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# Degenerate parabolic differential equations: Necessity of the well-posedness of the Cauchy problem

By

Masatake MIYAKE

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## §1. Introduction

We study in this note the following forward Cauchy problem;

$$\frac{\partial}{\partial t}u(x, t)=\sum_{j=0}^{2m} t^{n_j}\mathscr{L}_{2m-j}\left(x, t; \frac{\partial}{\partial x}\right)u(x, t), \tag{1.1}$$

$$u|_{t=0}=u_0(x)\in \mathscr{D}_{L^2}^{\infty}(R_x^n), \tag{1.2}$$

where $\mathscr{L}_{2m-j}\left(x, t; \frac{\partial}{\partial x}\right)=\sum_{|\alpha|=2m-j} a_{\alpha,j}(x, t)\left(\frac{\partial}{\partial x}\right)^{\alpha}$, $a_{\alpha,j}(x, t)\in \mathscr{E}_t^0(\mathscr{B}_x)$[1], $(x, t)\in R_x^n\times[0, 1]$ and $n_j\geqq 0$.

Our purpose in this note is to seek a necessary condition of the $\mathscr{D}_{L^2}^{\infty}$-well-posedness for the Cauchy problem (1.1)-(1.2). Recently K. Igari [4] has studied this problem, but our research is different from it. For instance, our research is based on the *modified order*[2] of the

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1) $\mathscr{D}_{L^2}^{\infty}(R_x^n)=\left\{u(x);\ \left(\frac{\partial}{\partial x}\right)^{\alpha}u(x)\in L^2(R_x^n) \text{ for any } \alpha\right\}$

$\mathscr{B}_x(R_x^n)=\left\{u(x)\in C^{\infty}(R_x^n);\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}u(x)\right|\leqq M_{\alpha} \text{ for some } M_{\alpha}\geqq 0 \text{ for any } \alpha\right\}$

$u(x, t)\in\mathscr{E}_t^0(\mathscr{B}_x)$ means that $u(x, t)\in\mathscr{B}_x$ for any fixed $t$ and continuous in $t$ in the usual topology of $\mathscr{B}_x$.

2) We say that the modified order at $t=0$ of $t^{n_j}\mathscr{L}_{2m-j}$ is $\frac{2m-j}{n_j+1}$.

differential operator. The notion of the modified order was introduced when we considered Cauchy-Kowalevski's theorem. And also this notion will be used when we shall study the hypoellipticity of degenerate parabolic differential equations. (M. Miyake [5]).

Now we give our theorem: let us assume the following conditions,

$$\text{i)}\quad \frac{2m-j_0}{n_{j_0}+1} = \max_{0\leq j\leq 2m} \frac{2m-j}{n_j+1} \text{ for some } j_0\in\{0, 1,\dots, 2m-1\},$$

$$\text{(C. I)}\quad \text{ii)}\quad \frac{2m-j_0}{n_{j_0}+1} > \frac{2m-j}{n_j+1} \text{ for any } j=0, 1,\dots, j_0-1,$$

$$\text{iii)}\quad \mathrm{Re}\,\mathscr{L}_{2m-j_0}(0, 0; 2\pi i\xi^0) = \delta>0 \text{ for some } \xi^0\in R^n_\xi,\ |\xi^0|=1,$$

where $\mathrm{Re}\,a$ means the real part of $a$, $|\xi|=\sqrt{\sum_{j=1}^n \xi_j^2}$, $\xi=(\xi_1,\dots, \xi_n)\in R^n_\xi$ and $i=\sqrt{-1}$.

Then we have

**Theorem 1.** *Let us assume* (C. I), *then the Cauchy problem* (1.1)–(1.2) *is not* $\mathscr{D}^\infty_{L^2}$*-well-posed in any neighborhood of* $t=0$.

We give now the definition of $\mathscr{D}^\infty_{L^2}$-well-posedness of the Cauchy problem for the equation (1.1).

**Definition.** *We say that the Cauchy problem for the equation* (1.1) *is uniformly* $\mathscr{D}^\infty_{L^2}$*-well-posed in* $[0, 1]$, *if for any* $u_0(x)\in\mathscr{D}^\infty_{L^2}$ *and any initial-time* $s\in[0, 1)$, *there exists a unique solution* $u(x, t)\in \mathscr{E}^1_t(\mathscr{D}^\infty_{L^2})$[3] *in* $t\geq s$ *satisfying* $u|_{t=s}=u_0(x)$, *and the mapping* $u_0(x)\to u(x, t)$ *is continuous. More precisely for any non-negative integer* $l$, *there exist a non-negative integer* $h$ *and a constant* $C$ *independent of* $s$ *such that*

3) $\|u(x)\|_h^2=\sum_{|\alpha|\leqq h}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha u(x)\right\|_{L^2}^2$. $\|u(x, t)\|_h$ denotes the norm in $x$-variable and $t$ is considered as a parameter. We also note that $\mathscr{D}^\infty_{L^2}$ is a Fréchet space with semi-norms $\|u(x)\|_h$, $(h=0, 1, 2,\dots)$.
$\mathscr{E}^1_t(\mathscr{D}^\infty_{L^2})\ni u(x, t)\Leftrightarrow u(x, t)\in\mathscr{D}^\infty_{L^2}$ for any fixed $t$ and it is continuously defferentiable with respect to $t$ in the topology of $\mathscr{D}^\infty_{L^2}$.

(1.3) $$\max_{s \leqq t \leqq 1} \|u(x, t)\|_l < C\|u_0(x)\|_h .$$

We shall prove our theorem from § 3 on, and the method of the proof rely on that of S. Mizohata ([1], [2]). In the case where the coefficients of the equation (1.1) depend only on $t$, we shall give sufficient conditions of $\mathscr{D}_{L^2}^{\infty}$-well-posedness in § 2.

