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On SO(3)-actions on homotopy 7-spheres

By

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§ 0. Introduction and Notations

In this paper, we shall study smooth actions of the rotation group $SO(3)$ on homotopy 7-spheres. Our category is the smooth category.

As for actions of $SO(3)$ on the n-sphere S^n, there are several works by D. Montgomery, H. Samelson and R, W. Richardson [5], [6] etc. In [5], Montgomery and Samelson proved that every smooth action of $SO(3)$ on the 7-sphere S^7 has an orbit of dimension less than three (Theorem 4 of [5]). The proof of this theorem uses only the differentiability and the homology properties, so that it holds also for homotopy 7-spheres. Our study is based on this result.

We wish to classify all smooth $SO(3)$-actions on homotopy 7-spheres. But in this paper, only partial answer will be given, that is in the case with 2 or 3 orbit types.

In §2 we will construct one type of $SO(3)$-actions on the n-sphere S^n for $n \geqslant 7$ which will be called type (A) (Theorem I and II). In §3 we offer more two types of $SO(3)$-actions on homotopy 7-spheres which will be called, type (B) and type (C). The method of these constructions is that of the orbit triple due to W. C. Hsiang and W. Y. Hsiang.

Our main result will be stated in §4 (Theorem III) and will be proved in §5. From this theorem, it follows that our constructions cover all of smooth $SO(3)$-actions on homotopy 7-spheres with two or three orbit types.

The closed subgroups of $SO(3)$ are known ([8]) and we use the following notations;

($\mathbf{Z}_k$) : the conjugate class of the cyclic group of order k

($\mathbf{D}_k$) : the conjugate class of the dihedral group of order $2k$

(**T**) : the conjugate class of the tetrahedral group

(**O**) : the conjugate class of the octahedral group

(**I**) : the conjugate class of the icosahedral group

(**N**) : the conjugate class of the normalizer of $SO(2)$.

With respect to the real representations of $SO(3)$, we use the following notations;

α : the 3-dimensional irreducible representation

β : the 5-dimensional irreducible representation

θ : the 1-dimensional trivial representation.

Finally, in general we denote by (M, φ) a smooth manifold M with $SO(3)$-action φ.

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§ 1. Preliminaries

In this section, we recall a classification theorem of G-manifolds with two orbit types due to W. C. Hsiang and W. Y. Hsiang [3]. This will be often used later.

Let G be a compact Lie group. Let H and K be two closed subgroups of G such that $H \subset K$. We assume that the homogeneous space K/H is diffeomorphic to the k-sphere S^k for some k. Let X be a paracompact contractible manifold with boundary ∂X. Then there are smooth G-manifolds $\{M\}$ such that

1) the orbit space M/G is X,
2) the isotropy subgroup types are (H) and (K) and Int$X$ is the image of the orbits of type (G/H) and ∂X is the image of the set of the orbits of type (G/K)

Let $N(H)$ and $N(K)$ be the normalizer of H and K respectively in G. Let $N(H)\cap N(K)\backslash N(H)$ be the right coset space.

Classification Theorem : *The set of the equivariant diffeomorphism classes of the above G manifolds is in one to one correspondence with the set*

$$[\partial X\ ,\ N(H)\cap N(K)\backslash N(H)]/\pi_0(N(H)/H)$$

where, [,] *denotes the set of the homotopy classes and* $\pi_0 N(H)/H$ *is the group of the arc components of* $N(H)/H$. $\pi_0(N(H)/H)$ *acts on* [,] *by the right translation action of* $N(H)/H$ *on* $N(H)\cap N(K)/N(H)$. (*See Bredon G. E.* [2] *V* 5, *for example*).

§ 2. A construction, type (A)

First, we give a short description of the 5-dimensional irreducible real representation of $SO(3)$ which we have denoted by β in § 0. Let us consider the space S of all symmetric 3×3 real matrices of trace 0. Note that this is a real vector space of dimension 5, i. e. $S\approx R^5$. For $g\in SO(3)$ and $s\in S$ define $\beta(g)\cdot s=gsg^{-1}$, where the right hand side is the matrix multiplication. Then we have a linear action of $SO(3)$ on R^5 and this is β. Define a norm on S by $||s||^2=$trace of ss for $s\ni S$. This norm is $SO(3)$-invariant.

Now let (S^4, β) be the restriction of β on the unit sphere S^4. It has two isotropy subgroup types $(\mathbf{D}_2)$ and $(\mathbf{N})$. The orbit space $S^4/SO(3)$ is an arc and the interior points of the arc correspond to the orbits of type $(SO(3)/\mathbf{D}_2)$ and the two endpoints to the orbits of type $(SO(3)/\mathbf{N})$. Now by the classification theorem in §1 we know that those $SO(3)$-manifolds are classified by elements of the set $[S^0, \mathbf{D}_4\backslash\mathbf{O}]/\pi_0(\mathbf{O}/\mathbf{D}_2)$, where S^0 denotes the 0-sphere (we note that $N(\mathbf{D}_2)=\mathbf{O}$, $N(\mathbf{N})=\mathbf{N}$ and $\mathbf{N}\cap\mathbf{O}=\mathbf{D}_4$). Let S_3 denote the symmetric group of 3 letters. Then $\mathbf{O}/\mathbf{D}_2$ is isomorphic to S_3 and $\mathbf{D}_4\backslash\mathbf{O}$ is isomorphic to $\mathbf{Z}_2\backslash S_3$ as sets. Now we have

$$[S^0, \mathbf{D}_4\backslash\mathbf{O}]/\pi_0(\mathbf{O}/\mathbf{D}_2)$$

$\approx (\mathbf{Z}_2\backslash S_3) \times (\mathbf{Z}_2\backslash S_3)/S_3$

where S_3 acts on $(\mathbf{Z}_2\backslash S_3) \times (\mathbf{Z}_2\backslash S_3)$ diagonally. Therefore the above set consists of two elements, the trivial one represented by (1, 1) ($1 \in S_3$ the unit) and nontrivial one represented by $(1, g)$ (some $g \in S_3$ of order 3).

Lemma 2, 1: *(S^4, β) corresponds to the nontrivial element.*

Proof: Let (M, φ) be an $SO(3)$-manifold corresponding to the trivial element. Then the fixed point set of $\mathbf{D}_2$ in M consists of three disjoint 1-spheres. But the fixed point set of $\mathbf{D}_2$ in (S^4, β) is a 1-sphere. It follows that $(M, \varphi) \neq (S^4, \beta)$. Q. E. D.

