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The integral cohomology ring of F_4/T and E_6/T

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§0. Introduction and statement of the result.

Let $\boldsymbol{G}$ be a compact, connected, simple Lie group and $\boldsymbol{T}$ its maximal torus. As is well known [7], $\boldsymbol{G}/\boldsymbol{T}$ has no torsion and its Poincaré poylnomial is

$$P(\boldsymbol{G}/\boldsymbol{T}; t)=\prod_{i=1}^{l} \frac{1-t^{2m_i}}{1-t^2}$$

where $(2m_1-1, \ldots, 2m_l-1)$ indicates the degrees of the primitive elements of $H^*(\boldsymbol{G}; \boldsymbol{Q})$. Thus the additive structure of $H^*(\boldsymbol{G}/\boldsymbol{T}; \boldsymbol{Z})$ is known. Furthermore its ring structure is known for $\boldsymbol{G}=\boldsymbol{U}(n), \boldsymbol{Sp}(n)$, $\boldsymbol{G}_2$ [2], [7] and probably for $\boldsymbol{G}=\boldsymbol{SO}(n)$. The purpose of this paper is to determine the ring structure of $H^*(\boldsymbol{G}/\boldsymbol{T}; \boldsymbol{Z})$ for $\boldsymbol{G}=\boldsymbol{F}_4$ and $\boldsymbol{E}_6$, where $\boldsymbol{F}_4$ and $\boldsymbol{E}_6$ are simply connected, compact exceptional Lie groups of rank 4 and 6 respectively.

Throughout the paper $H^*(X)$ always denotes the integral cohomology ring of X and

$$\sigma_i(t_1, t_2, \ldots, t_n)$$

is the i-th elementary symmetric function on n variables $t_1, t_2, \ldots, t_n$. Then our main results are stated as follows.

Theorem A.

$$H^*(\boldsymbol{F}_4/\boldsymbol{T})=\boldsymbol{Z}[t_1, t_2, t_3, t_4, \gamma_1, \gamma_3, w]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_6, \rho_8, \rho_{12})$$

where $t_1, t_2, t_3, t_4, \gamma_1 \in H^2, \quad \gamma_3 \in H^6, \quad w \in H^8$ *and*

$$\rho_1 = c_1 - 2\gamma_1, \quad \rho_2 = c_2 - 2\gamma_1^2, \quad \rho_3 = c_3 - 2\gamma_3,$$

$$\rho_4 = c_4 - 2c_3\gamma_1 + 2\gamma_1^4 - 3w, \quad \rho_6 = -c_4\gamma_1^2 + \gamma_3^2,$$

$$\rho_8 = 3c_4\gamma_1^4 - \gamma_1^8 + 3w(w + c_3\gamma_1), \quad \rho_{12} = w^3$$

for $$c_i = \sigma_i(t_1, t_2, t_3, t_4).$$

Theorem B.

$$H^*(\boldsymbol{E}_6/\boldsymbol{T}) = \boldsymbol{Z}[t_1, t_2, \dots, t_6, \gamma_1, \gamma_3, \gamma_4]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12})$$

where $t_1, \dots, t_6, \gamma_1 \in H^2, \quad \gamma_3 \in H^6, \quad \gamma_4 \in H^8$ *and*

$$\rho_1 = c_1 - 3\gamma_1, \quad \rho_2 = c_2 - 4\gamma_1^2, \quad \rho_3 = c_3 - 2\gamma_3,$$

$$\rho_4 = c_4 + 2\gamma_1^4 - 3\gamma_4, \quad \rho_5 = c_5 - c_4\gamma_1 + c_3\gamma_1^2 - 2\gamma_1^5,$$

$$\rho_6 = 2c_6 - c_4\gamma_1^2 - \gamma_1^6 + \gamma_3^2,$$

$$\rho_8 = -9c_6\gamma_1^2 + 3c_5\gamma_1^3 - \gamma_1^8 + 3\gamma_4(\gamma_4 - c_3\gamma_1 + 2\gamma_1^4),$$

$$\rho_9 = -3w^2t + t^9, \quad \rho_{12} = w^3 + 15w^2t^4 - 9wt^8$$

for $$c_i = \sigma_i(t_1, t_2, \dots, t_6), \quad t = \gamma_1 - t_1$$

and $$w = \gamma_4 - c_3\gamma_1 + 2\gamma_1^4 + (\gamma_3 - 2\gamma_1^3 + \gamma_1^2 t - \gamma_1 t^2 + t^3)t.$$

Let p be the projection of $\boldsymbol{E}_6/\boldsymbol{T}$ to the irreducible symmetric space $\boldsymbol{EIII} = \boldsymbol{E}_6/\boldsymbol{D}_5 \cdot \boldsymbol{T}^1$. Then t and w in the above theorem generate the image of the injective homomorphism $p^*\colon H^*(\boldsymbol{EIII}) \to H^*(\boldsymbol{E}_6/\boldsymbol{T})$, and we have the following ring structure of $H^*(\boldsymbol{EIII})$ which is additively determined in [9].

Corollary C. $H^*(\boldsymbol{EIII}) = \boldsymbol{Z}[t, w]/(t^9 - 3w^2t, w^3 + 15w^2t^4 - 9wt^8)$.

The paper is organized as follows. In §1 we describe how we calculate $H^*(G/T)$ from the information of the invariants of the Weyl group $\Phi(G)$. §2 is used to determine $H^*(SO(n)/T)$ which is needed in §6. In §3 we discuss low dimensional cohomology of G/T. The Weyl groups of F_4 and E_6 are explained in §4 and the rational cohomology rings of $F_4/T, E_6/T$ and $EIII$ are determined in §5. The final section §6 completes the proof of our main results.

§1. Sketch of the argument.

Let G be a compact connected Lie group and let U be a connected subgroup of G which contains a maximal torus T of G. The behavior of the rational cohomology rings of spaces related these Lie groups are well known [2]. The rational cohomology ring of G is an exterior algebra of odd dimensional generators:

$$H^*(G; Q)=\Lambda(x_{2m_1-1},\dots, x_{2m_l-1}), \qquad x_{2m_i-1}\in H^{2m_i-1}.$$

By Borel's transgression theorem

(1.1) $$H^*(BG; Q)=Q[x_{2m_1},\dots, x_{2m_l}], \qquad x_{2m_i}\in H^{2m_i},$$

in particular, $(2m_1,\dots, 2m_l)=(4, 12, 16, 24)$ *for* $G=F_4$

$=(4, 10, 12, 16, 18, 24)$ *for* $G=E_6$.

The rational cohomology spectral sequence associated with the fibering

$$G/T \xrightarrow{\iota_G} BT \xrightarrow{\rho_G} BG$$

collapses. Furthermore the image of ρ_G^* coincides with the subalgebra of $H^*(BT; Q)$ which consists of the elements invariant under the action of the Weyl group $\Phi(G)=N(T)/T$ of G. Thus

(1.2) $$\rho_G^*\colon H^*(BG; Q)\cong H^*(BT; Q)^{\Phi(G)}$$

and $$\iota_G^*\colon H^*(BT; Q)/(\mathrm{Im}\,\rho_G^+)\cong H^*(G/T; Q),$$

where $(\mathrm{Im}\,\rho_G^+)$ *indicates the ideal generated by* $\mathrm{Im}\,\rho_G^+=\rho_G^*H^+(BG; Q)$ $=H^+(BT; Q)^{\Phi(G)}$.

Consider three fiberings

$$G/T \longrightarrow G/U \longrightarrow BU, \quad G/U \longrightarrow BU \xrightarrow{\rho} BG$$

and $$U/T \xrightarrow{i} G/T \xrightarrow{p} G/U.$$

In the first one, the rational cohomologies of G/T and BU vanish for the odd dimensions. Thus the spectral sequence collapses. Then the same holds for the second and the third fiberings, and we have

$$(1.3) \qquad H^*(G/U; Q) \cong H^*(BU; Q)/(\mathrm{Im}\,\rho^+)$$
$$\cong H^*(BT; Q)^{\Phi(U)}/(H^+(BT; Q)^{\Phi(G)})$$

and the homomorphism $p^*: H^*(G/U; Q) \to H^*(G/T; Q)$ *is injective and equivalent to that induced by the inclusion of* $H^*(BT; Q)^{\Phi(U)}$ *into* $H^*(BT; Q)$.

For the integral cohomology the most important result is the following ([7]):

(1.4) $H^*(G/T)$ *has no torsion and vanishing odd dimensional part.*

In the following we shall consider the cases that the following (1.5), (iii) holds.

(1.5) *The following conditions are equivalent.*

(i) *The integral cohomology spectral sequence associated with the fibering* $U/T \xrightarrow{i} G/T \xrightarrow{p} G/U$ *collapses.*

(ii) $i^*: H^*(G/T) \to H^*(U/T)$ *is surjective.*

(iii) $H^*(G/U)$ *has no torsion and vanishing odd dimensional part.*

(1.5), (i) implies

(1.6) $p^*: H^*(G/U) \to H^*(G/T)$ *is injective and* $\mathrm{Ker}\, i^* = (p^*H^+(G/U))$.

Describe the rings $H^*(U/T)$ and $H^*(G/U)$ by generators and relations:

$$H^*(\boldsymbol{U}/\boldsymbol{T})=\boldsymbol{Z}[\alpha_i']/(r_k) \quad \text{and} \quad H^*(\boldsymbol{G}/\boldsymbol{U})=\boldsymbol{Z}[\beta_j]/(s_l),$$

and denote by the same symbol β_j its image in $H^*(\boldsymbol{G}/\boldsymbol{T})$ under the injection p^* of (1.6). Since i^* is surjective there are elements α_i of $H^*(\boldsymbol{G}/\boldsymbol{T})$ such that

$$i^*(\alpha_i)=\alpha_i'.$$

Then from (1.5) the following lemma follows easily.

Lemma 1.1. *Let $\rho_k=\rho_k(\alpha_i, \beta_j)\in \boldsymbol{Z}[\alpha_i, \beta_j]$ be a polynomial such that it vanishes in $H^*(\boldsymbol{G}/\boldsymbol{T}; \boldsymbol{Q})$ and that $(i^*\rho_k=)\rho_k(\alpha_i', 0)\equiv r_k$ modulo the ideal of $\boldsymbol{Z}[\alpha_i']$ generated by r_j for $j<k$, then*

$$H^*(\boldsymbol{G}/\boldsymbol{T})=\boldsymbol{Z}[\alpha_i, \beta_j]/(\rho_k, s_l).$$

§2. $H^*(SO(m)/T)$.

Put $\boldsymbol{B}_n=\boldsymbol{SO}(2n+1)/\boldsymbol{SO}(2n-1)\times \boldsymbol{T}^1$. First we see

(2.1) $H^*(\boldsymbol{B}_n)=\boldsymbol{Z}[t, e]/(t^n-2e, e^2)$ *where* $t\in H^2$ *and* $e\in H^{2n}$.