## § 2. Sufficiency of the well-posedness

In this section we only consider the following equation;

(2.1) $$\frac{\partial}{\partial t} u(x, t) = \sum_{j=0}^{2m} t^{n_j} \mathscr{L}_{2m-j}\left(t; \frac{\partial}{\partial x}\right) u(x, t) .$$

In this case, we have easily sufficient conditions of the well-posedness for the equation (2.1), and an elementary result is the following

**Theorem 2.** *Let us assume that the coefficients of* $\mathscr{L}_{2m-j}$ *are continuous and*

(C. II)

i) $$\frac{2m}{n_0+1} = \max_{0 \leqq j \leqq 2m} \frac{2m-j}{n_j+1} ,$$

ii) $$\operatorname{Re} \mathscr{L}_{2m}(t; 2\pi i\xi) \leqq -\delta|\xi|^{2m} \quad \textit{for any } \xi \in R_{\xi}^n .$$

*Then the forward Cauchy problem for the equation* (2.1) *is uniformly* $\mathscr{D}_{L^2}^{\infty}$*-well-posed in* [0, 1].

In order to prove our theorem we use a fundamental inequality.

**Lemma.** *If* $\frac{2m}{n_0+1} = \max_{0 \leqq j \leqq 2m} \frac{2m-j}{n_j+1}$, *then we have*

(2.2) $$(t^{n_j+1} - s^{n_j+1})^{2m} \leqq C(t^{n_0+1} - s^{n_0+1})^{2m-j}, \qquad 0 \leqq s < t \leqq 1$$

*for some positive constant* $C$.[4)]

*Proof of the lemma.* We prove (2.2) dividing into three cases; i) $s=0$, ii) $0<s<t \leqq 2s$ and iii) $0<s<2s<t$. In the first case, (2.2)

4) In the sequel, we shall denote by the same symbol $C$ any one of various different constants.

is obvious from the assumption. Now we prove (2.2) in the second case. $t^{n_j+1}-s^{n_j+1}=(n_j+1)\int_s^t \tau^{n_j}\,d\tau<(n_j+1)t^{n_j}(t-s)\leqq c\frac{t-s}{s}s^{n_j+1}$. Thus

$$(t^{n_j+1}-s^{n_j+1})^{2m}\leqq \text{const.}\left(\frac{t-s}{s}\right)^{2m}s^{2m(n_j+1)}.$$

Next, it is obvious that $(t^{n_0+1}-s^{n_0+1})>(n_0+1)\frac{t-s}{s}s^{n_0+1}$, then we have

$$(t^{n_0+1}-s^{n_0+1})^{2m-j}\geqq \text{const.}\left(\frac{t-s}{s}\right)^{2m-j}s^{(2m-j)(n_0+1)}.$$

Since $\frac{t-s}{s}\leqq 1$, it holds $\left(\frac{t-s}{s}\right)^{2m}\leqq\left(\frac{t-s}{s}\right)^{2m-j}$. On the other hand, $2m(n_j+1)\geqq(2m-j)(n_0+1)$ from the assumption, therefore we get $s^{2m(n_j+1)}\leqq s^{(2m-j)(n_0+1)}$, $(0<s\leqq 1)$. It proves the inequality (2.2). Finally let us consider the third case. It is obvious that $(t^{n_j+1}-s^{n_j+1})^{2m}<t^{2m(n_j+1)}$. And the condition, $(0<s<2s<t)$ implies $t^{n_0+1}-s^{n_0+1}>\text{const. }t^{n_0+1}$. Hence we have $(t^{n_0+1}-s^{n_0+1})^{2m-j}>\text{const.}\,t^{(2m-j)(n_0+1)}$. These imply the inequality (2.2). q.e.d.

*Proof of the theorem.* Let $E_x(t,s)$ be an elementary solution of the Cauchy problem for the equation (2.1), that is,

$$(2.3)\qquad \frac{\partial}{\partial t}E_x(t,s)=\sum_{j=0}^{2m}t^{n_j}\mathscr{L}_{2m-j}\left(t;\frac{\partial}{\partial x}\right)E_x(t,s),\qquad 1\geqq t\geqq s\geqq 0.$$

$$(2.4)\qquad E_x|_{t=s}=\delta_x,\quad \delta_x \text{ means Dirac's distribution.}$$

Now let $\hat{E}(t,s;\xi)$ be a Fourier transform of $E_x(t,s)$ with respect to $x$, then, due to Petrowski's theorem ([3], Th. 5.2) the necessary and sufficient condition of the uniformly $\mathscr{D}^\infty_{L^2}$-well-posedness in $[0,1]$ is that $\hat{E}(t,s;\xi)$ satisfies the following inequality,

$$(2.5)\qquad |\hat{E}(t,s;\xi)|\leqq C(1+|\xi|)^p,\qquad (t\geqq s),$$

where $C$ and $p$ are positive constants independent of $t$ and $s$.

Since

$$\hat{E}(t,s;\xi)=\exp\left[\int_s^t\sum_{j=0}^{2m}\tau^{n_j}\mathscr{L}_{2m-j}(\tau;2\pi i\xi)\,d\tau\right],$$

it holds

$$|\hat{E}(t, s; \xi)| \leqq \exp\left[-\frac{\delta}{n_0+1}(t^{n_0+1}-s^{n_0+1})|\xi|^{2m} + C\sum_{j=1}^{2m}(t^{n_j+1}-s^{n_j+1})|\xi|^{2m-j}\right].$$

Considering Lemma, we have

$$(2.6)\qquad |\hat{E}(t, s; \xi)| \leqq \exp\left[-\frac{\delta}{n_0+1}(t^{n_0+1}-s^{n_0+1})|\xi|^{2m} + C\sum_{j=1}^{2m}(t^{n_0+1}-s^{n_0+1})^{\frac{2m-j}{2m}}|\xi|^{2m-j}\right].$$

Let $X=(t^{n_0+1}-s^{n_0+1})^{\frac{1}{2m}}|\xi|$, then $-\frac{\delta}{n_0+1}X^{2m}+C\sum_{j=1}^{2m}X^{2m-j}<C'$ for some positive constant $C'$. This completes the proof. q.e.d.

