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On the group $\mathrm{Ext}^{1,*}_{A^{BP}}(BP^*(S^0), BP^*(S^0))$ and the Hurewicz Image of BP/S^0

By

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§ 0. Introduction

In [11] Novikov showed that the Adams spectral sequence of MU-theory (complex cobordism theory) is an important method for studying the stable homotopy groups of spheres, and he also showed that for studying the p-primary component of the stable homotopy groups of spheres it is convenient to use BP-theory, which is the direct summand of $MUZ_{(p)}$-theory, where $MUZ_{(p)}$ is the spectrum localized at a prime p of MU-spectrum.

However in general, it is difficult to calculate the E_2 terms of the Adams spectral sequence of MU- or BP-theory.

In [7] Buȟstaber announced that there exists a tri-graded spectral sequence $\{E_r^{*,*,*}, d_r\}$ such that $E_1^{0,*,*} = Ext^{*,*}_{A^{MU}}(MU^*(S^0), MU^*(S^0))$, and $E_\infty^{*,*,*} = E_\infty^{0,0,0} = Z$. Using this spectral sequence he gave an interpretation of $Ext^{1,*}_{A^{MU}}(MU^*(S^0), MU^*(S^0))$ in terms of the integral homology group of MU and its Hurewicz image.

We shall show that there is a BP- analogy of Buȟstaber's interpretation of $Ext^{1,*}_{A^{MU}}(MU^*(S^0), MU^*(S^0))$. However our method is quite different from his.

Let X be a CW spectrum with basepoint and $BP^*(X)$ the reduced BP-cohomology of X. Let $\Lambda = BP^*(S^0)$, the reduced BP- cohomology of the sphere spectrum S^0, and A^{BP} be the algebra of primary operations of BP- cohomology theory. Let $Z_{(p)}$ be the integers localized at p. Our main results are as follows;

Theorem 2-7. *There exists a tri-graded multiplicative spectral sequence* $\{E_r^{*,*,*}, d_r\}$ $(1 \leqq r \leqq \infty)$ *such that*

i) $E_1^{u,s,t} = H_u(BP; Z) \otimes_{Z_{(p)}} Ext_{A^{BP}}^{s,t}(\Lambda, \Lambda)$,

ii) d_r: $E_r^{u,s,t} \to E_r^{u-r,s+1,t+r}$ *is an anti-derivation,*

iii) $E_\infty^{u,s,t} = \begin{cases} \pi_u(BP) & (if\ s=t=0), \\ 0 & (otherwise), \end{cases}$

iv) *the edge homomorphism* $E_\infty^{u,0,0} \to E_1^{u,0,0}$ *coincides with the Hurewicz homomorphism* h_u: $\pi_u(BP) \to H_u(BP; Z)$,

v) A^{BP} *acts canonically on* $\{E_r, d_r\}$.

Corollary 3-3.

$$Ext_{A^{BP}}^{1,t}(\Lambda, \Lambda) \cong N_t/Im h_t, \quad for \quad t>0.$$

where h_t: $\pi_t(BP) \to H_t(BP; Z)$ *is the Hurewicz homomorphism of* BP *and* N_t *is a certain subgroup of* $H_t(BP; Z)$, *which is algebraically determined by the actions of* A^{BP} *on* $H_t(BP; Z)$.

We also obtained the geometrical interpretation of N_t.

Theorem 4-1. *Let* p *be an odd prime. Then*

$$N_t \cong Im\ h_t', \ for\ t>0,$$

where h_t': $\pi_t(BP/S^0) \to H_t(BP/S^0; Z) = H_t(BP; Z)$ *is the Hurewicz homomorphism of* BP/S^0.

Consider the cofiber sequence; $S^0 \xrightarrow{i} BP \xrightarrow{\pi} BP/S^0$, where i is the canonical inclusion map and π is the canonical projection map. Associated with this there exists a short exact sequence for $*>1$;

$$0 \longrightarrow \pi_*(BP) \xrightarrow{\Phi} \pi_*(BP/S^0) \xrightarrow{\partial} \pi_{*-1}(S^0) \longrightarrow 0,$$

where $\Phi = \pi_*$. Let $\{v_i\}$ be a system of generators of $\pi_*(BP)$ with dim $v_i = 2(p^i - 1)$. Then we get the following;

Corollary 4-4. *Let* p *be an odd prime. Let* $v \in \pi_{nq}(BP)$, *where*

$n>0$ and $q=2(p-1)$. Then $\Phi(v)$ is divisible by p in $\pi_{nq}(BP/S^0)$ if and only if v belongs to the subgroup, $p\pi_{nq}(BP)+Z_{(p)}(v_1{}^n)$. Moreover $v_1{}^n$ is divisible by $p^{\nu_p(n)+1}$ and it is best possible, where $\nu_p(n)$ is the power of p in the expansion of n, i.e., $n=p^{\nu_p(n)}\ s$ and g.c.d. $(p,s)=1$.

The above result is originally obtained by L. Smith [14], in which he used K-theoretic characteristic numbers.

Remark. We can also calculate the spectral sequence of Theorem 2-7 up to dim 45 for $p=3$. Then the first obstruction for further calculations lies in dim 48, where I can not determine if $\alpha\varphi=0$ or not. For $p>3$, calculations could be done beyond this range by using informations on the behaviors of Massey product. These results will be published elsewhere.

This paper is organized as follows. In § 1 we list up the properties of the Brown-Peterson spectrum BP. In § 2 the spectral sequence which relates $Ext_{A^{BP}}^{*,*}(BP^*(X), BP^*(S^0))$ to the integral homology of X will be constructed. In § 3 we shall obtain some corollaries and some differential formulas of the spectral sequence. In § 4 the Hurewicz image of BP/S^0 will be determined. In § 5 we shall prove the multiplicativity of the spectral sequence.

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§ 1. Brown-Peterson spectrum.

Let p be a fixed prime and L the original Brown-Peterson spectrum at p, defined by Browm-Peterson [6]. Let $\mathfrak{R}$ be the set of sequences of integers $(e_1, e_2, \cdots)$ such that $e_i\geqq 0$ and $e_i=0$ for almost all i. If $E=(e_1, e_2, \cdots)$, let $|E|=2\sum e_i(p^i-1)$. If $F=(f_1, f_2, \cdots)$, let $E+F=(e_1+f_1, e_2+f_2, \cdots)$. Let $\Delta_i=(0, \cdots, 0, 1, 0, \cdots)$, where 1 takes i-th place. We denote the sum of n-copies of E by nE. Let BP be the Brown-Peterson spectrum localized at p of L. BP has many nice properties as follows.

Theorem 1. *(D. Quillen)* [13] [1]. *BP is the CW ring spectrum, which has the following properties;*

i) $H_*(BP; Z) = Z_{(p)}[m_1, m_2, \cdots]$ *degree* $m_i = 2(p^i - 1)$,

ii) $\pi_*(BP) = Z_{(p)}[v_1, v_2, \cdots]$ *degree* $v_i = 2(p^i - 1)$,

where $Z_{(p)}$ *denotes the localization at* p *of integers* Z.

iii) the Hurewicz homomorphism $h: \pi_*(BP) \to H_*(BP; Z)$ *is monomorphic and we can choose the generators* $\{v_i\}$ *so that they satisfy the following inductive formula* [4],

$$h(v_n) = pm_n - \sum_{1 \leq i \leq n-1} m_{n-i}(h(v_i))^{p^{n-i}},$$

iv) the Steenrod algebra $A^{BP} = BP^*(BP)$ *is a Hopf algebra over* $\pi_*(BP)$. A^{BP} *is isomorphic to* $\pi_*(BP) \widehat{\otimes}_{Z_{(p)}} R$, *where* R *is a free module over* $Z_{(p)}$ *with generators* $\{\gamma_E\}$, $E \in \mathfrak{R}$ *and degree* $\gamma_E = |E|$, *and* $\widehat{\otimes}_{Z_{(p)}}$ *means the completed tensor product over* $Z_{(p)}$. $\{\gamma_E\}$ *are characterized by the following properties:*

a) (Cartan formula) For $x, y \in H_*(BP; Z)$,

$$\gamma_E(xy) = \sum_{F+G=E} \gamma_F(x)\gamma_G(y)$$

where $F, G \in \mathfrak{R}$,

b) $$\gamma_E(m_n) = \begin{cases} m_{n-i} & \text{if } E = p^{n-i}\Delta_i, \\ 0 & \text{otherwise,} \end{cases}$$

c) (R. Zahler) [15] *If* $|E| = |F|$, $E, F \in \mathfrak{R}$, *then*

$$\gamma_E(m^F) = \begin{cases} 1 & (E = F), \\ 0 & (E \neq F), \end{cases}$$

where m^F *means* $m_1^{f_1} m_2^{f_2} \cdots$.