The Stiefel manifold $\boldsymbol{V}=\boldsymbol{SO}(2n+1)/\boldsymbol{SO}(2n-1)$ is a $\boldsymbol{T}^1$-bundle over $\boldsymbol{B}_n$. Then $\boldsymbol{V}$ is equivalent to a fibre of a map $\boldsymbol{B}_n \xrightarrow{f} \boldsymbol{BT}^1$ classifying the $\boldsymbol{T}^1$-bundle. As is well known $H^*(\boldsymbol{BT}^1)=\boldsymbol{Z}[t]$, $t\in H^2$, and $H^q(\boldsymbol{V})\cong \boldsymbol{Z}$ $(q=0, 4n-1)$, $\cong \boldsymbol{Z}_2$ $(q=2n)$ and $H^q(\boldsymbol{V})=0$ $(q\neq 0, 2n, 4n-1)$. By dimensional reason, the spectral sequence associated with the fibering $\boldsymbol{V}\longrightarrow \boldsymbol{B}_n\longrightarrow \boldsymbol{BT}^1$ collapses for total degree$<4n-1$. Thus $H^i(\boldsymbol{B}_n)=0$ for odd i and we have the following exact sequences for $0\leq i\leq 2n-1$:

$$0\longrightarrow H^{2i}(\boldsymbol{BT}^1)\xrightarrow{f^*} H^{2i}(\boldsymbol{B}_n)\longrightarrow H^{2i-2n}(\boldsymbol{BT}^1)\otimes H^{2n}(\boldsymbol{V})\longrightarrow 0.$$

In particular $H^{2i}(\boldsymbol{B}_n)\cong \boldsymbol{Z}$ for $0\leq i\leq n-1$. Since $\boldsymbol{B}_n$ is an orientable manifold of dimension $4n-2$, $H^{2i}(\boldsymbol{B}_n)\cong H_{4n-2-2i}(\boldsymbol{B}_n)\cong \boldsymbol{Z}$ for $n\leq i\leq 2n-1$. This proves (2.1) for $t=f^*(t)$.

Take a maximal torus $\boldsymbol{T}^n$ of $\boldsymbol{SO}(2n+1)$ as usual: $\boldsymbol{T}^n\subset \boldsymbol{U}(n)\subset \boldsymbol{SO}(2n+1)$ and consider the following diagram

$$\begin{array}{ccccc} H^*(\boldsymbol{BSO}(2n+1)) & \longrightarrow & H^*(\boldsymbol{BT}^n) & \longrightarrow & H^*(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n) \\ \downarrow & \nearrow & & & \\ H^*(\boldsymbol{BU}(n)) & & & & \end{array}$$

where the homomorphisms are natural ones and $H^*(\boldsymbol{BT}^n)=\boldsymbol{Z}[t_1,\dots, t_n]$ for canonical generators t_i. $c_i=\sigma_i(t_1,\dots, t_n)$ is the image of the i-th Chern class $c_i\in H^{2i}(\boldsymbol{BU}(n))$. Consider the above diagram in mod 2 coefficient, then it follows from $c_i\equiv w_{2i}$ (mod 2) (*cf.* [4]) that the image of c_i in $H^*(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n)$ vanishes mod 2, that is, it is divisible by 2. Then it follows from (1.4)

(2.2) *Denoting by the same symbols* t_i, c_i *their images in* $H^*(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n)$ *we have the unique existence of elements* $e_{2i}\in H^{2i}(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n)$ *such that* $2e_{2i}=c_i$.

Theorem 2.1. $H^*(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n)=\boldsymbol{Z}[t_i, e_{2i}]/(c_i-2e_{2i}, e_{4i}+\sum_{0<j<2i}(-1)^j e_{2j}e_{4i-2j})$ *where* $i=1, 2,\dots, n$, $t_i\in H^2$ *and* $e_{2i}\in H^{2i}$; $e_{2k}=0$ *for* $k>n$.

Proof. We prove the theorem by induction on n. Clearly it holds for $n=1$. Let $n>1$ and consider the argument in § 1 for $\boldsymbol{U}=\boldsymbol{SO}(2n-1)\times \boldsymbol{T}^1$ and $\boldsymbol{G}=\boldsymbol{SO}(2n+1)$. By (2.1), $\boldsymbol{G}/\boldsymbol{U}=\boldsymbol{B}_n$ satisfies (1.5), (iii), and Lemma 1.1 can be applied. For $1\leq i\leq n-1$, $i^*(t_i)=t_i$, $i^*(c_i)=c_i$ $(i^*(t_n)=0)$, and it follows from (1.4) and (2.2) that $i^*(e_{2i})=e_{2i}$. We may choose t such that $p^*(t)=t_n$. Rational invariant forms for $\boldsymbol{G}=\boldsymbol{SO}(2n+1)$ are given by the Pontrjagin classes:

$$I_{2i}=(-1)^i p_i=\sum_{j=0}^{2i}(-1)^j c_j c_{2i-j}=4(e_{4i}+\sum_{0<j<2i}(-1)^j e_{2j}e_{4i-2j}).$$

Thus the relation $I_{2i}/4=0$ holds in $H^*(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n)$. Remark that $i^*(c_n)=i^*(e_{2n})=0$. Then it follows from Lemma 1.1

$$H^*(\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n)=\boldsymbol{Z}[t_i, e_{2i}, t_n, \bar{e}]/(c_i-2e_{2i}, I_{2i}/4, t_n^n-2\bar{e}, \bar{e}^2)$$

where $1\leq i\leq n-1$ and $\bar{e}=p^*(e)$. Put $c'_i=\sigma_i(t_1,\dots, t_{n-1})$, then $c_i=c'_i+c'_{i-1}t_n$ and $0=c'_n=\sum_{0\leq i\leq n}(-1)^i c_{n-i}t_n^i$. It follows, by putting

$$e_{2n}=e_{2n-2}t_n-\cdots+(-1)^n e_2 t_n^{n-1}+(-1)^{n+1}\tilde{e},$$

that $c_n-2e_{2n}\equiv(-1)^{n+1}(t_n^n-2\tilde{e}) \mod (c_i-2e_{2i},\ 1\leq i\leq n-1)$

and $I_{2n}/4=e_{2n}^2\equiv\tilde{e}^2 \mod (I_{2i}/4,\ 1\leq i\leq n-1,\ t_n^n-2\tilde{e})$.

Then we have the assertion of the theorem. Q.E.D.

Corollary 2.2. *$H^*(\boldsymbol{SO}(2n)/\boldsymbol{T}^n)=\boldsymbol{Z}[t_i, t_n, e_{2i}]/(c_i-2e_{2i}, c_n, e_{4i}+\sum_{0<j<2i}(-1)^j e_{2j}e_{4i-2j})$ where $1\leq i\leq n-1$, and $t_i, t_n\in H^2$, $e_{2i}\in H^{2i}$; $e_{2k}=0$ for $k\geq n$.*

Proof. Since $H^*(\boldsymbol{SO}(2n)/\boldsymbol{T}^n)$ has vanishing odd part, the spectral sequence associated with the fibering $\boldsymbol{SO}(2n)/\boldsymbol{T}^n\longrightarrow\boldsymbol{SO}(2n+1)/\boldsymbol{T}^n\xrightarrow{p}S^{2n}$ collapses. Since c_n is invariant under $\Phi(\boldsymbol{SO}(2n))$, c_n and also $e_{2n}=c_n/2$ vanish in $H^*(\boldsymbol{SO}(2n)/\boldsymbol{T}^n)$. Then e_{2n} generates $p^*H^+(S^{2n})$, and the corollary follows from Theorem 2.1 and (1.6).

§3. Considerations in low dimension.

Throughout this §, $\boldsymbol{G}$ stands for $\boldsymbol{F}_4$ and $\boldsymbol{E}_6$. The mod p cohomology rings of $\boldsymbol{F}_4$ and $\boldsymbol{E}_6$ are known [3], [1], [5], and they have the same structure for $\dim\leq 8$:

$$H^*(\boldsymbol{G};\boldsymbol{Z}_2)=\{1,\ x_3,\ Sq^2x_3,\ Sq^3x_3,\ x_3Sq^2x_3,\ x_3Sq^3x_3,\ (Sq^4Sq^2x_3),\dots\},$$

$$H^*(\boldsymbol{G};\boldsymbol{Z}_3)=\{1,\ x_3,\ \mathscr{P}^1x_3,\ \beta\mathscr{P}^1x_3,\ (x_9),\dots\}$$

and $$H^*(\boldsymbol{G};\boldsymbol{Z}_p)=\{1,\ x_3,\ (x_9),\dots\} \quad \text{for } p\geq 5,$$

where $x_3\in H^3$ and for $\boldsymbol{G}=\boldsymbol{E}_6$ $x_9, Sq^4Sq^2x_3\in H^9$. It follows

(3.1) *$H^3(\boldsymbol{G})\cong\boldsymbol{Z}$ generated by x_3, $H^6(\boldsymbol{G})\cong\boldsymbol{Z}_2$ generated by $x_6\equiv Sq^3x_3$* (mod 2), *$H^8(\boldsymbol{G})\cong\boldsymbol{Z}_3$ generated by $x_8\equiv\beta\mathscr{P}^1x_3$* (mod 3) *and $H^i(\boldsymbol{G})=0$ for $i=1, 2, 4, 5, 7$.*

Let $t_1,\dots,t_l$ be an additive base of $H^2(\boldsymbol{BT})$, then $H^*(\boldsymbol{BT})=\boldsymbol{Z}[t_1,\dots,t_l]$. We use the same symbols $t_1,\dots,t_l\in H^2(\boldsymbol{G}/\boldsymbol{T})$ for their images under the homomorphism $\iota^*: H^*(\boldsymbol{BT})\to H^*(\boldsymbol{G}/\boldsymbol{T})$ induced by the natural inclusion $\iota: \boldsymbol{G}/\boldsymbol{T} \rightarrow \boldsymbol{BT}$. We have a fibering

$$\boldsymbol{G} \longrightarrow \boldsymbol{G}/\boldsymbol{T} \xrightarrow{\iota} \boldsymbol{BT}.$$

Let $u \in H^4(\boldsymbol{BT})$ be the transgression image of $x_3 \in H^3(\boldsymbol{G})$. Obviously $\iota^*(u)=0$. Then we have the following

Lemma 3.1 *Let $\boldsymbol{G}=\boldsymbol{F}_4$ or $\boldsymbol{E}_6$. There exist elements $\gamma_3 \in H^6(\boldsymbol{G}/\boldsymbol{T})$ and $\gamma_4 \in H^8(\boldsymbol{G}/\boldsymbol{T})$ such that $2\gamma_3=\iota^*(y_6)$, $y_6 \equiv Sq^2 u \pmod 2$ and $3\gamma_4=\iota^*(y_8)$, $y_8 \equiv \mathscr{P}^1 u \pmod 3$ for some $y_6 \in H^6(\boldsymbol{BT})$ and $y_8 \in H^8(\boldsymbol{BT})$. For such elements the natural homomorphism*

$$\boldsymbol{Z}[t_1,\dots,t_l,\gamma_3,\gamma_4]/(u,\ y_6-2\gamma_3,\ y_8-3\gamma_4) \longrightarrow H^*(\boldsymbol{G}/\boldsymbol{T})$$

is an isomorphism onto for $\dim \leq 8$.

Proof. First remark that $u \neq 0$ since it is true in the rational coefficient. Consider the integral cohomology spectral sequence $(E_r^{p,q})$ associated with the above fibering, then $E_2^{p,q}=H^p(\boldsymbol{BT})\otimes H^q(\boldsymbol{G})$ and $d_4(1\otimes x_3)=u\otimes 1$. Since $E_2^{p,q}=0$ for odd p, the possible cases of non-trivial differential $d_r: E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ for $q \leq 8$ are d_4 for $q=3$, 6 and d_6 for $q=8$. d_4 for $q=3$ is equivalent to the multiplication of u in $H^*(\boldsymbol{BT})$, hence it is injecitve. It follows $d_4=0$ for $q=6$, $E_r^{p,3}=0$ for $r>4$, hence $d_6=0$ for $q=8$. Thus, in total degree $p+q \leq 8$, the non-trivial E_∞ terms are $E_\infty^{*,0} \cong H^*(\boldsymbol{BT})/(u)$, $E_\infty^{p,6} \cong H^6(\boldsymbol{G}) \cong \boldsymbol{Z}_2$ $(p=0, 2)$ and $E_\infty^{0,8} \cong \boldsymbol{Z}_3$.