Now let us weaken the assumption (C. II) as follows.

There exists a sequence $\{m_i\}_{i=0}^{k+1}$ satisfying

$$\text{i)}\quad 0=m_0<m_1<m_2<\cdots<m_k<m_{k+1}=m.$$

$$\text{(C. III)}\quad \text{ii)}\quad \frac{2(m-m_i)}{n_{2m_i}+1}=\max_{2m_i\leqq j\leqq 2m_{i+1}-1}\frac{2m-j}{n_j+1},\qquad (i=0, 1,\dots, k),$$

$$\text{iii)}\quad \mathrm{Re}\,\mathscr{L}_{2(m-m_i)}(t; 2\pi i\xi)<-\delta|\xi|^{2(m-m_i)},$$

$$(\delta>0,\ i=0,\ i=0, 1,\dots, k).$$

Then we have

**Corollary 1.** *Under the assumption* (C. III), *the Cauchy problem for the equation* (2.1) *is uniformly* $\mathscr{D}_{L^2}^{\infty}$*-well-posed in* [0, 1].

*Proof.* It is clear, since we may repeat the above reasoning for each brock of $\sum_{j=2m_i}^{2m_{i+1}-1}t^{n_j}\mathscr{L}_{2m-j}\left(t; \frac{\partial}{\partial x}\right)$. Precisely, we can show the following inequality

$$(2.7)\qquad \mathrm{Re}\int_s^t\{\sum_{j=2m_i}^{2m_{i+1}-1}\tau^{n_j}\mathscr{L}_{2m-j}(\tau; 2\pi i\xi)\}d\tau \leqq C,$$

($i=0, 1, \dots, k$) by the same way as the thoerem. q.e.d.

Finally, let us consider the case where $n_j$ are integers and the coefficients of $\mathscr{L}_{2m-j}\left(t; \frac{\partial}{\partial x}\right)$ are continuous in an interval $[-1, 1]$. Then we have

**Corollary 2.** *If we assume the condition* (C. II) *and* $n_0$ *is an even integer, then the Cauchy problem for the equation* (2.1) *is uniformly* $\mathscr{D}^{\infty}_{L^2}$*-well-posed in* $[-1, 1]$.

*Proof.* Under the assumption of the corollary, it holds that

$$\int_s^t |\tau^{n_j}| d\tau \leqq C(t^{n_0+1} - s^{n_0+1})^{\frac{2m-j}{2m}}, \qquad (-1 \leqq s < t \leqq 1).$$

Its proof is obvious in view of the proof of the lemma. Therefore, we can prove the corollary from the above inequality. q.e.d.

**Remark.** In the case where the coefficients are dependent only on $t$, we can obtain trivial extensions of our theorems. That is, instead of considering (2.1), we may consider the equation

$$(2.1)' \qquad \frac{\partial}{\partial t} u = \sum_{j=0}^{2m} \mathscr{L}_{2m-j}\left(t; \frac{\partial}{\partial x}\right) u,$$

where $\mathscr{L}_{2m-j}(t; 2\pi i\xi)$ is a homogeneous polynomial in $\xi$ of degree $2m-j$ with continuous coefficients.

In the assumption (C. I), it suffices to assume that $\mathrm{Re}\, \mathscr{L}_{2m-j}(t; 2\pi i\xi^0) = t^{n_j} \mathscr{L}'_{2m-j}(t; 2\pi i\xi^0)$ for some $\xi^0 \in R^n$, $(j=0, 1, \dots, 2m-1)$. And in the assumption (C. II) or (C. III), it suffices to assume that $\mathrm{Re}\, \mathscr{L}_{2m-j}(t; 2\pi i\xi) = t^{n_j} \mathscr{L}'_{2m-j}(t; 2\pi i\xi)$ for any $\xi \in R^n$, $(j=0, 1, \dots, 2m-1)$.

## §3. Localization of the equation

From this section on, we shall prove our theorem stated in §1. At first, we localize the equation (1.1). Let $\beta(x) \in C_0^\infty(R_x^n)$ satisfy that $\mathrm{supp}[\beta]$ is contained in a sufficiently small neighborhood of $x=0$, and apply $\beta(x)$ to the equation (1.1) then we have

$$\frac{\partial}{\partial t}(\beta u)=\sum_{j=0}^{2m} t^{n_j}\Big\{\mathscr{L}_{2m-j}\Big(x, t; \frac{\partial}{\partial x}\Big)(\beta u) + \sum_{1\leq|\mu|\leq 2m-j} \breve{\mathscr{L}}^{(\mu)}_{2m-j}\Big(x, t; \frac{\partial}{\partial x}\Big)(\beta^{(\mu)}u)\Big\}, \tag{3.1}$$

where $\breve{\mathscr{L}}^{(\mu)}_{2m-j}$ denotes differential operator of order $2m-j-|\mu|$ and $\beta^{(\mu)}=\Big(\frac{\partial}{\partial x}\Big)^{\mu}\beta$.

Since we may modify coefficients of the equation (3.1) outside of $\mathrm{supp}[\beta]$ in view of (3.1), we assume that the oscillations of coefficients are small as we desire. Let $\hat{\alpha}(\xi)\in C_0^\infty(R^n)$ be $\hat{\alpha}(\xi)=1$ in a neighborhood of $\xi=\xi^0$ and $\mathrm{supp}[\hat{\alpha}]$ is sufficiently small. Thus we may assume that $\inf_{\xi\in\mathrm{supp}[\hat{\alpha}]} \mathrm{Re}\,\mathscr{L}_{2m-j_0}(0, 0; 2\pi i\xi)>\frac{2}{3}\delta$.