§ 2. The spectral sequence.

Let X be a CW spectrum. In this section we shall establish a spectral sequence relating the integral homology of X with $Ext_{A^{BP}}^{*,*}(BP^*(X), \Lambda)$ under certain conditions of X.

To get the spectral sequence, we need some fundamental facts.

Let $X^{(r)}$ be the r-skeleton of X and $X^{(\infty)}=X$. Consider a chain complex $\{C_n, d_n\}$, where $C_n=H_n(X^{(n)}/X^{(n-1)}; Z)$ and d_n is the boundary homomorphism for a triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$. As is well known, the homology group of $\{C_n, d_n\}$ is the ordinary homology group of $X, H_*(X; Z)$. The following Lemma is easily proved.

Lemma 2-1. *If X is a (-1)-connected CW-spectrum such that $H_*(X; Z)$ is locally finitely generated and free, then there exist a (-1)-connected CW-spectrum K and a map $g: K\to X$ which satisfy the following conditions;*

i) $g: K\to X$ *is homotopy equivalent,*

ii) $H_n(K^{(n)}/K^{(n-1)}; Z) \underset{\cong}{\xleftarrow{\pi_*}} H_n(K^{(n)}; Z) \underset{\cong}{\xrightarrow{i_*}} H_n(K; Z)$,

where $K^{(n)}$ is the n-skeleton of K and i and π are the canonical maps.

Lemma 2-2. *Let K be the spectrum in Lemma 2-1. Then for any integers l, m, n such that $0\leq l\leq m\leq n\leq\infty$, there exists a short exact sequence;*

$$0\to BP^*(K^{(n)}/K^{(m)})\to BP^*(K^{(n)}/K^{(l)})\to BP^*(K^{(m)}/K^{(l)})\to 0.$$

Proof. By Lemma 2-1 it is clear that $H_*(K^{(n)}/K^{(m)}; Z)$, $H_*(K^{(n)}/K^{(l)}; Z)$ and $H_*(K^{(m)}/K^{(l)}; Z)$ are free and locally finitely generated. So each Atiyah-Hirzebruch spectral sequence of $K^{(n)}/K^{(m)}$, $K^{(n)}/K^{(l)}$ and $K^{(m)}/K^{(l)}$ collaspes. Therefore it is enough to show the existence of a short exact sequence;

$$0\to H^*(K^{(n)}/K^{(m)}; Z)\to H^*(K^{(n)}/K^{(l)}; Z)\to H^*(K^{(m)}/K^{(l)}; Z)\to 0.$$

However this is clear from the universal coefficient formula and the freeness of the above spectra, and from Lemma 2-1. q.e.d.

Thus Lemma 2-1 allows us to identify X with K which has the nice skeletal filtration. From now on we always identify X with K, so $X^{(r)}$ means $K^{(r)}$ under this identification.

Theorem 2-3. *Let X be a (-1)-connected CW spectrum with* *basepoint, such that* $H_*(X; Z)$ *is free and locally finitely generated. Then there exists a tri-graded spectral sequence* $\{E_r(X), d_r(X)\}$ *converging to* $Ext_{ABP}^{*,*}(BP^*(X), \Lambda)$ *such that*

i) $E_1^{u,s,t} \cong H_u(X; Z) \otimes_Z Ext_{ABP}^{s,t}(\Lambda, \Lambda)$.

ii) $d_r\colon E_r^{u,s,t} \to E_r^{u-r,s+1,t+r}$,

iii) $E_\infty^{u,s,t} \cong \mathcal{G}(Ext_{ABP}^{s,t+u}(BP^*(X), \Lambda)) = D^{u,s,t}/D^{u-1,s,t+1}$, *where* $D^{u,s,t} = \mathrm{Im}(i^*)\#\colon Ext_{ABP}^{s,t+u}(BP^*(X^{(u)}), \Lambda) \to Ext_{ABP}^{s,t+u}(BP^*(X), \Lambda)$, *and* $i\colon X^{(u)} \to X$ *is the cannical inclusion,* $i^*\colon BP^*(X) \to BP^*(X^{(u)})$ *is the induced homomorphism of* i *and* $(i^*)\#\colon Ext_{ABP}^{s,t+u}(BP^*(X^{(u)}), \Lambda) \to Ext_{ABP}^{s,t+u}(BP^*(X), \Lambda)$ *is the induced homomorphism of* i^* *by the functor* $Ext_{ABP}^{*,*}(\ ,\Lambda)$,

iv) the above spectral sequence is natural with respect to maps of X, *i.e., for another spectrum* X' *satisfying the same conditions, and a map* $f\colon X \to X'$, *the induced homomorphism* $f_{r*}\colon E_r(X) \to E_r(X')$ *are compatible with* d_r, *moreover* f_{1*} *comes from the homolohy induced homomorphism of* f *as*

$$f_* \otimes 1\colon H_*(X; Z) \otimes_Z Ext_{ABP}^{*,*}(\Lambda, \Lambda) \to H_*(X'; Z) \otimes_Z Ext_{ABP}^{*,*}(\Lambda, \Lambda).$$

Remark 1. From the (-1)-connectedness of X it is clear that $E_r^{u,s,t}(X) = 0$ if $u < 0$.

Remark 2. The theorem in the above cannot be applied for $X = BP$, since BP is a spectrum localized at p. However, for X_p which is the spectrum localized at p of X, defining the filtration $\{X_p^u\}$ so that $X_p^u = (X^{(u)})_p$, we obtain the same spectral sequence $\{E_r(X_p), d_r(X_p)\}$.

Theorem 2-4. *Under the same conditions of* X, *let* X_p *be the spectrum localized at* p *of* X, *then there exists a spectral sequence* $\{E_r(X_p), d_r(X_p)\}$ *converging to* $Ext_{ABP}^{*,*}(BP^*(X_p), \Lambda)$ *such that*

i) $E_1^{u,s,t}(X_p) \cong H_u(X_p; Z) \otimes_{Z(p)} Ext_{ABP}^{s,t}(\Lambda, \Lambda)$,

ii) $d_r(X_p)\colon E_r^{u,s,t}(X_p) \to E_r^{u-r,s+1,t+r}(X_p)$,

iii) $E_\infty^{*,*,*}$ *gives the quotient in the filtration of* $Ext_{ABP}^{*,*}(BP^*(X_p), \Lambda)$, *iv) the spectral sequence* $\{E_r(X_p), d_r(X_p)\}$ *is natural with respect to maps of* X_p.

Remark 3. Let $f: X \to Y_p$ be a map, where X and Y satisfy the assumption of Theorem 2-3. Then even in this case the naturality holds, i.e., there are homomorphisms f_r: $E_r(X) \to E_r(Y_p)$, which are compatible with d_r.

