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Sheaf cohomology theory on harmonic spaces

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The theory of harmonic functions has been extensively developed since M. Brelot introduced the axiomatic method. Brelot's axiomatic theory consists of a (complete) presheaf $\mathscr{H}$ of vector spaces of continuous functions such that there exist sufficiently many open sets for which Dirichlet problem is solvable, and such that Harnack's principle is satisfied for $\mathscr{H}_U$ on any open set U. But B. Walsh is the first who adapted the general sheaf theory to the study of harmonic functions. B. Walsh, as in the case of classical potential theory, investigated the cohomology groups of $\mathscr{H}$ (or $\mathscr{H}$ with certain limitation at infinity). He also proved, in the presence of the adjoint sheaf $\mathscr{H}^*$ of $\mathscr{H}$, a fundamental duality relation between $\mathscr{H}_c^1(X, \mathscr{H})$ and $\mathscr{H}_X^*$.

In this paper we shall study the theory of duality and cohomology of the sheaves $\mathcal{O}$ on a Brelot's harmonic space that are obtained from $\mathscr{H}$ by limiting it at infinity. (This is the general scheme of solution sheaves of an elliptic second order differential equation with various boundary conditions.) $\mathcal{O}$ is a sheaf on the one-point compactification $X \cup \{a\}$ of a Brelot's harmonic space $(X, \mathscr{H})$ such that, (i) $\mathcal{O}|X = \mathscr{H}$ (ii) there is a neighborhood system of a formed by open sets ω such that any continuous function on the boundary of ω is uniquely extended to a section of $\mathcal{O}$ on ω. $\mathcal{O}$ is no more a sheaf of continuous functions, and a germ in $\mathcal{O}_a$ is recognized as a local solution near the boundary of the above elliptic differential equation.

In section 1 we shall construct various resolutions of $\mathcal{O}$. One of those is the fine resolution due to B. Walsh;

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{R} \xrightarrow{d} \mathcal{P} \longrightarrow 0$$

where $\mathcal{R}$ is the sheaf of germs of differences of non-negative continous superharmonic functions and $\mathcal{P}$ is the sheaf of germs generated by local potentials. (As for the sheaf theory see Gunning-Rossi [4]). The other two resolutions are obtained from this if we take out from $\mathcal{P}$ the part which represents the singularity at infinity. This is the sheaf theoretical versoin of a method in potential theory which we can find in articles of Z. Kuramochi [9], F. Y. Maeda [12] and T. Kori [7]. More precisely the resolution $0 \to \mathcal{O} \to \mathcal{F} \xrightarrow{d} \mathcal{G} \to 0$ plays a fundamental role through this paper, where $\mathcal{F}|X = \mathcal{H}$ and $\mathcal{F}_a = \varinjlim_{\omega \ni a} \mathcal{H}_{\omega - a}$ and $\mathcal{G}_a$ is the germs at a of the differences of local potentials which are harmonic except at the point a (HS_0-functions if we follow the terminology in [9]), of course, $\mathcal{G}|X = 0$.

In section 2 we shall construct a sheaf which represents Dirichlet boundary condition if we use the terminology of boundary conditions of elliptic differential equations. This sheaf is found to be the minimal one among the sheaves $\mathcal{O}$ by an appropriate order.

In section 3 the cohomology groups of $\mathcal{O}$ are calculated:

$$H^q(Y, \mathcal{O}) = 0 \qquad (q \geqq 2),$$

$$H^1(Y, \mathcal{O}) = 0 \qquad \text{if} \quad 1 \notin \mathcal{O}_Y,$$

$$H^1(Y, \mathcal{O}) \cong R^1 \qquad \text{if} \quad 1 \in \mathcal{O}_y.$$

These are also obtained by B. Walsh. But our proofs are more simple and somewhat more suggestive. We also have $H^q(X, \mathcal{H}) = 0$ for $q \geqq 1$. This was classically proved by Mittag-Leffler's argument, where it was essential that any section of $\mathcal{H}$ on a relatively compact open subset U of X can be approximated by the sections on X. B. Walsh proved the vanishing of $H^1(X, \mathcal{H})$ for a Brelot's harmonic sheaf satisfying the approximation property. We do not assume this property. Our argument reduces to the fact $H^1(Y, \mathcal{F}) = 0$.

In section 4, in the presence of the adjoint sheaf $\mathcal{O}^*$ of $\mathcal{O}$ we shall prove

$$(\mathcal{O}_K)' \cong \mathcal{O}^*_{V-K}/\mathcal{O}^*_V = H^1_K(Y, \mathcal{O}^*),$$

where K is a compact subset of V. This fundamental relation classically due to Köthe-Tillmann-Grothendieck yeilds some duality relations, for example, we have the following exact sequences in duality;

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_a \longrightarrow H^1_c(X, \mathcal{H}) \longrightarrow H^1(Y, \mathcal{O}) \longrightarrow 0$$

$$0 \longleftarrow H^1(Y, \mathcal{O}^*) \longleftarrow H^1_a(Y, \mathcal{O}^*) \longleftarrow \mathcal{H}^*_X \longleftarrow \mathcal{O}^*_Y \longleftarrow 0$$

$$H^1_c(X, \mathcal{H})' \cong \mathcal{H}^*_X, \mathcal{O}'_a \cong H^1_a(Y, \mathcal{O}^*) \cong \mathcal{G}^*_a.$$

Section 5 is devoted to the decomposition of the singularity of $\mathcal{G}_a$. Here Kuramochi boundary with respect to $\mathcal{O}$, which was introduced in [7, 12], is important. Every germ in $\mathcal{G}_a$ is represented by the integration of extreme germs in $\mathcal{G}_a$, and the latter corresponds to the minimal boundary points. We shall introduce analogous normal derivatives of germs in $\mathcal{O}_a$ at the adjoint boundary (extremes in $\mathcal{G}^*_a$) and express the above duality by the integration of this normal derivative (Greeen-Stokes' Theorem). By virtue of thus formulated duality relation we can give a condition for the duality between $\mathcal{O}_a$ and $\mathcal{G}^*_a$ to be separated.

Some results of our previous paper [7] are quoted. If we view our present paper as the development of global theory of harmonic sheaves, our previous one may be interpretaed as the treatments of local theory.

§1. Resolutions of harmonic sheaves.

Let X be a non compact Brelot's harmonic space and $\mathcal{H}$ be the sheaf on X of harmonic functions. We assume that 1 is harmonic on X. Let $Y = X \cup \{a\}$ be the one-point compactification of X. We consider the following sheaves $\mathcal{O}$ on Y of linear spaces;

(1.1) $\mathcal{O}_x = \mathcal{H}_x$ for every $x \in X$,

(1.2) $\mathcal{O}_a$ is a linear subspace of the linear space $\varinjlim_{V \ni a} \mathcal{H}_{V \setminus a}$, which is the inductive limit of linear spaces $\mathscr{H}_{V\setminus a}$ as V ranges over a neighborhood system of a.

An open set $G\subset Y$ with the boundary ∂G contained in X is said to be *$\mathcal{O}$-regular* if any continuous function f on ∂G possesses a unique continuous extension $\bar{f}$ to $\bar{G}\setminus a$ such that the restriction $\bar{f}|G\setminus a$ coincides with a $u\in\mathcal{O}_G=\Gamma(G, \mathcal{O})$ on $G\setminus a$ and such that $\bar{f}$ is non-negative if f is non-negative.

For any $\mathcal{O}$-regular set G and $f\in C(\partial G)$, the above $u\in\mathcal{O}_G$ is uniquely determined and is denoted by $H^G f\equiv{}^{\mathcal{O}}H^G f$. For a $u\in\mathcal{O}_G$ we shall adapt the notation u_y to represent the germ of u at $y\in G$. On the other hand the value at $x\in G\setminus a$ of u viewed as a continuous function on $G\setminus a$ is denoted by $u(x)$. These notations are adapted for any sheaf on Y if its section on a set G can be considered as a continuous function on $G\setminus a$. Now, since the linear form $f\to H^G f(x)$, $x\in G\setminus a$, on $C(\partial G)$ is positive, it defines a Radon measure $H_x^G(dy)$ on $G\subset X$;

$$H^G f(x)=\int f(y)H_x^G(dy)$$

for any $f\in C(\partial G)$ and $x\in G\setminus a$.

Definition. A sheaf $\mathcal{O}$ on Y of linear spaces is called a *harmonic sheaf* it if satisfies (1.1), (1.2) and

(1.3) $\mathcal{O}$-regular sets form basis for the topology on Y.

(1.4) 1 is $\mathcal{O}$-superharmonic on Y.

Here we give explanations of these conditions. The condition (1.3) may be stated merely for a base of topology at a, because Brelot's axiom 2 and the fact $\mathcal{O}_x=\mathscr{H}_x$ for $x\in X$ imply the same condition stated on X. Following the terminology of [7], [12] we might say a full-harmonic sheaf rather than a harmonic sheaf, but we will not use this. $\mathcal{O}$-superharmonic functions are defined as follows.

Let G be an open set. A function s on $G-a$ is said to be $\mathcal{O}$-

superharmonic on G if; (i) s is lower semi-continuous, $>-\infty$ and is not identically equal to $+\infty$ on any connected component of $G-a$, and (ii) for any $\mathcal{O}$-regular set ω, $\bar{\omega}\subset G$, and any $f\in C(\partial\omega)$, the relation $f\leqq s$ on $\partial\omega$ implies the relation $H^{\omega}f\leqq s$ on $\omega-a$.

A non-negative $\mathcal{O}$-superharmonic function p on G which have the following property is called a *potential* on G: If an $\mathcal{O}$-superharmonic function u on G such that $u+p\geqq 0$ exists then $u=0$. A domain $V\subset Y$ is called a ($\mathcal{O}$-) small set if there is a non-zero potential on V. Every regular set is small and there is an exhaustion of X by relatively compact small sets, hence there is a cover $\{U, V\}$ of Y formed by small sets with $U\subset X$, $V\ni a$.

We note that Harnack's principle for the sheaf $\mathcal{O}$ are valid, that is, if h_n is an increasing sequence of sections of $\mathcal{O}$ on a domain G, then either $\sup h_n\equiv+\infty$ or $\sup h_n\in\mathcal{O}_G$.

We have the following criterion:

(1.6) *Y is small iff* $1\notin\mathcal{O}_Y$

The proof is as follows. Every non-negative $\mathcal{O}$-superharmonic function u on G has a unique decomposition $u=p+h$ with $h\in\mathcal{O}_G$ and p a potential on G. Let $1=p_0+h_0$ be the decomposition. If $1\notin\mathcal{O}_Y$ p_0 must not be identically equal to 0, hence $p_0>0$ (a consequence of Harnack's principle for $\mathcal{O}$). Thus Y is small. If $1\in\mathcal{O}_Y$, $H^{\omega}1=1$ on $\omega\backslash a$ for any regular neighborhood ω of a. Let K be a compact subset of X such that $K\supset Y\backslash\omega$, and let p be a non-zero potential on Y. We have $p\geqq H^{\omega}p\geqq cH^{\omega}1=c$ on $\omega\backslash a$, where $c=\inf_K p>0$. Therefore $p\geqq c$ on $Y\backslash a$. Since $-c$ is $\mathcal{O}$-superharmonic on Y and $p-c\geqq 0$, it follows $-c\geqq 0$ from the definition of a potential. This is absurd and there is no non-zero potential on Y.

After B. Walsh [18] (See also W. Hansen [5].) we shall give a fine resolution of the sheaf $\mathcal{O}$.

For any open set G, let $\mathcal{P}_G^+$ be the convex cone of potentials on G that are continuous on $G\backslash a$. Let $\mathcal{P}_G=\mathcal{P}_G^+-\mathcal{P}_G^+=\{p-q;\ p, q\in\mathcal{P}_G^+\}$ and let $\mathcal{R}_G=\mathcal{O}_G+\mathcal{P}_G$. $\mathcal{P}_G$ and $\mathcal{R}_G$ are R-modules and $\mathcal{R}_G$ is a direct sum (as modules) of $\mathcal{O}_G$ and $\mathcal{P}_G$. Let $i_G\colon\mathcal{O}_G\to\mathcal{R}_G$ and

j_G: $\mathscr{P}_G \to \mathscr{R}_G$ be the natural injections, and $d_G: \mathscr{R}_G \to \mathscr{P}_G$ be the projection; $d_G \circ j_G$=Identity on $\mathscr{P}_G$. If we denote by $r_V^U: \mathscr{R}_V \to \mathscr{R}_U$ $(V \supset U)$ the restriction map, $(\mathscr{R}_U, r_V^U)$ forms a presheaf of R-module. $(\mathscr{P}_U, \rho_V^U = d_U \circ r_V^U \circ j_V)$ is also a presheaf of R-module and the next diagram commutes;

$$\begin{array}{ccc} \mathscr{R}_V & \xrightarrow{d_V} & \mathscr{P}_V \\ {\scriptstyle r_V^U}\downarrow & & \downarrow{\scriptstyle \rho_V^U} \\ \mathscr{R}_U & \xrightarrow{d_U} & \mathscr{P}_U . \end{array}$$

The sequence

$$0 \longrightarrow \mathscr{O}_U \xrightarrow{i_U} \mathscr{R}_U \xrightarrow{d_U} \mathscr{P}_U \longrightarrow 0 \tag{1.6}$$

of R-modules is exact. Let $\mathscr{R}$ and $\mathscr{P}$ be the associated sheaves to the presheaves $(\mathscr{R}_U, r_V^U)$ and $(\mathscr{P}_U, \rho_V^U)$ rsepectively, and let i and d be the induced homomorphisms of i_U and d_U respectively. We have the exact sequence;

$$0 \longrightarrow \mathscr{O} \xrightarrow{i} \mathscr{R} \xrightarrow{d} \mathscr{P} \longrightarrow 0 . \tag{1.7}$$

We can verify easily that $f \in \Gamma(U, \mathscr{R})$ iff, for any $x \in U$, there exist $s_1, s_2 \in \mathscr{S}_V$, $\geqq 0$, on a neighborhood V of x, $V \subset U$, such that $r_U^V f = s_1 - s_2$ on $V \backslash a$. Where $\mathscr{S}_V$ is the convex cone of $\mathscr{O}$-superharmonic functions on V which are continuous on $V \backslash a$. Here we note that the definition in [18] of the sheaf $\mathscr{R}$ is incorrect. There $f \in \Gamma(U, \mathscr{R})$ for $U \ni a$ iff there is a neighborhood V of a in U such that $f | V \backslash a = \Sigma \alpha_i s_i$, where $\{\alpha_i\}$ are scalars and the $\{s_i\}$ non-negative $\mathscr{O}$-subharmonic functions on V which are continuous on $V \backslash a$. But the existence of nonnegative $\mathscr{O}$-subharmonic functions is not assured for certain $\mathscr{O}$. For example, take the minimal harmonic sheaf $\bar{\mathscr{O}}$ in section 2 such that $1 \notin \bar{\mathscr{O}}_Y$.

The following results are due to B. Walsh ([5], [18]).