Lemma 2, 2: *Let f be an equivariant diffeomorphism of (S^4, β) such that the induced map of the orbit space is the identity. Then f is the identity map.*

Proof: Let $F(\mathbf{D}_2)$ be the fixed point set of $\mathbf{D}_2$. $F(\mathbf{D}_2)$ is a 1-sphere and contain exactly 6 points whose isotropy subgroups are conjugate to $\mathbf{N}$. Let l be an arc in $F(\mathbf{D}_2)$ such that the isotropy subgroups of the two endpoints are conjugate to $\mathbf{N}$ and those of the interior points are $\mathbf{D}_2$. Then l is a cross section of (S^4, β). As f is equivariant, f fix the two endpoints of l. By the assumption, l must be pointwise fixed by f. Since $SO(3)l = S^4$, f is the identity map. Q. E. D.

Now let X be a comact contractible manifold with boundary ∂X. We assume the dimension of $X = n \geqslant 4$. The boundary ∂X is a Z-homology $(n-1)$-sphere. Let F^{n-2} be a $\mathbf{Z}$-homology $(n-2)$-sphere embedded in ∂X. Let $\mathbf{R}$ and $\mathbf{R}^+$ denote the interval $(-\infty\ +\infty)$ and the interval $[0, +\infty)$ respectively. Let $F \times \mathbf{R} \times \mathbf{R}^+$ be an open tubular neighborhood of F in X such that $F \times 0 \times 0 = F$ and $F \times \mathbf{R} \times \mathbf{R}^+ \cap \partial X = F \times \mathbf{R} \times 0$.

Let (Σ_0, φ_0) be a smooth $SO(3)$-manifold such that

1) the orbit space $\Sigma_0/SO(3)$ is $X-F$,
2) the principal isotropy subgroup type is $(\mathbf{D}_2)$ and $\mathrm{Int}(X-F)$ is the image of the set of the principal orbits and
3) the singular isotropy subgroup type is $(\mathbf{N})$ and $(X-F)\cap\partial X$ is the image of the set of the singular orbits.

By the classification theorem in §1, those $SO(3)$-manifolds are classified by elements of the set

$$[\partial X-F,\ \mathbf{D}_4\backslash\mathbf{O}]/\pi_0(\mathbf{O}/\mathbf{D}_2).$$

Now by the Alexander duality, $\partial X-F$ consists of two connected components. Under the restriction the above set coincides with the set

$$[F\times(\mathbf{R}-0)\times 0,\ \mathbf{D}_4\backslash\mathbf{O}]/\pi_0(\mathbf{O}/\mathbf{D}_2)=[S^0,\ \mathbf{D}_4\backslash\mathbf{O}]/\pi_0(\mathbf{O}/\mathbf{D}_2).$$

As was shown before, this set consists of two elements. We assume that $(\Sigma_0,\ \varphi_0)$ corresponds to the nontrivial element. Then by the orbit structure and Lemma 2, 1 and the covering homotopy property ([2], II, 7), we see that the part of Σ_0 over $(F\times\mathbf{R}\times\mathbf{R}^+-F)$ is equivariantly diffeomorphic to $F\times S^4\times\mathbf{R}=F\times(\mathbf{R}^5-0)$ where $SO(3)$ acts on S^4 and $\mathbf{R}^5-0$ by β and trivially on F and R.

Now let $F\times\mathbf{R}^5$ be the $SO(3)$-manifold such that $SO(3)$ acts on $\mathbf{R}^5$ by β and trivially on F. Then we may patch together Σ_0 and $F\times\mathbf{R}^5$ by an equivariant diffeomorphism of $F\times(\mathbf{R}^5-0)$ over $(F\times\mathbf{R}\times\mathbf{R}^+-F)$. But by Lemma 2, 2, such an equivariant diffeomorphism must be the identity map. Therefore we obtain a unique equivariant diffeomorphism class of $SO(3)$-manifold $(\Sigma,\ \varphi)$.

Theorem I: *Let $n\geqslant 4$ be an integer. Let $(\Sigma^{n+3},\ \varphi)$ be an $(n+3)$-dimensional $SO(3)$-manifold such that*

1) *the isotropy subgroup types are $(\mathbf{D}_2)$, $(\mathbf{N})$ and $(SO(3))$*
2) *the orbit space $\Sigma/SO(3)$ is a compact contractible manifold with boundary and the boundary corresponds to the singular orbits and*
3) *the fixed point set F is a $\mathbf{Z}$-homology $(n-2)$-sphere.*

Then the set of the equivariant diffeomorphism classes of such $SO(3)$-manifolds is in one to one correspondence with the set of the diffeomorph-

ism classes of the pairs (X, F) such that

1) *X is an n-dimensional compact contractible manifold with boundary and*

2) *F is an $(n-2)$-dimensional $\mathbf{Z}$-homology sphere embedded in the boundary of X.*

The correspondencee is given by taking the orbit space of the given $SO(3)$-manifold.

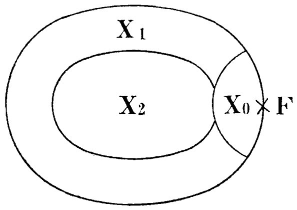
Proof: As above, we have constructed one class of $SO(3)$-manifold for a given pair (X, F). Now conversely let (Σ, φ) be an $SO(3)$-manifold satisfying the above condition. Then by its isotropy subgroup structure, we see that the slice representation of $SO(3)$ at a fixed point is $\beta\oplus(n-2)\theta$. Hence by the condition of the orbit space, there is an equivariant tubular neighborhood of F which is equivariantly diffeomorphic to $F\times\mathbf{R}^5$, where $SO(3)$ acts on $\mathbf{R}^5$ by β and trivially on F. Now $\Sigma-F$ has two orbit types and by the classification theorem in §1, it follows that (Σ, φ) coincides with a manifold constructed as above. Q. E. D.

We note that if (X, F) is (D^n, S^{n-2}), then the corresponding $SO(3)$-manifold is $(S^{n+3}, \beta\oplus(n-1)\theta)$.

Now we denote by $\mathscr{A}$ the set of the $SO(3)$-manifolds in Theorem I.

Lemma 2, 3: : *If (Σ, φ) is in $\mathscr{A}$, then Σ is a homotopy sphere.*

Proof: We decompose X into three parts X_0, X_1 and X_2 as follows,



X_0: a closed neighborhood of F in X which is a trivial half 2-disc bundle over F,

X_1: a closed neighborhood of $\overline{\partial X-X_0}$ in $\overline{X-X_0}$ which is a half 1-disc bundle over $\overline{\partial X-X_0}$ and X_2: $\overline{X-X_0\cup X_1}$.

