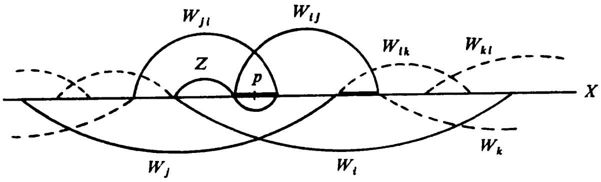
for some neighbourhood $W \Subset W_i\cap W_j$ of p, where $A\Subset B$ means that the closure of A is included in the interior of B. Narrowing W if necessary we may assume that

$$\varepsilon_i f_i\circ\theta_{ip}(W)\Subset\theta_{ip}(W_i\cap W_j)\,,$$

$$\varepsilon_j f_j\circ\theta_{jp}(W)\Subset\theta_{jp}(W_i\cap W_j)\,.$$

Let Z be the connected part of $W_i\cap W_j\cap W^c$ closer to $W_i^c\cap W_j$ and put

$$W_{ij}=W_i \cap Z^c \cap (\bigcup_{k\neq i,j} W_k)^c.$$



By Lemma 4.3 there exists a function $f_{ij}(t)\in M^r[\theta_{ip}(W_{ij})]$ such that

$$f_{ij}(t)=\begin{cases} \varepsilon_i f_i(t) & (t\in\theta_{ip}(W)) \\ t & (t\in\theta_{ip}(W_{ij}\cap W_j^c)). \end{cases}$$

putting

$$\theta_{ij}(x)=f_{ij}\circ\theta_{ip}(x) \qquad (x\in W_{ij})$$

we have charts $\{(W_{ij},\ \theta_{ij}(x))\}_{i\neq j;W_i\cap W_j\neq\phi}$. Then it follows that

$$\theta_{ij}(x)=\varepsilon_i\varepsilon_j\ \theta_{ij}(x) \qquad (x\in W_{ij}\cap W_{ji}),$$

$$\theta_{ij}(x)+\theta_i(p_{ij})=\theta_{ik}(x)+\theta_i(p_{ik}) \qquad (x\in W_{ij}\cap W_{ik}).$$

Hence $\{(W_{ij},\ \theta_{ij}(x))\}_{i\neq j;W_i\cap W_j\neq\phi}$ is a basis of a C^r-structure $\mathscr{D}$. $\theta_{ij}(x)$ is C^r-dependent upon $\theta_{ip}(x)$, $\theta_i(x)$ and thus upon $\varphi_\lambda(x)$. $\mathscr{D}$ is a required C^r-structure on X, *q.e.d.*

Using this theorem and (3.1) again we have the following:

Corollary 4.5. *Let $\mathscr{D}$ and $\mathscr{D}'$ be $({}_1\boldsymbol{E}_r^{-1}\circ{}_1\boldsymbol{E}_r)$-equivalent C^r-structures on a 1-dimensional manifold X. Then there exists a third C^r-structure $\mathscr{D}''$ that is ${}_1\boldsymbol{E}_r$-dependent on $\mathscr{D}$ and $\mathscr{D}'$. And hence, if (I G) is answered in the affirmative for $\mathscr{D}$ and $\mathscr{D}'$, (II G) is also.*

Let us consider the homeomorphism group of X and obtain another kind of global versions of local theorems. Let $\{(U_\lambda,\ \varphi_\lambda(x))\}_{\lambda\in\Lambda}$ and $\{(V_\mu,\ \psi_\mu(x))\}_{\mu\in M}$ be $\mathbf{G}$-bases of $\mathbf{G}$-structures $\mathscr{D}$ and $\mathscr{D}'$ on X and $\mathbf{K}$ be a subset of $\mathbf{C}$ satisfying

$$(\dagger\dagger) \qquad \mathbf{G}\circ\mathbf{K}\circ\mathbf{G}=\mathbf{K}\,.$$

We define $\mathbf{K}[\mathscr{D}, \mathscr{D}']$ to be the set of all homeomorphisms $f\colon X\to X$ such that

$$(\widetilde{\psi_\mu\circ f\circ\varphi_\lambda^{-1}})_t\in\mathbf{K}$$

for all $\lambda\in\Lambda$, $\mu\in M$ and all $t\in\varphi(U_\lambda\cap f^{-1}(V_\mu))$. ($\dagger\dagger$) assures that this definition is independent of the choice of λ and μ. If $\mathbf{K}$ and $\mathbf{K}'$ satisfy ($\dagger\dagger$) we have

$$\mathbf{K}[\mathscr{D}, \mathscr{D}']\circ\mathbf{K}'[\mathscr{D}', \mathscr{D}'']\subset(\mathbf{K}\circ\mathbf{K}')[\mathscr{D}, \mathscr{D}''],$$

$$\mathbf{K}[\mathscr{D}, \mathscr{D}']^{-1}=\mathbf{K}^{-1}[\mathscr{D}', \mathscr{D}]\,.$$