Proof of Theorems 2-3 and 2-4. We use the method of [10]. Recall that the functor $Ext_{A^{BP}}^{*,*}(\ ,\Lambda)$ is a half exact functor on the category of A^{BP}-modules and A^{BP}-homomorphisms. So if there is a short exact sequence of A^{BP}-modules; $0 \to M \to N \to L \to 0$, then there is a long exact sequence;

$$\cdots \to Ext_{A^{BP}}^{s-1,t}(M, \Lambda) \xrightarrow{\Delta} Ext_{A^{BP}}^{s,t}(L, \Lambda) \to Ext_{A^{BP}}^{s,t}(N, \Lambda) \to Ext_{A^{BP}}^{s,t}(M, \Lambda) \to \cdots,$$

where Δ is the connecting homomorphism induced by the above short exact sequence.

Therefore we can define $\{E_r^{u,s,t} = Z_r^{u,s,t}/B_r^{u,s,t}, d_r\}$ by virtue of Lemma 2-2 as follows;

$$Z_r^{u,s,t} = \mathrm{Im}\{Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-r)}), \Lambda) \xrightarrow{j} Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-1)}), \Lambda)\},$$

$$B_r^{u,s,t} = \mathrm{Im}\{Ext_{A^{BP}}^{s-1,t+u}(BP^*(X^{(u+r-1)}/X^{(u)}), \Lambda) \xrightarrow{\Delta} Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-1)}), \Lambda)\}.$$

We define $E_r^{u,s,t} = Z_r^{u,s,t}/B_r^{u,s,t}$. Then the differential d_r: $E_r^{u,s,t} \to E_r^{u-r,s+1,t+r}$ is induced by the composition $\Delta' \circ j^{-1}$, where

$$\Delta': Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-r)}), \Lambda) \to Ext_{A^{BP}}^{s+1,t+u}(BP^*(X^{(u-r)}/X^{(u-r-1)}), \Lambda)$$

is the connecting homomorphism induced by the short exact sequence;

$$0 \to BP^*(X^{(u)}/X^{(u-r)}) \to BP^*(X^{(u)}/X^{(u-r-1)}) \to BP^*(X^{(u-r)}/X^{(u-r-1)}) \to 0.$$

Also we define $E_\infty^{u,s,t} = Z_\infty^{u,s,t}/B_\infty^{u,s,t}$, where

$$Z_\infty^{u,s,t} = \mathrm{Im}\{Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}), \Lambda) \longrightarrow Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-1)}), \Lambda)\},$$

$$B^{u,s,t} = \mathrm{Im}\{Ext_{A^{BP}}^{s-1,t+u}(BP^*(X/X^{(u)}), \Lambda) \xrightarrow{\Delta} Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-1)}), \Lambda)\}.$$

Then we can easily prove that d_r is well-defined, $H(E_r^{u,s,t})=E_{r+1}^{u,s,t}$ and $E_\infty^{u,s,t}\cong D^{u,s,t}/D^{u-1,s,t+1}$.

In order to show the convergence, we need

Lemma 2-5 (*Novikov Th. 3-1* [*11*]). *Let Y be a* $(k-1)$-*connected spectrum and* $H_*(Y;Z)$ *be free and locally finitely generated over* Z *or* $Z_{(p)}$. *Then*

$$Ext_{A^{BP}}^{s,t}(BP^*(Y),\Lambda)=0 \text{ for } t-2(p-1)s<k.$$

Using this Lemma we can easily prove that $E_r^{u,s,t}\cong E_\infty^{u,s,t}$ for r sufficiently large. So the spectral sequence $\{E_r(X),d_r(X)\}$ converges to $Ext_{A^{BP}}^{*,*}(BP^*(X),\Lambda)$.

Now we prove i). By definition $E_1^{u,s,t}=Ext_{A^{BP}}^{s,t+u}(BP^*(X^{(u)}/X^{(u-1)}),\Lambda)$. From the construction of the Atiyah-Hirzebruch spectral sequence [5], $BP^*(X^{(u)}/X^{(u-1)})$ is isomorphic to the cochain complex $C^u(X;BP^{*-u}(S^0))$. From Lemma 2-1, $C^u(X;BP^{*-u}(S^0))$ is isomorphic to $H^u(X;BP^{*-u}(S^0))$ which is the E_2-terms of the Atiyah-Hirzebruch spectral sequence. By the universal coefficient theorem and by the assumption that $H_*(X;Z)$ is free and locally finitely generated, $H^u(X;BP^{*-u}(S^0))$ is isomorphic to $Hom_Z(H_u(X;Z),BP^{*-u}(S^0))$ as A^{BP}-module.

Notice that the all above isomorphisms are functorial.

Lemma 2-6. *If M is Z-free and of finite type, then*

$$Ext_R^{s,t}(Hom_Z(M,\Lambda),\Lambda)\cong M\otimes_Z Ext_R^{s,t}(\Lambda,\Lambda),$$

where Λ *is an R-module and R is a graded commutative ring. Further this isomorphism is functorial, that is, for a morphism* $f\colon M\to N$, *the next diagram commutes*;

$$\begin{array}{ccc} Ext_R^{s,t}(Hom_Z(M,\Lambda),\Lambda) & \cong & M\otimes_Z Ext_R^{s,t}(\Lambda,\Lambda) \\ \downarrow (f^*)^\# & & \downarrow f\otimes 1 \\ Ext_R^{s,t}(Hom_Z(N,\Lambda),\Lambda) & \cong & N\otimes_Z Ext_R(\Lambda,\Lambda), \end{array}$$

where N is Z-free and of finite type.

Proof. Let $\{C_s,d_s\}$ be an R-free resolution of Λ, then $\{Hom_Z(M,$

$C_s), d_{s*}\}$ is an R-free resolution of $Hom_Z(M, \Lambda)$, because M is Z-free and of finite type. We define homomorphisms φ_s: $M\otimes_Z Hom_R(C_s, \Lambda) \to Hom_R(Hom_Z(M, C_s), \Lambda)$ by $\varphi_s(a\otimes f)(g)=f(g(a))$, where $a\in M$, $f\in Hom_R(C_s, \Lambda)$ and $g\in Hom_Z(M, C_s)$. Then it is obvious that φ_s is an R-isomorphism, and that the following diagram commutes;

$$\begin{array}{ccc} Hom_R(Hom_Z(M, C_s), \Lambda) & \xleftarrow{\varphi_s} & M\otimes_Z Hom_R(C_s, \Lambda) \\ \downarrow (d_{s*})^{\#} & & \downarrow 1\otimes d_s^{\#} \\ Hom_R(Hom_Z(M, C_{s+1}), \Lambda) & \xleftarrow{\varphi_{s+1}} & M\otimes_Z Hom_R(C_{s+1}, \Lambda). \end{array}$$

So taking the homology, we obtain Lemma 2-5. The functoriality is clear from the construction of $\{\varphi_s\}$. q.e.d.

Using Lemma 2-6, we get a functorial isomorphism;

$$Ext_{ABP}^{s,t+u}(BP^*(X^{(u)}/X^{(u-1)}), \Lambda)\cong Ext_{ABP}^{s,t+u}(Hom_Z(H_u(X; Z), BP^{*-u}(S^0)), \Lambda)$$

$$\cong H_u(X; Z)\otimes_Z Ext_{ABP}^{s,t+u}(BP^{*-u}(S^0), \Lambda)\cong H_u(X; Z)\otimes_Z Ext_{ABP}^{s,t}(\Lambda, \Lambda).$$

So i) was proved. iv) follows from the facts that the skeleton filtration $\{X^{(r)}\}$ has the functorial properties and that the above construction of the spectral sequence is functorial. Thus we complete a proof of Theorem 2-3.

Theorem 2-4 immediately follows by the following facts;

1) The chain complex $\{H_u(X_p{}^u/X_p{}^{u-1}; Z), \partial_u\}$ gives the homology group $H_*(X_p; Z)$ of X_p.