Let p : $\Sigma \longrightarrow X$ be the orbit map. Then we have 1) $p^{-1}(X_0) \approx F\times D^5$, 2) $p^{-1}(X_1)$ is homotopically equivalent to $p^{-1}\ \overline{(\partial X-X_0)} \approx \overline{(\partial X-X_0)}\times P^2 \simeq (\partial X-F\times I)\times P^2$, where $P^2=SO(3)/\mathbf{N}$ denotes the real projective plane and I the interval [0, 1] and 3) $p^{-1}(X_2)\approx X_2 \times (SO(3)/\mathbf{D}_2)\simeq SO(3)/\mathbf{D}_2$ since X is contractible.

Now we denote $p^{-1}(X_1\cup X_2)$ by Y. Then the Mayer-Vietoris sequence (with integer coefficients)

$$\cdots\rightarrow H_i(p^{-1}(X_1))\oplus H_i(p^{-1}(X)_2)\rightarrow H_i(Y)\xrightarrow{\partial *} H_{i-1}(p^{-1}(X_1\cap X_2))\rightarrow\cdots$$

shows that $H_4(Y;\ \mathbf{Z})=\mathbf{Z}$ and $H_i(Y;\ \mathbf{Z})=0$ for $i\neq 4$.

The Mayer-Vietoris sequence (with integer coefficients)

$$\cdots\rightarrow H_i(Y)\oplus H_i(p^{-1}(X_0))\rightarrow H_i(\Sigma)\xrightarrow{\partial *} H_{i-1}(Y\cap p^{-1}(X_0))\rightarrow\cdots$$

shows that Σ is a homology sphere.

It remains to show that Σ is simply connected. First we calculate $\pi_1(p^{-1}(X_0\cup X_1))$. By the van-Kampen's theorem and the above observation 1), 2) and 3),

$$\pi_1(p^{-1}(X_0\cup X_1))=\pi_1(p^{-1}(\partial X))$$
$$=\pi_1(Z_0\times P^2)*_{\pi_1(F\times 0\times P2)}\pi_1(F)*_{\pi_1(F\times 1\times P2)}\pi_1(Z_1\times P^2)$$

where $\partial X=Z_0\cup F\times I\cup Z_1$, $Z_i\cap F\times I=\partial Z_i=F\times i$ for $i=0,\ 1$, $Z_0\cap Z_1=\phi$ and * denotes the amalgamated product. Now the factor $\pi_1(P^2)$ in $\pi_1(F\times i\times P^2)$ is mapped isomorphically onto the factor $\pi_1(P^2)$ in $\pi_1(Z_i\times P^2)$ for $i=0$, 1 and trivially into $\pi^1(F)$. Hence we may cansel $\pi_1(P^2)$ in the above product. Then,

$$\pi_1(p^{-1}(X_0\cup X_1))=\pi_1(Z_0)*_{\pi_1(F\times 0)}\pi_1(F)*_{\pi_1(F\times 1)}\pi^1(Z_1)=\pi_1(\partial X)$$

by the van-Kampen's theorem. Now

$$\pi_1(\Sigma)=\pi_1(p^{-1}(X_0\cup X_1))*_{\pi_1(p^{-1}(X_0\cup X_1)\cap p^{-1}(X_2))}\pi_1(p^{-1}(X_2))$$
$$=\pi_1(\partial X)*_{\pi_1(\partial X)\times\pi_1(SO(3)/D_2)}\pi_1(SO(3)/\mathbf{D}_2),$$

where $\pi_1(\partial X)\times\pi_1(SO(3)/\mathbf{D}_2)$ is mapped onto $\pi_1(\partial X)$ and $\pi_1(SO(3)/\mathbf{D}_2)$ by the projection onto the first and the second factor respectively. Hence this product is trivial. Q. E. D.

Theorem II : *Let $n\geqslant 7$ be an integer. If $(\Sigma^n,\ \varphi)$ is in $\mathscr{A}$, then Σ^n is the standard n-sphere S^n. If $(S^n,\ \varphi)$ is in $\mathscr{A}$, then there is some*

$(S^{n+1}, \tilde{\varphi})$ in $\mathscr{A}$ such that (S^n, φ) can be equivariantly embedded in $(S^{n+1}, \tilde{\varphi})$. If $n \geqslant 8$, $(S^{n+1}, \tilde{\varphi})$ may be chosen so that it is equivariantly embedded in $(S^{n+2}, \beta \oplus (n-2)\theta)$.

Proof: Let (X^{n-3}, F^{n-5}) be the orbit pair of (Σ^n, φ). Let I be the closed unit inierval [0, 1]. Then $X \times I$ is an $(n-2)$-dimensional contractible manifold. Now we can find an embedding f: $\partial X \longrightarrow \partial(X \times I)$ such that
$f(\partial X) \cap X \times \{1/2\} = F \times \{1/2\}$ and $f(\partial X)$ intersects transversally with $X \times \{1/2\}$. Let $\tilde{F}$ be $f(\partial X)$. Then the pair $(X \times I, \tilde{F})$ satisfy the coditions of Theorem I and we obtain an $SO(3)$-manifold $(\Sigma^{n+1}, \tilde{\varphi})$ in $\mathscr{A}$. Identifying (X, F) with $(X \times \{1/2\}, F \times \{1/2\})$ we see that (Σ^n, φ) can be equivariantly embedded in $(\Sigma^{n+1}, \tilde{\varphi})$. Now by Lemma 2, 3, $\Sigma^n$and $\Sigma^{n+1}$are both homotopy spheres. Hence Σ^n is the standard sphere S^n. The same procedure as above for the pair $(X \times I, \tilde{F})$ gives us a pair $(X \times I \times I, 2X)$ where $2X = \partial (X \times \mathrm{I})$ is the double of X. But $\pi_1(\partial(X \times I \times I)) = 1 = \pi_1(2X)$ hence by Smale's h-cobordism theorem we see that $X \times I \times I \approx D^{n-1}$ and $2X \approx S^{n-3}$ if $n \geq 8$. Therefore if $n \geqslant 8$, $(\mathrm{S}^{n+1}, \tilde{\varphi}) = (\Sigma^{n+1}, \tilde{\varphi})$ can be embedded in $(\mathrm{S}^{n+2}, \beta \oplus (\mathrm{n}-2)\theta)$. Q. E. D.

By Mazur's result ([4]), there are infinitely many compact contractible manifolds which are not diffeomorphic to D^4. Hence we obtain infinitely many distinct $SO(3)$-actions on the standard 7-sphere S^7. We call these actions as of type (A).

§3. More two constructions, type (B) and type (C)

In this section, we state two more types of $SO(3)$-actions on homotopy 7-spheres.