Now we define a convolution operator $\alpha(D)$ as follows.

$$\alpha(D)u=\mathscr{F}_\xi^{-1}[\hat{\alpha}(\xi)\hat{u}(\xi, t)], \qquad \hat{u}(\xi, t)=\mathscr{F}_x[u(x, t)]. \tag{3.2}$$

Obviously $\alpha(D)u$ is rewritten by $\alpha(D)u=\alpha(x)\underset{(x)}{*}u(x, t)$, where $\alpha(x)=\mathscr{F}_\xi^{-1}[\hat{\alpha}(\xi)]$ and $\underset{(x)}{*}$ denotes the convolution. Hereafter we use the following notations.

$$\hat{\alpha}_n(\xi)=\hat{\alpha}\Big(\frac{\xi}{n}\Big), \qquad \alpha_n(D)u=\mathscr{F}_\xi^{-1}[\hat{\alpha}_n(\xi)\hat{u}(\xi, t)].$$

Let us apply $\alpha_n(D)$ to the equation (3.1), then we have

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha_n(D)\beta u)=&\sum_{j=0}^{2m} t^{n_j}\mathscr{L}_{2m-j}\Big(x, t; \frac{\partial}{\partial x}\Big)(\alpha_n(D)\beta u)\\ &+\sum_{j=0}^{2m} t^{n_j}\Big[\alpha_n(D), \mathscr{L}_{2m-j}\Big(x, t; \frac{\partial}{\partial x}\Big)\Big](\beta u)\\ &+\sum_{j=0}^{2m} t^{n_j}\Big\{\sum_{1\leq|\mu|\leq 2m-j} \breve{\mathscr{L}}^{(\mu)}_{2m-j}\Big(x, t; \frac{\partial}{\partial x}\Big)(\alpha_n(D)\beta^{(\mu)}u)\Big\}\\ &+\sum_{j=0}^{2m} t^{n_j}\Big\{\sum_{1\leq|\mu|\leq 2m-j}\Big[\alpha_n(D), \breve{\mathscr{L}}^{(\mu)}_{2m-j}\Big(x, t; \frac{\partial}{\partial x}\Big)\Big](\beta^{(\mu)}u)\Big\},\end{aligned} \tag{3.3}$$

where $[\alpha_n(D), \mathscr{L}_{2m-j}]u=\alpha_n(D)(\mathscr{L}_{2m-j}u)-\mathscr{L}_{2m-j}(\alpha_n(D)u)$. Thus we have

$$(3.3)' \quad \frac{\partial}{\partial t}(\alpha_n(D)\beta u)=\sum_{j=0}^{2m} t^{n_j}\mathscr{L}_{2m-j}\left(x, t; \frac{\partial}{\partial x}\right)(\alpha_n(D)\beta u)+f_n(x, t),$$

where we denote by $f_n(x, t)$ the terms following the second term in the right hand side of (3.3).

In the following we shall prove an energy inequality for the equation (3.3)$'$. At first, we remark that

$$(3.4) \quad \frac{d}{dt}\|\alpha_n(D)\beta u\|^2=2\sum_{j=0}^{2m} t^{n_j}\operatorname{Re}\left(\mathscr{L}_{2m-j}\left(x, t; \frac{\partial}{\partial x}\right)(\alpha_n(D)\beta u), \alpha_n(D)\beta u\right) +2\operatorname{Re}(f_n(x, t), \alpha_n(D)\beta u),$$

where $\|\cdot\|$ and ( , ) denote the $L^2$-norm and the inner product of $L^2$ in $x$-variable.

Then we shall show an energy inequality when $t$ is small,

$$(3.5)^{5)} \quad \frac{d}{dt}\|\alpha_n(D)\beta u\|>g_n(t)\|\alpha_n(D)\beta u\|-\|f_n\|,$$

where $g_n(t)=\frac{\delta}{2}t^{n_{j_0}}n^{2m-j_0}-C\sum_{\substack{0\leq j\leq 2m\\ j\neq j_0}} t^{n_j}n^{2m-j}$ for some positive constant $C$.

In fact, we prove (3.5) dividing (3.4) into two parts; i) $j=j_0$ and ii) $j\neq j_0$. At first we investigate the case i).

$$\operatorname{Re}\left(\mathscr{L}_{2m-j_0}\left(x, t; \frac{\partial}{\partial x}\right)(\alpha_n(D)\beta u), \alpha_n(D)\beta u\right)$$

$$=\operatorname{Re}\left(\mathscr{L}_{2m-j_0}\left(0, 0; \frac{\partial}{\partial x}\right)\alpha_n(D)\beta u, \alpha_n(D)\beta u\right)$$

$$+\operatorname{Re}\left(\left\{\mathscr{L}_{2m-j_0}\left(x, t; \frac{\partial}{\partial x}\right)-\mathscr{L}_{2m-j_0}\left(0, 0; \frac{\partial}{\partial x}\right)\right\}(\alpha_n(D)\beta u), \alpha_n(D)\beta u\right)$$

5) Instead of the inequality (3.5) we have

$$(3.5)' \quad \frac{d}{dt}\|\alpha_n(D)\beta u\|^2>g_n(t)\|\alpha_n(D)\beta u\|^2-\frac{\text{const.}}{\sum_{0\leq j\leq 2m} t^{n_j}n^{2m-j}}\|f_n\|^2.$$

For the simplicity we use (3.5) in the sequel. The singular part of the second term of the right hand side of (3.5)$'$ does not trouble in view of (4.6).

$=I+II$.

