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# Homomorphisms of differentiable dynamical systems

By

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## Introduction.

In this paper we consider the following problems.

Let $(M, \varphi_t)$ and $(N, \psi_t)$ be differentiable dynamical systems (D.D.S.). Assume that there exists a homomorphism, *i.e.* differentiable mapping $\pi: M \rightarrow N$ such that $\pi \cdot \varphi_t = \psi_t \cdot \pi$ for all $t$. Under this assumption, what relation can exist between the structures of $(M, \varphi_t)$ and $(N, \psi_t)$?

The following examples motivate our problems.

**Example 1.** Let $(M, \mu, \varphi_t)$ be a classical dynamical system, *i.e.* $M$ a differentiable manifold, $\mu$ a measure on $M$ defined by a continuous positive density, and $\varphi_t: M \rightarrow M$ a one-parameter group of measure-preserving diffeomorphisms.

In [1], we showed the following:

Let $(M, \mu, \varphi_t)$ be ergodic and $M$ be compact. If there exist eigen-values $\lambda_1, \lambda_2, \ldots, \lambda_r$ of the $(M, \mu, \varphi_t)$ which are rationally independent and whose eigen-functions are $C^\rho$-differentiable ($\rho \geqq 1$), then $M$ is the total space of a locally trivial fibre space over an $r$-dimensional torus $T^r$, whose fibres are $C^\rho$-submanifolds. The flow $(\varphi_t)$ is fibre-preserving and the flow which is naturally induced on the base space $T^r$ is a quasi-periodic motion with frequencies $\lambda_1, \lambda_2, \ldots, \lambda_r$.

In addition, if $(\varphi_t)$ has a pure point spectrum (discrete spectrum),

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then $(M, \mu, \varphi_t)$ is $C^\rho$-isomorphic to a quasi-periodic motion as classical dynamical systems.

The arguments of these results depend on the existence of a homomorphism $\pi$ of $(M, \varphi_t)$ to a quasi-periodic motion $(T^r, \tau_t)$ with frequencies $\lambda_1, \lambda_2, \ldots, \lambda_r$ (see below for the definition).

**Example 2.** Let $(N, \psi_t)$ be a D.D.S. and $(F, \{\chi_{y,t}\}_{y\in N})$ be a family of D.D.S.'s depending differentiably on the parameter $y$ which varies on the manifold $N$. We call the D.D.S. $(M, \varphi_t)$ a skew product D.D.S. of $(N, \psi_t)$ and $(F, \{\chi_{y,t}\}_{y\in N})$, if $M = N \times F$: direct product manifold of $N$ and $F$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(y, z)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(y)\Big|_{t=0} \times \frac{\mathrm{d}}{\mathrm{d}t}\chi_{y,t}(z)\Big|_{t=0},$$

in the case that $T = R$, and

$$\varphi_1(y, z) = (\psi_1(y), \chi_{y,1}(z)),$$

in the case that $T = Z$, for $(y, z) \in N \times F = M$.

The natural projection $\pi$ of $M$ onto $N$ is clearly a surjective homomorphism of $(M, \varphi_t)$ to $(N, \psi_t)$.

It is natural to ask whether the converse is true or not: Let $\pi$ be a surjective homomorphism of the system $(M, \varphi_t)$ to the system $(N, \psi_t)$. Under what additional conditions $(M, \varphi_t)$ becomes the skew product D.D.S. of $(N, \psi_t)$ and some $(F, \{\chi_{y,t}\}_{y\in N})$?

We consider this question in § 1.

**Example 3.** Let the system $(N, \psi_t)$ has an invariant submanifold $M \subset N$; $\psi_t(M) = M$ for all $t$, then the identity mapping $\pi$ of $M$ to $N$ is an injective homomorphism of $(M, \varphi_t)$ to $(N, \psi_t)$, where $(\varphi_t)$ is the restriction of $(\psi_t)$ to $M$.

We consider the related problems in § 2.

Here we enumerate necessary definitions.

**Definition 1.** Let $M$ be a differentiable connected manifold and $(\varphi_t)_{t\in T}$ (where $T=R$ or $T=Z$) be a one-parameter group of diffeomorphisms of $M$. We call $(M, \varphi_t)$ a *differentiable dynamical system* (D.D.S.).