Construction, type (B) : Let X be a 5-dimensional compact contractible manifold with boundary ∂X. Let D^3 be the unit 3-disc. Let $SO(3)$ act on $X \times D^3$ such as trivially on X and by α

on D^3. Then by the h-cobordism theorem, $\partial(X\times D^3)$ is the standard 7-sphere S^7 and we have an $SO(3)$-action on S^7. If X is the 5-disc D^5 then the action is $\alpha\oplus 5\theta$. There are infinitely many compact contractiple 5-manifolds such that $\pi_1(\partial X)\neq 1$. Hence we have infinitely many distinct $SO(3)$-actions on S^7.

We call these $SO(3)$-actions as of type (B).

Construction, type (C)

First, we consider the 5-sphere with $SO(3)$-action $\alpha\oplus\alpha$, $(S^5, \alpha\oplus\alpha)$. It has two isotropy subgroup types (1) and $(SO(2))$. The orbit space is the 2-disc D^2 whose boundary S^1 is the image of the S^2 orbits. Now by the classification theorem in §1, those $SO(3)$-manifolds are classified by elements of the set

$$[S^1, N\backslash SO(3)]/\pi_0(SO(3))$$
$$=[S^1, P^2]=H^1(S^1; \mathbf{Z}_2)=\mathbf{Z}_2,$$

where P^2 denotes the real projective plane.

Lemma 3, 1: *$(S^5, \alpha\oplus\alpha)$ corresponds to the nontrivial element of $H^1(S^1; \mathbf{Z}_2)=\mathbf{Z}_2$.*

Proof: Let M be an $SO(3)$-manifold corresponding to the non-trivial element. Then the fixed point set of $SO(2)$ in M consists of two disjoint 1-spheres. But that of $(S^5, \alpha\oplus\alpha)$ is a 1-sphere. Hence $M\neq(S^5, \alpha\oplus\alpha)$. Q. E. D.

Now let X be a compact contractible 4-manifold with boundary ∂X. Let S^1 be a 1-sphere embedded in ∂X. We assume that the double cover of ∂X branched at S^1 is a $\mathbf{Z}$-homology 3-sphere. Let $\mathbf{R}$ and $\mathbf{R}^+$ be the interval $(-\infty, +\infty)$ and $[0, +\infty)$ respectively. Let $S^1\times\mathbf{R}^2\times\mathbf{R}^+$ be an open tubular neighborhood of S^1 in X such that $S^1\times 0\times 0=S^1$ and $S^1\times\mathbf{R}^2\times\mathbf{R}^+\cap\partial X=S^1\times\mathbf{R}^2\times 0$ is an open tubular neighborhood of S^1 in ∂X.

Let (Σ_0^7, φ_0) be a 7-dimensional $SO(3)$-manifold such that

1) the orbit space is $X-S^1$,

2) there are two isotropy subgroup types (1) and $(SO(2))$ and $\partial X - S^1$ is the image of the S^2 orbits.

By the classification theorem in § 1, those $SO(3)$-manifolds are classified by elements of the set

$[\partial X - S^1,\ N\backslash SO(3)]/\pi_0(SO(3)) = [\partial X - S^1,\ P^2]$.

Lemma 3, 2 $[\partial X - S^1,\ P^2] = H^1(\partial X - S^1;\ \mathbf{Z}_2) = \mathbf{Z}_2$.

Proof: Since $\partial X - S^1$ is an open 3-manifold, it has a homotopy type of a 2-complex, K. Let P^∞ be the infinite real projective space which is $K(Z_2, 1)$. It suffices to show that the map induced by the inclusion $P^2 \subset P^\infty$, i: $[K, P^2] \longrightarrow [K, P^\infty]$ is bijective. As K is a 2-complex, i is surjective. Let K^1 be the 1-skeleton of K. By the following commutative diagram containing the Puppe sequence associated to the cofibration: $K^1 \to K \to \vee S^2$,

$$\begin{array}{ccccccc} [SK^1, P^2] & \longrightarrow & [\vee\ S^2, P^2] & \longrightarrow & [K, P^2] & \longrightarrow & [K^1\ P^2] \\ & & & & \downarrow & & \| \\ & & & & [K, P^\infty] & \longrightarrow & [K^1, P^\infty] \end{array}$$

it suffices to show that $[SK^1, P^2] \longrightarrow [\vee S^2, P^2]$ is onto. As K^1 is connected, we have $[SK^1, P^2] = [SK^1, S^2]$ and $[\vee S^2, P^2] = [\vee S^2, S^2]$. The surjectivity follows from the following commutative diagram

$$\begin{array}{ccccl} [SK^1, P^2] & \longrightarrow & [\vee S^2, P^2] & & \\ \| & & \| & & \\ [SK^1, S^2] & \longrightarrow & [\vee S^2, S^2] & \to & [K, S^2] = H^2(K;\ \mathbf{Z}) = 0. \end{array}$$

We note that $H^2(K; \mathbf{Z}) = H^2(\partial X - S^1;\ \mathbf{Z}) = 0$ by the Alexander duality. Q. E. D.

Now we assume that (Σ_0^7, φ_0) corresponds to the non-trivial element of $H^1(\partial X - S^1;\ \mathbf{Z}_2)$. Under the inclusion $(S^1 \times (\mathbf{R}^2 - 0)) \longrightarrow (\partial X - S^1)$, the non-trivial element of $H^1(\partial X - S^1; \mathbf{Z}_2)$ corresponds to the non-trivial element of $H^1(\mathbf{R}^2 - 0;\ \mathbf{Z}_2) \subset H^1(S^1 \times (\mathbf{R}^2 - 0);\ \mathbf{Z}_2)$. Hence by the orbit structure and Lemma 3, 1 and the covering homotopy property ([2]) we see that the part of (Σ_0^7, φ_0) over

$(S^1\times\mathbf{R}^2\times\mathbf{R}^+-S^1)$ is equivariantly diffeomorphic to $(S^1\times S^5\times\mathbf{R}$, $1\times(\alpha\oplus\alpha)\times 1)$, where 1 denotes the trivial action. Hence we may patch together (Σ^7_0, φ_0) and $(S^1\times\mathbf{R}^6, 1\times(\alpha\oplus\alpha))$ by an equivariant diffeomorphism f of $S^1\times(\mathrm{R}^6-0)$ over $(S^1\times\mathbf{R}^2\times\mathbf{R}^+-S^1)$. Then we obtain an $SO(3)$-manifold
$(\Sigma^7_f, \varphi)=(\Sigma^7_0, \varphi_0)\cup_f(\mathrm{S}^1\times\mathbf{R}^6, 1\times(\alpha\oplus\alpha))$.