(1.8) *For any open set G, $\Gamma(G, \mathscr{R})$ becomes a ring as the multiplication is defined pointwisely on $G \backslash a$.*

(1.9) *$\mathscr{R}$ is a fine sheaf.*

Now let $\mathscr{B}$ be the sheaf of bounded measureable functions on Y. To give a $\mathscr{B}$-module structure for the sheaf $\mathscr{P}$ and to introduce some important subsheaves of $\mathscr{P}$, we shall here study potential kernels on a small set following T. Kori [7]. As for small set $U \subset X$, such a potential kernel is now well-known [14], so we may assume $U \ni a$. Let

$$\mathscr{G}_U^+ = \mathscr{P}_U^+ \cap \mathscr{O}_{U \setminus a} = \{p \in \mathscr{P}_U^+ ;\ p \mid U \setminus a \in \mathscr{O}_{U-a} = \mathscr{H}_{U-a}\} .$$

Let $p \prec q$ be an order defined in P_U by the cone P_U^+. By this order $\mathscr{P}_U$ becomes a lattice [6, 7]. Let

$\mathscr{I}_U^+ = \{p \in \mathscr{P}_U^+$; if there is a $u \in \mathscr{G}_U^+$ such that $u \prec p$, then $u = 0)$. Then $\mathscr{P}_U^+$ is the direct sum of $\mathscr{G}_U^+$ and $\mathscr{I}_U^+$. The $\mathscr{G}_U^+$-part of $p \in \mathscr{P}_U^+$ is denoted as $B^U p$; $p - B^U p \in \mathscr{I}_U^+$.[1)]

Theorem 3.16 of [7] states the following:

For any $p \in \mathscr{P}_U^+$ *there is a unique kernel* $K(x, dy) \equiv K^p(x, dy) \equiv K^{p,U}(x, dy)$ *on* $U \setminus a$ *that satisfies*;

(i) $K1 = p - B^U p$,

(ii) *for any bounded measurable function* f *on* $U \setminus a$, $Kf \in \mathscr{I}_U^+$ *and* Kf *is bounded continuous on* $U \setminus a$, *and belongs to* $\mathscr{O}_{U - \mathrm{Supp}[f]}$.

Lemma 1.1. *Let* V *be a small set and* U *be an open set such that* $a \in U \subset \overline{U} \subset V$, *and let* $p \in \mathscr{P}_V^U$. *We have,*

(i) $\rho_V^+(B^V p) = B^U(\rho_V^+ p)$

(ii) $\rho_V^U(p - B^V p) = \rho_V^U p - B^U(\rho_V^U p)$

(iii) $\rho_V^U(K^p f) = K^q(f \mid U \setminus a)$, *where* $q = \rho_V^U p$ *and* f *is a bounded measurable function on* $V \setminus a$.

1) There is a strictly positive function on $\mathscr{G}_U^+$ for any small set U containing a. Moreover, for any small set U and any point $y \in U$, there is a strictly positive function of $\mathscr{P}_U^+$ which is harmonic on $U - y$. (Thm. 161. of [6], Proposition 5.1 of [7])

Proof. Let $p=q_1+h_1$ be the decomposition of $r_V^U p \in \mathscr{S}_U^+$ to $h_1 \in \mathscr{O}_U$ and $q_1 \in \mathscr{P}_U^+$; $q_1=\rho_V^U p$. By the same way, let $B^V p=q_2+h_2$, $q_2=\rho_V^U B^V p \in \mathscr{P}_U^+$ and $h_2 \in \mathscr{O}_U^+$, and let $p-B^V p=q_3+h_3$, $h_3 \in \mathscr{O}_U^+$, $q_3=\rho_V^U(p-B^V p) \in \mathscr{P}_U^+$. We have $q_1=q_2+q_3$. q_2 is obviously harmonic on $U\backslash a$, so $q_2 \in \mathscr{G}_U^+$. Set $t=p-B^V p \in \mathscr{I}_V^+$. For any compact subset K of X such that $\partial U \subset \mathring{K}$ (the interior of K), let

$$(t)_{V-K}=\inf\{u \in \mathscr{P}_V^+;\ u=t+s \text{ on } V-K \text{ for some } s \in \mathscr{S}_{V-K}\},$$

and

$$(q_3)_{U-K}=\inf\{u \in \mathscr{P}_U^+;\ u=q_3+s \text{ on } U-K \text{ for some } s \in \mathscr{S}_{U-K}\}.$$

From [12], $t_{V-K} \in \mathscr{P}_V^+$ and there is a $w \in \mathscr{P}_V^+ \cap \mathscr{O}_{V-K}$ such that $t=t_{V-K}+w$, hence $q_3=\rho_V^U t=\rho_V^U(t_{V-K})+\rho_V^U w$ and $\rho_V^U w \in \mathscr{O}_{U-K}$, that is, $\rho_V^U(t_{V-K})$ belongs to the bracket of the right-hand side which defines $(q_3)_{U-K}$. We have $(q_3)_{U-K} \leqq \rho_V^U(t_{V-K})$. Now since we know

$$B^U q_3=\inf(\{q_3)_{U-K};\ K \text{ is a compact set of } X \text{ such that } \mathring{K} \supset \partial U\}$$

from [12], we have

$$\begin{aligned} B^U q_3 &\leqq \inf\{\rho_V^U(t_{V-K});\ K \text{ compact in } X,\ \mathring{K} \supset \partial U\} \\ &\leqq \inf(t_{V-K})=B^V t, \end{aligned}$$

which equals to zero because of $t \in \mathscr{I}_V^+$, so $B^U q_3=0$ and $q_3 \in \mathscr{I}_U^+$. Thus we have the decomposition $q_1=q_2+q_3$, $q_2 \in \mathscr{G}_U^+$, $q_3 \in \mathscr{I}_U^+$. From the uniqueness of the decomposition we have the first and the second equalities. To prove the third, it is enough to show that $\rho_V^U(K^p 1)=K^q 1$ and $\rho_V^U(K^p f) \in \mathscr{O}_{U-\mathrm{Supp}[f]}$ for any $f \in C_c(U)$. From these the third assertion follows by virtue of the uniqueness of the potential kernel. But $\rho_V^U(K^p 1)=\rho_V^U(p-B^V p)=\rho_V^U p-B^U \rho_V^U p=K^q 1$, and $\rho_V^U(h) \in \mathscr{O}_G$ for any $h \in \mathscr{O}_G$ and any open set $G \subset \bar{G} \subset U$. These prove the above.

For any bounded measurable function f on U and $p \in \mathscr{P}_U$; $=p^+-p^-$, $p^{\pm} \in \mathscr{P}_U^+$, we shall define the multiplication as follows; $f\circ p=K^{p^+}(f|U\backslash a)-K^{p^-}(f|U\backslash a)+f(a)B^U p^+-f(a)B^U p^-$.
Lemma 1.1 shows that $\rho_V^U(f\circ p)=(r_V^U f)\circ(\rho_V^U p)$, where r_V^U: $\mathscr{B}_U \to \mathscr{B}_V$ is

the restriction map. Thus $\{(\mathscr{B}_U, \mathscr{P}_U), (r_V^U, \rho_V^U)\}$ is a presheaf with the presheaf $(\mathscr{B}_U, r_V^U)$ as its coefficient. By taking inductive limit we have;

(1.10) *$\mathscr{P}$ is a sheaf of $\mathscr{B}$-modules.*

Theorem 1.2. (B. Walsh). *$\mathscr{P}$ is a fine sheaf.*

The proof is given in Proposition 1.6.

In the following we shall give other resolutions of $\mathscr{O}$ which will simplify the calculus of cohomology groups of $\mathscr{O}$.

Lemma 1.3. *Let U be a small set and V, W be open subsets of U such that $\overline{W} \subset V$. For any $s \in \mathscr{S}_V$ there are $p, q \in \mathscr{P}_U^+$ such that $q = s + p$ on W and $p \in \mathscr{O}_W$.*

Lemma 1.4. *Let U, V be as in the above. If $s \in \mathscr{S}_V$ has as its carrier a compact subset of V, that is, if $s \in \mathscr{S}_V$ is $\mathscr{O}$-harmonic out of some compact set in 0, there is a unique $p \in \mathscr{P}_U^+$ whose carrier is the same as that of s such that $p - s \in \mathscr{O}_V$.*

Proof of Lemma 1.3. The case when $U \subset X$ is proved in Theorem 13.1 of [6]. First we suppose that $s \geqq 0$. Let D_i $(i = 1, 2)$ be an open set such that $V \supset \overline{D}_1 \supset D_1 \supset \overline{D}_2 \supset D_2 \supset W$. Let $s < M$ on ∂D_2 and $s < m$ on ∂D_1, and, for any $\varepsilon > 0$, let a continuous function f on $\partial D_1 \cup \partial D_2$ be defined as $f = M + \varepsilon$ on ∂D_2, $= m - \varepsilon$ on ∂D_1. There are $p_1, p_2 \in \mathscr{P}_U^+$ such that $|-f + (p_1 - p_2)| < \varepsilon$ on $\partial D_1 \cup \partial D_2$. (For example, see Proposition 1.7 in [7]). Moreover p_i $(i = 1, 2)$ is chosen so that $p_i \in \mathscr{O}_W$. Then the function

$$q = \begin{cases} s + p_2 & \text{on} \quad D_2 \\ \inf(s + p_2, p_1) & \text{on} \quad D_1 \backslash D_2 \\ p_1 & \text{on} \quad U \backslash D_1 \end{cases}$$

belongs to $\mathscr{P}_U^+$, and $q = s + p_2$ on W. In general we proceed as follows; (i) If there is an $\mathscr{O}$-regular set D such that $\overline{W} \subset D \subset \overline{D} \subset V$ and such that $\inf_{\partial D} s = m > 0$, then we have $s \geqq mH^D 1 \geqq 0$ on $D \backslash a$ from the minimum principle. Applying the above to the functions $u = s - mH^D 1 \in \mathscr{S}_D^+$ and $v = mH^D 1 \in \mathscr{S}_D^+$, we have the desired result. (ii) If for any regular set D such that $\overline{W} \subset D \subset \bar{D} \subset V$ it holds $\inf_{\partial D} s < 0$, choose such a D and let $m = \inf_{\partial D} s$. By the minimum principle $t = s - m \in \mathscr{S}_D^+$, so we may apply the above to t and $-m \in \mathscr{S}_D^+$, and have our result.

The proof of Lemma 1.4 is carried in the same manner as in Theorem 13.2 of [6] by virtue of Lemma 1.3 and the results of [7, § 3].

Let $\mathscr{G}_U = \mathscr{G}_U^+ - \mathscr{G}_U^+$ for any $U \ni a$. For $U \subset X$, since $\mathscr{G}_U^+ = \mathscr{P}_U^+ \cap \mathscr{H}_U = \{0\}$, we set $\mathscr{G}_U = \{0\}$. Let $\mathscr{S}_U = \mathscr{S}_U^+ - \mathscr{S}_U^+$, $U \subset Y$. Obviously $\mathscr{S}_U = \mathscr{P}_U$ for $U \subset X$. As is easily verified $(\mathscr{S}_U, \rho_U^V)$ and $(\mathscr{G}_U, \rho_U^V)$ are presheaves, hence $\mathscr{S}_y = \varinjlim_{U \ni y} (\mathscr{S}_U, \rho_U^V)$ and $\mathscr{G}_y = \varinjlim_{U \ni y} (\mathscr{G}_U, \rho_U^V)$ are subsheaves of $\mathscr{P}$. From Lemma 1.1 we have the following commutative diagrams of presheaves:

$$\begin{array}{ccc} \mathscr{P}_U & \xrightarrow{B^U} & \mathscr{G}_U \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ \mathscr{P}_V & \xrightarrow{B^V} & \mathscr{G}_V \end{array} , \qquad \begin{array}{ccc} \mathscr{P}_U & \xrightarrow{C^U} & \mathscr{S}_U \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ \mathscr{P}_V & \xrightarrow{C^V} & \mathscr{S}_V \end{array} ,$$

where $C^U p = p - B^U p$ for $p \in \mathscr{P}_U$. Passing to the inductive limits there exist sheaf homomorphisms $b: \mathscr{P} \to \mathscr{G}$ and $c: \mathscr{P} \to \mathscr{S}$ such that the diagrams

$$\begin{array}{ccc} \mathscr{P}_U & \xrightarrow{B^U} & \mathscr{G}_U \\ \rho_U \downarrow & & \downarrow \rho_U \\ \mathscr{P} & \xrightarrow{b} & \mathscr{G} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathscr{P}_U & \xrightarrow{C^U} & \mathscr{S}_U \\ \rho_U \downarrow & & \downarrow \rho_U \\ \mathscr{P} & \xrightarrow{c} & \mathscr{S} \end{array}$$

commute.

Proposition 1.5. (i) $\mathscr{G}_x = \{0\}$ *and* $\mathscr{S}_x = \mathscr{P}_x$ *for any* $x \in X$.
(ii) $0 \longrightarrow \mathscr{G} \underset{b}{\rightleftarrows} \mathscr{P} \xrightarrow{c} \mathscr{S} \longrightarrow 0$ *is a split exact sequence.*
(iii) *For any small set* $U \ni a$, $\Gamma(U, \mathscr{G}) \cong \mathscr{G}_U$.

Proof. (i) is obvious. (ii) follows from the fact that $\mathscr{P}_U$ is the direct sum of $\mathscr{G}_U$ and $\mathscr{S}_U$. As for (iii), since $\mathscr{G}_U \xrightarrow[\rho_U]{} \Gamma(U, \mathscr{G})$ is injective, we shall prove it is surjective. For any $M\in\Gamma(U,\mathscr{G})$ there is a neighborhood ω of a, $\omega\subset U$, such that $M|\omega=\rho_\omega p$ for a $p\in\mathscr{G}_\omega$. If we note that the carrier of p is the point a, Lemma 1.3 yields the existence of a $u\in\mathscr{G}_U$ such that $u-p\in\mathcal{O}_\omega$. $\rho_U u=\rho_\omega\rho_U^\omega u=\rho_\omega p=M$ on ω. On the other hand $\rho_U u=0=M$ on $U-\omega$, so $\rho_U u=M$ on U, this proves that ρ_U is surjective, hence is a linear isomorphism.