If there is no proper nonempty closed invariant subset of $M$ for the system $(M, \varphi_t)$, we call the system $(M, \varphi_t)$ a *minimal* system.

Let $T^n=\{(x^1, x^2,\dots, x^n);\ x^i\in R \pmod 1,\ i=1, 2,\dots, n\}$ be a $n$-dimensional torus, and

$$\tau_t\colon (x^1,\dots, x^n)\longmapsto(x^1+\omega^1 t,\dots, x^n+\omega^n t), \qquad \pmod 1.$$

The system $(T^n, \tau_t)$ is minimal if and only if $\omega^1,\dots, \omega^n$ $(\omega^1,\dots, \omega^n, 1)$ are rationally independent, when $T=R$ $(T=Z)$. In this case, we call $(T^n, \tau_t)$ a *quasi-periodic motion* with frequencies $\omega^1,\dots, \omega^n$.

**Definition 2.** Let $(M, \varphi_t)$ and $(N, \psi_t)$ be D.D.S.'s. A differentiable mapping

$$\pi\colon M\longrightarrow N$$

is called a *homomorphism* of $(M, \varphi_t)$ to $(N, \psi_t)$, if it satisfies the relation $\pi\cdot\varphi_t=\psi_t\cdot\pi$ for all $t\in T$.

## §1. Homomorphisms to minimal systems.

Let us begin with some remarks.

If $(\psi_t)$ is trivial, *i.e.* $\psi_t=$identities for all $t$, then the homomorphism $\pi$ of $(M, \varphi_t)$ to $(N, \psi_t)$ is a vector-valued first integral of the system $(M, \varphi_t)$. Conversely, if there exist $n$ integrals $\pi_1(x),\dots, \pi_n(x)$ of $(M, \varphi_t)$, then

$$\pi\colon\ M\longrightarrow N=\{y=(\pi_1(x),\dots, \pi_n(x))\in R^n;\ x\in M\}$$

$$x\longmapsto(\pi_1(x),\dots, \pi_n(x))$$

is a homomorphism. Moreover, if the integrals $\pi_1(x),\dots, \pi_n(x)$ are functionally independent everywhere on $M$, then $(M, \varphi_t)$ becomes a skew product D.D.S. of $(N, \{id.\})$ and some $(F, \{\chi_{y,t}\}_{y\in N})$, where $\{y\}\times F$

$(y\in N)$ are integral manifolds. In this case, we have also an imbedding homomorphism $\iota_y$ for each $y\in N$

$$\iota_y\colon (\{y\}\times F,\ \chi_{y,t})\simeq(F,\ \chi_{y,t})\longrightarrow(M,\ \varphi_t).$$

Now, we consider the question stated in the example 2. We obtain the following

**Theorem 1.** *Let $(M, \varphi_t)$ and $(N, \psi_t)$ be D.D.S.'s and $\pi$ be a homomorphism of $(M, \varphi_t)$ to $(N, \psi_t)$.*

*If $M$ is compact and the system $(N, \psi_t)$ is minimal, then $\pi$ is a surjective mapping of maximal rank, and as a consequent of it, $M$ is the total space of a locally trivial fibre space over $N$, the system $(\varphi_t)$ preserves the fibres, and the naturally induced system on the base space is isomorphic to $(N, \psi_t)$.*

**Proof:** **a)** $\pi$ is surjective: For any $x\in M$, we have

$$\pi(C_M(x))=C_N(\pi(x)),$$

where $C_M(x)$ is the trajectory through $x$ of $(M, \varphi_t)$, *i.e.*

$$C_M(x)=\bigcup_{t\in T}\varphi_t(x),$$

$C_N(\pi(x))$ is defined analogously.

By $\pi(M)\supset\pi(C_M(x))$, the compactness of $M$, and the minimality of $(N, \psi_t)$, we have

$$\pi(M)\supset\overline{\pi(C_M(x))}=N.$$

Where $\bar{A}$ denotes the closure of $A$.