2) The filtration $\{X_p{}^r\}$ satisfies Lemma 2-2, i.e., there is a short exact sequence; $0\to BP^*(X_p{}^n/X_p{}^m)\to BP^*(X_p{}^n/X_p{}^l)\to BP^*(X_p{}^m/X_p{}^l)\to 0$.

3) Let f: $X_p\to X_p'$, then there exists a map g: $X_p\to X_p'$ such that g is homotopic to f and $g(X_p{}^r)\subset X_p'^r$. q.e.d.

Applying Theorem 2-4 for $X=BP$, we obtain the following;

Theorem 2-7. *There exists a tri-graded spectral sequence* $\{E_r{}^{*,*,*}, d_r\}$ *such that*

i) $E_1^{u,s,t}=H_u(BP; Z)\otimes_{Z_{(p)}} Ext_{ABP}^{s,t}(\Lambda, \Lambda)$,

ii) d_r: $E_r^{u,s,t}\to E_r^{u-r,s+1,t+r}$, *iii)* $E_\infty^{u,s,t}=\begin{cases}\pi_u(BP) & \text{if } s=t=0,\\ 0 & \text{otherwise,}\end{cases}$

iv) the edge homomorphism $E^{u,0,0}=\pi_u(BP)\to E_1^{u,0,0}=H_u(BP;Z)$ *coincides with the Hurewicz homomorphism* $h\colon \pi_u(BP)\to H_u(BP;Z)$,

v) there exist pairings $\prod_r\colon E_r^{u,s,t}\otimes E_r^{u',s',t'}\to E_r^{u+u',s+s',t+t'}$ *such that* $\prod_{r+1}$ *is induced by* $\prod_r$, *moreover* $\prod_1$ *and* $\prod_\infty$ *are the standard product induced from the ring spectrum structure of BP.* d_r *is an anti-derivation with respect to this product* $\prod_r$, *that is, for* $a\in E_r^{u,s,t}$, $b\in E_r^{u',s',t'}$,

$$d_r(ab)=d_r(a)b+(-1)^{u+t-s}a\cdot d_r(b),$$

vi) A^{BP} *acts naturally on this spectral sequence, i.e., for any* $\gamma\in A^{BP}$, *there are homomorphisms* $\gamma_{r*}\colon E_r^{u,s,t}\to E_r^{u-\dim\gamma,s,t}$, *which are compatible with* d_r. *Moreover* γ_{1*} *is derived from the homology induced homomorphism* $\gamma_*\otimes 1\colon H_*(BP;Z)\otimes_{Z_{(p)}}Ext_{A^{BP}}^{*,*}(\Lambda,\Lambda)\to H_*(BP;Z)\otimes_{Z_{(p)}}Ext_{A^{BP}}^{*,*}(\Lambda,\Lambda)$.

Proof. i), ii) and vi) are already proved in Theorem 2-4. The proof of v) is very long, so we postpone it in the section 5. Proof of iii). By Theorem 2-4, $E_\infty^{u,s,t}$ is a quotient in the filtration of $Ext_{A^{BP}}^{s,t+u}(BP^*(BP),\Lambda)$. Since $BP^*(BP)$ is exactly A^{BP}, it is clear that $E_\infty^{u,s,t}=0$ unless $s=0$. Meanwhile, by definition $E_\infty^{u,0,t}$ is a quotient of a certain subgroup of $E_1^{u,0,t}=H_u(BP;Z)\otimes_{Z_{(p)}}Ext_{A^{BP}}^{0,t}(\Lambda,\Lambda)$. It is well known that $Ext_{A^{BP}}^{0,t}(\Lambda,\Lambda)=0$ unless $t=0$. So $E_\infty^{u,s,t}=0$ unless $s=t=0$. In the case $s=t=0$, we assert that $E_\infty^{u,0,0}=D^{u,0,0}=\pi_u(BP)$. The first equality follows from the facts that $E_\infty^{u,0,0}=D^{u,0,0}/D^{u-1,0,1}$, $D^{-1,0,u+1}=0$ and $E_\infty^{r,0,u-r}=0$ $(u\neq r)$. From the properties of the filtration of BP and from Lemma 2-5 we see that the homomorphism: $Ext_{A^{BP}}^{0,u}(BP^*(BP^u),\Lambda)\to Ext_{A^{BP}}^{0,u}(BP^*(BP),\Lambda)$ is an isomorphism, where BP^u is the spectrum localized at p of $L^{(u)}$, the u-skeleton of the original Brown-Peterson spectrum L. Then clearly the second equality holds. q.e.d. Proof of iv). Let $\alpha\in\pi_u(BP)$ and $f\colon S^u\to BP$ be a representative of α. Then by naturality of the spectral sequence there exists a commutative diagram;

$$\begin{array}{ccccccc} D^{u,0,0}(S^u) & \longrightarrow & E_\infty{}^{u,0,0}(S^u) & \longrightarrow & E_1{}^{u,0,0}(S^u) & = & H_u(S^u;Z) \\ \downarrow f_* & & \downarrow f_{\infty *} & & \downarrow f_{1*} & & \downarrow f_* \\ D^{u,0,0}(BP) & \longrightarrow & E_\infty{}^{u,0,0}(BP) & \longrightarrow & E_1{}^{u,0,0}(BP) & = & H_u(BP;Z). \end{array}$$

It is clear that $D^{u,0,0}(S^u)=Ext_{A^{BP}}^{0,u}(BP^*(S^u),A)=\pi_u(s^u)$ and $D^{u,0,0}(S^u)\cong E_\infty{}^{u,0,0}(S^u)\cong E_1{}^{u,0,0}(S^u)$. Then the commutativity of the diagram implies iv). q.e.d.

§ 3. Corollaries and some differential formulas.

In this section we shall obtain some results and some differential formulas of the spectral sequence in Theorem 2-7.

Lemma 3-1. *In the spectral sequence* $\{E_r{}^{*,*,*}(BP),\ d_r{}^{*,*,*}\}$, *for* $1\leq r\leq t$ *there exists a canonical monomorphism*: $E_r{}^{u,1,t}\subsetneq E_1{}^{u,1,t}$. *Especially* $E_r{}^{0,1,t}=E_1{}^{0,1,t}$ *if* $1\leq r\leq t$.

Proof. By definition, $E_r{}^{u,1,t}=\ker d_{r-1}^{u,1,t}/\mathrm{Im}\ d_{r-1}^{u+r-1,0,t-r+1}$, where $d_{r-1}^{u,1,t}$: $E_{r-1}^{u,1,t}\rightarrow E_{r-1}^{u-r+1,2,t+r-1}$ and $d_{r-1}^{u+r-1,0,t-r+1}$: $E_{r-1}^{u+r-1,0,t-r+1}\rightarrow E_{r-1}^{u,1,t}$ are differentials. Recall that $E_{r-1}^{u+r-1,0,t-r+1}$ is a quotient in the filtration of $E_1{}^{u+r-1,0,t-r+1}=H_{u+r-1}(BP;Z)\otimes_{Z_{(p)}}Ext_{A^{BP}}^{0,t-r+1}(A,A)$. Since $Ext_{A^{BP}}^{0,*}(A,A)=0$ unless $*=0$, for $1\leq r\leq t$, $E_r{}^{u,1,t}=\ker d_{r-1}^{u,1,t}\subset E_1{}^{u,1,t}$. By induction we easily obtain that $E_r{}^{u,1,t}\subset E_1{}^{u,1,t}$. Especially, if $u=0$ then from Remark 1 in § 2 we see that $\ker d_{r-1}^{0,1,t}=E_{r-1}^{0,1,t}$. Therefore we obtain that $E_r{}^{0,1,t}=E_1{}^{0,1,t}$. q.e.d.

Lemma 3-2. *Let* $z\in E_r{}^{n,0,0}$. *Then* $d_r(z)=0$ *in* $E_r{}^{n-r,1,r}$ *if and only if* $d_r(\gamma_E(z))=0$ *in* $E_r{}^{0,1,r}$ *for any* $E\in\mathfrak{R}$ *with* $|E|=n-r$.