We shall show that the sheaves $\mathscr{P},\mathscr{I},\mathscr{G}$, are fine. As we have mentioned in (1.10), $\mathscr{P}$ is a sheaf of $\mathscr{B}$-modules by the multiplication $\mathscr{B}\times\mathscr{P}\in(f,p)\to f\circ p\ni\mathscr{P}$, where the sheaf homomorphism $f\circ\colon\mathscr{P}\to\mathscr{P}$ is defined by

$$\begin{array}{ccc} \mathscr{P}_U & \xrightarrow{f\circ} & \mathscr{P}_U \\ {\scriptstyle\rho_U}\big\downarrow & & {\scriptstyle\rho_U}\big\downarrow \\ \mathscr{P} & \xrightarrow{f\circ} & \mathscr{P} \end{array}.$$

From Lemma 1.1 it can be verified easily that $f\circ M\in\mathscr{I}_y$ (resp. $\mathscr{G}_y$) whenever $M\in\mathscr{I}_y$ (resp. $\mathscr{G}_y$), so $\mathscr{I}$ and $\mathscr{G}$ are also sheaves of $\mathscr{B}$-modules. The sheaf homomorphism $f\circ$ is a zero-map on the stalk $\mathscr{P}_y$ for $y\in Y-\mathrm{Supp}[f]$, because $f\circ p\in\mathcal{O}_{V-\mathrm{Supp}[f]}$ for any $p\in\mathscr{P}_V$ and any small set V. Let $(U_i)_{i=1}^\infty$ be a locally finite cover of Y and $\{\varphi_i\}_{i=1}^\infty$ be a family of bounded measurable functions on Y such that $\varphi_i=0$ on an open neighborhood of $Y-U_i$. $\sum\varphi_i$ is a well defined bounded measurable function on Y, and for any $M\in\mathscr{P}_y$, $y\in Y$, we have $(\sum\varphi_i)\circ M=\rho_V^y((\sum\varphi_i)\circ p)$, where V is a small neighborhood of y which meets only finitely many U_i and $M=\rho_V^y p$. Since the support of the kernel $K^{p,V}$ is contained in V and $B^V p\neq 0$ only if $a\in V$, we have $(\sum\varphi_i)\circ p=K^p(\sum\varphi_i)+(\sum\varphi_i)(a)B^V p=\sum K^p\varphi_i+\sum\varphi_i(a)B^V p=\sum_{i=1}^n\varphi_i\circ p$, for large n. Hence $(\sum\varphi_i)\circ M=\rho_V^y((\sum\varphi_i)\circ p)=\sum_{i=1}^n\rho_V^y(\varphi_i\circ p)=\sum_{i=1}^n\varphi_i\circ M$. But, if $\mathrm{Supp}[\varphi_i]\cap\bar{V}=\emptyset$ then $\varphi_i\circ M=0$ from the above remark, so the last expression of the above equality becomes $\sum_{i=1}^\infty(\varphi_i\circ M)$;

$$(\sum\varphi_i)\circ M=\sum(\varphi_i\circ M).$$

The same relations are valid for the sheaves $\mathscr{I}$ and $\mathscr{G}$. By the argument we have done hitherto, we get the following:

Proposition 1.6. *$\mathscr{P}$, $\mathscr{I}$ and $\mathscr{G}$ are fine sheaves.*

Now we shall proceed to resolutions of the sheaf O other than (1.7). Let $\mathscr{F}_U=\mathcal{O}_U+\mathscr{G}_U$ and $\mathscr{K}_U=\mathcal{O}_U+\mathscr{I}_U$ for any open set U. These are seen to be direct sums. We have commutative diagrams:

$$\begin{array}{ccc} \mathscr{F}_U & \xrightarrow{d_U} & \mathscr{G}_U \\ {\scriptstyle r_U^V}\downarrow & & \downarrow{\scriptstyle \rho_U^V} \\ \mathscr{F}_V & \xrightarrow{d_V} & \mathscr{G}_V \end{array}, \qquad \begin{array}{ccc} \mathscr{K}_U & \xrightarrow{d_U} & \mathscr{I}_U \\ {\scriptstyle r_U^V}\downarrow & & \downarrow{\scriptstyle \rho_U^V} \\ \mathscr{K}_V & \xrightarrow{d_V} & \mathscr{I}_V \end{array}.$$

For example, $r_U^V\mathscr{K}_U\subset\mathscr{K}_V$ is verified as follows:
Let $f\in\mathscr{K}_U$ be written as $f=h_1+p_1$, $h_1\in\mathcal{O}_U$, $p_1\in\mathscr{I}_U$. If we write $r_U^Vp_1\in\mathscr{R}_V$ as $r_U^Vp_1=h_2+p_2$, $h_2\in\mathcal{O}_V$, $p_2\in\mathscr{P}_V$, we have $r_U^Vf=r_U^Vh_1+h_2+p_2$, $p_2=\rho_U^Vp_1$. But $B^Vp_2=\rho_U^V(B^Up_1)=0$ (from Lemma 1.1) implies $p_2\in\mathscr{I}_V$, thus $r_U^Vf\in\mathscr{K}_V$. From htis observation, if we proceed to the inductive limits of presheaves $(\mathscr{F}_U, r_U^V)$ and $(\mathscr{K}_U, r_U^V)$, we get the following sheaf exact sequences:

$$0\longrightarrow\mathcal{O}\longrightarrow\mathscr{F}\longrightarrow\mathscr{G}\longrightarrow 0, \tag{1.11}$$

and

$$0\longrightarrow\mathcal{O}\longrightarrow\mathscr{K}\longrightarrow\mathscr{I}\longrightarrow 0, \tag{1.12}$$

where $\mathscr{F}_y=\varinjlim_{V\ni y}(\mathscr{F}_U, r_U^V)$, $\mathscr{K}_y=\varinjlim_{V\ni y}(\mathscr{K}_V, r_U^V)$. It is obvious that $\mathscr{F}_x=\mathscr{K}_x=\mathcal{O}_x$, and $\mathscr{K}_x=\mathscr{R}_x$, for any $x\in X$.

Now if we consider the homomorphism of presheaves $C^U\circ d_U$: $\mathscr{R}_U\to\mathscr{I}_U$, we find that the kernel is the direct sum of $\ker d_U=\mathcal{O}_U$ and $\{j_Uf; f\in\operatorname{Ker}C^U\}=\mathscr{G}_U$, hence $\ker C^U\circ d_U=\mathscr{F}_U$. $C^U\circ d_U$ induces the sheaf homomorphism $c\circ d$: $\mathscr{R}\to\mathscr{I}$;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_U & \longrightarrow & \mathscr{R}_U & \xrightarrow{C^U\circ d_U} & \mathscr{I}_U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle r_U} & & \downarrow{\scriptstyle \rho_U} & & \\ 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{R} & \xrightarrow[c\circ d]{} & \mathscr{I} & \longrightarrow & 0. \end{array} \tag{1.13}$$

The last sequence is exact. Similarly we have the exact sequence;

$$0\longrightarrow\mathscr{K}\longrightarrow\mathscr{R}\xrightarrow{b\circ d}\mathscr{G}\longrightarrow 0. \tag{1.14}$$

Proposition 1.7. *$\Gamma(U, \mathscr{F}) = \mathscr{F}_U$ for a small set U.*

This follows from $\Gamma(U, \mathscr{G}) = \mathscr{G}_U$ for small U.

Proposition 1.8. $\Gamma_a(U, \mathscr{I}) = 0$,

$$\Gamma_a(U, \mathscr{P}) = \Gamma(U, \mathscr{G}) .$$

Proof. We may assume $U \ni a$. Let $M \in \Gamma_a(U, \mathscr{I})$. $M = \rho_\omega(p-q)$ for some $p, q \in \mathscr{I}^+_\omega$, $a \in \omega \subset U$. $M = 0$ on $\omega \backslash a$ implies $h = p - q \in \mathcal{O}_{\omega-a}$. From [7], [12] we have $p = p - Bp = \inf\{u \in \mathscr{P}^+_\omega ; u = p + s$ on $\omega \backslash a$ for some $s \in \mathscr{S}_{\omega-a}\}$. Since q satisfies the condition in the bracket, $p \leqq q$. Similarly $q \leqq p$, hence $h = 0$ and $M = 0$. $\Gamma_a(U, \mathscr{I}) = 0$ is proved. From Proposition 1.5 we have

$$0 \longrightarrow \Gamma_a(U, \mathscr{G}) \longrightarrow \Gamma_a(U, \mathscr{P}) \longrightarrow \Gamma_a(U, \mathscr{I}) \longrightarrow 0 .$$

Therefore $\Gamma_a(U, \mathscr{P}) = \Gamma_a(U, \mathscr{G}) = \Gamma(U, \mathscr{G})$.

§2. Minimal harmonic sheaf

We shall show that the collection of harmonic sheaves $\mathcal{O}$ on Y contains a minimal one, which can be constructed from $\mathscr{H}$.

For any open neighborhood V of a and $f \in C(\partial V)$, let $\bar{H}^V f(x) = \inf\{s(x);\ s$ is a superharmonic function on $V \backslash a$ such that

$$\left.\begin{array}{l} \liminf\limits_{V \ni y \to \xi} s(y) \geqq f(\xi) \quad \text{on} \quad \partial V, \\ \text{and} \\ \liminf\limits_{V \ni y \underset{\neq}{\to} a} s(y) \geqq 0 \end{array}\right\} .$$

$\bar{H}^V f$ is a harmonic function on $V \backslash a$, and $\bar{H}^V f = -\bar{H}^V(-f)$ for any $f \in C(\partial V)$. Following P. Loeb [10] we see that, for any outer regular compact subset K of X,

$$\lim_{V \ni y \to \xi} \bar{H}^V f(y) = f(\xi) \qquad \text{on} \quad \partial V,$$

where $V=Y\backslash K$, and that $\mathscr{V}=\{Y-K;\ K$ is an outer regular compact subset of $X\}$ forms a fundamental system of neighborhoods of a.

For any open set $V\ni a$, we set

$$\bar{\mathcal{O}}_V=\{h\in\mathscr{H}_{V-a}:\ \text{there is an } \omega\in\mathscr{V},\ \bar{\omega}\subset V, \text{ such that } h=\bar{H}^{\omega}[h|\partial\omega] \text{ on } \omega-a\},$$

and

$$\bar{\mathcal{O}}_a=\varinjlim_{V\ni a}\bar{\mathcal{O}}_V\,.$$

We define a harmonic sheaf $\bar{\mathcal{O}}$ on Y be letting $\bar{\mathcal{O}}_y=\bar{\mathcal{O}}_a$ if $y=a$, and $\bar{\mathcal{O}}_y=\mathscr{H}_y$ if $y\in X$. The quantities corresponding to $\bar{\mathcal{O}}$ are indicated by the line over the letters ($\bar{H}^{\omega}f$, $\bar{\mathscr{S}}_G$, $\bar{\mathscr{R}}$, $\bar{\mathscr{P}}$, $\bar{\mathscr{K}}$, $\bar{\mathscr{I}}$, etc.);
are exact. We shall list some elementary properties:

$$0\longrightarrow\bar{\mathcal{O}}\longrightarrow\bar{\mathscr{R}}\longrightarrow\bar{\mathscr{P}}\longrightarrow 0\,,\ \text{etc.}$$

(2.1) *Let U be an open neighborhood of a. Then every non-negative superharmonic function on $U\backslash a$ is $\bar{\mathcal{O}}$-superharmonic on U.*

(2.2) *Every $U\in\mathscr{V}$ is an $\mathcal{O}$-regular set for any harmonic sheaf $\mathcal{O}$ on Y.* ([8], [18])

(2.3) *Let $\mathcal{O}$ be a harmonic sheaf on Y. We have $H^{\omega}f\geqq\bar{H}^{\omega}f$ on $\omega\backslash a$ for any $\omega\in\mathscr{V}$ and $f\in C(\partial\omega)$, $\geqq 0$.*

(2.4) *Every non-negative $\mathcal{O}$-superharmonic function on G is $\bar{\mathcal{O}}$-super harmonic on G.* This follows from (2.3).

(2.5) *Every $\mathcal{O}$-potential on G is an $\bar{\mathcal{O}}$-potential on G.* For, let K be a compact subset of G, then we have

$$\inf\{s\in\bar{\mathscr{S}}_+(V);\ s\geqq p \text{ on } V-K\}\leqq \inf\{s\in\mathscr{S}_+(V);\ s\geqq p \text{ on } V-K\},$$

from (2.4). If p is an $\mathcal{O}$-potential the last quantity tends to 0 as

K runs through all compact subsets of V, hence p is also an $\bar{\mathcal{O}}$-potential.

Here we shall introduce an order between the harmonic sheaves on Y and show that $\bar{\mathcal{O}}$ is the minimal sheaf. For any two harmonic sheaves $\mathcal{O}_1$ and $\mathcal{O}_2$ we denote $\mathcal{O}_1 \succ \mathcal{O}_2$ if; for every $\omega \in \mathscr{V}$ and every $f \in C(\partial\omega)$, $\geqq 0$, we have $H^{1,\omega}f \geqq H^{2,\omega}f$ on $\omega \backslash a$. Here $H^{i,\omega}$, $i=1, 2$, have the same meaning as in section 1 for $\mathcal{O}_i$, and it is well defined by virtue of (2.2). This relation defines an order on harmonic sheaves and $\bar{\mathcal{O}}$ is a minimal one.

The harmonic space $(X, \mathscr{H})$ is said to be *hyperbolic* if, for some (any) $\omega \in \mathscr{V}$, the lower envelope $H(s, \omega)$ of functions in $\{u \in \mathscr{S}_{\omega-a}$: $\liminf_{\omega \backslash a \ni x \to \xi} u(x) \geqq 0$ on $\partial\omega$ and $\liminf_{\omega \backslash a \ni x \to a} u(x) \geqq 1\}$ is not zero, and *parabolic* if $H(a, \omega)=0$ for some (any) $\omega \in \mathscr{V}$.

Proposition 2.1. *The following conditions are equivalent.*

(i) $(X, \mathscr{H})$ *is parabolic.*

(ii) $\bar{H}^{\omega}1=1$ *for any* $\omega \in \mathscr{V}$.

(iii) $\mathscr{P}_X^+=(0)$

(iv) $1 \in \bar{\mathcal{O}}_Y$

(v) $\bar{\mathscr{P}}_Y=(0)$.

Proof. The equivalence of conditions (i), (ii) and (iii) is found in [10]. Conditions (ii), (iv) and (v) are obviously equivalent.

Proposition 2.2. *If* $(X, \mathscr{H})$ *is hyperbolic, then*; (i) *the* $\bar{\mathcal{O}}$-*superharmonic functions on* Y *coincide with the non-negative superhamonic functions on* X *and they are* $\bar{\mathcal{O}}$-*potentials on* Y, (ii) $\mathscr{S}_Y^+=\mathscr{P}_X^+$, $\mathscr{G}_Y^+=\mathscr{H}_X^+$, (iii) *we have* $\liminf_{\omega \cap X \ni x \to a} \bar{H}^{\omega}1(x)=0$ *for any regular neighborhood* ω *of* a.

Proof. (i) and (ii) are verified easily (Thm. 11. of [7]). To prove (iii) let $\alpha=\liminf_{\omega \cap X \ni x \to a} \bar{H}^{\omega}1(x)$. $\alpha \geqq 0$ is obvious. Suppose $\alpha>0$. Since X

is small there is a $p \in \mathscr{P}_X^+$ such that $p=\bar{H}^\omega 1$ on $\omega' \cap X$ for some $a \in \omega' \subset \bar{\omega}' \subset \omega$ (Lemma 1.3), and such that $p \geqq \frac{\alpha}{2}$ on $\omega' \cap X$. $p-\frac{\alpha}{2}$ being a superharmonic function on X, $p \geqq \frac{\alpha}{2}$ follows from the minimum principle. We have seen $p \in \bar{\mathscr{S}}_Y^+$, $\frac{\alpha}{2} \in \bar{\mathscr{G}}_Y^+$ and $p \succ \frac{\alpha}{2}$, that is $p-\frac{\alpha}{2} \in \mathscr{S}_X^+=\mathscr{P}_Y^+$, hence $\frac{\alpha}{2}$ must be zero. This is a contradiction and we have $\alpha=0$.

Theorem 2.3. *If* $(X, \mathscr{H})$ *is parabolic then* $\bar{\mathcal{O}}$ *is the unique harmonic sheaf on* Y.

Proof. Let $\mathcal{O}$ be a harmonic sheaf on Y. From (2.3) we have $1 \geqq H^\omega 1 \geqq \bar{H}^\omega 1=1$, so $1 \in \mathcal{O}_Y$. Let $f \in C_+(\partial\omega)$ and $u=H^\omega f-\bar{H}^\omega f$. $u \in \mathscr{H}_{\omega-a}$ and $0 \leqq u \leqq MH^\omega 1-m\bar{H}^\omega 1=M-m$, where $M=\sup f$ and $m=\inf f$. By the definition we have $-u \geqq (m-M)H(s, \omega)$, but the latter is equal to zero since $(X, \mathscr{H})$ is parabolic. Hence $u=0$ and $H^\omega f=\bar{H}^\omega f$. This shows $\mathcal{O}=\bar{\mathcal{O}}$.

§3. Cohomology groups of harmonic sheaves

In this section we shall calculate various cohomology groups of the sheaf $\mathcal{O}$, for example, $H^q(\omega, \mathcal{O})$ for a small set ω, $H^q(X, \mathscr{H})$ and $H^q(Y, \mathcal{O})$. The vanishing of the cohomology $H^1(Y, \mathscr{F})$ is fundamental.