From the assumption that $\inf_{\xi\in\mathrm{supp}[\hat{\alpha}]} \mathrm{Re}\,\mathscr{L}_{2m-j_0}(0, 0; 2\pi i\xi) > \frac{2}{3}\delta$, we have

$$I = \mathrm{Re}(\mathscr{L}_{2m-j_0}(0, 0; 2\pi i\xi)\hat{\alpha}_n(\xi)\widehat{\beta u}(\xi, t), \hat{\alpha}_n(\xi)\widehat{\beta u}(\xi, t))_\xi$$

$$> \frac{2}{3}\delta n^{2m-j_0}\|\alpha_n(D)\beta u\|^2 .$$

On the other hand, since the oscillation of the coefficients are small, we have the following inequality when $t$ is small.

$$|II| < \varepsilon \sum_{|\alpha|=2m-j_0} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha \alpha_n(D)\beta u\right\| \cdot \|\alpha_n(D)\beta u\| ,$$

where $\varepsilon$ is a sufficiently small positive constant. Therefore we have $|II| < \varepsilon' \cdot n^{2m-j_0}\|\alpha_n(D)\beta u\|^2$ for some sufficiently small positive consatnt $\varepsilon'$. Combining the above two inequalities, we have

$$2\mathrm{Re}\left(\mathscr{L}_{2m-j_0}\left(x, t; \frac{\partial}{\partial x}\right)(\alpha_n(D)\beta u),\ \alpha_n(D)\beta u\right) > \delta n^{2m-j_0}\|\alpha_n(D)\beta u\|^2 .$$

In the case of ii), we have easily

$$|(\mathscr{L}_{2m-j}(\alpha_n(D)\beta u),\ \alpha_n(D)\beta u)| \leqq \mathrm{const.}\, n^{2m-j}\|\alpha_n(D)\beta u\|^2$$

since the order of $\mathscr{L}_{2m-j}$ is $2m-j$. This proves the inequality (3.5).

## §4. Proof of the theorem

We shall prove the theorem by contradiction. Let $\{\varphi_n(x)\}_{n=1}^\infty \subset \mathscr{D}_{L^2}^\infty(R_x^n)$ be a sequence of Cauchy data satisfying $\hat{\varphi}_n(\xi) = \hat{\varphi}(\xi - n\xi^0)$, where $\hat{\varphi}(\xi) \in C_0^\infty(R_\xi^n)$ and $\hat{\varphi}(\xi) = 1$ in a neighborhood of $\xi = 0$ and $\mathrm{supp}[\hat{\varphi}]$ is sufficiently small. That is,

$$\varphi_n(x) = e^{2\pi i <x, n\xi^0>}\varphi(x), \tag{4.1}$$

$\varphi(x) = \mathscr{F}_\xi^{-1}[\hat{\varphi}(\xi)]$ and $<x, \xi> = \sum_{j=1}^n x_j\xi_j$. And now let $\{u_n(x, t)\}_{n=1}^\infty \subset \mathscr{E}_t^1(\mathscr{D}_{L^2}^\infty)$ be a sequence of solutions with Cauchy data $\{\varphi_n(x)\}_{n=1}^\infty$ at

$t=0$, that is,

$$\frac{\partial}{\partial t}u_n(x,t)=\sum_{j=0}^{2m}t^{nj}\mathscr{L}_{2m-j}\Big(x,t;\frac{\partial}{\partial x}\Big)u_n(x,t), \tag{4.2}$$

$$u_n(x,0)=\varphi_n(x). \tag{4.3}$$

If we assume that the forward Cauchy problem (4.2)–(4.3) is well-posed, $u_n(x,t)$ should satisfy

$$\max_{0\leqq t\leqq 1}\|u_n(x,t)\|\leqq C\|u_n(x,0)\|_h\leqq\tilde{C}n^h, \tag{4.4}$$

for some positive constants $C$ and $\tilde{C}$, and non-negative integer $h$, where we may assume without loss of generality that $h\geqq 2m$.

It is easy to see

$$\|\alpha_n(D)\beta u_n(x,0)\|>c_0 \tag{4.5}$$

for some postive constant $c_0$. (see S. Mizohata [1]).

We shall prove the following inequality for $f_n(x,t)$ appeared in (3.5) substituting $u$ for $u_n$,

$$\|f_n(x,t)\|<h_n(t)\Big\{\sum_{1\leqq|\mu|+|\nu|\leqq h}\|\alpha_n^{(\nu)}(D)n^{-|\mu|}\beta^{(\mu)}u_n\|+\frac{1}{n}\Big\}, \tag{4.6}$$

where $h_n(t)=C\sum_{j=0}^{2m}t^{nj}n^{2m-j}$, ($C$ is a sufficiently large constant) and $\alpha_n^{(\nu)}(D)u_n=\{x^\nu\alpha_n(x)\}\underset{(x)}{*}u_n(x,t)$.

At first, we consider the term of $\Big[\alpha_n(D),\mathscr{L}_{2m-j}\Big(x,t;\frac{\partial}{\partial x}\Big)\Big](\beta u_n)$.

$$\Big[\alpha_n(D),a_{\alpha,j}(x,t)\Big(\frac{\partial}{\partial x}\Big)^\alpha\Big](\beta u_n)$$

$$=\int\{a_{\alpha,j}(y,t)-a_{\alpha,j}(x,t)\}\alpha_n(x-y)\Big(\frac{\partial}{\partial y}\Big)^\alpha(\beta u_n)(y,t)\mathrm{d}y$$

$$=\sum_{1\leqq|\nu|\leqq h}\frac{(-1)^{|\nu|}a_{\alpha,j}^{(\nu)}(x,t)}{\nu!}\alpha_n^{(\nu)}(D)\Big\{\Big(\frac{\partial}{\partial x}\Big)^\alpha(\beta u_n)\Big\}$$

$$+\sum_{|\nu|=h+1}\frac{(-1)^{h+1}}{\nu!}\int a_{\alpha,j,\nu}(x, y, t)\,(x-y)^{\nu}\alpha_n(x-y)$$
$$\left(\frac{\partial}{\partial y}\right)^{\alpha}(\beta u_n)(y, t)\mathrm{d}y,$$

where $|\alpha|=2m-j$, $a_{\alpha,j}^{(\nu)}(x, t)\in\mathscr{E}_t^0(\mathscr{B}_x)$ and $a_{\alpha,j,\nu}\in\mathscr{E}_t^0(\mathscr{B}_{x\times y})$.