Theorem I′: *Let M be a 7-dimensional $SO(3)$-manifold such that*
1) *the isotropy subgroup types are* (1), $(SO(2))$ *and* $(SO(3))$,
2) *the orbit space X is a compact contractible 4-manifold with boundary ∂X and the boundary is the image of the set of the singular orbits, and*
3) *the fixed point set is a 1-sphere S^1 and the double cover of ∂X branched at S^1 is a* **Z**-*homology* 3-*sphere.*

Then M is equivalent to one of (Σ^7_f, φ) constructed as above and it is a homotopy sphere.

Proof: By the isotropy subgroup structure of M, we see that the slice representation of $SO(3)$ at a fixed point is $\alpha\oplus\alpha$. Hence by the condition of the orbit space, there is an equivariant tubular neighborborhood of S^1 which is equivariantly diffeomorphic to $(S^1\times\mathbf{R}^6, 1\times(\alpha\oplus\alpha))$. $M-S^1$ has two orbit types and by the classification theorem in § 1, it follows that M is equivalent to one of $\{(\Sigma^7_f, \varphi)\}$. Now let O (3) be the orthogonal group. According to G. E. Bredon ([2], V Theorem 11, 5, and VI Theorem 7, 2) for a pair (X, S^1), we can construct a unique homotopy 7-sphere Σ with $O(3)$-action such that
1) it has three isotropy subgroup types $(O(1))$, $(O(2))$ and $(O(3))$,
2) the orbit space is X and ∂X is the image of the singular set and
3) the fixed point set is S^1.

Now we restrict the $O(3)$-action on Σ to $SO(3)$. This $SO(3)$-manifold is one of those constructed as above by the above argument. If we remove a tubular neighborhood of the fixed point set which is equivariantly diffeomorphic to $(S^1\times\mathbf{R}^6,\ 1\times(\alpha\oplus\alpha))$ and reattach it by an $SO(3)$ equivariant diffeomorphism over $S^1\times(\mathbf{R}^2-0)\times\mathbf{R}^+$, we see that any $(\Sigma^7_f,\ \varphi)$ with orbit space $(X,\ S^1)$ can be obtained in this way. The reattaching does not change the fundamental group and the homology properties of the total space. Hence Σ^7_f is a homotopy sphere. Q. E. D.

Remark: The double cover of ∂X branched at S^1 is the submanifold of $(\Sigma^7_f,\ \varphi)$ fixed by $SO(2)$.

We call the above $SO(3)$-actions on homotopy 7-spheres as of type (C).

§ 4. On SO(3)-actions on homotopy 7-spheres

In § 4 and § 5, we denote $SO(3)$ by G. Let $(\Sigma^7,\ \varphi)$ be an $SO(3)$-action on a homotopy 7-sphere Σ^7. For a closed subgroup H of G, $F(H)$ denotes the fixed point set of H. $F(H)$ is a smooth submanifold of Σ^7 and if $K\supset H$ then $F(K)\subset F(H)$. $F(G)$ is denoted simply by F.

Now $F(\mathbf{Z}_2)$ and $F(\mathbf{D}_2)$ are both $\mathbf{Z}_2$-homology spheres by P. A. Smith's theorem ([1], III). Since all the elements of order 2 in G are mutually conjugate, it follows from a theorem of A. Borel ([1] p 175), that

$$7-\dim. F(\mathbf{D}_2)=3\ \dim.\ F(\mathbf{Z}_2)-3\dim. F(\mathbf{D}_2).$$

Therefore we have $\dim. F(\mathbf{Z}_2)=5$ and $\dim. F(\mathbf{D}_2)=4$ or $\dim. F(\mathbf{Z}_2)=3$ and $\dim. F(\mathbf{D}_2)=1$.

We separate our study into four cases ;

Case 1: $\dim. F(\mathbf{Z}_2)=5$, $\dim. F(\mathbf{D}_2)=4$ and $F(\mathbf{D}_2)\neq F(\mathbf{N})$,

Case 2: $\dim. F(\mathbf{Z}_2)=5$, $\dim. F(\mathbf{D}_2)=4$ and $F(\mathbf{D}_2)=F(\mathbf{N})$,

Case 3: $\dim. F(\mathbf{Z}_2)=3$, $\dim. F(\mathbf{D}_2)=1$ and $F(\mathbf{D}_2)=F(SO(2))$,

Case 4: $\dim. F(\mathbf{Z}_2)=3$, $\dim. F(\mathbf{D}_2)=1$ and $F(\mathbf{D}_2)\neq F(SO(2))$.

The linear model of each case is as follows, Case 1: $\beta\oplus 3\theta$, Case

2: $\alpha\oplus 5\theta$, Case 3: $2\alpha\oplus 2\theta$ and Case 4: $\alpha\oplus\beta$, $\gamma\oplus\theta$, where γ is the 7-dimensional irreducible real representation of $SO(3)$.

Theorem III: *Let* (Σ^7, φ) *be a smooth* $SO(3)$*-action on a homotopy 7-sphere. Then*
in Case 1, (Σ^7, φ) *is equivalent to one of type* (A)
in Case 2, (Σ^7, φ) *is equivalent to one of type* (B)
in Case 3, (Σ^7, φ) *is equivalent to one of type* (C)
and in Case 4, (Σ^7, φ) *has more than three orbit types.*

This theorem will be proved in the next section §5.

Corollary: *If* (Σ^7, φ) *has two or three orbit types and* Σ^7 *is an exotic sphere, then* (Σ^7, φ) *is of type* (C).

Proof: This is an immediate consequence of the above theorem and Theorem II in § 2.

§ 5. Proof of Theorem III

First we note that the orbit space $X=\Sigma^7/G$ is simply connected. This is a consequence of the fact
$\pi_1(\Sigma^7)=1=\pi_0(G)$ (Bredon [2], p. 91, Corollary 6, 3).

Case 1

By theorem 4 of Montgomery and Samelson [5], $F(SO(2))$ is not empty (this theorem is proved for the standard sphere S^7 in [5], but as was noted in § 0, it holds also for homotopy 7-spheres).

Lemma 5,1: *Let* (P) *be the principal isotropy subgroup type, then* (P) *must be* $(\mathbf{D}_{2k})$ *for some* k *and if* $k\geqslant 2$, *then* $F=\phi$.