As we have seen in section 1, $\mathcal{O}$ has the fine resolution $0 \to \mathcal{O} \to \mathscr{R} \xrightarrow{d} \mathscr{P} \to 0$, so we have $H^q(Y, \mathcal{O})=0$ for $q \geqq 2$, and $H^1_\Phi(Y, \mathcal{O})=\dfrac{\Gamma_\Phi(Y, \mathscr{P})}{d\Gamma_\Phi(Y, \mathscr{R})}$ where Φ is any support family.

Theorem 3.1. *If* $1 \notin \mathcal{O}_Y$ *then* $H^1(Y, \mathcal{O})=0$ *and* $\mathcal{O}_Y=\{0\}$.

Proof. The facts that $\mathcal{O}_Y=0$ and Y is small are easily proved. Take $M \in \Gamma(Y, \mathscr{P})$. For each $x \in Y$ there are domains U_x, V_x with $x \in V_x \subset U_x$ such that $M|V_x=\rho_{U_x}(p_x)$ on V_x for some $p_x \in \mathscr{P}_{U_x}$. We can choose a finite set $x_1, \dots, x_n$ and corresponding $V_i=V_{x_i}$, $U_i=U_{x_i}$ with the above properties such that $Y=\bigcup_{i=1}^n V_i$. Moreover we may assume $a \in V_n$ and $a \notin V_i$, $1 \leqq i \leqq n-1$. Let $s_i, t_i \in \mathscr{P}_{V_i}^+$ be such that

$s_i - t_i = \rho_{U_i}^{V_i}(p_{x_i})$; $M|V_i = \rho_{vV_i}(s_i - t_i)$. Now let $\{\varphi_i\}$ be a partition of unity (for the sheaf $\mathscr{P}$) subordinate to $\{V_i\}$. We have $\varphi_i \circ M = \rho_{V_i}(\varphi_i \circ s_i - \varphi_i \circ t_i)$ on V_i. Since Y is small and the carrier of $\varphi_i \circ s_i$ is a compact subset of V_i, there is a $p_i \in \mathscr{P}_Y^{\dagger}$ with the same carrier such that $p_i - \varphi_i \circ s_i \in \mathcal{O}_{V_i}$ (Lemma 1.4). Similarly there is a $q_i \in \mathscr{P}_Y^{\dagger}$ such that the carrier is compact in V_i and such that $q_i - \varphi_i \circ t_i \in \mathcal{O}_{V_i}$. Hence $\varphi_i \circ M = \rho_{V_i}(\rho_Y^{V_i} p_i - \rho_Y^{V_i} q_i) = \rho_Y(p_i - q_i)$ on V_i. Evidently $\varphi_i \circ M = 0 = \rho_Y(p_i - q_i)$ out of V_i, and so $\varphi_i \circ M = \rho_Y(p_i - q_i)$. Let $f = \sum_{i=1}^{n} (p_i - q_i)$. $f \in \mathscr{P}_Y$ and $\rho_Y f = \sum_{i=1}^{n} \varphi_i \circ M = M$, that is, $df = M$ if f and $j_Y f \in \Gamma(Y, \mathscr{R})$ are identified. This proves $H^1(Y, \mathcal{O}) = 0$.

Given a sheaf $\mathscr{A}$ on Y. For any open two cover $\{U, V\}$, $U \cup V = Y$, we have the following exact sequence:

$$(3.1) \qquad 0 \longrightarrow \mathscr{A}_Y \xrightarrow{\alpha} \mathscr{A}_U \times \mathscr{A}_V \xrightarrow{\beta} \mathscr{A}_{U \cap V} \longrightarrow H^1(Y, \mathscr{A}),$$

where $\alpha f = (f|U, f|V)$, $f \in \mathscr{A}_Y$, and

$$\beta(g, h) = g|U \cap V - h|U \cap V, \qquad (g, h) \in \mathscr{A}_U \times \mathscr{A}_V.$$

This is Mayer-Vietoris theorem. If (3.1) is applied to $\mathcal{O}$ with $1 \notin \mathcal{O}_Y$, the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{O}_U \times \mathcal{O}_V \xrightarrow{\beta} \mathcal{O}_{U \cap V} \longrightarrow 0$$

follows. But in case $1 \in \mathcal{O}_Y$, $\mathcal{O}_{U \cap V}$ does not coincide with the image of β. M. Nakai and B. Walsh introduced the concept of flux functionals to characterize the image of β when $1 \in \mathcal{O}_Y$. Here we summarize it.

Theorem 3.8 of [18] states that, *for any open cover $\{U, V\}$ of Y such that $V \ni a$ and $U \subset X$, there corresponds a linear functional $\Psi^{(U,V)}$ (called the flux functional) on $\mathcal{O}_{U \cap V}$ such that:*

(1) *$s \in \mathcal{O}_{U \cap V}$ is in the image of β iff $\Psi^{(U,V)}[s] = 0$.*

(2) *If $\{U', V'\}$ is another open cover of Y such that $U' \subset U$, $V' \subset V$, then $\Psi^{(U',V')}[s|U' \cap V'] = \Psi^{(U,V)}[s]$ for $s \in \mathcal{O}_{U \cap V}$.*

(3) *If $s \in \mathcal{O}_{U \cap V}$ is a restriction of an $\mathcal{O}$-superharmonic function p*

on U (resp. V) then $\Psi^{(U,V)}\mathcal{O}[s]\geqq 0$ (resp. $\Psi^{(V,U)}[s]\leqq 0$), equality holding iff p is $\mathcal{O}$-harmonic in U (resp. in V).

Lemma 3.2. *Suppose $1\in\mathcal{O}_Y$ and let $\{U, V\}$ be an open cover of Y such that either U or V is contained in X and V is small. For any $y\in V\backslash U$ and $h\in\mathscr{H}_{U\cap V}$ we can find $p\in\mathscr{P}_V^+$, $f\in\mathcal{O}_V$ and $g\in\mathcal{O}_U$ with the properties;*

(i) $h=\Psi^{(U,V)}[h]\cdot p+f-g$ *on* $U\cap V$,

(ii) *the carrier of p is the point $\{y\}$ and $\Psi^{(U,V)}[p|U\cap V.]=1$.*

Proof. Take a $q\in\mathscr{P}_V^+$ with the carrier $\{y\}$. Property (3) of flux functionals implies $\Psi^{(U,V)}[q|U\cap V]>0$ or <0 according to $V\subset X$ or $V\ni a$. Let $p=(\Psi^{(U,V)}[q|U\cap V])^{-1}\cdot q$ (resp. $-(\Psi^{(U,V)}[q|U\cap V])^{-1}\cdot q$), and set $u=h-\Psi^{(U,V)}[h]\cdot(p|U\cap V)$. The flux $\Psi^{(U,V)}[u]$ being 0, u is of the form $u=f-g$ for $f\in\mathcal{O}_V$ and $g\in\mathcal{O}_U$. Lemma is proved.

Before we proceed to the cohomology $H^1(Y, \mathcal{O})$ we shall investigate some fundamental properties enjoyed by the sheaf $\mathscr{F}$: $\mathscr{F}$ depends only on the initial harmonic space $(X, \mathscr{H})$ and $H^q(Y, \mathscr{F})=0$ for $q\geqq 1$.

Proposition 3.3. (i) *$\Gamma(U, \mathscr{F})=\mathscr{H}_{U\searrow a}$ for any open set U, consequently $\mathscr{F}_y=\varinjlim_{U\ni y}\mathscr{H}_{U-y}$ for any $y\in Y$ and $\mathscr{F}$ does not depend on the choice of $\mathcal{O}$.*

(ii) $H_a^1(Y, \mathscr{F})=0$

(iii) *$H_\Phi^q(Y, \mathscr{F})=0$ for $q\geqq 2$ and any support family Φ.*

Proof. By virtue of the fine resolution (1.18) of $\mathscr{F}$ we have $H_\Phi^q(Y, \mathscr{F})=0$ for any Φ and $q\geqq 2$, and we have $H_a^1(Y, \mathscr{F})=\dfrac{\Gamma_a(Y, \mathscr{S})}{(c\circ d)\Gamma_a(Y, \mathscr{R})}$. From Proposition 1.8 $H_a^1(Y, \mathscr{F})$ vanishes. By the same reason $H_a^1(U, \mathscr{F})=0$ for any open set U. Hence we have the exact sequence;

$$0=\Gamma_a(U, \mathscr{F})\longrightarrow\Gamma(U, \mathscr{F})\longrightarrow\Gamma(U-a, \mathscr{F})\longrightarrow 0.$$

Therefore $\Gamma(U, \mathscr{F})=\Gamma(U-a, \mathscr{F})=\mathscr{H}_{U\searrow a}$.

Theorem 3.4. $H^1(Y, \mathscr{F})=0$

Proof. By the above proposition we have the exact sequence; $0\to\bar{\mathcal{O}}\to\mathscr{F}\to\bar{\mathscr{G}}\to 0$. If $(X, \mathscr{H})$ is hyperbolic we have $H^1(Y, \bar{\mathcal{O}})=0$ from Theorem 3.1. Hence the exactness of $0=H^1(Y, \bar{\mathcal{O}})\to H^1(Y, \mathscr{F})\to H^1(Y, \bar{\mathscr{G}})=0$ yields $H^1(Y, \mathscr{F})=0$. Next suppose that $(X, \mathscr{H})$ is parabolic. In this case $\bar{\mathcal{O}}$ is the unique harmonic sheaf on Y. We shall prove $H^1(Y, \mathscr{F})=\frac{\Gamma(Y, \mathscr{I})}{(\bar{c}\circ\bar{d})\Gamma(Y, \mathscr{R})}=0$. (In the following, to the ends of this proof, we shall omit the upper lines of the letters which indicate the quantities related to the minimal harmonic sheaf $\bar{\mathcal{O}}$.) Take a $M\in\Gamma(Y, \mathscr{I})$. As in Theorem 3.1. we can choose a finite cover $\{U_i\}_{i=1}^n$ of Y and $q_i\in\mathscr{I}_{U_i}$ for each i such that $a\in U_n$, $a\notin U_i$ $(1\leqq i\leqq n-1)$ and $M|U_i=\rho_{U_i}q_i$. Let $\{V_i\}_{i=1}^n$ be an open cover of Y such that $\bar{V}_i\subset U_i$, $i=1,\dots,n$ and no proper subfamily covers Y. Let $\{\varphi_i\}_{i=1}^n$ be a partition of unity for the sheaf $\mathscr{I}$ subordinate to V_i, which was explained in section 1. We have $\varphi_i\circ M=\rho_{U_i}(\varphi_i\circ q_i)$ on U_i. Take a $y_i\in V_i-\bigcup_{j\neq i}V_j$ for each $i=1,\dots,n-1$, and $y_n=a$. Lemma 3.2 applied to $\varphi_i\circ q_i|U_i\setminus\bar{V}_i\in\mathscr{H}_{U_i\setminus\bar{V}_i}$ yields the existence of $w_i\in\mathscr{P}_{U_i}$ with $\mathrm{carr.}(w_i)=\{y_i\}$, $h_i\in\mathcal{O}_{U_i}$ and $u_i\in\mathcal{O}_{Y-\bar{V}_i}$ such that $\varphi_i\circ q_i=\alpha_i w_i+h_i-u_i$ on $U_i-\bar{V}_i$, where $\alpha_i=\Psi^{(U_i, Y-\bar{V}_i)}[\varphi_i\circ q_i|U_i-\bar{V}_i]$. In particular $w_n\in\mathscr{G}_{U_n}$. Let

$$f_i=\begin{cases}\varphi_i\circ q_i-\alpha_i w_i-h_i & \text{on } U_i\\ -u_i & \text{on } Y-\bar{V}_i.\end{cases}$$

Then $f_i\in\Gamma(Y, \mathscr{R})$ for each i. It is easy to verify that $(c\circ d)f_i=df_i=\rho_{U_i}(\varphi_i\circ q_i-\alpha_i w_i)$ on V_i, $=0$ on $Y-V_i$, for each $i=1,\dots,n-1$, and that $(c\circ d)f_n=\rho_{U_n}(\varphi_n\circ q_n)$ on V_n, $=0$ on $Y-V_n$. If we set $N=M-\sum_{i=1}^n(c\circ d)f_i\in\Gamma(Y, \mathscr{I})$ we have $N_y=\sum_{i=1}^n(\varphi_i\circ M-(c\circ d)f_i)_y=\rho_{U_i}^y(\alpha_i w_i)$ for $y\in V_i$, $i=1,\dots,n-1$, and $N|V_n=0$. Take a small set $W\subset X$ containing $\bigcup_{i=1}^{n-1}\bar{V}_i$. Lemma 1.4 yields, for each $i=1,\dots,n-1$, the existence of a $v_i\in\mathscr{P}_W$ such that the carrier of v_i is $\{y_i\}$ and $v_i-w_i\in\mathcal{O}_{V_i}$. Since $\sum_{i=1}^{n-1}\alpha_i v_i$ is harmonic on $W\cap V_n$, again applying Lemma 3.2, we have $\sum_{i=1}^{n-1}\alpha_i v_i=p+g-h$ on $W\cap V_n$ for some $p\in\mathscr{G}_{V^n}$, $g\in\mathcal{O}_{V_n}$ and $h\in\mathcal{O}_W$. Let

$$f=\begin{cases}\sum_{i=1}^{n-1}\alpha_i v_i+h & \text{on } W\\ p+g & \text{on } V_n\end{cases}$$

It follows that $f\in\Gamma(Y,\mathscr{R})$ and $(c\circ d)f=\rho_W^y(\sum_{i=1}^{n-1}\alpha_i v_i)$ on W, $=0$ on V_n. On the other hand, by the choice of w_i, $i=1,\dots,n-1$, we have $\rho_W^y(\sum^{n-1}\alpha_k v_k)=\rho_{U_i}^y(\alpha_i w_i)$ if $y\in V_i$, $1\leqq i\leqq n-1$, and $=0$ if $y\in W\backslash\bigcup_{j=1}^{n-1}V_j$. Thus we have shown $(c\circ d)f=N$ on Y. Therefore $M=(c\circ d)(f+\sum_{i=1}^{n}f_i)\in(c\circ d)\Gamma(Y,\mathscr{R})$.

Theorem 3.5. *$H^1(U,\mathscr{F})=0$ for any open set U, in particular $H^1(X,\mathscr{H})=0$.*

This follows from Excision Theorem;

$$0=H^1(Y,\mathscr{F})\longrightarrow H^1(U,\mathscr{F})\longrightarrow H^2_{Y-U}(Y,\mathscr{F})=0$$

If we take $U=X$ we have $H^1(X,\mathscr{H})=0$.

$H^1(X,\mathscr{H})=0$ was proved by B. Walsh [19] under an additional hypothesis; X possesses an exhaustion by small relatively compact open sets $\{U_i\}_{i=1}^{\infty}$ such that $\bar{U}_i\subset U_{i+1}$ and every element of $\Gamma(U_{i+1},\mathscr{H})$ can be uniformly approximated on $\bar{U}$ by restrictions of functions in $\Gamma(U_{i+2},\mathscr{H})$ to U_{i+1}. The argument there was the classical one due to Mittag-Leffler [4].

Theorem 3.6. *$H^1(\omega,\mathscr{O})=0$ for any small set ω.*

Proof. Since $\mathscr{F}|X=\mathscr{O}|X=\mathscr{H}$, we have $H^1(\omega,\mathscr{O})=0$ for any open set $\omega\subset X$. Now suppose that ω is a small neighborhood of a. From Proposition 1.5 (iii) we have $\Gamma(\omega,\mathscr{G})\cong\mathscr{G}_\omega$. Let $M\in\Gamma(\omega,\mathscr{G})$ and let $p\in\mathscr{G}_\omega$ be such that $\rho_\omega p=M$. Then $f=r_\omega p\in\Gamma(\omega,\mathscr{F})$ and $df=M$. This shows the exactness of $\Gamma(\omega,\mathscr{F})\xrightarrow{d}\Gamma(\omega,\mathscr{G})\to 0$. On the other hand we know that the exactness of the following sequence holds:

$\Gamma(\omega,\mathscr{F})\to\Gamma(\omega,\mathscr{G})\to H^1(\omega,\mathscr{O})\to H^1(\omega,\mathscr{F})=0$, which follows from

(1.11) and the above Theorem. Hence $H^1(\omega, \mathcal{O})=0$.