If f is a homeomorphism of X and $\mathscr{D}$ is a $\mathbf{G}$-structure on X then there exists a unique $\mathbf{G}$-structure $f^{\#}\mathscr{D}$ (*pull back* of $\mathscr{D}$) such that

$$f\in\mathbf{G}[f^{\#}\mathscr{D}, \mathscr{D}]\,.$$

Theorem 4.6. *Let* $\mathbf{H}$ *be a subsemigroup of* ${}_n\mathbf{C}$ *and* $\mathbf{G}, \mathbf{G}'$ *be subgroups of* ${}_n\mathbf{C}$ *such that* $\mathbf{H}\circ\mathbf{H}^{-1}\subset\mathbf{H}^{-1}\circ\mathbf{H}, \mathbf{G}\subset\mathbf{G}'$ *and* $\mathbf{H}\cap\mathbf{H}^{-1}=\mathbf{G}$. *Suppose that*:

($\dagger\dagger\dagger$) *If two* $\mathbf{G}$*-structures* $\mathscr{D}$ *and* $\mathscr{D}'$ *generates the same* $\mathbf{G}'$*-structure then there exists a third G-structure* $\mathscr{D}''$ $\mathbf{H}$*-dependent both on* $\mathscr{D}$ *and* $\mathscr{D}'$ *such that* $\mathbf{G}[\mathscr{D}, \mathscr{D}'']\neq\phi$.
Then we have

$$\mathbf{G}'[\mathscr{D}, \mathscr{D}]\subset\mathbf{H}^{-1}[\mathscr{D}, \mathscr{D}]\circ\mathbf{H}[\mathscr{D}, \mathscr{D}]\subset(\mathbf{H}^{-1}\circ\mathbf{H})[\mathscr{D}, \mathscr{D}]$$

for any $\mathbf{G}$*-structure* $\mathscr{D}$ *on* X.

Proof. If $f\in\mathbf{G}'[\mathscr{D}, \mathscr{D}]$, $\mathscr{D}$ and $f^{\#}\mathscr{D}$ induce the same $\mathbf{G}'$-structure. Hence there is a $\mathbf{G}$-structure $\mathscr{D}'$ $\mathbf{H}$-dependent on $\mathscr{D}$ and $f^{\#}\mathscr{D}$ such that $\mathbf{G}[\mathscr{D}, \mathscr{D}']\neq\phi$. If we take $g\in\mathbf{G}[\mathscr{D}, \mathscr{D}']$,

$$f=(f\circ id\circ g)\circ(g^{-1}\circ id)\in\mathbf{H}^{-1}[\mathscr{D}, \mathscr{D}]\circ\mathbf{H}[\mathscr{D}, \mathscr{D}]$$

by the following commutative diagram. This proves the theorem.

$$
\begin{tikzcd}
(X, \mathscr{D}) \arrow[r, "id \in \mathbf{H}[\mathscr{D}, \mathscr{D}']"] \arrow[d, "f"'] \arrow[dr, "id"] & (X, \mathscr{D}') & (X, \mathscr{D}) \arrow[l, "g \in \mathbf{G}[\mathscr{D}, \mathscr{D}']"'] \\
(X, D) & (X, f^{\#}\mathscr{D}') \arrow[l, "f \in \mathbf{G}[f^{\#}\mathscr{D}, \mathscr{D}]"] \arrow[u, "id \in \mathbf{H}[f^{\#}\mathscr{D}, \mathscr{D}']"'] &
\end{tikzcd}
$$

Corollary 4.7. $\boldsymbol{D}_r^*[\mathscr{D}, \mathscr{D}] = (\boldsymbol{E}_r^*[\mathscr{D}, \mathscr{D}])^{-1} \circ \boldsymbol{E}_r^*[\mathscr{D}, \mathscr{D}]$
This follows from (4.2) and (4.6).

Corollary 4.8. *If X is one dimensional C^r-manifold, we have*

$$
{}_1\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}] \circ {}_1(\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}])^{-1} \subset ({}_1\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}])^{-1} \circ {}_1\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}]
$$
$$
= ({}_1\boldsymbol{E}_r^{-1} \circ {}_1\boldsymbol{E}_r)[\mathscr{D}, \mathscr{D}].
$$

$({}_1\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}])^{-1} \circ {}_1\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}]$ *is the subgroup of* ${}_1\boldsymbol{C}[\mathscr{D}, \mathscr{D}]$ *generated by* ${}_1\boldsymbol{E}_r[\mathscr{D}, \mathscr{D}]$.