Proof. If $d_r(z)=0$ in $E_r{}^{n-r,1,r}$, then by naturality of the spectral sequence we get that $d_r(\gamma_E(z))=\gamma_E d_r(z)=0$ in $E_r{}^{0,1,r}$ for any $E\in\mathfrak{R}$. Conversely if $d_r(\gamma_E(z))=0$ in $E_r{}^{0,1,r}$ for any $E\in\mathfrak{R}$ with $|E|=n-r$, by Lemma 3-1 we can set

$$d_r(z)=\sum_{|E|=n-r}m^E\otimes\lambda_E,$$

where $m^E\otimes\lambda_E\in E_1{}^{n-r,1,r}=H_{n-r}(BP;Z)\otimes_{Z_{(p)}}Ext_{A^{BP}}^{1,r}(A,A)$. Applying $\gamma_F\in A^{BP}$ with $|F|=n-r$, and using Theorem 1-iv)-c), we obtain

$$\lambda_F=\gamma_F(\sum_{|E|=n-r} m^E\otimes\lambda_E)=\gamma_F d_r(z)=d_r(\gamma_F(z))=0.$$

Therefore $d_r(z)=0$. q.e.d.

Let $N_k=\{x\in H_k(BP;Z)\,|\,\gamma_E(x)\in \mathrm{Im}\,h$ for any $E\in\mathfrak{R}$ such that $|E|\neq 0\}$, where $h\colon \pi_*(BP)\to H_*(BP;Z)$ is the Hurewicz homomorphism of BP. Then, we obtain the following result which is a BP-analogy of Buȟstaber's result [7].

Corollary 3-3. $Ext^{1,t}_{ABP}(A,A)\cong N_t/\mathrm{Im}\,h_t$.

Proof. By Lemma 3-1 we know that $Ext^{1,t}_{ABP}(A,A)=E_1^{0,1,t}=E_t^{0,1,t}$. Consider the sequence: $E_t^{t,0,0}\to E_t^{0,1,t}\to E_t^{-t,2,2t}$, where $H(E_t^{0,1,t})=E^{0,1,t}_{t+1}$. Since $E^{0,1,t}_{t+1}=E_\infty^{0,1,t}=0$, and since $E_t^{-t,2,2t}=0$, we obtain $E_t^{0,1,t}=E_t^{t,0,0}/\ker d_t^{t,0,0}$. But $\ker d_t^{t,0,0}$ is clearly $E_\infty^{t,0,0}=\mathrm{Im}\,h_t$. On the other hand, by Lemma 3-2 and by induction, it is easily proved that $x\in H_t(BP;Z)$ belongs to $E_t^{t,0,0}$ if and only if $x\in N_t$. Therefore we obtain

$$Ext^{1,t}_{ABP}(A,A)\cong E_t^{0,1,t}\cong E_t^{t,0,0}/\ker d_t^{t,0,0}\cong N_t/\mathrm{Im}\,h_t. \qquad \text{q.e.d.}$$

Proposition 3-4. *Let $x\in E_r^{u,0,0}$. Then,*

$$d_r(x)=\sum_{\substack{|E|=u-r\\ E\in\mathfrak{R}}} m^E\otimes d_r(\gamma_E(x)),$$

where $d_r(\gamma_E(x))\in E_r^{0,1,r}=Ext^{1,r}_{ABP}(A,A)$.

Proof. This is trivial from Lemma 3-2. q.e.d.

Theorem 3-5. *Let p be an odd prime. Then,*

$$N_k/\mathrm{Im}\,h_k\cong\begin{cases} Z_{p^{\nu_p(t)+1}} \text{ with a generator } p^{t-\nu_p(t)-1}m_1^{\,t}, & \text{if } k=tq,\\ 0 \text{ otherwise,} & \end{cases}$$

where $q=2(p-1)$ and $\nu_p(t)$ is an integer defined by the requirement that $t=p^{\nu_p(t)}s$ and g.c.d. $(p,s)=1$.

Theorem 3-5 is a BP-analogy of Panov's result [12]. (See also [9]). In order to show Theorem 3-5, first we define an order on exponent sequences $\mathfrak{R}$ as follows. For $E=(e_1,e_2,\cdots)$ and $F=(f_1,f_2,\cdots)$, define $E<F$ if $|E|<|F|$ or if $|E|=|F|$ and there is an i such that $e_i<f_i$ and $e_j=f_j$ for any $j>i$. It is clear that $<$ is a linear ordering. Let $x\in\pi_*(BP)$. We define that type$(x)=E$, if $\gamma_E(x)\not\equiv 0$ (mod $p\pi_*(BP)$) and $\gamma_E(x)\equiv 0$ (mod $p\pi_*(BP)$) for any $F>E$.

Lemma 3-6. *Let* $x, y\in\pi_*(BP)$. *If type*$(x)=E$ *and type*$(y)=F$, *then, type*$(xy)=E+F$.

Proof. It is clear that if $E_1<E$ and $E_1<F$, then $E_1+F_1<E+F$. So by the Cartan formula we have

$$\gamma_G(xy)=\sum_{E_1+F_1=G}\gamma_{E_1}(x)\gamma_{E_1}(y)\equiv\sum_{\substack{F_1\leqq F\\ E_1\leqq E\\ E_1+F_1=G}}\gamma_{E_1}(x)\gamma_{E_1}(y)\quad(\mathrm{mod}\ p\pi_*(BP)).$$

It is clear that if $G>E+F$, then $\gamma_G(xy)\equiv 0$ (mod $p\pi_*(BP)$) and that $\gamma_{E+F}(xy)\equiv\gamma_E(x)\gamma_F(y)\not\equiv 0$ (mod $p\pi_*(BP)$). So we obtain that type$(xy)=E+F$. q.e.d.

Lemma 3-7. *Let* $\{v_i\}$ *be the ring generators of* $\pi_*(BP)$. *Then,*

$$type(v_n)=\begin{cases}\Delta_0 & if\ n=1,\\ p\Delta_{n-1} & if\ n\geqq 2,\end{cases}$$

where Δ_0 *is the zero sequence* $(0, 0, 0, \cdots)$.

Proof. Recall the formula (Theorem 1);

$$h(v_n)=pm_n-\sum_{i=1}^{n-1}m_{n-i}(h(v_i))^{p^{n-i}}.$$

It is clear that type$(v_1)=\Delta_0$. From Theorem 1 we have that $\gamma_{p\Delta_{n-1}}(v_n)=pm_1=v_1\not\equiv 0$ (mod $p\pi_*(BP)$). On the other hand, $p\Delta_{n-1}$ is the largest sequence in dimension $2(p^n-p)$, so, if $E>p\Delta_{n-1}$ then $|E|\geqq 2(p^n-p)+2(p-1)=\dim v_n$. Therefore by iv)-c) in Theorem 1, it is easily proved that if $E>p\Delta_{n-1}$, then $\gamma_E(v_n)\equiv 0$ (mod $p\pi_*(BP)$). q.e.d.

Proposition 3-8. *If type*$(x)=\Delta_0$, *then* $x\equiv\lambda v_1^t$ (mod $p\pi_*(BP)$), *where* $x\in\pi_{tq}(BP)$ *and* $\lambda\in Z_{(p)}$.

Proof. Let $x=\sum_{i=1}^k\lambda_i w_i$, where $\lambda_i\in Z_{(p)}$ and w_i are the monomials of v_i which form an additive basis of $\pi_{tq}(BP)$ over $Z_{(p)}$. Then by Lemma 3-6 and 3-7, we can order $\{w_i\}$ by its type. Assume that type $(w_i)<$ type (w_{i+1}). Then, clearly $w_1=v_1^t$. Let type $(w_i)=E_i$, then $\lambda_j\gamma_{E_k}(w_k)\equiv\gamma_{E_k}(x)\equiv 0 \pmod{p\pi_*(BP)}$. Since $\gamma_{E_k}(w_k)\not\equiv 0 \pmod{p\pi_*(BP)}$, we get that $\lambda_k\equiv 0 \pmod{pZ_{(p)}}$. By induction we see that $\lambda_i \equiv 0 \pmod{pZ_{(p)}}$ if $i\geqq 2$. Therefore $x\equiv\lambda v_1^t \pmod{p\pi_*(BP)}$. q.e.d.