**b)** Let $r(x)=$rank of $\pi$ at $x\in M$. Clearly $r(x)$ is constant on the trajectory $C_M(x)$:

$$r(\varphi_t(x))=r(x) \qquad \text{for all} \quad t\in T.$$

**c)** $r(x)=n$ on $M$ ($n=$dimension of $N$):

Let $K=\{x\in M;\ r(x)<n\}$, critical points of $\pi$. If $K\neq\phi$, then there

exists a point $x_0 \in K$. By b) and the closedness of $K$, we have

$$\overline{C_M(x_0)} \subset K.$$

As is $M$ compact, we can easily show that

$$\pi(\overline{C_M(x_0)}) \supset \overline{C_N(\pi(x_0))}.$$

By the minimality of $(N, \psi_t)$, we have

$$\overline{C_N(\pi(x_0))} = N,$$

so

$$\pi(K) \supset \pi(\overline{C_M(x_0)}) \supset N. \tag{$*$}$$

But, by the well known Sard's theorem, if $\pi$ is sufficiently smooth (for instance, if $\pi$ is of $C^m$-class ($m$=dimension of $M$)) measure of $\pi(K)=0$. This is clearly contradict to ($*$), so $K=\phi$.

This is to be proved. **q.e.d.**

## §2. Homomorphic images of minimal systems.

Let us begin with some examples.

**Example 4–0.** Let $M$ be 0-dimensional space, *i.e.* $M$ consists in one point, then the existence of a homomorphism $\pi$ of $(M, \{id.\})$ to $(N, \psi_t)$ merely means the existence of a fixed point of $(N, \psi_t)$; $\pi(M)$ is the fixed point.

**Example 4–1.** Let $M$ be a circle, $M=S^1$, and $\varphi_t$ be a rotation of it. Then, if $\pi(M)$ is not of one-point (if $\pi(M)$ is of one-point, $\pi(M)$ is a fixed point of $(N, \psi_t)$), the homomorphism $\pi$ is an imbedding and $\pi(M)$ is a periodic solution of $(N, \psi_t)$.

More generally we obtain the following

**Theorem 2.** *Let $\pi$: $T^m \to N$ be a homomorphism of a quasi-periodic motion $(T^m, \tau_t)$ to D.D.S. $(N, \psi_t)$, and $r$=rank of $\pi$. Then $\pi(T^m)$,* *image of $\pi$ is an r-dimensional invariant submanifold of N, which is homeomorphic to an r-dimensional torus $T^r$, and the restricted system of $(N, \psi_t)$ to $\pi(T^m)\subset N$, $(\pi(T^m), \psi_t|_{\pi(T^m)})$ is $C^0$-isomorphic to some quasi-periodic motion $(T^r, \tilde{\tau}_t)$, i.e. there exists a homeomorphism h of $T^r$ to $\pi(T^m)$ such that*

$$h\cdot\tilde{\tau}_t=\psi_t|_{\pi(T^m)}\cdot h \qquad \text{for all} \quad t.$$

**Proof:** **a)** $r(x)$=rank of $\pi$ at $x$

$$=r \qquad\qquad \text{for } \forall x\in T^m:$$

This is clear, because, $r(x)$ is constant along the trajectory, and the set

$$K=\{x\in T^m;\ r(x)<r\}$$

is closed, and every trajectory of $(T^m, \tau_t)$ is dense on $T^m$.

**b)** $\forall x\in T^m$, $\exists U(x)$; nbd. of $x$, and
$\exists$ local coordinates $\bar{x}^1, \bar{x}^2,\dots, \bar{x}^m$ of $U(x)$, and
$\exists$ local coordinates $y^1, y^2,\dots, y^n$ at $\pi(x)\in N$, such that

$$y^i\cdot\pi=\bar{x}^i, \qquad i=1, 2,\ \ r,$$

$$y^j\cdot\pi=0, \qquad j=r+1, r+2,\dots, n,$$

and

$$\bar{x}^i(x)=0 \quad (i=1, 2,\ , m), \qquad y^j(\pi(x))=0 \qquad (j=1, 2,\dots, n).$$

Therefore $\pi(U(x))$ is an $r$-dimensional submanifold of $N$:

This follows from a) and the implicit function theorem by standard arguments.