This follows from (4.4), (4.6) and the fact that two homeomorphic one dimensional C^r manifolds are diffeomorphic. The latter half is obvious.

5. Diffeomorphism classes of $\mathbf{D}^{*+}$-equivalent C^∞-structures

Finally we make a remark on the diffeomorphism classes of two $\mathbf{D}_\infty^*$-equivalent C^∞-structures. $\mathbf{D}_\infty^*$-equivalence, or a homeomorphism which is a diffeomorphism except at a discrete subset A, is a weaker notion than Munkres's diffeomorphism mod 0-dimensional A in [5]. Here we interpret a part of Wilson's theory in [13]. We restrict ourselves to C^∞ case for the sake of simplicity and so we omit the right subscript ∞ of $\mathbf{D}_\infty^*$, $\mathbf{E}_\infty^{**}$ etc..

By the induced map of the n-th local homology group, an element of ${}_n\mathbf{C}$ is distinguished to be either orientation preserving or orientation reversing. If $\mathbf{G}$ is a subgroup of ${}_n\mathbf{C}$, its orientation preserving elements form a subgroup $\mathbf{G}^+$ of $\mathbf{G}$ of index one or two. In this section we make the following agreements on terms. If X and Y are orientable, we fix their orientations and all homeomorphisms of their open sets are assumed to be orientation preserving. A C^∞-structure on X means an $\mathbf{E}^{*+}$-structure whose admissible charts are orientation preserving.

Let $\mathscr{D}$ be a C^∞-structure on an n-dimensional manifold X with basis $\{(U_\lambda, \varphi_\lambda(x))\}_{\lambda\in\Lambda}$. Take an element $f\in\mathbf{D}^{*+}$ and its representative $f(t)$ which is a diffeomorphism on $R^n-\{0\}$(1). If $(V, \psi(x))$ is an admissible chart centered at $p\in X$, then

$$\{(U_\lambda-\{p\},\ \varphi_\lambda(x))\}_{\lambda\in\Lambda}\cup\{(V, f\circ\psi(x))\}$$

defines a new C^∞-structure $\mathscr{D}'$ on X. Let $\bar{\mathscr{D}}'$ denotes the diffeomorphism class of $\mathscr{D}'$. By the theorem of Palais and Cerf, the diffeomorphism class $\bar{\mathscr{D}}'$ of $\mathscr{D}'$ is determined by $\bar{\mathscr{D}}$ and f. Thus we write $\bar{\mathscr{D}}'=\tau_f\bar{\mathscr{D}}$. Let $\bar{\Sigma}(X)$ be the set of all diffeomorphism classes of C^∞-structures on X. It is easy to see the following:

Proposition 5.1. *$\mathbf{D}^{*+}$ is a transformation group on $\bar{\Sigma}(X)$ by the operation τ_f and it holds that $\tau_{f\circ g}=\tau_{g\circ f}$. The $\mathbf{D}^{*+}$-orbit of $\bar{\mathscr{D}}$ is the set of the classes of the C^∞-structures differing from $\mathscr{D}$ only on a single point (or a finite subset) of X.*

Let ${}_nA$ be the set of diffeomorphism classes of those C^∞-structures on topological n-sphere which, minus one point, is diffeomorphic to $\mathbf{R}^n$. Milnor [4] has shown this is an abelian group with respect to connected sum(2). The class $\bar{\mathscr{D}}_0$ of the standard n-sphere is the identity of ${}_nA$. We define a map $c\colon {}_n\mathbf{D}^{*+}\to{}_nA$ by $cf=\tau_f\bar{\mathscr{D}}_0$. It is easy to see that this is a group homomorphism. Let ${}_n\mathbf{K}$ be the group of the elements of ${}_n\mathbf{D}^{*+}$ with representative $f(t)$ which is a diffeomorphism of $\mathbf{R}^n$ except at 0 and has a compact support ($f(t)=t$ outside some compact subset of $\mathbf{R}^n$). For an element of ${}_n\mathbf{K}$ such a support can be chosen arbitrary small.