Proof of Theorem 3-5. It is clear that the group $N_{tq}/\mathrm{Im}\, h_{tq}$ is a finite abelian group. Let $z\in N_{tq}$ such that $pz\in\mathrm{Im}\, h_{tq}$, then Proposition 3-8 implies $pz\equiv\lambda v_1^t$ (mod. $p\pi_*(BP)$), which implies that the group $N_{tq}/\mathrm{Im}\, h_{tq}$ consists of only one generator. So, in order to show Theorem 3-5 it is sufficient to show that $p^{t-\nu_p(t)-1}m_1^t\in N_{tq}$ and $P^{t-\nu_p(t)-2}m_1^t\notin N_{tq}$. By iv)-a) and b) in Theorem 1, if $E\neq i\Delta_1$ then $\gamma_E(m_1^t)=0$ and $\gamma_i\Delta_1(m_1^t)=\binom{t}{i}m_1^{t-i}$ so $\gamma_i\Delta_1(p^k m_1^t)=\binom{t}{i}p^k m_1^{t-i}$. But $\max_{1\leq i<t}\left\{t-i-\nu_p\left(\binom{t}{i}\right)\right\}=t-\nu_p(t)-1$, and hence $p^k m_1^t\in N_{tq}$ if and only if $k\geqq t-\nu_p(t)-1$. This completes the proof of Theorem 3-5.

q.e.d.

From Corollary 3-3 and Theorem 3-5 we see that $d_{tq}(p^{t-\nu_p(t)-1}m_1^t)$ is well defined and it is a generator of $Ext_{ABP}^{1,tq}(A, A)$. Let $\alpha_t^{(\nu_p(t))}=d_{tq}(p^{t-\nu_p(t)-1}m_1^t)$. Then we obtain an explicit differential formula on $E_r^{u,0,0}$. In order to describe this formula we define a ring homomorphism ρ: $H_*(BP;Z)\rightarrow Z_{(p)}$ by

$$\rho(m_n)=p^{(p^n-1/p-1)-n}.$$

Lemma 3-9. *If* $x\in N_{tq}$, *then*

$$x\equiv\rho(x)m_1^t \pmod{\mathrm{Im}\, h_{tq}}.$$

Proof. By Theorem 3-5, $x=\lambda p^{t-\nu_p(t)-1}m_1^t+v$, where $\lambda\in Z_{(p)}$ and $v\in\mathrm{Im}\, h_{tq}$. Recall the formula iii) of Theorem 1. We assert that $\rho(v_1)=p$, and $\rho(v_i)=0$ if $i\geqq 2$. It is clear that $\rho(v_1)=\rho(pm_1)=p$. By induction, using the formula iii) in Theorem 1, we get

$$\rho(v_n)=p\rho(m_n)-\sum_{i=1}^{n-1}\rho(m_{n-i})(\rho(v_i))^{p^{n-i}}=p\rho(m_n)-\rho(m_{n-1})\rho(v_1)^{p^{n-1}}$$

$=P^{(p^n-1/p-1)-n+1}-p^{p^{n-1}}p^{(p^{n-1}-1/p-1)-n+1}=0$

So $\rho(x)=\lambda p^{t-\nu_p(t)-1}+\rho(v)$. Therefore $\rho(x)\equiv\lambda p^{t-\nu_p(t)-1}\pmod{p^tZ_{(p)}}$. This implies that $x\equiv\rho(x)m_1{}^t\pmod{\mathrm{Im}\,h_{tq}}$. q.e.d.

As an immediate corollary we obtain the explicit differential formula on $E_r^{u,0,0}$.

Proposition 3-10. *If* $x\in E_r^{u,0,0}$, *then*

$$d_r(x)=\sum_{\substack{|E|=u-r\\ E\in\mathfrak{R}}}\frac{\rho(\gamma_E(x))}{p^{r-\nu_p(r)-1}}\,m^E\otimes\alpha_r{}^{(\nu_p(r))}.$$

Using the above Proposition, we obtain

Proposition 3-11.

$E_r^{u,0,0}=\{x\in H_u(BP;\ Z)\mid$ *for any* $E\in\mathfrak{R}$ *such that* $|E|>u-r$, $\gamma_E(x)\in\mathrm{Im}\ h\}$.

§ 4. Hurewicz homomorphism of *BP*/S^0

In this section we shall determine the Hurewicz image of BP/S^0. Let h': $\pi_*(BP/S^0)\to H_*(BP/S^0;Z)$ be the Hurewicz homomorphism of BP/S^0. It is clear that if $n>0$, $H_n(BP/S^0;Z)\cong H_n(BP;Z)$. Therefore we identify $H_n(BP/S^0;Z)$ with $H_n(BP;Z)$ if $n>0$. Then,

Theorem 4-1. *If p is an odd prime, and $n>0$,*

$$\mathrm{Im}\ h_n'\cong N_n\subset H_n(BP;Z).$$

Proof. By the cellular approximation theorem and the choice of filtration $\{BP^r\}$, we see that there is a commutative diagram;

$$\begin{array}{ccc}
\pi_n(BP/S^0) & \xrightarrow{h'} & H_n(BP/S^0;Z)\\
\uparrow\cong & & \uparrow\cong\\
\pi_n(BP^n/S^0) & \longrightarrow & H_n(BP^n/S^0;Z)\\
\downarrow & & \downarrow\cong\\
\pi_n(BP^n/BP^0) & \longrightarrow & H_n(BP^n/BP^0;Z)\\
\downarrow j_* & & \downarrow\cong\\
\pi_n(BP^n/BP^{n-1}) & \xrightarrow{\cong} & H_n(BP^n/BP^{n-1};Z)\cong H_n(BP;Z),
\end{array}$$

where j: $BP^n/BP^0 \to BP^n/BP^{n-1}$ is the canonical map. So, in order to determine the image of h', it is sufficient to determine the image of $j_*: \pi_n(BP^n/BP^0) \to \pi_n(BP^n/BP^{n-1})$. There are two Adams-Novikov spectral sequences, $\{E_r^{*,*}(BP^n/BP^0)\}$ and $\{E_r^{*,*}(BP^n/BP^{n-1})\}$, which converge to $\pi_*(BP^n/BP^0)$ and $\pi_*(BP^n/BP^{n-1})$, respectively. By naturality of the Adams-Novikov spectral sequence, there exist homomorphisms j_*^r: $E_r^{*,*}(BP^n/BP^0) \to E_r^{*,*}(BP^n/BP^{n-1})$ for $2 \leq r \leq \infty$ and $j_*(F_t^i(BP^n/BP^0)) \subset F_t^i(BP^n/BP^{n-1})$, where $F_t^i(X)$ means the i-th filtration of $\pi_t(X)$. It is obvious that $F_n^i(BP^n/BP^{n-1}) = 0$ for $i \neq 0$ and $E_2^{0,n}(BP^n/BP^{n-1}) = E_\infty^{0,n}(BP^{n-1}) = F_n^0(BP^n/BP^{n-1}) = \pi_n(BP^n/BP^{n-1})$. So we obtain that $\mathrm{Im}\, j_* = \mathrm{Im}\, j_*^\infty$. We assert that $\mathrm{Im}\, j_*^\infty = \mathrm{Im}\, j_*^2$. In order to prove this assertion, we need the following lemma.