In the following we shall show $H^1(Y, \mathcal{O})=R^1$ when the constant 1 is $\mathcal{O}$-harmonic on Y. Let $M\in\Gamma(Y, \mathcal{G})$ and let $\{U, \omega\}$ be a covering of Y with ω small and $U\subset X$. There is a $p\in\mathcal{G}_\omega$ such that $\rho_\omega p= M|\omega$. We shall define $\Psi(M)=\Psi^{(U,\omega)}[p|U\cap\omega]$. This is well defined, that is, does not depend on the choice of (U, ω) and $p\in\mathcal{G}_\omega$. For, let $\{U', \omega'\}$ be another pair such that $a\in\omega'\subset\omega$ and $U'\subset U$, then $M|\omega'=\rho_{\omega'}p'$ with $p'=\rho_\omega^{\omega'}p\in\mathcal{G}_{\omega'}$. We have $p-p'\in\mathcal{O}_{\omega'}$ and, from the above property (3) of flux functionals, $\Psi^{(\omega',U')}[p'|\omega'\cap U']=\Psi^{(\omega',U')}[p|\omega'\cap U']+\Psi^{(\omega',U')}[(p'-p)|\omega'\cap U']=\Psi^{(\omega',U')}[p|\omega'\cap U']$. This equals to $\Psi^{(\omega,U)}[p|\omega\cap U]$ from the property (2).

Lemma 3.7. *Let $M\in\Gamma(Y, \mathcal{G})$. There is a $t\in\Gamma(Y, \mathcal{F})$ such that $dt=M$ iff $\Psi(M)=0$.*

Proof. Suppose $M=dt$ with $t\in\Gamma(Y, \mathcal{F})$. Take a small neighborhood ω of a. From Proposition 1.7, $\Gamma(\omega, \mathcal{F})=\mathcal{F}_\omega$, so $r_Y^\omega t$ has the decomposition $r_Y^\omega t=h+p$ with $h\in\mathcal{O}_\omega$ and $p\in\mathcal{G}_\omega$ Since $M|\omega= dr_Y^\omega t=\rho_\omega(d_\omega r_Y^\omega t)=\rho_\omega p$, we have $\Psi(M)=\Psi^{(\omega,U)}[p]$ with a cover $\{U, \omega\}$ of Y, $U\subset X$. On the other hand from the property (3) of the flux functional it follows that $\Psi^{(\omega,U)}[t|\omega\cap U]=\Psi^{(\omega,U)}[h|\omega\cap U]=0$. Hence $\Psi(M)=\Psi^{(\omega,U)}[p]=\Psi^{(\omega,U)}[t]-\Psi^{(\omega,U)}[h]=0$. Conversely let $M\in\Gamma(Y, \mathcal{G})$ be such that $\Psi(M)=0$. For a small open set $\omega\ni a$, let $p\in\mathcal{G}_\omega$ be such that $M|\omega=\rho_\omega p$. Then $\Psi^{(\omega,U)}[p|\omega\cap U]=\Psi(M)=0$, and the above property (1) yields $p=f-g$ on $\omega\cap U$ for some $f\in\mathcal{O}_\omega$ and $g\in\mathcal{O}_U$. Let

$$t=\begin{cases} p-f & \text{on} \quad \omega \\ -g & \text{on} \quad U. \end{cases}$$

Then $t\in\Gamma(Y, \mathcal{F})$ and $dr_Y^\omega t=dr_Y^\omega(p-f)=\rho_\omega p=M$ on ω. Obviously $dt= 0=M$ on U, so $dt=M$ on Y.

Lemma 3.8. *There is an $M_0\in\Gamma(Y, \mathcal{G})$ with $\Psi(M_0)>0$.*

Proof. As was noted in section 1 there is a non-zero function $p\in\mathscr{G}_V^+$ for any small set $V\ni a$. Let $M_0\in\Gamma(Y,\mathscr{G})$ be defined as $-\rho_V p$ on V and 0 on $Y-V$. We have $\Psi(M_0)>0$ from the property (3) of flux functionals.

Theorem 3.9. *If* $1\in\mathcal{O}_Y$, *then* $\mathcal{O}_Y\cong R^1$ *and* $H^1(Y,\mathcal{O})\cong R^1$

Proof. $H^1(Y,\mathcal{O})\cong\dfrac{\Gamma(Y,\mathscr{G})}{d\Gamma(Y,\mathscr{F})}$ follows from (1.11) and $H^1(Y,\mathscr{F})=0$. Lemmas 3.7 and 3.8 yield $\Gamma(Y,\mathscr{G})\supsetneqq\{M\in\Gamma(Y,\mathscr{G});\ \Psi(M)=0\}=d\Gamma(Y,\mathscr{F})$. Hence $H^1(Y,\mathcal{O})\cong\left\{\dfrac{\Psi(M)}{\Psi(M_0)}M_0\right\}\cong R^1$. $\mathcal{O}_Y\cong R^1$ is easily proved.

Theorem 3.10. $H_a^1(Y,\mathcal{O})\cong\Gamma(Y,\mathscr{G})$.

Proof. From the exactness of $0=\Gamma_a(Y,\mathscr{F})\to\Gamma_a(Y,\mathscr{G})\to H_a^1(Y,\mathcal{O})\to H_a^1(Y,\mathscr{F})=0$, we have the assertion because $\Gamma_a(Y,\mathscr{G})=\Gamma(Y,\mathscr{G})$.

§4. Duality of harmonic sheaves

In this section we shall introduce the adjoint harmonic sheaf $\mathcal{O}^*$ of $\mathcal{O}$, and shall investigate some fundamental duality relation between $\mathcal{O}$ and $\mathcal{O}^*$. In particular it can be shown that the space of germs of $\bar{\mathcal{O}}$-harmonic functions at infinity (equipped with an inductive topology) has as its dual the space of adjoint harmonic functions on X, whenever $(X,\mathscr{H})$ is hyperbolic. This is a generalization of a classical result obtained by H. G. Tillmann [17] and A. Grothendieck [3]. First we shall explain some additional hypotheses and their consequences.

(4.1) For any small set U every potential of $\mathscr{S}_U^+$ with the one-point carrier is proportional to each other; if $p,q\in\mathscr{S}_U^+\cap\mathcal{O}_{U-\{y\}}$, $y\in U$, then there is a constant $c=c(y)$ such that $p=c(y)q$.

Note that the above y is necessarily in $U\backslash a$, for any potential with one-point carrier $\{a\}$ belongs to $\mathscr{G}_U$.

Under assumption (4.1) there exist kernel functions on small sets. More precisely, if V is a small set, one can find, for each $y\in V\backslash a$, a potential $p_y(\cdot)\in\mathscr{S}_U^+$ in such a way that p_y has carrier $\{y\}$ and that the function $(x,y)\to p_y(x)$ on $(V\backslash a)\times(V\backslash a)$ is a lower semicontinuous function, continuous off the diagonal [6, Prop. 18.1] [7, Prop. 5.8]. From Theorem 5.9 of [7] every $p \in \mathscr{I}_V$ has a unique integral representation:

$$p(x) = \int p_y(x)\mu(dy) \qquad \text{on} \quad V \backslash a,$$

by a Radon measure μ on $V\backslash a$ supported by the carrier of p.

After B. Walsh we patch together local potentials to give a system (on Y) of normalized kernels.

(4.2) (B. Walsh [19] Thm. 1.6.) *Let $\{V_i\}_{i\in I}$ be a cover of Y by small regions. There exists a corresponding set of kernel functions $\{p_y^i(\cdot)\}_{i\in I}$ such that for each ordered pair (i, j) of indices with $V_i \cap V_j \neq \phi$ and each region $U \subset V_i \cap V_j$, the relation $\rho_{V_i}^U p_y^i = \rho_{V_j}^U p_y^j$ in $\mathscr{P}_U$ holds for all $y \in U\backslash a$, and this quantity gives a kernel function on U.*

(4.3) ([19], Prop. 1.8.) *For any small set V there is a unique kernel function $p_y(\cdot)$ on V such that for every $y \in V\backslash a$ and every $i \in I$ with $y \in V_i$ there exists a neighborhood U of y in $V \cap V_i$ on which $\rho_V^U p_y = \rho_{V_i}^U p_y^i$.*

Though B. Walsh stated the aboves for small sets contained in X, we can verify them in the above form by virtue of some results from [7]. The pair $(\{V_i\}, \{p_y^i\})$ is called a *normalization* for $\mathscr{O}$ and the above p_y on V is called a *normalized kernel* (with respect to the given normalization).

Adjoint sheaf of $\mathscr{O}$

The adjoint sheaf of $\mathscr{O}$ is introduced in the same manner as it was done by R. M. Hervé and B. Walsh for the case of Brelot's harmonic sheaf $\mathscr{H}$. Though we must modify them to introduce the germ at infinity of the adjoint sheaf, but it is easy and here we shall not give proofs.

Definition. A subdomain G of a small set V is called a *c.d.* *≪complètement déterminant≫ set* if ${}^V R^{V-G} p = p$ for every potential p on V which is harmonic on G. Here ${}^V R^A s$, the *reduced function* of nonnegative $\mathcal{O}$-superharmonic function s on $A\subset V$, is defined as

inf$\{t$; t is a non-negative $\mathcal{O}$-superharmonic function on V such that $t\geqq s$ on $A\}$.

We shall, henceforth, for the rest of this paper, make the following hypothesis.

(4.4) There is a basis for the topology on Y composed of c. d. sets.

Let V be a small set, and ω a relatively compact open subset of V. Let p_y be a kernel function on V. Since ${}^{V}R^{V-\omega}p_y\in\mathcal{S}_V^+$ and is $\mathcal{O}$-harmonic off $\partial\omega$ for any $y\in V\backslash a$, there is a unique measure ${}^{*}H_y^{\omega}(dz)\geqq 0$ on $\partial\omega$ which represent ${}^{V}R^{V-\omega}p_y$:

$$ {}^{V}R^{V-\omega}p_y=\int p_x(\cdot){}^{*}H_y^{\omega}(dz) \qquad \text{on} \quad V-a\,. $$

The *adjoint presheaf* ${}^{V}\mathcal{O}$ *(formed on* V *by using* p_y*)* is defined as follows;

$$ ({}^{V}\mathcal{O}^{*})_U=\{f\in C(U\backslash a);\; f(y)=\int f(z){}^{*}H_y^{\omega}(dz) \text{ for every c. d. set } \omega\subset\overline{\omega}\subset U \text{ and every } y\in\omega\}\,. $$

${}^{V}\mathcal{O}^{*}$ is seen to be a complete presheaf on V. ${}^{V}\mathcal{O}^{*}$-regular sets are defined as in section 1 and the ${}^{V}\mathcal{O}^{*}$-regular sets coincide with the c. d. sets in V, hence forms a basis for the topology on V. Harnack's principle for ${}^{*}\mathcal{O}^{V}$ also holds. The proofs are carried in the same way as [6]. If ${}^{U}\mathcal{O}^{*}$, $U\subset V$, is the adjoint sheaf formed on U by $q_y(\cdot)=\rho_V^U p_y$, we have ${}^{V}\mathcal{O}^{*}|U={}^{U}\mathcal{O}^{*}$. This is proved with the aid of the following property of reduced functions [19, Prop. 1.14 and Lemma 1.15].

(4.5) *Let* U *be a small set and* V, W *open subsets of* U *with* $W\subset\overline{W}\subset V$. *Let* s *be a potential on* V *with the carrier a compact subset of* W. *Then* $({}^{U}\hat{R}^{U-W}s)|V={}^{V}\hat{R}^{V-W}(s|V)$.

(4.6) [19, Thm. 1.16] *If* $(\{V_i\}_{i\in I}, \{p_y^i\}_{i\in I})$ *is a normalization of* $\mathcal{O}$ *on* Y, *then* ${}^{i}\mathcal{O}^{*}|V_i\cap V_j={}^{j}\mathcal{O}^{*}|V_i\cap V_j$ *for any indices* i *and* j. *Here*

${}^{i}\mathcal{O}^*$ *denotes the adjoint sheaf on* V_i *formed by* p_y^i.

We define the *global adjoint sheaf* $\mathcal{O}^*$ of $\mathcal{O}$ by the following; for any open set $G\subset Y$, $f\in C(G\backslash a)$ belongs to $\mathcal{O}_G^*$ iff $f|V_i\cap G\in({}^{i}\mathcal{O}^*)_{V_i\cap G}$ for each index i with $V_i\cap G\neq\phi$

Let V be a small domain and p_y be the normalized kernel on V given in (4.3). Then the adjoint sheaf induced on V by p_y is precisely $\mathcal{O}^*|V$, [19, Prop. 1.18]. If we set $\mathscr{H}^*=\mathcal{O}^*|X$, then $(X, \mathscr{H}^*)$ 2) satisfies Brelot's axioms 1, 2, 3, and $\mathcal{O}^*$ satisfies our hypothesis (1.1)~(1.3). Moreover, by taking an appropriate normalization, we may assume that $1\in\mathscr{H}_X^{**}$ and the hypothesis (1.4) are satisfied. $(X, \mathscr{H}^*)$ is hyperbolic iff $(X, \mathscr{H})$ is so.

Now we shall establish a duality between $\mathcal{O}_K=\varinjlim_{U\subset K}\mathcal{O}_U$ and $\mathcal{O}^*_{V\smallsetminus K}$ for any small set V and its compact subset K.

Let $\mathcal{O}_K\in f$. f is $\mathcal{O}$-harmonic on a neighborhood $U\subset V$ of K. Take open sets ω_1, ω_2 such that $K\subset\omega_1\subset\omega_2\subset U$, each ω_i is relatively compact in the following. Since $\mathscr{R}$ is a fine sheaf there is a $\varphi\in\Gamma(V, \mathscr{R})$ which equals to 1 on ω_1 and 0 on $V\backslash\omega_2$. Let $t=f\cdot\varphi$ on U, $=0$ on $V-U$, then $t\in\Gamma(V, \mathscr{R})$. There is a $p\in\mathscr{P}_V$ with $dt=\rho_V p$. Since $dt=0$ on ω_1, $p\in\mathscr{S}_V$ follows if $K\ni a$, and if $K\not\ni a$ we may suppose $V\not\ni a$, so p is also in $\mathscr{S}_V$ in this case. p has the integral representation

$$p=\int p_y\mu(dy)$$

be a measure μ supported by $\bar{\omega}_2\backslash\omega_1$, for p is $\mathcal{O}$-harmonic on $(V\backslash\bar{\omega}_2)\cup\omega_1$. By the way, $t-p\in\mathcal{O}_V$ follows from the definitions of t and p. But the facts that $t=0$ near ∂V and $p\in\mathscr{S}_V$ yield $t-p=0$ (Minimum principle), consequently, $f=p$ on ω_1.

Lemma 4.1. *$f=0$ in $\mathcal{O}_K$ iff the corresponding measure μ satisfies $\mu^*H^G=0$ 3) for some open set $G\subset\bar{G}\subset V\backslash K$ with* $\operatorname{Supp}\mu\subset G$.