Now let us consider the last term in the above equality.

$$\int a_{\alpha,j,\nu}(x, y, t)(x-y)^{\nu}\alpha_n(x-y)\left(\frac{\partial}{\partial y}\right)^{\alpha}(\beta u_n)(y, t)\mathrm{d}y$$
$$=(-1)^{|\alpha|}\sum_{\alpha'\leqq\alpha}C_{\alpha,\alpha'}\int\left(\frac{\partial}{\partial y}\right)^{\alpha-\alpha'}a_{\alpha,j,\nu}(x, y, t)$$
$$\times\left(\frac{\partial}{\partial y}\right)^{\alpha'}\{(x-y)^{\nu}\alpha_n(x-y)\}\times(\beta u_n)(y, t)\mathrm{d}y.$$

Using Hausdorff-Young's inequality for each term of the right hand side, we have

$$\|\text{each term}\|_{L^2}\leqq\text{const.}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha'}\{x^{\nu}\alpha_n(x)\}\right\|_{L^1}\cdot\|\beta u_n\|_{L^2}.$$

It is easy to show $\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha}(x^{\nu}\alpha_n(x))\right\|_{L^1}=n^{|\alpha|-|\nu|}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha}(x^{\nu}\alpha(x))\right\|_{L^1}$, since $\alpha_n(x)=n^{\dim(R^n)}\alpha(nx)$. And on the other hand, we know that $\|u_n\|=O(n^h)$ from (4.4), hence we have

$$\|\text{each term}\|_{L^2}\leqq\text{const.}\,n^{2m-j-1}.$$

Since $\left\|\alpha_n^{(\nu)}(D)\left(\frac{\partial}{\partial x}\right)^{\alpha}(\beta u_n)\right\|\leqq\text{const.}\,n^{2m-j}\|\alpha_n^{(\nu)}\beta u_n\|$ in view of $|\alpha|=2m-j$, it holds

$$\left\|\left[\alpha_n(D), \mathscr{L}_{2m-j}\left(x, t;\frac{\partial}{\partial x}\right)\right](\beta u_n)\right\|<C\Big\{\sum_{1\leqq|\nu|\leqq h}n^{2m-j}\|\alpha_n^{(\nu)}(D)\beta u_n\|+n^{2m-j-1}\Big\}. \tag{4.7}$$

Next, it is obvious

$$(4.8)\qquad \|\tilde{\mathscr{L}}^{(\mu)}_{2m-j}(\alpha_n(D)\beta^{(\mu)}u_n)\| \leqq \text{const. } n^{2m-j-|\mu|}\ \|\alpha_n(D)\beta^{(\mu)}u_n\|\,,$$

because of the order of $\tilde{\mathscr{L}}^{(\mu)}_{2m-j}$ is at most $2m-j-|\mu|$.

Finally for the term of $[\alpha_n(D), \tilde{\mathscr{L}}^{(\mu)}_{2m-j}](\beta^{(\mu)}u_n)$, we have

$$(4.9)\qquad \|[\alpha_n(D), \tilde{\mathscr{L}}^{(\mu)}_{2m-j}](\beta^{(\mu)}u_n)\|$$

$$\leqq \text{const. } \Big\{\sum_{1\leqq|\nu|\leqq h} n^{2m-j-|\mu|}\|\alpha_n^{(\nu)}(D)\beta^{(\mu)}u_n\| + n^{2m-j-|\mu|-1}\Big\}$$

by the similar way as the first term. And also we know that

$$(4.10)\qquad \|\alpha_n^{(\nu)}(D)n^{-|\mu|}\beta^{(\mu)}u_n\| \leqq \frac{\text{const.}}{n} \quad \text{if} \quad |\mu|+|\nu| \geqq h+1\,.$$

Hence combining (4.7)~(4.10) we have the inequality (4.6). Therefore

$$(4.11)\qquad \frac{d}{dt}\|\alpha_n(D)\beta u_n\| > g_n(t)\|\alpha_n(D)\beta u_n\|$$

$$-h_n(t)\Big\{\sum_{1\leqq|\mu|+|\nu|\leqq h} \|\alpha_n^{(\nu)}(D)n^{-|\mu|}\beta^{(\mu)}u_n\| + \frac{1}{n}\Big\}\,,$$

in view of (3.5).