Lemma 4-2. *The following diagram commutes;*

$$\begin{array}{ccccc} 0 \longrightarrow & \pi_n(BP^n) & \longrightarrow & \pi_n(BP^n/BP^0) & \\ & \downarrow d_{BP} & \mathrm{I} & \downarrow d_{BP} & \\ 0 \to & Hom^n_{A^{BP}}(BP^*(BP^n), A) & \to & Hom^n_{A^{BP}}(BP^*(BP^n/BP^0), A) & \end{array}$$

$$\begin{array}{ccc} \xrightarrow{\partial} & \pi_{n-1}(BP^0) & \longrightarrow 0 \\ \mathrm{II} & \downarrow e_{BP} & \\ \xrightarrow{\Delta} & Ext^{1,n}_{A^{BP}}(A, A) & \longrightarrow 0, \end{array}$$

where d_{BP} is the Adams d-invariant, e_{BP} is the Adams e-invariant in BP-theory [3], *and the lower sequence is the short exact sequence induced by the short exact sequence;*

$$0 \to BP^*(BP^n/BP^0) \to BP^*(BP^n) \to BP^*(BP^0) = A \to 0.$$

Proof. The diagram (I) is clearly commutative. Commutativity of the diagram (II) is shown as follows. Let $\alpha \in \pi_n(BP^n/BP^0)$, f: $S^n \to BP^n/BP^0$ be a representative of α, g: $S^{n-1} \to BP^0$ be a representative of $\partial\alpha$, and C_f be the mapping cone of f, then it is easily proved that there exists a map h: $C_g \to BP^n$ such that the following diagram commutes;

$$\begin{array}{ccccccccc} S^{n-1} & \xrightarrow{g} & BP^0 & \longrightarrow & C_g & \longrightarrow & S^n & \longrightarrow & \Sigma BP^0 \\ & & \| & & \downarrow \exists h & & \downarrow f & & \| \\ & & BP^0 & \longrightarrow & BP^n & \longrightarrow & BP^n/BP^0 & \longrightarrow & \Sigma BP^0. \end{array}$$

Applying the functor $BP^*(\ \ )$ to the above diagram, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & BP^*(S^n) & \longrightarrow & BP^*(C_g) & \longrightarrow & BP^*(BP^0) & \longrightarrow & 0 \\ & & \uparrow f^* & & \uparrow h^* & & \| & & \\ 0 & \longrightarrow & BP^*(BP^n/BP^0) & \longrightarrow & BP^*(BP^n) & \longrightarrow & BP^*(BP^0) & \longrightarrow & 0. \end{array}$$

Applying the functor $Ext^{*,n}_{ABP}(\ \ ,A)$, we obtain the following commutative diagram;

$$\begin{array}{ccc} \cdots\to Hom^n_{ABP}(BP^*(S^n),A) & \xrightarrow{\Delta'} & Ext^{1,n}_{ABP}(BP^*(BP^0),A)\to\cdots \\ \downarrow (f^*)\# & & \| \\ \cdots\to Hom^n_{ABP}(BP^*(BP^n/BP^0),A) & \xrightarrow{\Delta} & Ext^{1,n}_{ABP}(BP^*(BP^0),A)\to\cdots. \end{array}$$

From the definition of d_{BP} and e_{BP} it is clear that $(f^*)\#(1)=d_{BP}(\alpha)$, and $\Delta'(1)=e_{BP}(g)=e_{BP}(\partial\alpha)$, where $1\in Hom^n_{ABP}(BP^*(S^n),A)=Hom^0_{ABP}(A,A)$. Therefore we obtain that $\Delta d_{BP}(\alpha)=e_{BP}(\partial\alpha)$. q.e.d.

Lemma 4-3. *Let p be an odd prime. Then, $d_{BP};\pi_n(BP^n/BP^0)\to Hom^n_{ABP}(BP^*(BP^n/BP^0),A)$ is an epimorphism.*

Proof. It is a famous theorem of Novikov [11] that $e_{BP}:\pi_{n-1}(BP^0)={}_p\pi_{n-1}(S^0)\to Ext^{1,n}_{ABP}(BP^*(BP^0),A)=Ext^{1,n}_{ABP}(A,A)$ is an epimorphism if p is odd. So, in order to prove Lemma 4-3 it is sufficient to show that $d_{BP}:\pi_n(BP^n)\to Hom^n_{ABP}(BP^*(BP^n),A)$ is epic. But this is clear from the next commutative diagram:

$$\begin{array}{ccc} \pi_n(BP^n) & \longrightarrow & Hom^n_{ABP}(BP^*(BP^n),A) \\ i_*\downarrow\cong & & \cong\downarrow(i^*)\# \\ \pi_n(BP) & \xrightarrow{\cong} & Hom^n_{ABP}(BP^*(BP),A). \end{array}$$

In fact, $(i^*)\#:\ Hom^n_{ABP}(BP^*(BP^n),A)\to Hom^n_{ABP}(BP^*(BP),A)$ is an isomorphism. (Cf. Lemma 2-5). q.e.d.

Lemma 4-3 implies that in the Adams-Novikov spectral sequence of BP^n/BP^0, $F_n^0(BP^n/BP^0)/F_n^1(BP^n/BP^0)=E_\infty^{0,n}(BP^n/BP^0)\cong Hom^n_{ABP}(BP^*(BP^n/BP^0),A)$. So it is clear that $\mathrm{Im}\, j_*^2=\mathrm{Im}\, j_*^\infty$. Meanwhile by the definition of our spectral sequence in Theorem 2-7, the image of j_*^2: $Hom^{0,n}_{ABP}(BP^*(BP^n/BP^0),A)\to Hom^{0,n}_{ABP}(BP^*(BP^n/BP^{n-1}),A)=$

$H_n(BP; Z) \otimes Ext^{0,n}_{ABP}(A, A)$ is exactly N_n. This completes a proof of Theorem 4-1.

As a corollary of Theorem 4-1 we obtain the following result. Consider the cofiber sequence; $S^0 \xrightarrow{i} BP \xrightarrow{\pi} BP/S^0$, where i is an inclusion map, and π is a projection map. Associated with this, there is a short exact sequence for $*>1$;

$$0 \to \pi_*(BP) \to \pi_*(BP/S^0) \to \pi_{*-1}(S^0) \to 0,$$

where Φ is π_*.

Corollary 4-4. *Let p be an odd prime. Let $v \in \pi_{nq}(BP)$, where $n>0$ and $q=2(p-1)$. Then $\Phi(v)$ is divisible by p in $\pi_{nq}(BP/S^0)$ if and only if v belongs to the subgroup $p\pi_{nq}(BP)+Z_{(p)}(v_1^n)$. Moreover v_1^n is divisible by $p^{\nu_p(n)+1}$, and it is best possible.*

Proof. Consider the commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{nq}(BP) & \xrightarrow{\Phi} & \pi_{nq}(BP/S^0) & \xrightarrow{\partial} & \pi_{nq-1}(S^0) & \longrightarrow & 0 \\ & & \| & & \downarrow h' & & \downarrow e_{BP} & & \\ 0 & \longrightarrow & \pi_{nq}(BP) & \xrightarrow{h} & N_{nq} & \xrightarrow{\varphi} & Ext^{1,nq}_{ABP}(A, A) & \longrightarrow & 0, \end{array}$$

where both horizontal sequences are short exact sequences, and φ is an epimorphism of Corollary 3-3. If $\Phi(v)=pw$ for some $w \in \pi_{nq}(BP/S^0)$, from the above diagram we see that $h(v) \in pN_{nq}$. Using Proposition 3-8, we obtain

$$h(v) \equiv \lambda v_1^n \pmod{p\pi_{nq}(BP)}, \text{ where } \lambda \in Z_{(p)}.$$