Proof: There are exactly three subgroups of $SO(3)$, $\mathbf{N}$, $\mathbf{N}_1$ and $\mathbf{N}_2$ which contain $\mathbf{D}_2$ and are of infinite order. $\mathbf{N}$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are mutually conjugate. Hence dim. $F(\mathbf{N})=$dim. $F(\mathbf{N}_1)=$dim. $F(\mathbf{N}_2)$

$\leqslant 3$ by the assumption. Let x be a point of $F(\mathbf{D}_2)-(F(\mathbf{N})\cup F(\mathbf{N}_1)\cup F(\mathbf{N}_2))$. Then the isotropy subgroup of x, G_x, is a finite subgroup containing $\mathbf{D}_2$. Since the normalizer of $\mathbf{D}_2$, $N(\mathbf{D}_2)=\mathbf{O}$, is finite, there are at most finitely many elements g of G such that $g\mathbf{D}_2g^{-1}\subset G_x$. Hence $Gx \cap F(\mathbf{D}_2)$ consists of finite points. As $\dim.G=3$ and $\dim.(F(\mathbf{D}_2)-(F(\mathbf{N})\cup F(\mathbf{N}_1)\cup F(\mathbf{N}_2)))=4$, we have $\dim.GF(\mathbf{D}_2)=7$. Hence $GF(\mathbf{D}_2)=\Sigma^7$, and it follows that $(P)\geqslant(\mathbf{D}_2)$. Therefore $(P)=(\mathbf{D}_{2k})$ or $(\mathbf{T})$ or $(\mathbf{O})$ or $(\mathbf{I})$. Now if F is not empty, we have a slice representation of G at a fixed point. But the principal isotropy subgroup type of 7-dimensional real representation of G is (1) or $(\mathbf{D}_2)$. Hence if $(P)\neq(\mathbf{D}_2)$, then F must be empty. But if $(P)=(\mathbf{T})$ or $(\mathbf{O})$ or $(\mathbf{I})$, then $F(SO(2))=F$ and this is impossible. Hence $(P)=(\mathbf{D}_{2k})$. Q. E. D.

Now the natural representation of $\mathbf{N}$: $\mathbf{N}\xrightarrow{\subset}SO(3)$ has a 2-dimensional invariant subspace and this induces a 2-dimensional representation $\delta:\mathbf{N}\longrightarrow\mathbf{O}(2)$.

Lemma 5, 2: *$F(\mathbf{N})$ is a connected 3-manifold. and for each point x of Σ^7, $(G_x)=(\mathbf{D}_{2k})=(P)$ or $(G_x)\geqslant(\mathbf{N})$.*

Proof: As $(P)=(\mathbf{D}_{2k})$, we have $F(SO(2))=F(\mathbf{N})$ and $GF(\mathbf{N})$ is the singular set. As $F(\mathbf{N})\subsetneqq F(\mathbf{D}_2)$, $\dim.F(\mathrm{N})\leqslant 3$. If $\dim.F(\mathbf{N})\leqslant 2$, then $\dim.GF(\mathrm{N})\leqslant 4$ and (P) must be (1) by theorem 2 of Montgomery and Samelson [5] (this theorem holds also for homotopy spheres in the same reason as theorem 4 of [5]). Hence we have $\dim.F(\mathbf{N})=3$. As $F(SO(2))=F(\mathbf{N})$, it is a $\mathbf{Z}$-homology sphere by P. A Smith's theorem, hence connected.

Now a 4-dimensional real representation of G has at least 1-dimensional trivial subspace, hence we see $\dim.\mathrm{F}\neq 3$. Therefore $F\subsetneqq F(\mathbf{N})$. Let y be a point of $F(\mathbf{N})-F$. As (D_{2k}) is principal and $\dim.F(\mathbf{N})=3$, the slice representation of $\mathbf{N}$ at y is of the form $\tilde{\delta}\oplus 3\theta$, where $\tilde{\delta}$ is given by the homomorphism $\mathbf{N}\rightarrow\mathbf{N}/\mathbf{Z}_{2k}\approx\mathbf{N}\xrightarrow{\delta}\mathbf{O}$

(2). From this fact it follows that if $p^s \nmid 2k$ and $p^s \geqslant 3$ for an integer s and a prime p, then $F(\mathbf{N})$ is a connected component of $F(\mathbf{Z}_{p^s})$ (we note that $F(\mathbf{Z}_{p^s}) \cap GF(\mathbf{N}) = F(\mathbf{N})$). But $F(\mathbf{Z}_{p^s})$ is a $\mathbf{Z}_p$-homology sphere and $\dim. F(\mathbf{Z}_{p^s}) \geqslant \dim.\ F(\mathbf{N}) = 3$, hence $F(\mathbf{Z}_{p^s})$ is connected and $F(\mathbf{Z}_{p^s}) = F(\mathbf{N})$. Now let x be a point of Σ^7. If $(G_x) \gneqq (\mathbf{D}_{2k}) = (P)$, then $g\ G_x\ g^{-1} \supsetneqq \mathbf{D}_{2k}$ for some $g \ni G$. We can choose an integer s and a prime p such that $\mathbf{Z}_{p^s} \subset g\ G_x\ g^{-1}$ and $\mathbf{Z}_{p^s} \not\subset \mathbf{D}_{2k}$ (hence $p^s \nmid 2k$) and $p^s \geqslant 3$. Then we have $F(g\ G_x\ g^{-1}) \subset F(\mathbf{Z}_{p^s}) = F(\mathbf{N})$ and $(G_x) \geqslant (\mathbf{N})$. Q. E. D.

Lemma 5, 3: *F is not empty.*

Proof: Let us assume that F is empty. Then $GF(\mathbf{N}) = F(\mathbf{N}) \times P^2$, where P^2 denotes the real projective plane. The orbit space X is a compact 4-manifold with boundary $\partial X \approx F(\mathbf{N})$. $\Sigma^7 - \mathrm{GF}(\mathbf{N})$ is a fibre bundle over IntX with fibre $G/\mathbf{D}_{2k}$ and structure group $\mathbf{N}(\mathbf{D}_{2k})/\mathbf{D}_{2k}$ (this is finite) by Lemma 5, 2. As X is simply connected, $\Sigma^7 - GF(N) = \mathrm{Int} X \times (G/\mathbf{D}_{2k})$. By the Alexander-duality we have

$$H_1(\mathrm{Int}\ X \times (G/\mathbf{D}_{2k})\ ;\ \mathbf{Z}) = H^5(F(\mathbf{N}) \times \mathrm{P}^2;\ \mathbf{Z}).$$

The first group is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the second to $\mathbf{Z}_2$. This is a cotradiction. Q. E. D.