Let γ_3 and γ_4 be representatives of the permanent cycles $1\otimes x_6$ and $1\otimes x_8$ respectively. Since $\operatorname{Im}\iota^*=E_\infty^{*,0}$, $E_\infty^{0,6}=H^6(\boldsymbol{G}/\boldsymbol{T})/\operatorname{Im}\iota^*$ and $E_\infty^{0,8}+E_\infty^{2,6} \cong H^8(\boldsymbol{G}/\boldsymbol{T})/\operatorname{Im}\iota^*$, the existence of y_6 and y_8 satisfying the required relation and the last assertion of the lemma follow.

Next consider the spectral sequence in $\boldsymbol{Z}_2$-coefficient. By the naturality of Sq^2, $d_6(1\otimes Sq^2 x_3)=Sq^2 u\otimes 1$ which should be non-zero since $H^5(\boldsymbol{G}/\boldsymbol{T})=0$. Thus $Sq^2 u$ generates the kernel of $H^6(\boldsymbol{BT};\boldsymbol{Z}_2)/(u) \cong E_6^{6,0} \rightarrow H^6(\boldsymbol{G}/\boldsymbol{T};\boldsymbol{Z}_2)$. From the last assertion of the lemma the kernel is also generated by $y_6 \pmod 2$. Thus $y_6 \equiv Sq^2 u \pmod 3$. Similarly, $y_8 \equiv \mathscr{P}^1 u \pmod 3$. By (1.4), we see that the choice of the elements has no influence to the last assertion. Q.E.D.

§4. The Weyl group of F_4 and E_6.

(A) The case F_4.

Let T^4 be the maximal torus of $SO(9)$ defined as in §2. Then we have

$$H^*(BT^4)=Z[t_1, t_2, t_3, t_4],$$

where $t_i \in H^2$. Let $\mu: Spin(9) \to SO(9)$ be the universal covering. Then $T=\mu^{-1}(T^4)$ is a maximal torus of $Spin(9)$ and

$$H^*(BT)=Z[t_1, t_2, t_3, t_4, \gamma_1]/(c_1-2\gamma_1) \tag{4.1}$$

where $t_i, \gamma_1 \in H^2$ *and* $c_1=\sigma_1(t_1, t_2, t_3, t_4)=t_1+t_2+t_3+t_4$.

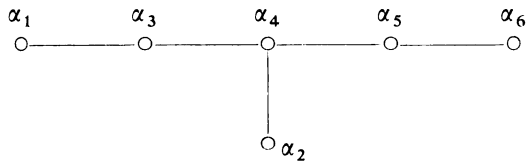
According to [6], we choose $Spin(9)$ as a subgroup of F_4 such that $F_4/Spin(9)$ may be identified with the Cayley plane Π. The torus T is maximal in F_4, and the Weyl group $\Phi(F_4)$ acts on $H^*(BT; Q)=Q[t_1, t_2, t_3, t_4]$ as follows (see [6, §19]). Let R be the reflection to the plane $t_1+t_2+t_3+t_4=0$:

$$R(t_i)=t_i-c_1/2 \qquad (i=1,\dots, 4). \tag{4.2}$$

Then $\Phi(F_4)$ is generated by R and the Weyl group $\Phi(Spin(9))$ of $Spin(9)$, where $\Phi(Spin(9))$ is the group of permutations of t_i's together with the changements of signs of t_i.

(B) The case E_6.

Let T be a maximal torus of E_6. We begin with choosing generators of $H^2(E_6/T)$. According to Bourbaki [8], the Schläfli diagram of E_6 is



where α_i's are the simple roots of E_6. The corresponding fundamental weights ϖ_i are given in page 261 of [8]. These elements form a basis of $H^2(\boldsymbol{E}_6/\boldsymbol{T})$ as explained in [6, § 14]. Let R_i denote the reflection to the hyperplane $\alpha_i=0$. Then

$$(4.3)\qquad R_i(\varpi_i)=\varpi_i-\sum_j<\alpha_i, \alpha_j>\varpi_j \quad \textit{and} \quad R_i(\varpi_j)=\varpi_j \qquad \textit{for } i\neq j.$$

Now we put

$$t_6=\varpi_6,$$

$$t_5=R_6(t_6)=\varpi_5-\varpi_6,$$

$$t_4=R_5(t_5)=\varpi_4-\varpi_5,$$

$$t_3=R_4(t_4)=\varpi_2+\varpi_3-\varpi_4,$$

$$t_2=R_3(t_3)=\varpi_1+\varpi_2-\varpi_3,$$

$$t_1=R_1(t_2)=-\varpi_1+\varpi_2$$

and $$x=\varpi_2=\frac{1}{3}c_1 \qquad \text{for} \quad c_1=t_1+t_2\cdots+t_6.$$

Then x and t_i, $1\leq i\leq 6$, span $H^2(\boldsymbol{E}_6/\boldsymbol{T})$ since ϖ_i are integral linear combinations of x and t_i's. $H^2(\boldsymbol{BT})$ is identified with $H^2(\boldsymbol{E}_6/\boldsymbol{T})$ since $\boldsymbol{E}_6$ is simply connected. Thus

$$(4.4)\qquad H^*(\boldsymbol{BT})=\boldsymbol{Z}[x, t_1,\dots, t_6]/(3x-c_1).$$

Denote by $\boldsymbol{U}$ the centralizer of the one dimensional torus $\boldsymbol{T}^1$ which is defined by

$$\alpha_i(t)=0 \qquad (2\leq i\leq 6,\ t\in \boldsymbol{T}).$$

Then $\boldsymbol{U}$ is a closed connected subgroup of maximal rank and of local type $\boldsymbol{D}_5\times\boldsymbol{T}^1$ with $\boldsymbol{D}_5\cap\boldsymbol{T}^1=\boldsymbol{Z}_4$. (See [10] for details.) The quotient manifold

$$\boldsymbol{EIII}=\boldsymbol{E}_6/\boldsymbol{U}$$

is the compact irreducible hermitian symmetric space of dimension 32.

The Weyl groups $\Phi(\boldsymbol{E}_6)$ and $\Phi(\boldsymbol{U})$ are generated by $R_1, R_2,\dots, R_6$ and $R_2,\dots, R_6$ respectively. From (4.3) we have the following table of the action of R_i's for the generators x and t_i.

(4.5)

	R_1	R_2	R_3	R_4	R_5	R_6
t_1	t_2	$x-t_2-t_3$				
t_2	t_1	$x-t_1-t_3$	t_3			
t_3		$x-t_1-t_2$	t_2	t_4		
t_4				t_3	t_5	
t_5					t_4	t_6
t_6						t_5
x		$-x+t_4+t_5+t_6$				

where the blanks indicate the trivial action.

Putting

(4.6) $t=x-t_1=\varpi_1$ and $t'_i=t_{i+1}-\frac{1}{2}t$ for $i=1,\dots,5$,

we have

$$H^*(BT;\boldsymbol{Q})=\boldsymbol{Q}[t_1,\dots,t_6]=\boldsymbol{Q}[t,t'_1,\dots,t'_5]$$

and the following table:

(4.7)

	R_2	R_3	R_4	R_5	R_6
t					
t'_1	$-t'_2$	t'_2			
t'_2	$-t'_1$	t'_1	t'_3		
t'_3			t'_2	t'_4	
t'_4				t'_3	t'_5
t'_5					t'_4

We shall consider the relation between the elements $t_i \in H^2(\boldsymbol{SO}(10)/\boldsymbol{T}^5)$ in §3 and the elements just defined. For the subgroup $\boldsymbol{T}^1 \subset \boldsymbol{T}$ of $\boldsymbol{U}=\boldsymbol{D}_5 \cdot \boldsymbol{T}^1$, put $\boldsymbol{T}_0=\boldsymbol{T}/\boldsymbol{T}^1$. Since $\boldsymbol{D}_5 \cap \boldsymbol{T}^1=\boldsymbol{Z}_4$, $\boldsymbol{U}/\boldsymbol{T}^1=\boldsymbol{SO}(10)/\boldsymbol{Z}_2$ and it contains $\boldsymbol{T}_0$ as a maximal torus. The inverse image $\boldsymbol{T}^5$ of $\boldsymbol{T}_0$ under the double covering $\mu: \boldsymbol{SO}(10) \to \boldsymbol{SO}(10)/\boldsymbol{Z}_2$ is a maximal torus of $\boldsymbol{SO}(10)$. Since every maximal tori are conjugate to each other, changing $\boldsymbol{SO}(10)$ by an inner automorphism, we may regard that the torus $\boldsymbol{T}^5$ is the canonical one. We have the following commutative diagram of natural maps

$$\text{(4.8)} \qquad \begin{array}{ccccccccc} & & \boldsymbol{U}/\boldsymbol{T} & \cong & (\boldsymbol{SO}(10)/\boldsymbol{Z}_2)/\boldsymbol{T}_0 & \cong & \boldsymbol{SO}(10)/\boldsymbol{T}^5 & & \\ & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} & & \\ \boldsymbol{BT}^1 & \xrightarrow{\lambda} & \boldsymbol{BT} & \xrightarrow{\pi} & \boldsymbol{BT}_0 & \xleftarrow{\mu} & \boldsymbol{BT}^5 & \xleftarrow{\nu} & \boldsymbol{BZ}_2 . \end{array}$$

The Weyl groups $\boldsymbol{\Phi}(\boldsymbol{U})$, $\boldsymbol{\Phi}(\boldsymbol{SO}(10)/\boldsymbol{Z}_2)$ and $\boldsymbol{\Phi}(\boldsymbol{SO}(10))$ are isomorphic and the action is compatible with π and μ, and also compatible with λ and ν for the trivial action on $\boldsymbol{BT}^1$ and $\boldsymbol{BZ}_2$. The action of $\boldsymbol{\Phi}(\boldsymbol{SO}(10))$ on $H^*(\boldsymbol{BT}^5)=\boldsymbol{Z}[t_1,\dots, t_5]$ is as usual, that is, same as (4.7) replacing t_i' by t_i. Since the sequences of both sides of $\boldsymbol{BT}_0$ in (4.8) are fiberings, we have exact sequences

$$0 \longrightarrow H^2(\boldsymbol{BT}_0) \xrightarrow{\pi^*} H^2(\boldsymbol{BT}) \xrightarrow{\lambda^*} H^2(\boldsymbol{BT}^1) \longrightarrow 0$$

and $$0 \longrightarrow H^2(\boldsymbol{BT}_0) \xrightarrow{\mu^*} H^2(\boldsymbol{BT}^5) \xrightarrow{\nu^*} \boldsymbol{Z}_2 \longrightarrow 0 .$$

Since λ^* is compatible with the action of $\boldsymbol{\Phi}(\boldsymbol{U})$, $\lambda^*(t_i)=\lambda^* R_{i+1}(t_{i+1})=\lambda^*(t_{i+1})$ for $i=2, 3, 4, 5$, $\lambda^*(t_2)=\lambda^*(2t_2+2t_3-t_4-t_5-t_6+(R_2x-x))=\lambda^*(-2R_2(t_1))=-2\lambda^*(t_1)$ and $\lambda^*(x)=\lambda^*(R_2t_1+t_2+t_3)=-3\lambda^*(t_1)$. It follows that $H^2(\boldsymbol{BT}^1) \cong \boldsymbol{Z}$ is generated by $\lambda^*(t_1)$ and the kernel of λ^* is generated by

$$t_{i+1}'-t_i'=t_{i+2}-t_{i+1} \quad \text{for} \quad i=1, 2, 3, 4 \quad \text{and} \quad t_2'+t_1'=t_1+t_2+t_3-x .$$