2) $(X, \mathscr{H}^*)$ is nothing but the Hervé's adjoint sheaf of $(X, \mathscr{H})$ [6], so it is determined from $\mathscr{H}$ and independent from $\mathcal{O}$.

3) $\mu^*H^G(f)=\int\mu(dy)^*H^G f(y)$ for any $f\in C(\partial G)$.

Proof. Since $f=p$ on $\omega_1, f=0$ in $\mathcal{O}_K$ implies $p=0$ on a neighborhood of K. Take G so as to satisfy $\mathrm{Supp}\,\mu\subset G\subset\bar{G}\subset V\backslash K$ and $p=0$ out of G. Let $p=p_+-p_-$, $p_\pm\in\mathscr{P}_V^+$. Then $p_+=p_-$ on $V\backslash G$, so ${}^V R^{V-G}p_+={}^V R^{V-G}p_-$. Hence

$$\int p_z(\cdot)\int\mu_+(dy)^*H_y^G(dz)=\int{}^V R^{V-G}p_y(\cdot)\mu_+(dy)={}^V R^{V-G}p_+$$

$$={}^V R^{V-G}p_-=\int p_z(\cdot)\int\mu_-(dy)^*H_y^G(dz)\,.$$

From the uniqueness of represention of potentials in $\mathscr{P}_V^+$, we have $\mu_+{}^*H^G=\mu_-{}^*H^G$, that is, $\mu^*H^G=0$. Conversely, from $\mu^*H^G=0$ follows ${}^V R^{V-G}p_+={}^V R^{V-G}p_-$, so $p_+=p_-$ on $V\backslash G$. Thus $p=0$ on a neighborhood of K and $f=0$ in $\mathcal{O}_K$.

We define a bilinear form of $h\in\mathcal{O}_K$ and $h^*\in\mathcal{O}_{V-K}^*$ by

$$\mathscr{b}(h, h^*)=\int h^*d\mu\,,$$

where μ is the above measure corresponding to h. This expression is in fact independent of the choice of μ and depends only on h and h^*. For, if μ' is another measure which also corresponds to h, then the Radon measure $\mu-\mu'$ represents the zero elements of $\mathcal{O}_K$ and by the above lemma there is a $G\subset\bar{G}\subset V\backslash K$ such that $\mathrm{Supp}(\mu-\mu')\subset G$ and $(\mu-\mu')^*H^G=0$. Since $h^*=H^{*G}h^*$ on G we have $\int h^*d(\mu-\mu')=0$.

Since $p_y(\cdot)\in\mathcal{O}_{V-\{y\}}$ for $y\in V\backslash K$ it defines a $k_y=r_V^K(p_y)\in\mathcal{O}_K$.[4)]

Lemma 4.2. (i) *The map $y\to k_y$ from $V\backslash K$ to $\mathcal{O}_K$ is continuous, where $\mathcal{O}_U$, for any open U, is equipped with the topology of uniform convergence on compact subset of U, and $\mathcal{O}_K$ is given the inductive limit topology $\mathcal{O}_K=\varinjlim_{U\supset K}\mathcal{O}_U$.*

(ii) *Let ω be a c.d. set, $\bar{\omega}\subset V-K$, and $y\in\omega\backslash a$. Take a finite partition $\pi=(\delta_j)_{j=1}^n$ of $\partial\omega$, $\partial\omega=\bigcup_{j=1}^n\delta_j$, and choose a point y_j from each δ_j. The sum*

$$\mathscr{S}_y^\pi=\sum_\pi k_{y_j}(\cdot)\int_{\delta_j}{}^*H_y^\omega(dz)$$

4) $r_V^K\colon\Gamma(V,\mathscr{R})\longrightarrow\Gamma(K,\mathscr{R})$.

converges to k_y *in* $\mathcal{O}_K$ *as* π *becomes finer.*

Proof. (i) Let $y_n \to y_0$ in $V \backslash K$. Take an open neighborhood ω of K such that $\bar{\omega} \not\ni \{y_n, y_0\}$. Then p_{y_n} converges to p_{y_0} uniformly on any compact subset of ω [7, Prop. 6.10]. Hence $k_{y_n} \to k_{y_0}$ in $\mathcal{O}_K$. (ii) Let G be a neighborhood of K with $\bar{G} \cap \bar{\omega} = \phi$. $I_y^\pi \in \mathcal{O}_G$ and $x \to \int p_z(x) * H_y^\omega(dz) \in \mathcal{O}_G$. The Riemann sum $I_y^\omega(x)$ converges to $\int p_z(x) * H_y^\pi(dz) = p_y(x)$ for each $x \in G$, so uniformly on compacta of G (consequence of Harnack's prinicple). Thus $I_y^\pi \to k_y$ in $\mathcal{O}_K$.

Lemma 4.3. *Let* ω, ω' *be open sets such that* $K \subset \omega' \subset \omega' \subset \omega$. *Then the closed linear hull of*

$$H = \{y \longrightarrow p_y(x)|_{y \in \omega - \bar{\omega}'}\,;\ x \in \omega' \backslash a\}$$

in $\mathcal{O}^*_{\omega - \bar{\omega}'}$ *contains the restriction* $\mathcal{O}^*_{V-K}|_{\omega - \bar{\omega}'}$.

Proof. It is enough to show that, for any measure ν on $U = \omega - \bar{\omega}'$ with compact support, the relation $\int p_y(x)\nu(dy) = 0$ for any $x \in \omega' \backslash a$ implies the relation $\int h^*(y)\nu(dy) = 0$ for any $h^* \mathcal{O}^*_{V \backslash K}$. But $r_V^K\left(\int p_y(x)\nu(dy)\right)$ defines an element of $\mathcal{O}_K$ which equals to zero from assumption, hence $\mathscr{I}(h, h^*) = \int h^* d\nu = 0$.

Lemma 4.4. *Let* $h^* \in \mathcal{O}^*_{V-K}$. $h^* = r_V^{V-K} f^*$ *for some* $f^* \in \mathcal{O}^*_V$ *iff* $\mathscr{I}(h, h^*) = 0$ *for any* $h \in \mathcal{O}_K$.

Proof. $\mathscr{I}(h, h^*)$ was defined as $\int h^* d\mu$ by the measure representing $p = \int p_y \mu(dy) \in \mathscr{S}_V$ such that $p = h$ on a neighborhood of K. Moreover as we have seen at that paragraph $p = 0$ near ∂V. This yields, as in the proof of Lemma 4.1, $\mu * H^G = 0$ for an open set $G \subset V$. Now let $h^* \in \mathcal{O}^*_{V-K}$ be such that $h^* = r_V^{V-K} f^*, f^* \in \mathcal{O}^*_V$. Then $\mathscr{I}(h, h^*) = \int f^* d\mu = \int {}^*H^G f^* d\mu = \int f^* d\mu * H^G = 0$. Conversely suppose that $\mathscr{I}(h, h^*) = 0$ for any $h \in \mathcal{O}_K$. Choose a c.d. set U such that $K \subset U \subset \bar{U} \subset V$ and $y \in$

$U\backslash K$. Let $\mu=\varepsilon_y-{}^*H_y^U$. Since $r_V^K(\int p_y(\cdot)\mu(dy))\in\mathcal{O}_K$, it follows from the assumption $\int h^*d\mu=0$, that is $h^*(y)={}^*H^Uh^*$(fi). Let $f^*=h^*$ on $V\backslash K$, and $={}^*H^Uh^*(\cdot)$ on U. Then f^* is a well defined element of $\mathcal{O}_V^*$, and $h^*=r_V^{V-K}f^*$.

Theorem 4.5. (i) *For every continuous linear form F on $\mathcal{O}_K$ equipped with the inductive topology, there is an $h_F^*\in\mathcal{O}_{V-K}^*$ such that $F(h)=\mathscr{I}(h, h_F^*)$ for any $h\in\mathcal{O}_K$.*

(ii) *Each linear form $h\to\mathscr{I}(h, h^*)$ induced by an element $h^*\in\mathcal{O}_{V-K}^*$ on $\mathcal{O}_K$ is continuous.*

(iii) *The dual of $\mathcal{O}_K$ is isomorphic to $\mathcal{O}_{V-K}^*/\mathcal{O}_V^*$ by the pairing induced by $\mathscr{I}(\cdot,\cdot)$.*

Proof. (i) For $y\in V\backslash K$, let $h_F^*(y)=F(k_y)$. From Lemma 4.2 (i) h_F^* is continuous on $V\backslash K$. From Lemma 4.2 (ii) $F(I_y^\pi)$ converges to h_F^* as π becomes finer, where I_y^π is the same as Lemma 4.2 for a c. d. set $\omega\subset\bar{\omega}\subset V\backslash K$ and $y\in\omega\backslash a$. The sum $F(I_y^\pi)=\sum_\pi h_F^*(y_j)^*H_y^\omega(\delta_j)$ converges to $\int h_F^*(z)^*H_y^\omega(dz)$. Hence $h_F^*(y)={}^*H^\omega h_F^*(y)$, and $h_F^*\in\mathcal{O}_{V\setminus K}^*$. Let $h\in\mathcal{O}_K$ and let μ be the corresponding measure; $h=r_V^K\left(\left(\int p_y\mu(dy)\right)\right)$, Supp. μ is compact in $V\backslash K$. Take a partition $\pi=(\delta_j)$ of the support μ; $\bigcup_{i=1}^n\delta_j=$Supp. μ, and from each δ_j take a $y_j\in\delta_j$. As in Lemma 4.2 (ii) it can be proved that the sum $\sum_\pi p_{y_j}(\cdot)\mu(\delta_j)$ converges to $\int p_y(\cdot)\mu(dy)$ in $\mathcal{O}_K$ as $\pi\downarrow 0$. Therefore $\sum_\pi h_F^*(y_j)\mu(\delta_j)$ converges to $F(h)$. On the other hand $\sum_\pi h_F^*(y_j)\mu(\delta_j)$ obviously converges to $\int h_F^*(y)\mu(dy)$, so we have $F(h)=\int h_F^*d\mu=\mathscr{I}(h, h_F^*)$.

(ii) We shall prove that $h\to\mathscr{I}(h, h^*)$ is a continuous linear form on $\mathcal{O}_U$ for any open set U containing K. Let ω be an open neighborhood of K with $\bar{\omega}\subset U$. By Lemma 4.3 we can find a sequence of functions on $U-\bar{\omega}$ of the form $y\to\sum_{j=1}^{n_k}\alpha_{jk}p_y(x_{jk})$ $(k=1, 2,\dots)$ where $\{x_{jk}\}\subset\omega\backslash a$, that converges uniformly on compacta in $U-\bar{\omega}$ to h^* as $k\to\infty$. We may suppose $r_U^Kh\in\mathcal{O}_K$ has the representation $h|\omega=$

$\left(\int p_y\mu(dy)\right)\Big| \to \omega$ with $\operatorname{supp}\mu\subset U-\bar{\omega}$. Therefore $\sum_{j=1}^{n_k}\alpha_{jk}h(x_{jk})$ converges to $\int h^*(y)\mu(dy)=\mathscr{E}(h, h^*)$ as $k\to\infty$. Since $h\to\sum_{j=1}^{n_k}\alpha_{jk}h(x_{jk})$ is a continuous linear form on $\mathcal{O}_U$ for each k, Banach-Steinhaus theorm yields the continuity of $h\to\mathscr{E}(h, h^*)$ on $\mathcal{O}_U$. Thus $\mathscr{E}(h, h^*)|_{h\in\mathcal{O}_U}\in(\mathcal{O}_U)'$ for each open neighborhood U of K, which proves (ii).

(iii) follows from Lemma 4.4.

Corollary 4.6. $(\mathcal{O}_K)'\cong H^1_K(V, \mathcal{O}^*)=H'_K(Y, \mathcal{O}^*)$

Proof. If we consider the Čech two cover $\{V, V\backslash K\}$ of V, we have $H^1_K(V, \mathcal{O}^*)=\mathcal{O}^*_{V-K}/\mathcal{O}^*_V$ by Leray's theorem and the fact $H^1(V, \mathcal{O}^*)=0$. The latter follows from a version of Theorem 3.6 for the adjoint sheaf. $H^1_K(V, \mathcal{O}^*)=H^1_K(Y, \mathcal{O}^*)$ comes from excision theorem.

Corollary 4.7. $(\mathcal{O}_a)'\cong H^1_a(Y, \mathcal{O}^*)=\Gamma(Y, \mathscr{G}^*)$.

Corollary 4.8. *If* $x\in X$ *is polar we have* $\dim(\mathcal{O}_x)'=1$.

Proof. We have $(\mathcal{O}_x)'\cong H^1_x(V, \mathcal{O}^*)=\Gamma_x(V, \mathscr{P}^*)$, where V is a small neighborhood of x. If $x\in X$ is polar then every ajoint potential on V with the one-point carrier $\{x\}$ is proportional to each other, so dim $(\mathcal{O}_x)'=1$.

Let $H^1_c(X, \mathscr{H}))$ be the cohomology group of $\mathscr{H}$ with compact supports. We have $H^1_c(X, \mathscr{H})=\dfrac{\Gamma_c(X, \mathscr{P})}{d\Gamma_c(X, \mathscr{R})}$. (Note $\mathscr{R}|X$, $\mathscr{P}|X$ and $d|X$ are determined from $\mathscr{H}$ only.) In the following we shall investigate the exact sequences that are in duality;

$$(4.7)\qquad 0\longrightarrow \mathcal{O}_Y \xrightarrow{\ i\ } \mathcal{O}_a \xrightarrow{\ \varphi\ } H^1_c(X, \mathscr{H}) \xrightarrow{\ \psi\ } H^1(Y, \mathcal{O}) \longrightarrow 0$$

$$\begin{array}{c} 0\longleftarrow H^1(Y, \mathcal{O}^*) \xleftarrow{\ \nu^*\ } H^1_a(Y, \mathcal{O}^*) \xleftarrow{\ \mu^*\ } \mathscr{H}^*_X \xleftarrow{\ i\ } \mathcal{O}^*_Y \longleftarrow 0\,, \\ \qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\quad \Gamma(Y, \mathscr{G}^*) \end{array}$$

where $\mathcal{O}$ is a harmonic sheaf on Y. First we define the map φ as follows. Let $h\in\mathcal{O}_a$. Let h be $\mathcal{O}$-harmonic on a neighborhood U

of a. Take open sets ω_1, ω_2 such that $a\in\omega_1\subset\omega_2\subset U$, each ω_i is relatively compact in the following. Then there is a $t\in\Gamma(Y, \mathscr{R})$ which equals to h on ω_1 and to 0 on $Y-\omega_2$. We have $dt|X\in\Gamma_c(X, \mathscr{P})$ and if we take another t' in the same manner as the above we have $t-t'\in\Gamma_c(X, \mathscr{R})$, hence dt is uniquely determined from h up to $d\Gamma_c(X, \mathscr{R})$. Thus $\varphi h=[dt]\in H_c^1(X, \mathscr{H})$ is well defined. If $h=r_Y^a g$ with a $g\in\mathcal{O}_Y$ then $t-g\in\Gamma_c(X, \mathscr{R})$ and $dt=d(t-g)$, so $\varphi h=0$. Conversely if $h\in\mathcal{O}_a$ satisfies $\varphi h=0$, there is a $t'\in\Gamma_c(X, \mathscr{R})$ such that $dt=dt'$. Let $g=t-t'$. Then $g\in\mathcal{O}_Y$ and $g=t=h$ near a, that is, $h=r_Y^a g$. We have proved the exactness of $0\to\mathcal{O}_Y\to\mathcal{O}_a\xrightarrow{\varphi}H_c^1(X, \mathscr{H})$.