By Lemma 5, 1, 5, 2 and 5, 3, we see that the principal isotropy subgroup type $(P) = (\mathbf{D}_2)$. Now the slice representation of G at a fixed point must be $\beta \oplus 2\theta$. Hence each component of F is 2-dimensional and there are three isotropy subgroup types $(\mathbf{D}_2)$, $(\mathbf{N})$ and (G). The orbit space X is a 4-dimensional manifold with boundary corresponding to $GF(\mathbf{N})$. $\Sigma^7 - GF(\mathbf{N})$ is a fibre bundle over IntX with fibre $G/\mathbf{D}_2$ and structure group $\mathbf{O}/\mathbf{D}_2 = S_3$, the symmetric group of 3 letters. As X is simply connected

$$\Sigma^7 - GF(\mathbf{N}) = \mathrm{Int} X \times (G/\mathbf{D}_2).$$

In the proof of the next two lemmas, homology and cohomology groups have always integer coefficients.

Lemma 5,4: *X is acyclic, hence contractibl.*

Proof: It suffices to show that $H_2(X)=H_3(X)=0$ and hence to show that H_5 $(\mathrm{Int}X\times(G/\mathbf{D}_2))=H_6(\mathrm{Int}X\times(G/\mathbf{D}_2))=0$. By the Alexander-duality H_i $(\mathrm{Int}X\times(G/\mathbf{D}_2))=H^{6-i}(GF(\mathbf{N}))$. Since $F(\mathbf{N})=F(SO(2))$ is a $\mathbf{Z}$-homology 3-sphere, we have $\tilde{H}^0(GF(\mathbf{N}))=0$. Now $GF(\mathbf{N})/G=F(\mathbf{N})$. As $H^1(P^2)=H^1(\text{point})=H^1(F(\mathbf{N}))=0$, we have $H^1(GF(\mathbf{N}))=0$ by Leray's spectral sequence ([1], III). Q. E. D.

From this lemma, it follows that $\tilde{H}^i(GF(\mathbf{N}))=\tilde{H}_{6-i}(G/\mathbf{D}_2)$. Hence $H^i(GF(\mathbf{N}))$ is as follows; $H^1=H^2=H^4=0$, $H^3=\mathbf{Z}$ and $H^5=\mathbf{Z}_2\oplus\mathbf{Z}_2$.

Lemma 5, 5: *F is a 2-sphere.*

Proof: As was shown before, each component of F is an orientable 2-manifold. It suffices to show that $H^1(F)=0$ and $H^2(F)=\mathbf{Z}$. Let V be a closed neighborhood of F in $F(\mathbf{N})$ which is diffeomorphic to $F\times I$ (I is the unit interval [0, 1]). Then $\overline{F(\mathbf{N})-V}\cap V$ is a disjiont union of two copies of F. $G(F(\mathbf{N})-V)=(F(\mathbf{N})-V)\times P^2$ and $G(V)$ is homotopically equivalent to F. Since $F(\mathbf{N})$ is a $\mathbf{Z}$-homology 3-sphere, we have rank $H^1(\overline{F(\mathbf{N})-V})=$ rank $H^1(F)$ and rank $H^1(\overline{F(\mathbf{N})-V})=$ rank $H^2(F)-1$ by the Alexander-duality. Now consider the Mayer-Vietoris sequence,
$\cdots\to H^i(GF(\mathbf{N}))\to H^i(G(\overline{F(\mathbf{N})-V}))\oplus H^i(GV)\to H^i(GV\cap G(\overline{F(\mathbf{N})-V}))$
$\longrightarrow\cdots$ Put $i=4$ and we have 2 rank $H^2(F)-$rank $H^2(\overline{F(\mathbf{N})-V})=2$. Hence rank $H^2(F)=1$ that is $H^2(F)=\mathbf{Z}$. Put $i=3$ and we have rank $H^1(F)=0$ that is $H^1(F)=0$. Q. E. D.

Consequently X is a contractible 4-manifold with boundary ∂X and F is a 2-sphere embedded in ∂X by the orbit map. (Σ^7, φ) has three isotropy subgroup types $(\mathbf{D}_2)$, $(\mathbf{N})$ and (G) corresponding to Int$X$, $\partial X-F$ and F respectively.

By Theorem I in § 2, (Σ^7, φ) is of type (A).

Case 2

First we will show that $F(\mathbf{N})=F$.

Assume that $F(\mathbf{N})\neq F$. Let x be a point of $F(\mathbf{N})-F$. Then $G_x=\mathbf{N}$ and $Gx\cap F(\mathbf{N})=x$. As Gx is P^2 and dim. $(F(\mathbf{N})-F)=4$, we have dim. $GF(\mathbf{N})=6$. It follows from a theorem about the dimension of singular set ([1], IX p. 117) that there is no three dimensional orbit. As a principal orbit is orientable, the principal isotropy subgroup type must be $(SO(2))$. If $F\neq\phi$, the slice representation of G at a fixed point has three isotropy subgroup types $(SO(2))$, $(\mathbf{N})$ and (G). But there is no such 7-dimensional representation, so this is impossible and $F=\phi$. Hence every orbit is 2-dimensional and we have a fibre bundle

$$F(SO(2))\longrightarrow\Sigma^7\longrightarrow P^2.$$

But $F(SO(2))$ is a $\mathbf{Z}$-homology 5-sphere by P. A. Smith's theorem, so this is impossible. We get $F(\mathbf{N})=F$.

Now F is 4-dimensional by the assumption. The slice representation of G at a fixed point must be $\alpha\oplus 4\theta$. Let X be the orbit space of (Σ^7, φ). X is a 5-dimensional manifold with boundary which corresponds to F.

Lemma 5,6: X *is acyclic, hence contractible.*

Proof: The quotient group $\mathbf{N}/SO(2)=Z_2$ acts on F $(SO(2))$, and the fixed point set of this $\mathbf{Z}_2$-action is F and the orbit space can be identified with X. F is a $\mathbf{Z}_2$-homology sphere and it separates $F(SO(2))$ into two diffeomorphic parts. Let g be the generator of $\mathbf{N}/SO(2)$ and let B be a subset of $F(SO(2))$ such that $B\cup gB=F(SO(2))$ and $B\cap gB=F$. Then B can be identified with X. Now it suffices to show that B is acyclic. Let $i_1: F\subset B$ and i_2: $F\subset gB$ be the inclusions. The diagram

$$\begin{array}{ccc} H_i(F\,;\,\mathbf{Z}) & \xrightarrow{(i_1)_*} & H_i(B\,;\,\mathbf{Z}) \\ & {\scriptstyle (i_2)_*}\searrow & \downarrow{\scriptstyle g_*} \\ & & H_i(gB\,;\,\mathbf{Z}) \end{array}$$

commutes. The Mayer-Vietoris sequence with integer coefficients for the triple $(F(SO(2)), B, gB)$,

$$\cdots\longrightarrow H_j\ (F)\longrightarrow H_j(B)\oplus H_j\ (gB)\longrightarrow H_j(F(SO(2))\longrightarrow\cdots$$

shows that $(i_1)_*$ and $(i_2)_*$ are both isomorphisms and $\tilde{H}_j\ (B)=\tilde{H}_j\ (gB)=0.$ Q. E. D.