So, as a subgroup of $H^2(\boldsymbol{BT})$ we have

$$\text{(4.9)} \qquad H^2(\boldsymbol{BT}_0)=\{\sum_{i=1}^5 a_i t_i' \mid a_i \in \boldsymbol{Z},\ a_i \equiv 0 \pmod 2\} .$$

Up to constant multiple, t'_1 is characterized by the property: $R_3R_2(t'_1)=-t'_1$ and $R_4(t'_1)=R_5(t'_1)=R_6(t'_1)=t'_1$. Same is true for t_1 with respect to $\boldsymbol{\Phi}(\boldsymbol{SO}(10))$. Since μ^* is compatible with the action, $\mu^*(2t'_1)=c\cdot t_1$ for some $c\in\boldsymbol{Z}$ and $\mu^*(2t'_i)=c\cdot t_i$ for $i=2, 3, 4, 5$, by applying R_{i+1}. So, $\mu^*(\sum a_it'_i)=\frac{c}{2}(\sum a_it_i)$. Since μ^* is an injection of the index 2, it follows $c=\pm 2$. Changing t_i *to* $-t_i$ if $c=-2$, we have

(4.10) $$\mu^*(\textstyle\sum a_it'_i)=\sum a_it_i \qquad (\sum a_i\equiv 0 \pmod 2)).$$

(4.8) induces the following commutative diagram:

$$\begin{array}{ccccc} H^*(\boldsymbol{BT}) & \xleftarrow{\pi^*} & H^*(\boldsymbol{BT}_0) & \xrightarrow{\mu^*} & H^*(\boldsymbol{BT}^5) \\ \downarrow{\scriptstyle \iota_0^*} & & \downarrow{\scriptstyle i_0^*} & & \downarrow{\scriptstyle i^*} \\ H^*(\boldsymbol{U}/\boldsymbol{T}) & \cong & H^*((\boldsymbol{SO}(10)/\boldsymbol{Z}_2)/\boldsymbol{T}_0) & \cong & H^*(\boldsymbol{SO}(10)/\boldsymbol{T}^5). \end{array}$$

For the right vertical map i^* we use the convention $i^*(t_i)=t_i$ and $i^*(c_i)=c_i=\sigma_i(t_1,\dots,t_5)$. Since $t=x-t_1\in H^2(\boldsymbol{BT})$ is $\boldsymbol{\Phi}(\boldsymbol{U})$-invariant, $\iota_0^*(t)=0$ and $\iota_0^*(x)=\iota_0^*(t_1)$ by (1.2), (1.4). Compute in rational coefficient: $\iota_0^*(t_{i+1})=\iota_0^*(t_{i+1}-t/2)=i_0^*(t'_i)=i^*\mu^*(t'_i)=i^*(t_i)=t_i$, and $\iota_0^*(t_1)=\iota_0^*((t_2+\cdots+t_6)/2-3t/2)=(t_1+\cdots+t_5)/2=e_2$. Thus $\sigma_i(t_1,\dots,t_6)=\sigma_i(t_2,\dots,t_6)+\sigma_{i-1}(t_2,\dots,t_6)t_1$ is mapped by i^* to $c_i+c_{i-1}e_2$. Consequently we have the following

(4.11) *The natural homomorphism* ι_0^*: $H^*(\boldsymbol{BT})=Z[t_1,\dots,t_6,x]/(c_1-3x)$ $\to H^*(\boldsymbol{U}/\boldsymbol{T})\cong H^*(\boldsymbol{SO}(10)/\boldsymbol{T}^5)$ *satisfies*

$$\iota_0^*(t)=0,\ \iota_0^*(x)=\iota_0^*(t_1)=e_2,\ \iota_0^*(t_{i+1})=t_i \qquad \text{for } i=1, 2, 3, 4, 5$$

and $$\iota_0^*(\sigma_i(t_1,\dots,t_6))=c_i+c_{i-1}e_2 \qquad \text{for } i=1, 2,\dots, 6\ (c_6=0),$$

where the elements in the right hand sides of the equalities are those in Corollary 2.2.

§5. Rational cohomology ring of F_4/T, E_6/T and *EIII*.

(A) $H^*(F_4/T; Q)$.

Choose generators $t_i\in H^*(\boldsymbol{BT}; \boldsymbol{Q})=\boldsymbol{Q}[t_1,\dots,t_4]$ as in §4, (*A*) and put

$$p_i=\sigma_i(t_1^2, t_2^2, t_3^2, t_4^2)\in H^{4i}(\boldsymbol{BT}; \boldsymbol{Q})$$

and $$s_i=t_1^i+t_2^i+t_3^i+t_4^i\in H^{2i}(\boldsymbol{BT}; \boldsymbol{Q}).$$

s_{2n}'s are written as polynomials of p_i's by use of Newton's formula:

(5.1) $$s_{2n}=\sum_{1\le i<n}(-1)^{i-1}p_is_{2n-2i}+(-1)^{n-1}np_n \qquad (p_n=0 \text{ for } n>4).$$

Consider a set

$$\{\pm t_i\pm t_j;\ 1\le i<j\le 4\}$$

of elements of $H^2(\boldsymbol{BT}; \boldsymbol{Q})$, which is obviously invariant under the action of $\boldsymbol{\Phi}(\boldsymbol{Spin}(9))$ and also under that of R by (4.2). Thus it is invariant under $\boldsymbol{\Phi}(\boldsymbol{F}_4)$ and so is

$$I_n=\sum_{i<j}((t_i+t_j)^n+(t_i-t_j)^n+(-t_i+t_j)^n+(-t_i-t_j)^n).$$

Since

$$\sum_n I_n/n!=\sum_{i<j}(e^{t_i+t_j}+e^{t_i-t_j}+e^{-t_i+t_j}+e^{-t_i-t_j})$$

$$=\frac{1}{2}[(\sum_i e^{t_i})^2+(\sum_i e^{-t_i})^2-\sum_i(e^{2t_i}+e^{-2t_i})]+\sum_i e^{t_i}\cdot\sum_i e^{-t_i}-4,$$

we have easily the following

(5.2) $$I_n\in H^{2n}(\boldsymbol{BT}; \boldsymbol{Q})^{\boldsymbol{\Phi}(\boldsymbol{F}_4)},\ I_0=24,\ I_n=0 \qquad \textit{for odd } n$$

and $$I_{2n}=(16-2^{2n})s_{2n}+2\sum_{0<i<n}\binom{2n}{2i}s_{2i}s_{2n-2i} \qquad \textit{for } n>0.$$

Lemma 5.1. $H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\boldsymbol{\Phi}(\boldsymbol{F}_4)}=\boldsymbol{Q}[I_2, I_6, I_8, I_{12}]$

and $$H^*(\boldsymbol{F}_4/\boldsymbol{T}; \boldsymbol{Q})=\boldsymbol{Q}[t_1, t_2, t_3, t_4]/(p_1, p_3, 12p_4+p_2^2, p_2^3).$$

Proof. Applying (5.1) to (5.2) we have the following relations. At first

$$s_2=p_1 \quad \text{and} \quad I_2=(16-4)s_2=12p_1.$$

Next considering in modulo I_2, we have

$$s_2 \equiv 0, \quad s_4 \equiv -2p_2, \quad s_6 \equiv 3p_3$$

and $$I_6 \equiv (16-64)s_6 \equiv -144p_3 \qquad \bmod (I_2).$$

Similarly

$$s_6 \equiv 0, \quad s_8 \equiv 2p_2^2 - 4p_4$$

and $$I_8 \equiv -240s_8 + 2\binom{8}{4}s_2^4 \equiv 80(12p_4 + p_2^2) \quad \bmod (I_2, I_6).$$

Finally we have

$$s_8 \equiv \frac{7}{3}p_2^2, \quad s_{12} \equiv -\frac{5}{2}p_2^3$$

and $$I_{12} \equiv -4080s_{12} + 4\binom{12}{4}s_8 s_4 \equiv 960p_2^3 \qquad \bmod (I_2, I_6, I_8).$$

These show that I_2, I_6, I_8 and I_{12} are indecomposable. Since $H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\Phi(F_4)}$ is isomorphic to $H^*(\boldsymbol{BF}_4; \boldsymbol{Q}) = \boldsymbol{Q}[x_4, x_{12}, x_{16}, x_{24}]$, $x_i \in H^i$, we conclude that $H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\Phi(F_4)} = \boldsymbol{Q}[I_2, I_6, I_8, I_{12}]$ and $H^*(\boldsymbol{F}_4/\boldsymbol{T}; \boldsymbol{Q})$ is isomorphic to the quotient of $\boldsymbol{Q}[t_1, \ldots, t_4]$ by the ideal $(I_2, I_6, I_8, I_{12}) = (p_1, p_3, 12p_4 + p_2^2, p_2^3)$. Q.E.D.

(B) $H^*(E_6/T; Q)$ and $H^*(EIII; Q)$.

(4.7) shows that the action of $\boldsymbol{\Phi}(\boldsymbol{U})$ on $t'_1, \ldots, t'_5$ is same as the usual action of $\boldsymbol{\Phi}(\boldsymbol{SO}(10))$. Thus

(5.3) $$H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\Phi(U)} = \boldsymbol{Q}[t, q_1, q_2, d'_5, q_3, q_4]$$

where

$$d'_i = \sigma_i(t'_1, \ldots, t'_5) \in H^{2i} \text{ and } q_i = \sigma_i(t'^2_1, \ldots, t'^2_5) \in H^{4i}.$$

Since $\sum(-1)^i q_i = \prod(1 - t'^2_j) = \prod(1 - t'_j)(1 + t'_j) = \sum(-1)^i d'_i \sum d'_i$,

(5.4) $$q_i = \sum_{j+k=i}(-1)^{i+j} d'_j d'_k,$$

Next put

$$x_i = 2t_i - x \qquad \text{for} \quad i = 1, 2, \ldots, 6,$$

then it follows from the table (4.5) that the set

$$S=\{x_i+x_j,\ x-x_i,\ -x-x_i;\ i<j\}$$

is invariant under the action of $\Phi(E_6)$. Thus we have invariant forms

$$I_n=\sum_{y\in S} y^n \in H^{2n}(BT; Q)^{\Phi(E_6)}.$$

Consider the following elements ($J_i \in H^{2i}(BT; Q)$):

$$J_2=c_2-4x^2,$$

$$J_5=c_5-c_4x+c_3x^2-2x^5,$$

$$J_6=8c_6+c_3^2-4c_4x^2-4x^6,$$

$$J_8=-27c_6x^2+c_4^2-3c_4c_3x+19c_4x^4-15c_3x^5+31x^8,$$

$$J_9=-3w^2t+t^9,$$

and $$J_{12}=w^3+15w^2t^4-9wt^8,$$

where $$c_i=\sigma_i(t_1, t_2,\dots, t_6),$$

$$t=x-t_1$$

and $$w=\frac{1}{6}q_2+\frac{9}{16}t^4.$$

Then we have the following

Lemma 5.2. (i) $H^*(BT; Q)^{\Phi(E_6)}=Q[I_2, I_5, I_6, I_8, I_9, I_{12}]$

and $$H^*(E_6/T; Q)=Q[t_1,\dots, t_6]/(J_2, J_5, J_6, J_8, J_9, J_{12}).$$

(ii) *Identifying* $H^*(EIII; Q)$ *with the image of the injection* $p^*: H^*(EIII; Q)\to H^*(E_6/T; Q)$ *we have*

$$H^*(EIII; Q)=Q[t, w]/(J_9, J_{12}).$$

Proof. Put

$$c'_i=\sigma_i(t_2,\dots,t_6) \quad \text{and} \quad R=\boldsymbol{Q}[t_1, c'_1,\dots, c'_5].$$

R is a subalgebra of $H^*(\boldsymbol{BT};\boldsymbol{Q})$ containing c_i, d'_i, q_i, $x=c_1/3$, $t=x-t_1$, and $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(E_6)}$, $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(U)}$. Denote by

$$\mathfrak{a}_i\subset R \ (\text{resp. } \mathfrak{b}_i\subset H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(U)})$$

the ideal of R (resp. of $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(U)}$) generated by I_j's for $j<i$, $j\in\{2, 5, 6, 8, 9, 12\}$.