If $1\notin\mathcal{O}_Y$ then $\mathcal{O}_Y=H^1(Y, \mathcal{O})=0$ and Y is small. Let $M\in\Gamma_c(X, \mathscr{P})$. We can think $M\in(Y, \mathscr{P})$ by letting it equal to 0 at a. $M=\rho_Y p$ for some $p\in\mathscr{P}_Y$. Obviously $p\in\mathcal{O}_{Y-\mathrm{Supp}[M]}$ and $h=r_Y^a p\in\mathcal{O}_a$. $\varphi h=[M]$ in $H_c^1(X, \mathscr{H})$ is verified from the definition. This proves that φ is a surjection and the exactness of the upper line of (4.7) follows.

For the case $1\in\mathcal{O}_Y$, we shall define the flux of an $M\in\Gamma_c(X, \mathscr{P})$ and shall define the map ψ: $H_c^1(X, \mathscr{H})\to H^1(Y, \mathcal{O})=R^1$.

Let $V\subset X$ be a small set containing the support of M and ω be a neighborhood of a with $\omega\cup V=Y, \omega\cap\mathrm{Supp}(M)=\phi$. Let $\rho_V p$ with $p\in\mathscr{P}_V$. $p|\omega\cap V\in\mathscr{H}_{\omega\cap V}$ and by virtue of Properties (2) and (3) of flux functionals $\Psi^{(\omega, V)}[p|\omega\cap V]$ is independent from the choice of V, ω and p. We set $\Psi(M)=\Psi^{(\omega, V)}[p|\omega\cap V]$. Suppose $M=df$ for some $f\in\Gamma_c(X, \mathscr{R})$. Let V be a small neighborhood of $K=\mathrm{Supp}(f)\supset\mathrm{Supp}(M)$. We have $M|V=\rho_V p$ for a $p\in\mathscr{P}_V$ and $h=f-p\in\mathscr{H}_V$, hence $p|V-K$ is the restriction of h on $V-K$. From Property (3) of flux functionals $\Psi(M)=0$ follows. By these arguments, for any $\alpha\in H_c^1(X, \mathscr{H})$, $\psi\alpha=\Psi(M)$ is a well defined real number, here $M\in\Gamma_c(X, \mathscr{P})$ is a representative of α.

We shall prove; (i) ψ is surjective, (ii) $\psi\circ\varphi=0$, (iii) if $\psi\alpha=0$, $\alpha\in H_c^1(X, \mathscr{H})$, then $\alpha=\varphi h$ for some $h\in\mathcal{O}_a$.

Proof of (i). It is enoguh to show the existence of a $M_0\in\Gamma_c(X, \mathscr{P})$ with $\Psi(M_0)<0$. Fix an arbitrary point $z\in X$. Let $(\{V_i\}, \{p_y^i\})$ be a normalization for $\mathcal{O}$, ((4.2), (4.3)). Take a V_i containing z and let $M_0=\rho_{V_i}(p_z^i)$ on V_i, $=0$ on $X-V_i$. $M_0\in\Gamma_c(X, \mathscr{P})$ and from (4.3)

M_0 is defined independently from the choice of V_i containing z. The number $\Psi(M_0)$ is strictly positive from Property (3) of flux functionals.

Proof of (ii). Let $h \in \mathcal{O}_a$ and ω_i, $i=1, 2$, and $t \in \Gamma(Y, \mathscr{R})$ be as in the previous, that is, $t=h$ on a neighborhood ω_1 of a and $\varphi h = [dt] \in H_c^1(X, \mathscr{H})$. Let W be a small set in X with $W \cup \omega_1 = Y$, and let $p \in \mathscr{P}_W$ be such that $dt|W = \rho_W p$. $g=(t-p)|W$ is harmonic on W and we have $p=h-g$ on $W \cap \omega_1$. Hence $\Psi^{(W,\omega_1)}[p|W \cap \omega_1]=0$, that is, $\psi \circ \varphi(h) = \Psi[dt]=0$.

Proof of (iii). Suppose $\psi\alpha=0, \alpha \in H_c^1(X, \mathscr{H})$. Let $M \in \Gamma_c(X, \mathscr{P})$ be a representative of α. Take a small set V with $K=\mathrm{Supp}(M) \subset V \subset X$. Let $M|V=\rho_V p$ with $p \in \mathscr{P}_V$. Since $\Psi^{(V,V-K)}[p|V-K]=\Psi(M)=0$, p can be written as $p=h-g$ on $V-K$ with $h \in \mathcal{O}_{Y-K}$ and $g \in \mathscr{H}_V$. Let $k \in \Gamma(Y, \mathscr{R})$ be defined by $k=p+g$ on V, and $=h$ on $Y-K$. If $t \in \Gamma(Y, \mathscr{R})$ is chosen as in the above to satisfy $t=h$ near a and $\varphi(r^a_{Y-K}h)=[dt]$, we have $(t-k)|X \in \Gamma_c(X, \mathscr{R})$ and $M=dt+d(k-t)$. Therefore $\alpha=[M]=[dt]=\varphi(r^a_{Y-K}h)$.

Remark: Given a normalization $(\{V_i\}, \{p_y^i\})$, we define $M'_y \in \Gamma_c(X, \mathscr{P})$ for each $y \in X$, by $M'_y = \rho_{V_i}(p_y^i)$ on V_i, $=0$ on $Y-V_i$. If we let $q_y^i = (\Psi(M'_y))^{-1} p_y^i$, the renormalization $(\{V_i\}, \{q_y^i\})$ satisfies $\Psi(M_y)=1$, here $M_y=\rho_{V_i}(q_y^i)$ on V_i, $=0$ on $Y-V_i$. In the sequal we shall deal with the normalization of $\mathcal{O}$ with $\Psi(M_y)=1$ for any $y \in X$. This implies that the constant functions are in H_X^* because $y \to \Psi(M_y) \in \mathscr{H}_X^*$ can be verified as in Lemma 4.2.

Proposition 4.9. *We have the exact sequence;*

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathscr{H}_X \xrightarrow{\mu} H_a^1(Y, \mathcal{O}) = \Gamma(Y, \mathscr{G}) \xrightarrow{\nu} H^1(Y, \mathcal{O}) \longrightarrow 0.$$

Proof. Since $\mathscr{H}_X \cong \Gamma(Y, \mathscr{F})$ from Proposition 3.3, we can define $\mu: \mathscr{H}_X \to \Gamma(Y, \mathscr{G})$ as the composition of the maps $\mathscr{H}_X \to \Gamma(Y, \mathscr{F})$ and $d: \Gamma(Y, \mathscr{F}) \to \Gamma(Y, \mathscr{G})$. If $1 \notin \mathcal{O}_Y$ we have $\mathscr{F}_X \cong \Gamma(Y, \mathscr{G})$ as is easily verified. In case $1 \in \mathcal{O}_Y$ the flux $\Psi(M)$ of a $M \in \Gamma(Y, \mathscr{G})$ was defined in section 3. We shall define the map $\nu: \Gamma(Y, \mathscr{G}) \to H^1(Y, \mathcal{O}) = R^1$ by $\nu(M)=\Psi(M)$. Lemma 3.7 and some observation yield the exactness of the above sequence.

From the discussion we have made now the exactness of the sequences of (4.7) follows. Now we come to the duality assertion of these sequences. Again it is easy to show the duality for the case $1\notin\mathcal{O}_Y$. In this case we have $H_c^1(X,\mathscr{H})\cong\mathcal{O}_a$ and $\mathscr{H}_X^*\cong\Gamma(Y,\mathscr{G}^*)$. Hence Corollary 4.7 implies $H_c^1(X,\mathscr{H})'\cong\mathscr{H}_X^*$. Here the topology of $H_c^1(X,\mathscr{H})$ is defined as the inductive topology by the map $\varphi:\mathcal{O}_a\rightarrow H_c^1(X,\mathscr{H})$.

In the following we assume $1\in\mathcal{O}_Y$. The duality between $\mathcal{O}_a$ and $\Gamma(Y,\mathscr{G}^*)$ is denoted by $<h, M^*>$. Remember that $<h, M^*>=\mathscr{E}(h, h^*)$, where $M^*|V={}^*\rho_V p^*$ with a $p^*\in\mathscr{G}_V^*$ for a small set $V\in a$ and $h^*=p^*|V-a\in\mathscr{H}_{V-a}^*$.

For any $\alpha\in H_c^1(X,\mathscr{H})$ and $h^*\ni\mathscr{H}_X$ we define the bilinear form

$$[\alpha, h^*]=\int h^*dm,$$

where the measure m is defined as follows; let $M\in\Gamma_c(X,\mathscr{P})$ be a representative of α. For a small set $V\subset X$ containing the support of M, there is a $p\in\mathscr{P}_V$ with $M|V=\rho_V p$. m is the measure of the representation of p,

$$p=\int p_y m(dy),$$

for the unique kernel function p_y on V subordinate to the normalization stated before ((4.3)). From [19] this is a well defined belinear form on $H_c^1(X,\mathscr{H})\times\mathscr{H}_X^*$.

Here we shall show $<h, \mu^*h^*>=[\varphi h, h^*]$ for any $h\in\mathcal{O}_a$ and $h^*\in\mathscr{H}_X^*$. Let ω_i, $i=1, 2$, and $t\in\Gamma(Y,\mathscr{R})$ be the same notation as we have used frequently; $t=h$ on ω_1, $\varphi h=[dt]$. Let V be a small neighborhood of ω_2 and W be a small subset of X such that $W\cup\omega_1=Y$. $<h, \mu^*h^*>$ was defined as $\mathscr{E}(h, h^*|V-a)=\int h^*dn$ with the aid of a measure n such that $dt|V=\rho_V(\int^V p_y n(dy))$, and the support of n is contained in $\bar{\omega}_2-\omega_1$. On the other hand, as we have seen in the above, $[\varphi h, h^*]=\int h^* dm$ for a measure such that $dt|W=\rho_W(\int {}^W p_y m(dy))$. The support of m is also contained in $\bar{\omega}_2-\omega_1$. Since $\rho^V{}_W\rho^W({}^W p_y)=\rho^V{}_V\rho^W({}^V p_y)\equiv q_y$ for any $y\in W\cap V$, we have $dt|W\cap V=\rho_{W\cap V}(\int q_y m(dy))$ $=\rho_{W\cap V}(\int q_y n(dy))$. Hence $m=n$ and $<h, \mu^* h^*>=[\varphi h, h^*]$.

Theorem 4.10. *The exact sequences of* (4.7) *are in duality. In particular* $H_1^c(X, \mathscr{H})'\cong\mathscr{H}_X^*$.

Proof. We have seen that,

$$\mathcal{O}_a \longrightarrow H_c^1(X, \mathscr{H}) \longrightarrow R^1 \longrightarrow 0,$$

and

$$\Gamma(Y, \mathscr{G}^*) \longleftarrow H_X^* \longleftarrow R^1 \longleftarrow 0$$

are exact and that $[\varphi h, h^*]=<h, \mu^* h^*>$. If we prove $\psi\alpha=[\alpha, 1]$, $\alpha\in H_c^1(X, \mathscr{H})$, the relation $H_c^1(X, \mathscr{H})'\cong\mathscr{H}_X^*$ follows from $\mathcal{O}'_a\cong\Gamma(Y, \mathscr{G}^*)$. Let $M\in\Gamma_c(X, \mathscr{P})$ be a representative of α and $M|W=\rho_W(\int {}^W p_y m(dy))$. Let ω be a neighborhood of a with $\mathrm{Supp}(m)\subset W\cap\omega$. As we have remarked before $\Psi(M_y)=\Psi^{(W,\omega)}[{}^W p_y|W\cap\omega]=1$ for $y\in W\cap\omega$. Since the flux functional $\Psi^{(W,\omega)}$ is continuous on $\mathscr{H}_{W\cap\omega}$[19], we have $\Psi(M)$ $=\int\Psi(M_y)m(dy)=\int m(dy)=[\alpha, 1]$. Hence $\psi\alpha=[\alpha, 1]$.

Corollary 11. $\psi\alpha=[\alpha, 1]$,

$$v^* M=<h_0, M> \textit{ for some } h_0\in\mathcal{O}_a.$$

We need to prove the second. Fix a small sets V, V' such that $V\ni a$, $V'\subset X$, and $V\cup V'=Y$. Take a refinement of this cover composed by c.d. sets $\omega\subset V$ and $\omega'\subset V'$. In [18] the flux functional on $\mathscr{H}_{V\cap V'}^*$ is defined by ${}^*\Psi^{(V,V')}[h|V\cap V']=\int h d(n-n^* H^{\omega'})$ for a measure n on $\partial\omega$ satisfying $n=n^* H^{\omega'} H^{\omega'}$. Let $h_0=r_V^a\left(\left(\int {}^V p_y n-n^* H^{\omega'}\right)(dy)\right)$. Then $h_0\in\mathcal{O}_a$ and, for any $M\in\Gamma(Y, \mathscr{G}^*)$, $<h_0, M>=\int g(y)d(n-n^* H^{\omega'})(y)=$ ${}^*\Psi^{(V,V')}[g|V\cap V']={}^*\Psi(M)$, where $g\in\mathscr{G}_V^*$, $M={}^*\sigma_V g$.

§5. Decomposition of $\mathscr{G}_a$. Further description of the duality of $\mathcal{O}_a$ and $H^1_a(Y, \mathcal{O}^*)$

In the following we shall blow up the point at infinity to obtain a fine structure of the stalk $\mathscr{G}_a$; every germ in $\mathscr{G}_a$ is represented as an integral of extreme germs. We shall introduce an ideal boundary, called Kuramochi boundary with respect to $\mathcal{O}$, in which the extreme germs of $\mathscr{G}_a$ are homeomorphically embedded. When the adjoint structure exists these are carried also for $\mathscr{G}^*_a$, and it can be shown that for any germ in $\mathcal{O}_a$ there corresponds a function (like a normal derivative) on the Kuramochi boundary with respect to $\mathcal{O}^*$. The duality $\mathscr{I}(h, h^*)$ of $\mathcal{O}_a$ and $H^1_a(Y, \mathcal{O}^*)$ is represented by an integral of these normal derivative-like functions, this is Green's formula. The condition for the duality to be separated is given.

Let V be a small neighborhood of a and $\{\omega\}$ with $a\in\omega\subset\bar{\omega}\subset V$ be ordered by inclusion relation. If each $\mathscr{G}_\omega$ is equipped with the topology of compact uniform convergence on $\omega\backslash a$, $(\mathscr{G}_\omega, \rho^{\omega}_{\omega'})$ forms a (strict) inductive system of locally covex topological vector spaces (moreover, nuclear spaces). $\mathscr{G}_a=\varinjlim \mathscr{G}_\omega$ is complete (and nuclear). Since $\mathscr{G}^+_\omega$ is a lattice with respect to the order defined by; $u\succ v$ iff $u-v\in\mathscr{G}^+_\omega$ (Thm. 1.6 of [7]), $\mathscr{G}^+_a=\bigcup_{a\in\omega}\rho^a_\omega\mathscr{G}^+_\omega$ is a lattice by the induced order and $\mathscr{G}_a$ ordered by the cone $\mathscr{G}^+_a$ is also a lattice; $\mathscr{G}_a=\mathscr{G}^+_a-\mathscr{G}^+_a$. Every $\mathscr{G}^+_\omega$ being a metrisable convex cone with a compact base, $\mathscr{G}^+_a$ is also a metrisable convex cone with a compact base, which is denoted by $\mathscr{K}$. Things being so we can apply Choquet's representation thoerem; every $M\in\mathscr{G}^+_a$ has a unique integral representa tion

$$M=\int m\lambda(dm) \tag{5.1}$$

with the aid of a measure λ on $\mathscr{K}$ supported by the extreme points of $\mathscr{K}$. Let $\mathscr{K}_\omega$ be a compact base of the convex cone $\mathscr{G}^+_\omega$ and $\mathscr{K}$ be the canonical image of $\mathscr{K}_\omega$; $\mathscr{K}=\rho^a_\omega\mathscr{K}_\omega$, which is obviously compact. It is easy to verify that ρ^a_ω gives a bijective correspondence of the extreme points of $\mathscr{K}_\omega$ to those of $\mathscr{K}$.