If we repeat the above reasonings by setting $\alpha_n(D)$ by $\alpha_n^{(\nu)}(D)$ and $\beta(x)$ by $n^{-|\mu|}\beta^{(\mu)}$, we have

$$(4.12)\qquad \frac{d}{dt}\|\alpha_n^{(\nu)}(D)n^{-|\mu|}\beta^{(\mu)}u_n\| > g_n(t)\|\alpha_n^{(\nu)}(D)n^{-|\mu|}\beta^{(\mu)}u_n\|$$

$$-\text{const. } h_n(t)\Big\{\sum_{\substack{|\nu|+|\mu|+1\\ \leqq|\nu'|+|\mu'|\leqq h}} \|\alpha_n^{(\nu')}(D)n^{-|\mu'|}\beta^{(\mu')}u_n\| + \frac{1}{n}\Big\}\,.$$

Now let us define $S_n(t)$ by

$$S_n(t) = \sum_{|\mu|+|\nu|\leqq h} C_0^{|\mu|+|\nu|}\|\alpha_n^{(\nu)}(D)n^{-|\mu|}\beta^{(\mu)}u_n\|\,.$$

Then (4.11) and (4.12) imply, if we give a sufficiently large constant $C_0$, (4.13) $$\frac{d}{dt}S_n(t) > \tilde{g}_n(t)S_n(t) - \frac{C}{n}h_n(t),$$

where $\tilde{g}_n(t) = \delta_0 t^{n_0} n^{2m-j_0} - \text{const.} \sum_{\substack{0 \leq j \leq 2m \\ j \neq j_0}} t^{n_j} n^{2m-j}$, for some positive constant $\delta_0$.

Now let $\tilde{G}_n(t) = \int_0^t \tilde{g}(\tau)d\tau = \tilde{\delta}_0 t^{n_{j_0}+1} n^{2m-j_0} - \sum_{\substack{0 \leq j \leq 2m \\ j \neq j_0}} \tilde{c}_j t^{n_j+1} n^{2m-j}$, $\left(\tilde{\delta}_0 = \frac{\delta_0}{n_{j_0}+1}\right)$ then we have from (4.13)

$$\frac{d}{dt}\{\exp[-\tilde{G}_n(t)] \cdot S_n(t)\} > -\frac{C}{n}h_n(t) \cdot \exp[-\tilde{G}_n(t)].$$

Since $S_n(0) > c_0 > 0$ from (4.5), it holds

$$S_n(t) > c_0 \cdot \exp[\tilde{G}_n(t)] - \frac{C}{n}\exp[\tilde{G}_n(t)] \cdot \int_0^t h_n(\tau) \cdot \exp[-\tilde{G}_n(\tau)]d\tau.$$

We choose a positive constant $\varepsilon$ satisfying

(4.14) $$\varepsilon < \min_{0 \leq j \leq j_0-1} \frac{(2m-j_0)(n_j+1) - (2m-j)(n_{j_0}+1)}{j_0 - j},$$ [6]

then if $n$ is sufficiently large, we have

(4.15) $$S_n\left(n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}}\right) > \frac{c_0}{2}\exp\left[\frac{\tilde{\delta}_0}{2} n^{\frac{\varepsilon(2m-j_0)}{n_{j_0}+1+\varepsilon}}\right].$$

The proof of (4.15) will be given in §5, since it is long.

On the other hand, from the assumption of the well-posedness it must be $S_n(t) = O(n^h)$, $(0 \leq t \leq 1)$. This contradicts from (4.15), which proves the theorem. q.e.d.

## §5. Proof of (4.15)

In order to evaluate $\int_0^t \tau^{n_j} n^{2m-j} \exp[-\tilde{G}_n(\tau)]d\tau$, let us show

(5.1) $$-\tilde{G}_n(t) < -\frac{\tilde{\delta}_0}{2} t^{n_{j_0}+1} n^{2m-j_0} + C,$$

6) The existence of such $\varepsilon$ is guaranteed from the condition ii) of (C. I).

in $0\leqq t\leqq \min_{0\leqq j\leqq j_0-1}\left(\frac{\tilde{\delta}_0}{4j_0\tilde{c}_j}\right)^{\frac{1}{n_j-n_{j_0}}}\cdot n^{-\frac{j_0-j}{n_j-n_{j_0}}}$.

In fact, we prove (5.1) dividing into two cases.

i) the case where $j\geqq j_0+1$. Now let us consider

$$\tilde{G}_n^{(1)}(t)=\frac{\tilde{\delta}_0}{4}t^{n_{j_0}+1}n^{2m-j_0}-\sum_{j\geqq j_0+1}\tilde{c}_j t^{n_j+1}n^{2m-j}.$$

The condition i) of (C. I) implies $n_j+1\geqq\frac{(2m-j)(n_{j_0}+1)}{2m-j_0}$, hence

$$\tilde{G}_n^{(1)}(t)\geqq\frac{\tilde{\delta}_0}{4}t^{n_{j_0}+1}n^{2m-j_0}-\sum_{j\geqq j_0+1}\tilde{c}_j t^{\frac{(n_{j_0}+1)(2m-j)}{2m-j_0}}\cdot n^{2m-j},$$

$(0\leqq t\leqq 1)$. If we put $X=t^{\frac{n_{j_0}+1}{2m-j_0}}\cdot n$, we get

$$\tilde{G}_n^{(1)}(t)\geqq\frac{\tilde{\delta}_0}{4}X^{2m-j_0}-\sum_{j\geqq j_0+1}\tilde{c}_j X^{2m-j}>-C$$

because of that $X\geqq 0$.

ii) the case where $0\leqq j\leqq j_0-1$. We note that the condition ii) of (C. I) implies $n_j>n_{j_0}$ $(j<j_0)$, therefore in the interval

$$0\leqq t\leqq \min_{0\leqq j\leqq j_0-1}\left(\frac{\tilde{\delta}_0}{4j_0\tilde{c}_j}\right)^{\frac{1}{n_j-n_{j_0}}}\cdot n^{-\frac{j_0-j}{n_j-n_{j_0}}},$$

it holds

$$-\frac{\tilde{\delta}_0}{4j_0}t^{n_{j_0}+1}n^{2m-j_0}+\tilde{c}_j t^{n_j+1}n^{2m-j}\leqq 0.$$

These prove the inequality (5.1). Thus we have

$$\int_0^t \tau^{n_j}n^{2m-j}\exp\left[-\tilde{G}_n(\tau)\right]d\tau$$

$$\leqq \text{const.}\int_0^t \tau^{n_j}n^{2m-j}\exp\left[-\frac{\tilde{\delta}_0}{2}\tau^{n_{j_0}+1}n^{2m-j_0}\right]d\tau.$$