We assume the following sublemmas (5.5), (5.6), (5.7) which will be proved in the last half of this section.

(5.5) $$I_2=-2^4 3J_2, \qquad I_5\equiv -2^7 3\cdot 5J_5 \mod \mathfrak{a}_5,$$

$$I_6\equiv 2^7 3^2 J_6 \mod \mathfrak{a}_6 \quad \textit{and} \quad I_8\equiv 2^{12}5J_8 \mod \mathfrak{a}_8.$$

In $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(U)}=\boldsymbol{Q}[t, q_1, q_2, d'_5, q_3, q_4]$ we have

(5.6) $$I_2=6(4q_1+3t^2), \qquad I_5=-2^7 3\cdot 5d'_5+\textit{decomposable},$$

$$I_6=2^7 3^2 q_3+\textit{decomposable}, \qquad I_8=-2^{10}3\cdot 5q_4+\textit{decomposable}.$$

(5.7) $$I_9\equiv 2^{11}3^3 7J_9 \mod \mathfrak{b}_9 \quad \textit{and} \quad I_{12}\equiv -2^{15}3^4 5J_{12} \mod \mathfrak{b}_{12}.$$

By (5.6) and (5.7) we see that, for $i=2, 5, 6, 8, 9, 12$, I_i is not a polynomial of I_j's for $j<i$. Since $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(E_6)}\cong H^*(\boldsymbol{BE}_6;\boldsymbol{Q})=\boldsymbol{Q}[x_4, x_{10}, x_{12}, x_{16}, x_{18}, x_{24}]$, $x_i\in H^i$, it follows that $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(E_6)}=\boldsymbol{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}]$ and $H^*(\boldsymbol{E}_6/\boldsymbol{T};\boldsymbol{Q})=H^*(\boldsymbol{BT};\boldsymbol{Q})/(H^+(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(E_6)})=H^*(\boldsymbol{BT};\boldsymbol{Q})/(I_2, I_5,\dots, I_{12})$. By (5.5) and (5.7), $(I_2, I_5,\dots, I_{12})=(J_2, J_5,\dots, J_{12})$. Thus (i) of Lemma 5.2 is proved.

Next, by (1.3), $H^*(\boldsymbol{EIII};\boldsymbol{Q})$ is isomorphic to $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(U)}/(H^+(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(E_6)})=\boldsymbol{Q}[t, q_1, q_2, d'_5, q_3, q_4]/\mathfrak{b}_{13}$ and p^* is an injection equivalent to the natural correspondence. By (5.6), $H^*(\boldsymbol{BT};\boldsymbol{Q})^{\Phi(U)}=\boldsymbol{Q}[t, I_2, w, I_5, I_6, I_8]$. Thus by (5.7), $H^*(\boldsymbol{EIII};\boldsymbol{Q})=\boldsymbol{Q}[t, w]/(J_9, J_{12})$.

Q.E.D.

Proof of (5.5). We use the following notations:

$$s_n = x_1^n + \cdots + x_6^n \quad \text{and} \quad d_i = \sigma_i(x_1, x_2, \dots, x_6).$$

s_n is written as a polynomial on d_i's by use of Newton's formula

(5.8) $$s_n = \sum_{1 \leq i < n} (-1)^{i-1} s_i d_{n-i} + (-1)^{n-1} n d_n \qquad (d_n = 0 \quad \textit{for} \quad n > 6).$$

Note that

(5.8)′ $$d_1 = s_1 = 0$$

since $d_1 = \Sigma x_i = 2\Sigma t_i - 6x = 2(c_1 - 3x) = 0$.

From $\sum_n I_n/n! = \sum_{i<j} e^{x_i+x_j} + \sum_i e^{-x_i}(e^x + e^{-x}) = \frac{1}{2}(\sum e^{x_i})^2 - \frac{1}{2}\sum e^{2x_i}$ $+ 2\sum e^{-x_i} \sum x^{2j}/(2j)!$, it follows

(5.9) $$I_n = \frac{1}{2} \sum_{i+j=n} \binom{n}{i} s_i s_j - 2^{n-1} s_n + 2 \sum_{i+2j=n} (-1)^i \binom{n}{i} s_i x^{2j}.$$

First we have the following relations:

(5.10)
$$I_2 = -12(d_2 - x^2),$$
$$I_5 \equiv -60(d_5 + d_3 x^2) \qquad \bmod \mathfrak{a}_5,$$
$$I_6 \equiv 18(8d_6 - 8d_4 x^2 + d_3^2) \qquad \bmod \mathfrak{a}_6,$$
$$I_8 \equiv 80(-36 d_6 x^2 + d_4^2 + 22 d_4 x^4 + x^8) \qquad \bmod \mathfrak{a}_8$$
$$I_9 \equiv -2^{-1} 3^3 7 d_3^3 \qquad \bmod (x, \mathfrak{a}_9)$$
and $$I_{12} \equiv -2^{-3} 3^5 5 d_3^4 \qquad \bmod (x, \mathfrak{a}_9).$$

These are computed step by step as have seen in the proof of Lemma 5.1. We exhibit the data:

Step 1: $s_1 = d_1 = 0$, $s_2 = -2d_2$ *and* $I_2 = 6s_2 + 12x^2 = -12(d_2 - x^2)$.

Step 2 ($\bmod \mathfrak{a}_5$): $d_2 \equiv x^2$, $s_2 \equiv -2x^2$, $s_3 = 3d_3$, $s_4 \equiv 2x^4 - 4d_4$, $s_5 \equiv 5d_5 - 5d_3 x^2$ and $I_5 = -12 s_5 + 10 s_3 (s_2 - 2x^2) \equiv -60(d_5 + d_3 x^2)$.

Step 3 (mod $\mathfrak{a}_6$): $d_5 \equiv -d_3x^2$, $s_5 \equiv -10d_3x^2$, $s_6 \equiv -6d_6+6d_4x^2+3d_3^2 -2x^6$ and $I_6=-24s_6+15s_4(s_2+2x^2)+10s_3^2+30s_2x^4+12x^6 \equiv 18(8d_6 -8d_4x^2+d_3^2)$.

Step 4 (mod $\mathfrak{a}_8$): $d_6 \equiv d_4x^2-\frac{1}{8}d_3^2$, $s_6 \equiv \frac{15}{4}d_3^2-2x^6$, $s_8 \equiv 136(d_6 - d_4x^2)x^2+4d_4^2+2x^8$ and $I_8=-120s_8+28s_6(s_2+2x^2)+56s_5s_3+35s_4(s_4+4x^4)+56s_2x^6+12x^8 \equiv 80(-36d_6x^2+d_4^2+22d_4x^4+x^8)$.

Step 5 (mod $(x, \mathfrak{a}_9)$): $x \equiv d_2 \equiv d_5 \equiv d_4^2 \equiv 0$ and $d_6 \equiv -\frac{1}{8}d_3^2$. Then $I_9 \equiv a \cdot d_3^3$ and $I_{12} \equiv a' \cdot d_3^4$ for some $a, a' \in \boldsymbol{Q}$. So, we may consider modulo $(x, d_4, \mathfrak{a}_9)$. Then $s_n \equiv 0$ for $n \not\equiv 0 \pmod 3$, $s_3=3d_3$, $s_6 \equiv \frac{15}{4}d_3^2$, $s_9 \equiv \frac{33}{8}d_3^3$, $s_{12} \equiv \frac{147}{32}d_3^4$, and $I_9 \equiv -252s_9+84s_6s_3 \equiv -2^{-1}3^37d_3^3$, $I_{12} \equiv -2040s_{12} +220s_9s_3+462s_6^2 \equiv -2^{-3}3^55d_3^4$.

Next, we rewrite (5.10) in terms of c_i's. Since $\sum d_n=\prod(1+x_i) =\prod(1-x+2t_i)=\sum(1-x)^{6-i}2^ic_i$, we have

$$d_n=\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{6-i}{n-i}c_ix^{n-i}, \qquad c_1=3x. \tag{5.11}$$

For $n=2$, $d_2=15x^2-10c_1x+4c_2=4c_2-15x^2$ and $I_2=-12(d_2-x^2) =-48(c_2-4x^2)=-2^43J_2$.

Modulo $\mathfrak{a}_5=(I_2)=(J_2)$ we have $d_3 \equiv 8c_3-24x^3$, $d_4 \equiv 16c_4-24c_3x +51x^4$, $d_5 \equiv 32c_5-32c_4x+24c_3x^2-40x^5$ and $I_5 \equiv -60(d_5+d_3x^2) \equiv -1920 (c_5-c_4x+c_3x^2-2x^5)=-2^73\cdot5J_5$.

Similarly, modulo $\mathfrak{a}_6=(J_2, J_5)$, we have $d_6 \equiv 64c_6-16c_4x^2+24c_3x^3 -53x^6$ and $I_6 \equiv 18(8d_6-8d_4x^2+d_3^2) \equiv 2^73^2J_6$.

Finally we have directly $I_8 \equiv 2^{12}5J_8 \mod \mathfrak{a}_8$, completing the computation of (5.5).

By (5.11), $d_n \equiv 2^nc_n \mod (x)$. Then we have

$$I_9 \equiv -2^83^37c_3^3 \text{ and } I_{12} \equiv -2^93^55c_3^4 \qquad \mod (x, \mathfrak{a}_9). \tag{5.12}$$

Proof of (5.6). Since $I_8 \in H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\Phi(E_6)} \subset H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\Phi(U)}=\boldsymbol{Q}[t, q_1, q_2, d'_5, q_3, q_4]$, $I_8=aq_4+$*decomposable* for some $a \in \boldsymbol{Q}$. Take the following values of variables: $t=0$, $t'_i=\zeta^i$ for $i=1, 2, 3, 4$ and $t'_5=0$

where $\zeta=\exp(2\pi\sqrt{-1}/8)$. Obviously $q_1=q_2=d'_5=q_3=0$ and $q_4=1$ for such case. It is computed directly that $x=x_1=\frac{1}{2}(1+\zeta+\zeta^2+\zeta^3)$, $x_{i+1}=2\zeta^i-x$, $x_6=-x$ and $S=\{2\zeta^i, -2\zeta^i(1\leq i\leq 4); \zeta^j(1+\zeta+\zeta^2+\zeta^3), \zeta^j(1+\zeta-\zeta^2+\zeta^3)\ (1\leq j\leq 8); 0, 0, 0\}$. Here $1+\zeta\pm\zeta^2+\zeta^3=2\sqrt{1\pm\sqrt{1/2}}\,\xi^{2\pm1}$ for $\xi=\exp(2\pi\sqrt{-1}/16)$. Then we have $a=\sum_{y\in S}y^8=2^8(4+4-(1+\sqrt{1/2})^4-(1-\sqrt{1/2})^4)=-2^{10}3\cdot 5$, proving the last formula of (5.6).