**c)** $\pi(T^m)$ is a $(\psi_t)$-invariant compact set: Trivial.

**d)** Let $y\in\pi(T^m)$ and $x_1, x_2\in\pi^{-1}(y)\subset T^m$.
Then $\exists V(y)$, nbd. of $y$ in $N$ such that

$\pi(U(x_1)) \cap V(y) = \pi(U(x_2)) \cap V(y)$: As $(\tau_t)$ is a translation and every trajectory of $(\tau_t)$ is dense on $T^m$, we can take $t_1, t_2, \ldots, t_n, \ldots$ $(t_n \to \infty, n \to \infty)$ such that

$$\{\tau_{t_n}(x_i);\ n=1, 2, 3, \ldots\} \text{ is dense in}$$

$$U_i(x_i) \subset U(x_i), \text{ nbd. of } x_i\ (i=1, 2).$$

From $\pi(x_1) = \pi(x_2) = y$, and $\psi_{t_n}(y) = \psi_{t_n} \cdot \pi(x_i) = \pi \cdot \tau_{t_n}(x_i)$, $i=1, 2$, $n = 1, 2, 3, \ldots$, we obtain $\pi(\tau_{t_n}(x_1)) = \pi(\tau_{t_n}(x_2))$, $n=1, 2, 3, \ldots$. As $\pi$ is continuous, so $\pi(U_1(x_1)) = \pi(U_2(x_2))$.

**e)** $\pi(T^m)$ is an $r$-dimensional compact submanifold of $N$: This follows from a)$\sim$d).

**f)** $(\pi(T^m), \psi_t|_{\pi(T^m)})$ is minimal: Trivial.

**g)** With respect to the natural metric $d'$ on $T^m$, the translation $(\tau_t)$ is isometric. We define a metric $d$ on $\pi(T^m)$ compatible to the original topology, then $\pi$ is Lipschitz continuous because $\pi$ is differentiable and $T^m$ is compact. From these, $(\pi(T^m), \psi_t|_{\pi(T^m)})$ is equicontinuous with respect to the time $t$, *i.e.*

$$\forall \varepsilon > 0, \exists \delta > 0: d(y_1, y_2) < \delta, \quad y_1, y_2 \in \pi(T^m)$$

implies $d(\psi_t y_1, \psi_t y_2) < \varepsilon$ for all $t$.

**h)** By the theorem 3 of [1], we obtain the assertion of the theorem. **q.e.d.**

## §3. Remarks and some discussions.

**a)** Note that quasi-periodic motions are minimal. It is sure that in theorem 2, we can replace the quasi-periodic motion $(T^m, \tau_t)$ by a minimal D.D.S. $(M, \varphi_t)$:

Let $\pi$ be a homomorphism of $(M, \varphi_t)$ to $(N, \psi_t)$. If rank of $\pi = r$, and $(M, \varphi_t)$ is minimal, then $\pi(M)$ is an $r$-dimensional invariant submanifold of $(N, \psi_t)$ and the restricted system $(\pi(M), \psi_t|_{\pi(M)})$ is minimal, therefore by theorem 1, the mapping $\pi: M \to \pi(M)$ is maximal rank, and $M$ is a locally trivial fibre space over $\pi(M)$:

$$(M, \varphi_t) \xrightarrow{\pi} (\pi(M), \psi_{t|\pi(M)}) \xrightarrow{\iota} (N, \psi_t).$$

$(\varphi_t)$ preserves the fibres of $M$, and $\iota$ is the natural imbedding.

**b)** In theorem 1, can we we weaken the condition of the minimality of $(N, \psi_t)$ by the one of the ergodicity?

Unfortunately we can easily construct the counter-examples. But, if $(N, \psi_t)$ is uniquely ergodic and the unique ergodic measure has positive density, then the mapping $\pi$ is surjective. In this case it is open whether the similar results can be obtained or not.

**c)** In the case of flow, *i.e.* when $T=R$, we can weaken the assumptions, that is:

Let $X$, $Y$ be generators of the systems $(M, \varphi_t)$, $(N, \psi_t)$ respectively, *i.e.*

$$X(x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)|_{t=0} \in \mathscr{X}(M),$$

$$Y(y) = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(y)|_{t=0} \in \mathscr{X}(N).$$

$\pi$ being a homomorphism of $(M, \varphi_t)$ to $(N, \psi_t)$ is equivalent to the condition

$$\pi_* X = Y.$$

The arguments of the proceeding results can be weakened: It is sufficient to assume that

$$\pi_*(X(x)) = f(x) Y(\pi(x))$$

where $f(x) \neq 0$, is a smooth function on $M$.

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