Lemma 5.2. *Let $(U, \varphi(x))$ be a chart on a standard n-sphere $(S^n, \mathscr{D}_0)$ centered at p. Then an element f of ${}_n\mathbf{R}^{*+}$ is contained in ${}_n\mathbf{K}$*

(1) Moreover, if $n\neq 4$, we can choose $f(t)$ such that $f(R^n)=R^n$ using the uniqueness of the diffeomorphism class of C^∞-structures on open cell [11].

(2) Let Γ_n be the group of C^∞-structures on the ordinary combinatorial n-sphere. If $n\neq 4$, ${}_nA={}_n\Gamma$ is finite and includes all the C^∞-structures on topological n-sphere. If $n=4$, ${}_4A\supset{}_4\Gamma=1$ [1].

if and only if the germ of the map $\varphi^{-1}\circ f\circ\varphi(x)$ at p has a representative which is a diffeomorphism of $(S^n, \mathscr{D}_0)$ except at p.

The proof is omitted. The following theorem is essentially due to Wilson.

Theorem 5.3. *We have the diagrams*

$$1 \longrightarrow {}_n\boldsymbol{K} \xrightarrow{i} {}_n\boldsymbol{D}^{*+} \xrightarrow{c} {}_nA \longrightarrow 1 \qquad (n=1, 2, 3,\dots)$$

of group homomorphisms which are exact, except the part ${}_4\boldsymbol{K} \xrightarrow{i} {}_4\boldsymbol{D}^{*+} \xrightarrow{c} {}_4A$ where only the triviality of $c\circ i$ is proved.

Proof. The surjectivity of c follows from the fact that the representatives of two elements of ${}_nA$ can be chosen to differ from each other only at one point. Triviality of $c\circ i$ is also straightforward.

Let $n\neq 4$ and $cf=1$. Let $(U, \varphi(x))$ be an admissible chart of the standard n-sphere $(S^n, \mathscr{D}_0)$. Take a C^∞ n-disc B in $U\cap\varphi^{-1}\circ f^{-1}\circ\varphi(U)$. By the affirmative answer [9] and [10] (cf. [3]) to the differentiable Schoenflies problem $(n\neq 4)$, $B'=\varphi^{-1}\circ f\circ\varphi(B)$ is a C^∞ n-disc on $(S^n, \mathscr{D}_0)$. Hence there is a diffeomorphism $g(x)$ of $(S^n, \mathscr{D}_0)$ such that $g(B)=B'$. Since $cf=1$, the diffeomorphism $(g^{-1}\circ\varphi^{-1}\circ f\circ\varphi|\partial B)(x)$ of ∂B can be extended to a diffeomorphism $h(x)$ of $(S^n, \mathscr{D}_0)$ (see [5] or [12]). Pasting $(h|S^n\text{-}\mathrm{Int}\,B)(x)$ and $(g^{-1}\circ\varphi^{-1}\circ f\circ\varphi|B)$ on ∂B we obtain a diffeomorphism of $(S^n, \mathscr{D}_0)$ except at p. Then its composition with $g(x)$ is a representative of the germ $\varphi^{-1}\circ f\circ\varphi$ defined on S^n. Thus $f\in{}_n\mathbf{K}$ by (5.2).

Corollary 5.4. *If $n\neq 4$, ${}_n\boldsymbol{K}$ includes the commutator subgroup of ${}_n\boldsymbol{D}^{*+}$.*

The $\tau_f\bar{\mathscr{D}}$ of a topological manifold X is just the connected sum of $\bar{\mathscr{D}}$ and the sphere cf. Thus we obtain the following:

Corollary 5.5. *If $cf=cg$ then $\tau_f=\tau_g$. Thus ${}_nA$ is a transformation group on $\bar{\Sigma}(X)$.*

Corollary 5.6. *Let $n\neq 4$ and ${}_nA={}_n\Gamma=0$ (e.g. $n=1, 2, 3, 5, 6, 12$).*

If two C^∞-structures $\mathscr{D}$ and $\mathscr{D}'$ differ from each other only on a discrete subset (i.e. ${}_n\boldsymbol{D}^{*+}$-equivalent), then $\bar{\mathscr{D}}=\bar{\mathscr{D}}'$.

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Added in Proof: The naive generalization of (3.1) does not hold i.e. ${}_n\mathbf{E}_r\circ{}_n\mathbf{E}_r^{-1}\not\subset{}_n\mathbf{E}_r^{-1}\circ{}_n\mathbf{E}_r$ $(n\geqq 2)$. (e.g. $(x+y, y^3)\circ(x, y^3)^{-1}\notin{}_2\mathbf{E}_1^{-1}\circ{}_2\mathbf{E}_1$.) Accordingly (2.4) and (2.5) have little meaning for $n\geqq 2$.