For I_5, take $t=0$, $t'_i=\zeta^i$ for $\zeta=\exp(2\pi\sqrt{-1}/5)$, then $S=\{2\zeta^i, -2\zeta^i, -2\zeta^i\ (1\leq i\leq 5),\ 2\zeta^i+2\zeta^j\ (1\leq i<j\leq 5), 0, 0\}$, and I_5 becomes $2^5(5-5-5+5(1+\zeta)^5+5(1+\zeta^2)^5)=-2^7 3\cdot 5$ which is the coefficient of d'_5.

For I_6, take $t=t'_4=t'_5=0$ and $t'_i=\omega^i$ $(i=1, 2, 3)$ for $\omega=\exp(2\pi\sqrt{-1}/3)$, then $S=\{2\omega^i, 2\omega^i, 2\omega^i, -2\omega^i, -2\omega^i, (i=1, 2, 3), 2\omega^i+2\omega^j\ (1\leq i<j\leq 3), 0,\dots, 0\}$ and the coefficient of q_3 is $2^6 3(3+2+1)=2^7 3^2$ since $\omega^6=(1+\omega)^6=1$.

I_2 is determined similarly or by a direct computation from the following (5.13) and $q_1=d_1'^2-2d'_2$.

Since $x=t+t_1$ and $\left(1+x-\frac{3}{2}t\right)\sum d'_n=\prod_{i=1}^{6}\left(1-\frac{t}{2}+t_i\right)=\sum\left(1-\frac{t}{2}\right)^{6-i}c_i$, we have

$$(5.13)\qquad d'_n+\left(x-\frac{3}{2}t\right)d'_{n-1}=\sum_{0\leq i\leq n}\left(-\frac{1}{2}\right)^{n-i}\binom{6-i}{n-i}c_i t^{n-i}.$$

Modulo $\mathfrak{a}_5=(c_2-4x^2)$, we have

$$d'_1=2x-\frac{3}{2}t,\qquad d'_2\equiv 2x^2-3xt+\frac{3}{2}t^2,$$

$$d'_3\equiv c_3-2x^3-2x^2t-\frac{3}{2}xt^2-\frac{1}{4}t^3,$$

and
$$d'_4\equiv c_4-c_3x+2x^4-x^3t+\frac{3}{2}x^2t^2-\frac{5}{4}xt^3+\frac{9}{16}t^4.$$

Since $q_2=d_2'^2-2d'_3d'_1+2d'_4$ we have directly

$$(5.14)\qquad w=\frac{1}{6}q_2+\frac{9}{16}t^4$$
$$\equiv\frac{1}{3}c_4+\frac{1}{2}c_3t+t^4-(c_3+t^3)x+t^2x^2-2tx^3+\frac{8}{3}x^4 \qquad \bmod \mathfrak{a}_5.$$

Proof of (5.7). Put $w_0=w-t^4$. Since I_9 and I_{12} belong to $H^*(\boldsymbol{BT}; \boldsymbol{Q})^{\Phi(U)}=\boldsymbol{Q}[t, q_1, q_2, d'_5, q_3, q_4]=\boldsymbol{Q}[t, I_2, w_0, I_5, I_6, I_8]$, we may put

$$I_9 \equiv -2^8 3^3 7(a_2 w_0^2 t + a_1 w_0 t^5 + a_0 t^9) \qquad \mathrm{mod}\ \mathfrak{b}_9$$

and $$I_{12} \equiv -2^9 3^5 5(b_3 w_0^3 + b_2 w_0^2 t^4 + b_1 w_0 t^8 + b_0 t^{12}) \qquad \mathrm{mod}\ \mathfrak{b}_9$$

for some $a_i, b_j \in \boldsymbol{Q}$. We consider these relations modulo

$$(x, \mathfrak{a}_9)=(x, J_2, J_5, J_6, J_8)=(x, c_2, c_5, 8c_6+c_3^2, c_4^2) \subset R.$$

By (5.14) and (5.12) we have

$$w_0 \equiv \frac{1}{3} c_4 + \frac{1}{2} c_3 t, \qquad c_3^3 \equiv a_2 w_0^2 t + a_1 w_0 t^5 + a_0 t^9$$

and $$c_3^4 \equiv b_3 w_0^3 + b_2 w_0^2 t^4 + b_1 w_0 t^8 + b_0 t^{12} \qquad \mathrm{mod}\ (x, \mathfrak{a}_9).$$

Now we assume the following (5.15) which will be proved in later.

(5.15) (i) $t^6 \equiv \frac{1}{8} c_3^2 - c_4 t^2 - c_3 t^3 \qquad \mathrm{mod}\ (x, \mathfrak{a}_9)$,

(ii) *$R/(x, \mathfrak{a}_9)$ has a basis* $\{c_3^i t^j, c_4 c_3^i t^j; i \geq 0, 5 \geq j \geq 0\}$.

Then $$w_0^2 t \equiv \frac{1}{3} c_4 c_3 t^2 + \frac{1}{4} c_3^2 t^3,$$

$$w_0 t^5 \equiv \frac{1}{16} c_3^3 - \frac{1}{2} c_4 c_3 t^2 - \frac{1}{2} c_3^2 t^3 + \frac{1}{3} c_4 t^5,$$

$$t^9 \equiv -\frac{1}{8} c_3^3 + c_4 c_3 t^2 + \frac{9}{8} c_3^2 t^3 - c_4 t^5,$$

and as the solution of $c_3^3 \equiv a_2 w_0^2 t + a_1 w_0 t^5 + a_0 t^9$ we have

$$a_2=24, \quad a_1=48 \quad \text{and} \quad a_0=16.$$

Thus $$I_9 \equiv -2^{11} 3^3 7(3 w_0^2 t + 6 w_0 t^5 + 2 t^9) \qquad \mathrm{mod}\ \mathfrak{b}_9$$

$$=2^{11} 3^3 7(-3 w^2 t + t^9)=2^{11} 3^3 7 J_9.$$

Similarly $w_0^3 \equiv \frac{1}{4} c_4 c_3^2 t^2 + \frac{1}{8} c_3^3 t^3$,

$$w_0^2 t^4 \equiv \frac{1}{32} c_3^4 - \frac{1}{4} c_4 c_3^2 t^2 - \frac{1}{4} c_3^3 t^3 + \frac{1}{3} c_4 c_3 t^5,$$

$$w_0 t^8 \equiv -\frac{1}{16} c_3^4 + \frac{13}{24} c_4 c_3^2 t^2 + \frac{9}{16} c_3^3 t^3 - \frac{5}{6} c_4 c_3 t^5,$$

$$t^{12} \equiv \frac{9}{64} c_3^4 - \frac{5}{4} c_4 c_3^2 t^2 - \frac{5}{4} c_3^3 t^3 + 2 c_4 c_3 t^5,$$

and we have $b_3 = b_0 = \frac{64}{3}$, $b_2 = 192$, $b_1 = 128$, and

$$\begin{aligned} I_{12} &\equiv -2^{15} 2^4 5 (w_0^3 + 9 w_0^2 t^4 + 6 w_0 t^8 + t^{12}) \qquad \mathrm{mod}\, \mathfrak{b}_9 \\ &= -2^{15} 3^4 5 (w^3 + 6 w^2 t^4 - 9 w t^8 + 3 t^{12}) \\ &\equiv -2^{15} 3^4 5 (w^3 + 15 w^2 t^4 - 9 w t^8) = -2^{15} 3^4 5 J_{12} \qquad \mathrm{mod}\, \mathfrak{b}_{12}. \end{aligned}$$

Finally we prove (5.15). Obviously c'_i satisfies $c_i = c'_i + c'_{i-1} t_1$ ($i = 1, 2, \dots, 6$; $c'_6 = 0$). Conversely, $c'_i = \sum_{j=0}^{i} c_j (-t_1)^{i-j}$. Thus $R = \boldsymbol{Q}[t_1, c'_i, \dots, c'_5]$ is generated by t_1 and $c_1, \dots, c_6$ in which the relation $\sum_{i=0}^{6} c_i (-t_1)^{6-i} = 0$ holds. So, there is a natural ring homomorphism of $\boldsymbol{Q}[t_1, c_1, \dots, c_6] / (\sum c_i (-t_1)^{6-i})$ onto R. By comparing the Poincaré polynomials, we see that this is an isomorphism, and we may identify

$$R = \boldsymbol{Q}[t_1, c_1, \dots, c_6] / (\sum c_i (-t_1)^{6-i}).$$

Since $t = x - t_1$ and $c_1 = 3x$, $R/(x) = \boldsymbol{Q}[t, c_2, \dots, c_6] / (c_6 + c_5 t + \cdots + c_2 t^4 + t^6)$. Then

$$\begin{aligned} R/(x, \mathfrak{a}_9) &= \boldsymbol{Q}[t, c_3, c_4, c_6] / (8 c_6 + c_3^2, c_4^2, c_6 + c_4 t^2 + c_3 t^3 + t^6) \\ &= \boldsymbol{Q}[t, c_3, c_4] / (c_4^2, -\frac{1}{8} c_3^2 + c_4 t^2 + c_3 t^3 + t^6), \end{aligned}$$

and (5.15) follows. Consequently, (5.15), (5.7) and Lemma 5.2 are established.

§6. Integral cohomology rings.

(A) $H^*(F_4/T)$.

For the subgroups $T \subset Spin(9)$ of F_4 in §4, (*A*), we have a fibering

$$(6.1) \qquad Spin(9)/T \xrightarrow{i} F_4/T \xrightarrow{p} \Pi = F_4/Spin(9).$$

The universal covering μ induces a homeomorphism of $Spin(9)/T$ onto $SO(9)/T^4$. Apply Theorem 2.1 to $H^*(Spin(9)/T) = H^*(SO(9)/T^4)$, then it has the generators t_i, e_{2i} $(i=1, 2, 3, 4)$ with the relations $2e_{2i} = c_i$ $(i=1, 2, 3, 4)$, $e_4 = e_2^2$, $e_8 = 2e_6e_2 - e_4^2$, $2e_8e_4 = e_6^2$ and $e_8^2 = 0$. Thus we have

$$(6.2) \qquad H^*(Spin(9)/T) = Z[t_1, t_2, t_3, t_4, e_2, e_6]/(r_1, r_2, r_3, r_4, r_6, r_8)$$

where $r_1 = c_1 - 2e_2$, $r_2 = c_2 - 2e_2^2$, $r_3 = c_3 - 2e_6$,

$r_4 = c_4 - 2c_3e_2 + 2e_2^4$, $r_6 = -c_4e_2^2 + e_6^2$

and $r_8 = 3c_4e_2^4 - e_2^8$.