In the following we shall summarize the results in [7, 12] concerning ideal boundary of $(X, \mathscr{H})$ with respect to $\mathcal{O}$, and shall translate (5.1) to the integral representation on the boundary.

Suppose that the proportionality hypothesis is satisfied on V, and let $p_y(x)$ be the kernel function on V introduced in (4.3). Let $x_0 \in V\backslash a$ and $\alpha_0(y)$ a continuous function on $V\backslash a$ such that $\alpha_0(y) = p_y(x_0)$ on $\omega\backslash a$ for some $a \in \omega \subset \bar{\omega} \subset V$, and let $k_y(x) = [\alpha_0(y)]^{-1} p_y(x)$. There is a locally convex topological vector space $\mathbf{E}$ which is a vector lattice with positive cone the set of potentials on V. Let $\mathscr{P}', \mathscr{S}'$ be the same as $\mathscr{P}_V^+$ and $\mathscr{S}_V^+$ except the elements being continuous on $V\backslash a$. $\mathscr{P}'$ is a complete metrisable convex cone with a compact base, and $\mathscr{G}_V^+$ is closed in P'. (The topology of $\mathbf{E}$ induced on $\mathscr{G}_V^+$ coincides with the topology of compact uniform convergence on $V\backslash a$.) Let $\mathscr{E}_i$ (resp. $\mathscr{E}_b$) be the set of extreme points of $\mathscr{S}' \cap \mathscr{K}'$ (resp. $\mathscr{G}_V^+ \cap \mathscr{K}'$) where $\mathscr{K}'$ is a compact base of $\mathscr{P}'$ such that $k_y \in \mathscr{K}'$. Then the map $y \to k_y$ gives a homeomorphism of $V\backslash a$ onto $\mathscr{E}_i$. The uniformity on $V\backslash a$ given by the fundamental system of entourages

$$\{(y_1, y_2);\ y_1, y_2 \in V\backslash a,\ |f_j(y_1)k_{y_1}(x_j) - f_j(y_2)k_{y_2}(x_j)| < \varepsilon,\ 1 \leqq j \leqq n\},$$

where $x_j \in V\backslash a$, and $f_j \in C_c(V\backslash x_j)$, $j = 1, 2, \ldots, n$, is the inverse image by the map $y \to k_y$ of the uniformity on $\mathbf{E}$ restricted on $\mathscr{E}_i$, and $y \to k_y$ gives an isomorphism of these two uniform spaces. Therefore the completion $\tilde{V}$ of $V\backslash a$ is homeomorphic to the closure $\bar{\mathscr{E}}_i$ of $\mathscr{E}_i$, and $y \to k_y$ is extended to be a continuous map $\xi \to k_\xi$ from $\tilde{V}$ to $\bar{\mathscr{E}}_i$. $\tilde{V}$ contains a point ξ_0 where $k_{\xi_0} = 0$, that is, if $y_n \in V\backslash a$ converges to a point of ∂V then $k_{y_n} \to 0$ in $\mathbf{E}$. We shall omit this point.

We have proved in [7] that the relation $\bar{\mathscr{E}}_b \subset \bar{\mathscr{E}}_i\backslash(\mathscr{E}_i \cup \{0\}) \subset \mathscr{G}_V^+$ holds. Let $\Delta = \tilde{V}\backslash(V\backslash a) \cup \{\xi_0\}$ and $\Delta_1 = \{\xi \in \Delta;\ k_\xi \in \mathscr{E}_b\}$. Δ_1 is a G_δ-set. Every $u \in \mathscr{G}_V^+$ has the unique integral representation

$$u(x) = \int k_\xi(x)\mu(d\xi). \qquad x \in V\backslash a,$$

by a measure μ on Γ_1. Δ and Δ_1 are unchanged if we carry the above procedure for any $\omega, a\in\omega\subset V$, and every $u\in\mathscr{G}_*^+$ has the representation;

$$u(x)=\int_{\varDelta_1}(\rho_V^\omega k_\xi)(x)\mu(d\xi),$$

this is a consequence of Theorem 7.8 of [7]. We call $\varDelta$ the $\mathcal{O}$-ideal boundary and $\varDelta_1$ the minimal points of $\varDelta$.

Let $\mathscr{K}=\{\rho_V^a g; g\in\mathscr{K}'\cap\mathscr{G}_V^+\}$. Then $\mathscr{K}$ is a compact base of the cone $\mathscr{G}_a^+$, and $\mathscr{K}'\cap\mathscr{G}_V^+$ is homeomorphic to $\mathscr{K}$ by the definition of the topology of $\mathscr{G}_a$ and the bijective property of ρ_V^a. As was mentioned previously the extreme points of $\mathscr{K}'\cap\mathscr{G}_V^+$ corresponds one to one and onto to the extreme points of $\mathscr{K}$, hence $\varDelta_1\cong$the extreme points of $\mathscr{K}$; $\xi\to\rho_V^a(k_\xi)$. Thus (5.1) becomes

$$M=\int_{\varDelta_1}\rho_V^a(k_\xi)\mu(d\xi), \tag{5.2}$$

where $M=\rho_V^a g$, $g\in\mathscr{G}_V^+$, and

$$g=\int_{\varDelta_1}k_\xi\mu(d\xi).$$

Now suppose that the adjoint sheaf $\mathcal{O}^*$ of $\mathcal{O}$ exists as in section 4. An appropriate quotient of any germ in $\mathcal{O}_a^*$ has a continuous extension over $\varDelta$. In fact the adjoint assertion of Lemma 1.3 shows that every $h\in\mathcal{O}_a^*$ is represented as $h=r_V^a\left(\int p_x^*\nu(dx)\right)$ by a measure ν on V with its support contained in a compact set of $V\backslash a$, that is,

$$h(y)=\int p_y(x)\nu(dx) \qquad \text{on} \quad \omega\backslash a$$

for some neighborhood ω of a. Hence

$$\frac{1}{p_y(x_0)}h(y)=\int k_y(x)\nu(dx) \qquad \text{on } \omega\backslash a.$$

If we let y converge to a point $\xi\in\varDelta$, the right-hand side integral converges to $\int k_\xi(x)\nu(dx)$, for k_y converges to k_ξ uniformly on any compact subset of $V\backslash a$. Therefore we have proved that, for every

$h \in \mathcal{O}_a^*$ the limit $\lim_{y \to \xi}\left(\frac{h}{p_{x_0}^*}\right)(y)$ exists at any $\xi \in \Delta$.

For the consideration below we must transpose the above statements to those related to the sheaf $\mathcal{O}^*$. Let Δ^* and Δ_1^* be the $\mathcal{O}^*$-ideal boundary and its minimal points, which are introduced in a similar manner as the above to obtain representations of functions in $\mathcal{G}_V^*$ or germs in $\mathcal{G}_a^*$, and let $k_\xi^*(y)$ be the corresponding kernel of $((V \backslash a) \cup \Delta^*) \times (V \backslash a)$ such that $k_x^*(y) = \frac{p_y(x)}{p_{x_0}(x)}$ in $\omega \backslash a$ for a neighborhood ω of a. Let $h^* \in \mathcal{H}_{V \setminus a}^* \cong \Gamma(V, \mathcal{F}^*)$. $d^*h^* \in \Gamma(V, \mathcal{G}^*)$ can be represented in the form

$$d^*h^* = {}^*\rho_V\left(\int_{\Delta_1^*} k_\xi^* \lambda(d\xi)\right).$$

Hence h^* has the representation

$$h^* = f^* + \int_{\Delta_1^*} k_\xi^* \lambda(d\xi) \tag{5.3}$$

on $V \backslash a$, where $f^* \in \mathcal{O}_V^*$

In parallel with the fact that $h^* \in \mathcal{O}_a^*$ has the limit $\lim_{x \to \xi}\left(\frac{h^*}{p_{x_0}^*}\right)(x)$ at $\xi \in \Delta$, every $h \in \mathcal{O}_a$ has the limit $Dh(\xi) \equiv \lim_{x \to \xi}\left(\frac{h}{p_{x_0}}\right)(x) = \int k_\xi^*(y)\mu(dy)$ at any $\xi \in \Delta^*$, where the measure μ is given by $h = r_V^a\left(\int p_y \mu(dy)\right)$ from Lemma 1.3.

Theorem 5.1. (*Green's formula*). *Let* $h \in \mathcal{O}_a$, $h^* \in \mathcal{O}_{V \setminus a}^*$. *Then* $\mathscr{I}(h, h^*) = \int_{\Delta_1^*} (Dh)(\xi)\lambda(d\xi)$, *where* λ *is the measure of* (5.3).

Proof. Let $h \in \mathcal{O}_a$ and $h = r_V^a\left(\int p_y \mu(dy)\right)$ with a measure of compact support in $V \backslash a$, and let h^* be represented by (5.3). Then $\mathscr{I}(h, h^*) = \mathscr{I}(h, f^*) + \int \mu(dy)\left(\int_{\Delta_1^*} k_\xi^*(y)\lambda(d\xi)\right) = \int_{\Delta_1^*} \lambda(d\xi)\left(\int k_\xi^*(y)\mu(dy)\right)$, here we used $\mathscr{I}(h, f^*) = 0$ since $f^* \in \mathcal{O}_V^*$, and Fubini's theorem. The last expression is nothing but the right-hand side of the equality to be proved.

Corollary 5.2. $\measuredangle(h, k_\alpha^*)=(Dh)(\alpha)$ *for any* $\alpha\in\Delta_1^*$.

Corollary 5.3. $\{h\in\mathcal{O}_a;\ \measuredangle(h, h^*)=0$ *for any* $h^*\in\mathcal{O}_{V\setminus a}^*\}=\{h\in\mathcal{O}_a;$ $Dh=0$ *on* $\Delta_1^*\}$.

Corollary 5.4. $\mathcal{O}_a$ *is a Hausdorff topological vector sace iff* $\{h\in\mathcal{O}_a;\ Dh=0$ *on* $\Delta_1^*\}=0$.

In [19] it was proved that if $H_c^1(X, \mathscr{H})$ is a Hausdorff topological vector space then every exhaustion of X by relatively compact subregions $\{U_k\}_{k=1}^\infty$ possesses a subsequence $\{U_{k_i}\}_{i=1}^\infty$ such that $\bar{U}_{k_i}\subset U_{k_{i+1}}$ and every element of $\Gamma(U_{k_{i+1}}, \mathscr{H}^*)$ can be approximated uniformly on $\bar{U}_{k_i}$ by restrictions of elements of $\Gamma(X, \mathscr{H}^*)$. In the next theorem $\bar{\Delta}_1^*$ is the minimal points of the ideal boundary with respect to $\bar{\mathcal{O}}^*$, the adjoint of the minimal sheaf $\bar{\mathcal{O}}$.

Theorem 5.5. *If* $(X, \mathscr{H})$ *is hyperbolic and every* $h\in\bar{\mathcal{O}}_a$ *with* $Dh=0$ *on* $\bar{\Delta}_1^*$ *is equal to zero, then the above approximation property holds.*

In fact $H_c^1(X, H)\cong\bar{\mathcal{O}}_a$ becomes a Hausdorff topological vector space.

Suppose $1\in\mathcal{O}_Y$ and let $h_0\in\mathcal{O}_a$ be as in Corollary 4.11; $<h_0, M>={}^*\Psi(M)$ for any $M\in\Gamma(Y, \mathscr{G}^*)$. Then $Dh_0(\xi)=<h_0, M_\xi>=\Psi(M_\xi)<0$ for any $\xi\in\Delta_1^*$. For every $t\in\Gamma(Y, \mathscr{F}^*)$ let $Nt(d\xi)$ be the measure on Δ_1^* which represents d^*t;

$$d^*t|V={}^*\rho_V\left(\int_{\Delta_1^*}k_\xi^*\lambda(d\xi)\right).$$

The following theorem which concludes from Lemma 3.7 is an analogy of Neumann-problem:

$$\begin{cases}\Delta u(z)=0 & \text{for } |z|<1\\ \dfrac{\partial u}{\partial n}(\xi)=f(\xi) & \text{for } |\xi|=1\end{cases}\Longleftrightarrow\int_{|\xi|=1}f(\xi)d\sigma(\xi)=0.$$

Theorem 5.2. *For a measure* λ *on* Δ^* *supported by* Δ_1^*, *there* *is a* $t \in \Gamma(Y, \mathscr{F}^*)$ *with* $Nt = \lambda$ *iff* $\int Dh_0(\xi)\lambda(d\xi) = 0$.

§6. Applications

Let $\Omega \subset R^n$ be a bounded domain. The following results (**1**) and (**2**) are due to *R.* M. Hervé and M. Hervé [21] and J. Bony [20] respectively.

(**1**) Let

$$L = -\sum_{i,j}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + \sum_i b_i \frac{\partial}{\partial x_i}$$

be a uniformly elliptic differential operator such that $a_{ij}(x)$ are measurable functions with $|a_{ij}| \leqq M < \infty$, and $b_i(x) \in L^r(\Omega)$, $r > n$, and $\sum_i \frac{\partial b_i}{\partial x_i} \leqq 0$. The continuous local solutions of $Lu = 0$ (in the sense of variational problem) gives a presheaf of harmonic functions that satisfies Brelot's axioms;

$$\mathscr{H}_U = \Big\{ f \in C(U) \cap W^{1,2}_{\mathrm{loc}}(U);$$

$$\int_U \Big\{ \sum_j \Big(\sum_i a_{ij} \frac{\partial u}{\partial x_i} \Big) \frac{\partial \varphi}{\partial x_j} + \Big(\sum_i b_i \frac{\partial u}{\partial x_i} \Big) \varphi \Big\} dx = 0$$

$$\text{for any } \varphi \in \mathscr{D}(U) \Big\}.$$

(**2**) Let

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

be a differential operator with C^∞-coefficients such that

(a) $\sum a_{ij}(x)\xi_i\xi_j \geqq 0$ for any $\xi \in R^n$, and

(b) $Lu = \sum_{k=1}^{r} (X_k)^2 u + Yu$

for some C^∞-vector fields $X_1, \ldots, X_r$, Y on Ω, and such that the Lie algebra generated by $X_1,\dots, X_r$ has rank n at every point of Ω. Then

$$\mathscr{H}_U=\{u\in C^\infty(U);\ Lu=0\},\qquad U \text{ open}\subset\Omega,$$

satisfies Brelot's axioms.

In both cases Green functions on appropriate subdomains are constructed, and such domains exhausts Ω. All our results are applicable. In particular $H^1(\Omega, \mathscr{H})=0$. This assures the global existence of the solution of $Lu=f$. Before we state this problem more precisely we note that the sheaf P is isomorphic to the following sheaf $\mathscr{L}$ of measures [19]; $\Gamma(U.\ \mathscr{L})\in\mu$ iff for every $x\in U$ there is a small neighborhood V of x and a compact neighborhood K of x with $K\subset V$ such that

$$x\longrightarrow\int_K V_{p_y}(x)d|\mu|(y)$$

is continuous on V.

Theorem 6.1. *Let L be either of the above differential operator. Then for any measure $\mu\in\Gamma(\Omega, \mathscr{L})$ there is a continuous solution u of $Lu=\mu$ (in the sense of variational problem or that of distribution). In particular, for any continuous function f on Ω, there is a continuous solution of $Lu=f$.*

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