Consequently X is a contractible 5-manifold with boundary ∂X and (Σ^7, φ) has two isotropy subgroup types $(SO(2))$ and (G). By the classification theorem in § 1, (Σ^7, φ) is of type (B).

Case 3

First we will show that $F(\mathbf{D}_2)=F(\mathbf{N})=F$. From the assumption $(F(\mathbf{Z}_2)=F(SO(2)))$, it follows that $F(\mathbf{D}_2)=F(\mathbf{N})$. It is a $\mathbf{Z}_2$-homology sphere (1-dimensional), hence a circle. For a point x of $F(SO(2))$, the isotropy subgroup G_x is $SO(2)$. Hence the principal isotropy subgroup type (P) must be $(\mathbf{Z}_k)$ for some k. But by theorem 1 of Montgomery and Samelson [5], it must be (1). Let x be a point of $F(\mathbf{N})$, then $G_x=\mathbf{N}$ or G. Assume that $G_x=\mathbf{N}$, then the slice representation of $\mathbf{N}$ at x have a 2-dimensional invariant subspace (the normal plane to GF(SO(2))) on which $\mathbf{N}$ acts freely (as $(P)=(1)$). But there is no such a 2-dimensional representation of $\mathbf{N}$, so this is impossible. Hence we have $F(\mathbf{D}_2)=F(\mathbf{N})=F$.

Lemma 5, 7: (Σ^7, φ) *has three isotropy subgroup types* (1), $(SO(2))$ *and* (G).

Proof: It remains to show that for any point of $(\Sigma^7-GF(SO(2)))$, its isotropy subgroup is trivial. Since $F(\mathbf{Z}_2)=F(SO(2))$, this group has no elements of order 2 and must be conjugate to $\mathbf{Z}_{2k+1}$ for some k. Now fix an odd prime p. Let x be a point of $F(\mathbf{Z}_p)-F(SO(2))$. Consider the slice representation of $\mathbf{Z}_p$ at x. As the principal isotropy subgroup is trivial, $\mathbf{Z}_p$ acts non trivially on the slice. Hence dim. $F(\mathbf{Z}_p)\leqslant 3$. But $F(\mathbf{Z}_p)\supseteqq F(SO(2))$ and dim. $F(SO(2))=3$, so that we have $F(\mathbf{Z}_p)=F(SO(2))$ (we note that $F(\mathbf{Z}_p)$ and $F(SO(2))$ are both connected). It follows that $(\Sigma^7-GF(SO(2)))$ consists of principal orbits. Q. E. D.

Now the slice representation of G at a fixed point must be $2\alpha\oplus\theta$. The orbit space X is a 4-dimensional manifold with boundary which corresponds to $GF(SO(2))$.

Lemma 5, 8: *X is acyclic, hence contractible.*

Proof: We will show that $H_2(X)=H_3(X)=0$. Throughout the proof homology and cohomology groups are understood to have integer coefficients. Since there is a fibre bundle, $G\longrightarrow(\Sigma^7-GF(SO(2)))\longrightarrow\text{Int}X$, it suffices to show that $H_5(\Sigma^7-GF(SO(2)))=H_6(\Sigma^7-GF(SO(2)))=0$. By the Alexander duality, $\tilde{H}_i(\Sigma^7-GF(SO(2)))\cong\tilde{H}^{6-i}(GF(SO(2)))$. As $F(SO(2))$ is connected, $\tilde{H}^0(GF(SO(2)))=0$. It remains to show that $H^1(GF(SO(2)))=0$. Now $\mathbf{N}/SO(2)=\mathbf{Z}_2$ acts on $F(SO(2))$ and $F(SO(2))/\mathbf{Z}_2=GF(SO(2))/G$. As $H^1(F(SO(2)))=0$, we have $H^1(F(SO(2))/Z_2)=H^1(GF(SO(2))/G)=0$. Since $H^1(S^2)=H^1(\text{point})=0$, we have $H^1(GF(SO(2)))=0$ by Leray's spectral sequence ([1], III). Q. E. D.

Consequently X is a contractible 4-manifold with boundary ∂X. F is a circle and embedded in ∂X by the orbit map. (Σ^7, φ) has three isotropy subgroup types (1), $(SO(2))$ and (G) corresponding to $\text{Int}X$, $\partial X-F$ and F respectively.

By Theorem I' in § 3, (Σ^7, φ) is of type (C).

Case 4

We note that $F(\mathbf{D}_2)$ and $F(SO(2))$ are both circles. $SO(2)$ and $\mathbf{N}$ act on $F(\mathbf{Z}_2)$. Let Y be the orbit space $F(\mathbf{Z}_2)/SO(2)$. It is an orientable 2-manifold with boundary corresponding to $F(SO(2))$. Since $F(\mathbf{Z}_2)$ is a $\mathbf{Z}_2$-homology 3-sphere and the map $H_1(F$

$(\mathbf{Z}_2)$; $\mathbf{Z}_2) \longrightarrow H_1(Y;\ \mathbf{Z}_2)$ is onto, we have $H_1(Y;\ \mathbf{Z}_2)=0$. Hence Y is a 2-disc. Now $\mathbf{N}/SO(2)=Z_2$ acts on Y. The image of $F(\mathbf{D}_2)$ in Y is just the fixed point set of this Z_2-action. By P. A. Smith's theorem ([1], III), it is acyclic over $\mathbf{Z}_2$, so that it is an arc with the end points in ∂Y. Now it follows that $F(\mathbf{D}_2)\cap F(SO(2))$ consists of two points.

For $x\in F(\mathbf{D}_2)-F(SO(2))$, G_x is a finite subgroup containing $\mathbf{D}_2$, and for $x\ni F(SO(2))-F(\mathbf{D}_2)$, G_x is $SO(2)$, and for $x\in F(SO(2))\cap F(\mathbf{D}_2)$, G_x is $\mathbf{N}$ or G. As $F(\mathbf{Z}_2)\cap GF(\mathbf{D}_2)$ is 2-dimensional $(F(\mathbf{Z}_2)-GF(\mathbf{D}_2))$ is not empty, and for $x\in(F(\mathbf{Z}_2)-GF(\mathbf{D}_2))$, G_x is a cyclic group. Hence (Σ^7, φ) has more than three orbit types.

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