Here we see that these t_i's are identified with those in §4, (*A*) and §5, (*A*) by the isomorphisms $\iota^*: H^2(BT) \cong H^2(F_4/T)$ and $i^*: H^2(F_4/T) \cong H^2(Spin(9)/T)$. As is well known

$$(6.3) \qquad H^*(\Pi) = Z[w]/(w^3), \qquad w \in H^8(\Pi).$$

Thus (1.5), (iii) is satisfied and we can apply (1.6) and Lemma 1.1. In particular

(6.4) *$i^*: H^j(F_4/T) \to H^j(Spin(9)/T)$ is bijective for $j<8$ and* Ker i^* *is generated by p^*w for $j=8$.*

In $H^*(BT; Z_p)$ $(p=2, 3)$ the following holds.

$$(6.5) \qquad Sq^2c_2 \equiv c_3 + c_2c_1 \quad (\text{mod } 2),$$

$$\mathscr{P}^1c_2 \equiv c_4 - c_3c_1 + c_2^2 + c_2c_1^2 \quad (\text{mod } 3).$$

For, $Sq^2c_2\equiv\sum_{i<j}Sq^2(t_it_j)\equiv\sum_{i<j}(t_i+t_j)t_it_j=c_2c_1-3c_3$ and $\mathscr{P}^1c_2\equiv\sum_{i<j}(t_i^2+t_j^2)t_it_j=c_2(c_1^2-2c_2)-c_3c_1+4c_4$.

Now apply Lemma 3.1 for $G=F_4$ where $H^*(BT)=Z[t_1,\dots,t_4,\gamma_1]/(c_1-2\gamma_1)=Z[t_1,t_2,t_3,\gamma_1]$ by (4.1). First we see, up to sign,

$$u=\rho_2=c_2-2\gamma_1^2$$

by (6.4) and (6.2). By (6.5), $Sq^2u\equiv Sq^2c_2\equiv c_3+2c_2\gamma_1\equiv c_3 \pmod 2$ and $\mathscr{P}^1u\equiv\mathscr{P}^1c_2-2\mathscr{P}^1\gamma_1^2\equiv c_4-2c_3\gamma_1+c_2^2+4c_2\gamma_1^2-4\gamma_1^4 \pmod 3$.

It follows from Lemma 3.1 the existence of elements

$$\gamma_3\in H^6(F_4/T) \quad\text{and}\quad \gamma_4\in H^8(F_4/T)$$

satisfying $$2\gamma_3=\iota^*c_3=c_3$$

and $$3\gamma_4=\iota^*(c_4-2c_3\gamma_1+c_2^2+4c_2\gamma_1^2-4\gamma_1^4)$$
$$=c_4-2c_3\gamma_1+8\gamma_1^4,$$

and that, by putting $w=\gamma_4-2\gamma_1^4$,

(6.6) *the nautral homomorphism* $Z[t_1,\dots,t_4,\gamma_1,\gamma_3,w]/(\rho_1,\rho_2,\rho_3,\rho_4)\longrightarrow H^*(F_4/T)$ *is an isomorphism for* $\dim\leq 8$, *where* $\rho_1,\dots,\rho_4$ *are given in Theorem A.*

Since $i^*(t_i)=t_i$, $i^*(c_i)=c_i$, and by (6.2), $2i^*(\gamma_1)=c_1=2e_2$, $2i^*(\gamma_3)=c_3=2e_6$ and $3i^*(w)=c_4-4e_6e_2+2e_2^4=0$ in $H^*(\mathbf{Spin}(9)/T)$. It follows from (1.4)

(6.7) $$i^*(\gamma_1)=e_2,\quad i^*(\gamma_3)=e_6\quad\text{and}\quad i^*(w)=0.$$

This defines a homomorphism

$$i^*\colon Z[t_1,t_2,t_3,t_4,\gamma_1,\gamma_3,w]\longrightarrow Z[t_1,t_2,t_3,t_4,e_2,e_6].$$

Then we have obviously

(6.8) $$i^*(\rho_i)=r_i \quad\textit{for}\quad i=1,\ 2,\ 3,\ 4,\ 6,\ 8.$$

It follows from (6.6) and (6.2) that the kernel of $i^*: H^8(\boldsymbol{F}_4/\boldsymbol{T}) \longrightarrow H^8(\boldsymbol{Spin}(9)/\boldsymbol{T})$ is generated by w. Thus (6.4) implies

(6.9) *We may choose the generator* w *of* (6.3) *such that* $p^*(w)=w$.

Proof of Theorem A.

Apply Lemma 1.1 to the fibering (6.1), then by (6.3), (6.8) and $\rho_{12}=w^3$ it is sufficient to prove that $\rho_1,\dots,\rho_4,\rho_6$ and ρ_8 are relations in $H^*(\boldsymbol{F}_4/\boldsymbol{T};\boldsymbol{Q})$. (6.6) shows that $\rho_1,\rho_2,\rho_3,\rho_4$ are relations. By Lemma 5.1, $H^*(\boldsymbol{F}_4/\boldsymbol{T};\boldsymbol{Q})=\boldsymbol{Q}[t_1,\dots,t_4]/(p_1,p_3,12p_4+p_2^2,p_2^3)$. As (5.4) the relation $p_i=\sum_{j+k=i}(-1)^{i+j}c_jc_k$ holds. Then we have

$$\rho_6=\gamma_3^2-c_4\gamma_1^2=\frac{1}{4}(c_3^2-2c_4c_2)=\frac{1}{4}p_3=0$$

and
$$\rho_8=\rho_8+4\rho_6\gamma_1^2+\rho_4\gamma_1^4=3w^2+3w(c_3\gamma_1-\gamma_1^4)+(c_3\gamma_1-\gamma_1^4)^2$$

$$=\frac{1}{4}(2c_3\gamma_1-2\gamma_1^4+3w)^2+\frac{3}{4}w^2=\frac{1}{48}(12p_4+p_2^2)=0$$

since $3w=c_4-2c_3\gamma_1+2\gamma_1^4=c_4-c_3c_1+\frac{1}{2}c_2^2=\frac{1}{2}p_2$ and $c_4^2=p_4$.

Q.E.D.

(B) $H^*(E_6/T)$ and $H^*(EIII)$.

Let $\boldsymbol{T}\subset\boldsymbol{U}$ be the subgroups of $\boldsymbol{E}_6$ defined in §4, (*B*), and consider the fibering

(6.10) $$\boldsymbol{U}/\boldsymbol{T}\xrightarrow{i}\boldsymbol{E}_6/\boldsymbol{T}\xrightarrow{p}\boldsymbol{EIII}=\boldsymbol{E}_6/\boldsymbol{U}.$$

The following (6.11) is essentially proved in [9].

(6.11) $H^*(\boldsymbol{EIII})$ *is multiplicatively generated by two elements* $t\in H^2$ *and* $w\in H^8$.

For, apply the Gysin exact sequence for $\boldsymbol{T}^1$-bundle $\boldsymbol{E}_6/\boldsymbol{D}_5\to\boldsymbol{EIII}$, where $H^*(\boldsymbol{E}_6/\boldsymbol{D}_5)=\boldsymbol{Z}[x_8,x_{17}]/(x_8^3,x_{17}^2)$ and $H^i(\boldsymbol{EIII})=0$ for odd i by Corollaries 4 and 5 of [9]. Then we have an exact sequence

$$H^{*-2}(\boldsymbol{EIII}) \xrightarrow{\times t} H^*(\boldsymbol{EIII}) \longrightarrow \boldsymbol{Z}[x_8]/(x_8^3) \longrightarrow 0$$

which implies (6.11).

As in §4, (B) we identify $\boldsymbol{U}/\boldsymbol{T}$ with $\boldsymbol{SO}(10)/\boldsymbol{T}^5$. Then Corollary 2.2 implies

(6.12) $H^*(\boldsymbol{U}/\boldsymbol{T})=\boldsymbol{Z}[t_1,\dots, t_5, e_2, e_6]/(r_1, r_2, r_3, r_4, c_5, r_6, r_8)$ *where* r_i's *are the same relations as* (6.2) *for* $c_i=\sigma_i(t_1,\dots, t_5)$.

Apply (4.11) to the commutativity of the diagram

$$\begin{array}{ccc} & H^*(\boldsymbol{BT})=\boldsymbol{Z}[t_1,\dots, t_6, x]/(c_1-3x) & \\ {\scriptstyle \iota^*}\swarrow & & \searrow{\scriptstyle \iota_0^*} \\ H^*(\boldsymbol{E}_6/\boldsymbol{T}) & \xrightarrow{\quad i^* \quad} & H^*(\boldsymbol{U}/\boldsymbol{T}) \end{array}$$

and use the notations:

$$\iota^*(x)=\gamma_1,\quad \iota^*(t)=t,\quad \iota^*(t_i)=t_i \quad \text{and} \quad \iota^*(c_i)=c_i$$

for $c_i=\sigma_i(t_1,\dots, t_6)$. Then we have

(6.13) $\quad i^*(t)=0,\quad i^*(\gamma_1)=i^*(t_1)=e_2,\ i^*(c_6)=c_5e_2,$

$$i^*(t_{i+1})=t_i \quad \textit{and} \quad i^*(c_i)=c_i+c_{i-1}e_2 \qquad \textit{for} \quad i=1,\dots, 5\,.$$

Now consider the element u in Lemma 3.1 which generates the kernel of $\iota^*\colon H^4(\boldsymbol{BT})\longrightarrow H^4(\boldsymbol{E}_6/\boldsymbol{T})$. By Theorem 5.2 and (1.2), in the rational coefficient the kernel of ι^* is generated by $J_2=c_2-4x^2$. J_2 is an integral class and not divisible in $H^4(\boldsymbol{BT})$. Thus $u=c_2-4\gamma_1^2$ up to sign.

By (6.5), $Sq^2u\equiv c_3 \pmod 2$ and $\mathscr{P}^1u\equiv c_4+c_2^2-14\gamma_1^4 \pmod 3$. Then Lemma 3.1 implies the existence of elements γ_3 and γ_4 such that

$$2\gamma_3=\iota^*(c_3)=c_3$$

and
$$3\gamma_4=\iota^*(c_4+c_2^2-14\gamma_1^4)=c_4+2\gamma_1^4$$

and that

(6.14) *the natural homomorphism* $\boldsymbol{Z}[t_1,\dots, t_6, \gamma_1, \gamma_3, \gamma_4]/(\rho_1, \rho_2, \rho_3, \rho_4)\to H^*(\boldsymbol{E}_6/\boldsymbol{T})$ *is bijective for* $\dim\leq 8$, *where* $\rho_1, \rho_2, \rho_3, \rho_4$ *are the relations in Theorem B.*

By (6.13) $2i^*(\gamma_3)=i^*(c_3)=c_3+c_2e_2=2e_6+2e_2^3$ and $3i^*(\gamma_4)=i^*(c_4+2\gamma_1^4)=c_4+c_3e_2+2e_2^4=6e_6e_2$. Then it follows from (1.4)

$$i^*(\gamma_3)=e_6+e_2^3 \quad \textit{and} \quad i^*(\gamma_4)=2e_6e_2\,. \tag{6.15}$$

Since $t=\gamma_1-t_1$ and $i^*(\gamma_4-2\gamma_3\gamma_1+2\gamma_1^4)=2e_6e_2-2(e_6+e_2^3)e_2+2e_2^4=0$, the following (6.15)$'$ is obtained easily.

(6.15)$'$ *The kernel of the homomorphism* $i^*\colon \boldsymbol{Z}[t_1,\dots, t_6, \gamma_1, \gamma_3, \gamma_4]\longrightarrow \boldsymbol{Z}[t_1,\dots, t_5, e_2, e_6]$ *defined by* (6.13) *and* (6.15) *is the ideal generated by* t *and* $\gamma_4-2\gamma_3\gamma_1+2\gamma_1^4$.