Next, let us evaluate $H_j(t)=\int_0^t \tau^{n_j}n^{2m-j}\exp\left[-\frac{\tilde{\delta}_0}{2}\tau^{n_{j_0}+1}n^{2m-j_0}\right]d\tau$,

$(j=0, 1,..., 2m)$.

a) the case where $0 \leqq j \leqq j_0 - 1$. We note that $n_j > n_{j_0}$, then obviously it holds

$$H_j(t) \leqq \text{const. } t^{n_j - n_{j_0}} n^{j_0 - j} \left\{ 1 - \exp\left[ -\frac{\tilde{\delta}_0}{2} t^{n_{j_0}+1} n^{2m-j_0} \right] \right\}.$$

b) the case where $j = j_0$.

$$H_{j_0}(t) = \text{const. } \left\{ 1 - \exp\left[ -\frac{\tilde{\delta}_0}{2} t^{n_{j_0}+1} n^{2m-j_0} \right] \right\}.$$

c) the case where $j \geqq j_0 + 1$. By the same way as i) of the proof of (5.1) we can prove that $-\frac{\tilde{\delta}_0}{2} t^{n_{j_0}+1} n^{2m-j_0} + t^{n_j+1} n^{2m-j} \leqq C$ for any $n$ and any $t \in [0, 1]$. Therefore we have

$$H_j(t) \leqq \text{const. } \int_0^t \tau^{n_j} n^{2m-j} \exp[-\tau^{n_j+1} n^{2m-j}] d\tau$$

$$= \text{const. } \{1 - \exp[-t^{n_j+1} n^{2m-j}]\}.$$

Now if we put together with the above inequalities, we have

$$\text{(5.2)} \qquad \frac{\text{const.}}{n} \exp[\tilde{G}_n(t)] \int_0^t \{ \sum_{j=0}^{2m} \tau^{n_j} n^{2m-j} \} \exp[-\tilde{G}_n(\tau)] d\tau$$

$$< \frac{\text{const.}}{n} \{ \sum_{j=0}^{j_0-1} t^{n_j - n_{j_0}} n^{j_0 - j} + 1 \} \exp[\tilde{G}_n(t)]$$

$$+ \frac{\text{const.}}{n} \{ \sum_{j=0}^{j_0-1} t^{n_j - n_{j_0}} n^{j_0 - j} + 1 \} \exp\left[ \tilde{G}_n(t) - \frac{\tilde{\delta}_0}{2} t^{n_{j_0}+1} n^{2m-j_0} \right]$$

$$+ \frac{\text{const.}}{n} \sum_{j=j_0+1}^{2m} \exp[\tilde{G}_n(t) - t^{n_j+1} n^{2m-j}].$$

Under the above preparations, we shall evaluate at $t = n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}}$

$$I_n(t) = c_0 \exp[\tilde{G}_n(t)] - \frac{C}{n} \exp[\tilde{G}_n(t)] \int_0^t h_n(\tau) \exp[-\tilde{G}_n(\tau)] d\tau.$$

At first, we note that if $n$ is sufficiently large, it holds

$$n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}} < \min_{0\leq j\leq j_0-1}\left(\frac{\tilde{\delta}_0}{4j_0\tilde{c}_j}\right)^{\frac{1}{n_j-n_{j_0}}} n^{-\frac{j_0-j}{n_j-n_{j_0}}},$$

in view of the determination of $\varepsilon$. And we can show

$$\tilde{G}_n(n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}}) = \tilde{\delta}_0 n^{\frac{\varepsilon(2m-j_0)}{n_{j_0}+1+\varepsilon}} + o(n^{\frac{\varepsilon(2m-j_0)}{n_{j_0}+1+\varepsilon}}), \tag{5.3}$$

as $n\to+\infty$. In fact, it suffices to see that when $j\neq j_0$, we have

$$t^{n_j+1}n^{2m-j}\Big|_{t=n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}}} = o(n^{\frac{\varepsilon(2m-j_0)}{n_{j_0}+1+\varepsilon}}) \quad \text{as} \quad n\longrightarrow+\infty,$$

in view of the condition (C. I) and the determination of $\varepsilon$.

Finally, in the case where $0\leqq j\leqq j_0-1$, it follows

$$t^{n_j-n_{j_0}}n^{j_0-j}\Big|_{t=n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}}} = o(1) \quad \text{as} \quad n\longrightarrow+\infty$$

because $(2m-j_0)(n_j-n_{j_0})-(n_{j_0}+1+\varepsilon)(j_0-j)>0$.

Thus, combining the above inequalities, it follows

$$I_n(n^{-\frac{2m-j_0}{n_{j_0}+1+\varepsilon}}) > \frac{c_0}{2}\exp\left[\frac{\tilde{\delta}_0}{2}n^{\frac{\varepsilon(2m-j_0)}{n_{j_0}+1+\varepsilon}}\right]$$

when $n$ is sufficiently large, which proves (4.15). q.e.d.

Department of Mathematics,
University of Tsukuba

## References

[1] Mizohata, S., Some remarks on the Cauchy problem. J. Math. Kyoto Univ. **1** (1961), 109–127.

[2] ———, On the evolution equations with finite propagation speed. Proc. Japan Acad., **46** (1970), 258–261.

[3] ———, The theory of partial differential equations. Iwanami, Tokyo (1965) (in Japanese, Camb. Univ. Press, 1973).

[4] Igari, K., Well-posedness of the Cauchy problem for some evolution equations. to appear in Publ. R.I.M.S. Kyoto Univ..

[5] Miyake, M., Degenerate parabolic differential equations: hypoellipticity, in preparation.