It is verified directly

$$i^*(\rho_i)\equiv r_i \quad \mathrm{mod}\,(r_j;\, j<i) \quad \textit{for } i=1, 2, 3, 4, 5, 6, 8 \ (r_5=c_5)\,. \tag{6.16}$$

For example, $i^*(c_6)=c_5e_2\equiv 0 \mod (c_5)$, $i^*(3c_5\gamma_1^3-\gamma_1^8)=3c_5e_2^3+3c_4e_2^4-e_2^8\equiv r_8 \mod (c_5)$ and $i^*(\gamma_4-c_3\gamma_1+2\gamma_1^4)\equiv i^*(\gamma_4-2\gamma_3\gamma_1+2\gamma_1^4)=0 \mod (r_2, r_3)$. Thus $i^*(\rho_8)\equiv r_8 \mod (r_2, r_3, c_5)$.

The kernel of the composite of i^* of (6.15)$'$ and the natural map onto $H^*(\boldsymbol{U}/\boldsymbol{T})$ is the ideal $(\rho_1,\dots, \rho_6, \rho_8, t, \gamma_4-c_3\gamma_1+2\gamma_1^4)$ by (6.15)$'$, (6.12) and (6.16). By (6.14), for $\dim\leq 8$, the ideal is the inverse image of the kernel of i^* in the following (6.17). Thus we have

(6.17) *The kernel of* $i^*\colon H^*(\boldsymbol{E}_6/\boldsymbol{T})\longrightarrow H^*(\boldsymbol{U}/\boldsymbol{T})$ *is the ideal* $(t, \gamma_4-c_3\gamma_1+2\gamma_1^4)$ *for* $\dim\leq 8$.

By (5.14), the element $w=\frac{1}{6}q_2+\frac{9}{16}t^4\in H^8(\boldsymbol{E}_6/\boldsymbol{T}; \boldsymbol{Q})$ is of the form

$$w=\gamma_4-c_3\gamma_1+2\gamma_1^4+(\gamma_3-2\gamma_1^3+\gamma_1^2t-\gamma_1t^2+t^3)t \tag{6.18}$$

which is contained in $H^*(\boldsymbol{E}_6/\boldsymbol{T})$. Then $\mathrm{Ker}\, i^*=(t, w)$ for $\dim\leq 8$

by (6.17). (6.11) means that (6.10) satisfies (1.5), (iii). Then by (1.6) and (6.11), $\mathrm{Ker}\, i^* = (p^*(t), p^*(w))$. Thus, up to sign,

$$p^*(t) = t \quad \text{and} \quad p^*(w) = w + ft \quad \text{for some} \quad f \in H^6(\boldsymbol{E}_6/\boldsymbol{T}).$$

(1.3) and (5.3) show that $p^*(w) = a'q_2 + b't^4 = a \cdot w + b \cdot t^4$ for some a', b', a, $b \in \boldsymbol{Q}$. Thus

$$\frac{1}{6}(a-1)q_2 = \left(f - \left(b + \frac{9}{16}(a-1)\right)t^3\right)t \qquad \text{in } H^*(\boldsymbol{E}_6/\boldsymbol{T}; \boldsymbol{Q}).$$

By Lemma 5.2, (5.5) and (5.6), $H^*(\boldsymbol{E}_6/\boldsymbol{T}; \boldsymbol{Q})$ is isomorphic to $\boldsymbol{Q}[t_1, \ldots, t_6]/(J_2) = \boldsymbol{Q}[t, t'_1, \ldots, t'_5]/(q_1) = \boldsymbol{Q}[t] \otimes (\boldsymbol{Q}[t'_1, \ldots, t'_5]/(q_1))$ for $\dim \leq 8$. It follows that if $gt = h$ for $g \in H^*(\boldsymbol{E}_6/\boldsymbol{T}; \boldsymbol{Q})$ and $h \in \boldsymbol{Q}[t'_1, \ldots, t'_5]/(q_1)$, $* \leq 6$, then $g = 0$. From the above equality we have

$$a = 1 \quad \text{and} \quad b \cdot t^3 = f \in H^6(\boldsymbol{E}_6/\boldsymbol{T}), \qquad b \in \boldsymbol{Q}.$$

Since $i^*(f) = b \cdot i^*(t)^3 = 0$ in the rational coefficient, $f \in \mathrm{Ker}\, i^*$ by (1.4). So, $bt^3 = f = f't$ for some $f' \in H^4(\boldsymbol{E}_6/\boldsymbol{T})$, and it follows $b \cdot t^2 = f'$. Similarly we have $b \cdot t = f'' \in H^2(\boldsymbol{E}_6/\boldsymbol{T})$ and $b \cdot 1 \in H^0(\boldsymbol{E}_6/\boldsymbol{T})$. Thus b has to be an integer, and we have obtained

(6.19) *The generators t and w of* (6.11) *can be chosen such that*

$$p^*(t) = t \quad \textit{and} \quad p^*(w) = w.$$

Proof of Corollary C.

By (1.4) and (1.6) the composite

$$H^*(\boldsymbol{EIII}) \xrightarrow{p^*} H^*(\boldsymbol{E}_6/\boldsymbol{T}) \longrightarrow H^*(\boldsymbol{E}_6/\boldsymbol{T}; \boldsymbol{Q})$$

is injective. Then it follows from Lemma 5.2 (i) and (6.19) that the relations $J_9 = J_{12} = 0$ hold in $H^*(\boldsymbol{EIII})$. Thus we have a homomorphism

$$\theta: \boldsymbol{Z}[t, w]/(J_9, J_{12}) \longrightarrow H^*(\boldsymbol{EIII})$$

which is surjective by (6.11). Put $v = -45w^3t + 26w^2t^5$, then we have easily $t^9 \equiv 3w^2t$, $w^3 \equiv -15w^2t + 9wt^8$, $w^3t \equiv 15v$, $w^2t^5 \equiv 26v$ and $vt^4 \equiv 0$

(mod (J_9, J_{12})). Thus $\boldsymbol{Z}[t, w]/(J_9, J_{12})$ is additively generated by $\{w^i t^j | 0 \le i < 3,\ 0 \le j < 9,\ 4i + j < 13\} \cup \{vt^i | 0 \le i < 4\}$. These generators are linearly independent in $H^*(\boldsymbol{EIII}; \boldsymbol{Q})$ by Lemma 5.2, (ii). Thus θ is injective. Q.E.D.

Proof of Theorem B.

By (6.11) and (1.5), we can apply Lemma 1.1 to the fibering (6.10), in which $H^*(\boldsymbol{U}/\boldsymbol{T})$ is given by (6.12), $H^*(\boldsymbol{EIII})$ by Corollary C, i^* by (6.13), (6.15) and p^* by (6.19). The correspondence of the relations between ρ_i and r_i is known by (6.16). w is given by (6.18). $\rho_1, \rho_2, \rho_3, \rho_4$ vanish in $H^*(\boldsymbol{E}_6/\boldsymbol{T})$ by (6.14). In $H^*(\boldsymbol{E}_6/\boldsymbol{T}; \boldsymbol{Q})$, we have

$$\rho_5 = c_5 - c_4\gamma_1 + c_3\gamma_1^2 - 2\gamma_1^5 = \iota^* J_5 = 0,$$

$$\rho_6 = 2c_6 + \frac{1}{4}c_3^2 - c_4\gamma_1^2 - \gamma_1^6 = \iota^*\Big(\frac{1}{4}J_6\Big) = 0$$

and
$$\rho_8 = -9c_6\gamma_1^2 + 3c_5\gamma_1^3 - \gamma_1^8 + (c_4 + 2\gamma_1^4)\Big(\frac{1}{3}c_4 - c_3\gamma_1 + \frac{8}{3}\gamma_1^4\Big)$$

$$= -9c_6\gamma_1^2 + 3c_5\gamma_1^3 + \frac{1}{3}c_4^2 - c_4c_3\gamma_1 + \frac{10}{3}c_4\gamma_1^4 - 2c_3\gamma_1^5 + \frac{19}{3}\gamma_1^8$$

$$= \iota^*\Big(\frac{1}{3}J_8 + 3J_5\gamma_1^3\Big) = 0.$$

Thus the assumptions of Lemma 1.1 are satisfied, and Theorem B follows from Lemma 1.1. Q.E.D.

(C) $\boldsymbol{H^*(G_2/T)}$.

As an appendix, we shall give an alternative proof of the result on $H^*(\boldsymbol{G}_2/\boldsymbol{T})$ in [7]. Lemma 1.1 can be applied to the fibering

$$\boldsymbol{SU}(3)/\boldsymbol{T} \xrightarrow{i} \boldsymbol{G}_2/\boldsymbol{T} \xrightarrow{p} S^6 = \boldsymbol{G}_2/\boldsymbol{SU}(3)$$

where $H^*(\boldsymbol{SU}(3)/\boldsymbol{T}) = \boldsymbol{Z}[t_1, t_2, t_3]/(c_1, c_2, c_3)$, $H^*(\boldsymbol{BT}) = \boldsymbol{Z}[t_1, t_2, t_3]/(c_1)$ for $t_i \in H^2$, $c_i = \sigma_i(t_1, t_2, t_3)$ and $H^*(S^6) = \boldsymbol{Z}[x_6]/(x_6^2)$, $x_6 \in H^6$. Since $H^*(\boldsymbol{G}_2)$ is naturally isomorphic to $H^*(\boldsymbol{F}_4)$ for $\dim \le 6$, Lemma 3.1

holds for $G=G_2$ and $\dim\leq 6$: $H^*(G_2/T)\cong Z[t_1, t_2, t_3, \gamma_3]/(c_1, u, y_6-2\gamma_3)$ for $\dim\leq 6$ and $Sq^2u\equiv y_6 \pmod 2$. Then it is easy to see that $u=\pm c_2$. So we can choose $y_6=c_3$ by (6.5) and $p^*(x_6)=\pm\gamma_3$. It follows from Lemma 1.1

$$(6.20)\qquad H^*(G_2/T)=Z[t_1, t_2, t_3, \gamma_3]/(c_1, c_2, c_3-2\gamma_3, \gamma_3^2)$$
$$=Z[t_1, t_2, \gamma_3]/(t_1^2+t_1t_2+t_2^2, t_2^3-2\gamma_3, \gamma_3^2).$$

Put $\alpha=t_1-t_2$ and $\beta=t_2$, then we have

$$(6.21)\qquad H^*(G_2/T)=Z[\alpha, \beta, \gamma_3]/(\alpha^2+3\alpha\beta+3\beta^2, \beta^3-2\gamma_3, \gamma_3^2)$$

which coincides with the result in [7].

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References

[1] S. Araki, *Cohomology modulo 2 of the exceptional groups E_6 and E_7*, J. Math. Osaka City Univ., 12 (1961), 43–65.
[2] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math., 57 (1953), 115–207.
[3] A. Borel, *La cohomologie mod 2 de certains espaces homogènes*, Comm. Math. Helv., 27 (1953), 216–240.
[4] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math., 76 (1954), 273–342.
[5] A. Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes*, Tôhoku Math. J., (2) 13 (1961), 216–240.
[6] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, Amer. J. Math., 80 (1958), 458–538.
[7] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math., 80 (1958), 964–1029.
[8] N. Bourbaki, *Groupes et algébre de Lie IV–VI*, 1968.
[9] L. Conlon, *On the topology of EIII and EIV*, Proc. Amer. Math. Soc., 16 (1965), 575–581.
[10] H. O. Singh Varma, *The topology of EIII and a conjecture of Atiyah and Hirzebruch*, Nederl. Akad. Wet. Indag. Math., 30 (1968), 67–71.