Therefore v belongs to $p\pi_{nq}(BP)+Z_{(p)}(v_1^n)$. On the other hand according to Adams [3] and Novikov [11], there exists an element $\alpha \in \pi_{nq-1}(S^0)$ such that $e_{BP}(\alpha)=\alpha_n^{(\nu_p(n))}$ and $p^{\nu_p(n)+1}\alpha=0$. Let $x \in \pi_{nq}(BP/S^0)$ such that $\partial(x)=\alpha$, then from the commutativity and from the definition that $\varphi(p^{n-\nu_p(n)-1}m_1^n)=\alpha_n^{(\nu_p(n))}$, we see that there exists an element $z \in \pi_{nq}(BP)$ such that $h'(x)=p^{n-\nu_p(n)-1}m_1^n+h(z)$. Let $y=x-\Phi(z)$, then $h'(y)=p^{n-\nu_p(n)-1}m_1^n$ and $\partial y=\alpha$. Consider the element $\Phi(v_1^n)-p^{\nu_p(n)+1}y \in \pi_{nq}(BP/S^0)$. We assert that $\Phi(v_1^n)=p^{\nu_p(n)+1}y$. Since $\partial(\Phi(v_1^n)$

$-p^{\nu_p(n)+1}y) = -p^{\nu_p(n)+1}\partial y = -p^{\nu_p(n)+1}\alpha = 0$, it is clear that there is an element $z' \in \pi_{nq}(BP)$ such that $\Phi(z') = \Phi(v_1{}^n) - p^{\nu_p(n)+1}y$. Applying h' to this equation we get

$$h(z') = h'(\Phi(z')) = h'(\Phi(v_1{}^n) - p^{\nu_p(n)+1}y) = h(v_1{}^n) - p^n m_1{}^n = 0.$$

But h is a monomorphism, so we obtain $z' = 0$. By the same argument it is easily proved that $\Phi(v_1{}^n)$ is not divisible by $p^{\nu_p(n)+2}$ in $\pi_{nq}(BP/S^0)$. q.e.d.

§ 5. The multiplicativity of the spectral sequence.

In this section we shall prove the multiplicativity of the spectral sequence $\{E_r{}^{*,*,*}(BP), d_r{}^{*,*,*}(BP)\}$. Moreover we shall prove the following theorem.

Theorem 5-1. *Let K be a* (-1)*-connected CW ring spectrum such that* $H_*(K; Z)$ *is free and locally finitely generated over Z or* $Z_{(p)}$. *We consider the spectral sequence* $\{E_r{}^{*,*,*}(K),\ d_r{}^{*,*,*}(K)\}$. *Then there exist pairings* $\prod_r: E_r{}^{u,s,t} \otimes E_r{}^{u',s',t'} \to E_r{}^{u+u',s+s',t+t'}$, *such that*

i) $\prod_r$ *maps*

$$Z_r{}^{u,s,t} \otimes Z_r{}^{u',s',t'} \to Z_r{}^{u+u',s+s',t+t'},$$

$$B_r{}^{u,s,t} \otimes Z_r{}^{u',s',t'} \to B_r{}^{u+u',s+s',t+t'},$$

$$Z_r{}^{u,s,t} \otimes B_r{}^{u',s',t'} \to B_r{}^{u+u',s+s',t+t'},$$

ii) d_r *is an anti-derivation with respect to* $\prod_r$, *i.e., for* $a \in E_r{}^{u,s,t}$, $b \in E_r{}^{u',s',t'}$, $d_r(ab) = d_r(a) \cdot b + (-1)^{u-s+t} a \cdot d_r(b)$,

iii) $\prod_{r+1}$ *is induced by* $\prod_r$,

iv) $\prod_1$ *is the canonical product induced by the ring structure of* $H_*(K; Z)$ *and* $Ext^{*,*}_{ABP}(\Lambda, \Lambda)$. *Here* $\otimes$ *means the tensor product over Z or* $Z_{(p)}$.

In order to prove Theorem 5-1 first we summarize the results from [8].

5-1. Let Λ be a commutative ring with unit, and A be a graded augumented projective Hopf algebra over Λ. Let M and N be graded A-modules. If $Tor_A^n(M, N)=0$ for any $n>0$, then there exists a pairing

$$U: Ext_A^{s,t}(M, \Lambda)\otimes Ext_A^{s',t'}(N, \Lambda)\to Ext_A^{s+s',t+t'}(M\underset{\Lambda}{\otimes}N, \Lambda).$$

This pairing is defined as follows: Let $\mathcal{X}$ be an A-projective resolution of M and $\mathcal{X}'$ an A-projective resolution of N. Under the above conditions, the complex $\mathcal{X}\underset{\Lambda}{\otimes}\mathcal{X}'$ is an $A\underset{\Lambda}{\otimes}A$-projective resolution of $M\underset{\Lambda}{\otimes}N$. Therefore we obtain a homomorphism:

$$Hom_A(\mathcal{X}, \Lambda)\otimes Hom_A(\mathcal{X}', \Lambda)\to Hom_{A\underset{\Lambda}{\otimes}A}(\mathcal{X}\underset{\Lambda}{\otimes}\mathcal{X}', \Lambda\underset{\Lambda}{\otimes}\Lambda).$$

Passing to homology, we obtain an external pairing:

$$Ext_A(M, \Lambda)\otimes Ext_A(N, \Lambda)\to Ext_{A\underset{\Lambda}{\otimes}A}(M\underset{\Lambda}{\otimes}N, \Lambda\underset{\Lambda}{\otimes}\Lambda).$$

The diagonal map D: $A\to A\underset{\Lambda}{\otimes}A$ induces the homomorphism:

$$Ext_{A\underset{\Lambda}{\otimes}A}(M\underset{\Lambda}{\otimes}N, \Lambda\underset{\Lambda}{\otimes}\Lambda)\to Ext_A(M\underset{\Lambda}{\otimes}N, \Lambda).$$

Composing this with the external pairing, we obtain the required pairing U:

$$U: Ext_A(M, \Lambda)\otimes Ext_A(N, \Lambda)\to Ext_A(M\underset{\Lambda}{\otimes}N, \Lambda).$$

5-2. Under the same conditions as 5-1, the pairing U is natural, that is, for A-homomorphisms f: $M\to M'$ and $g: N\to N'$ the following diagram commutes under the conditions $Tor_A^n(M, N)=Tor_A^n(M', N')=0$;

$$\begin{array}{ccc} Ext_A^{s,t}(M, \Lambda)\otimes Ext_A^{s',t'}(N, \Lambda) & \xrightarrow{U} & Ext_A^{s+s',t+t'}(M\underset{\Lambda}{\otimes}N, \Lambda) \\ \uparrow f\#\otimes g\# & & \uparrow (f\otimes g)\# \\ Ext_A^{s,t}(M', \Lambda)\otimes Ext_A^{s',t'}(N', \Lambda) & \xrightarrow{U} & Ext_A^{s+s',t+t'}(M'\underset{\Lambda}{\otimes}N', \Lambda). \end{array}$$

5-3. Let $0\to M\to M''\to M'\to 0$ be a short exact sequence of A-modules. If N is a A-flat A-module, then the following diagram commutes;

$$\begin{array}{ccc} Ext_A^{s,t}(M, A)\otimes Ext_A^{s',t'}(N, A) & \xrightarrow{U} & Ext_A^{s+s',t+t'}(M\underset{A}{\otimes}N, A) \\ \big\downarrow{\scriptstyle \Delta\otimes id} & & \big\downarrow{\scriptstyle \Delta'} \\ Ext_A^{s+1,t}(M', A)\otimes Ext_A^{s',t'}(N, A) & \xrightarrow{U} & Ext_A^{s+s'+1,t+t'}(M'\underset{A}{\otimes}N, A), \end{array}$$

where Δ is the connecting homomorphism induced by the short exact sequence; $0\to M\to M''\to M'\to 0$, and Δ' is the one induced by the short exact sequence; $0\to M\underset{A}{\otimes}N\to M''\underset{A}{\otimes}N\to M'\underset{A}{\otimes}N\to 0$.

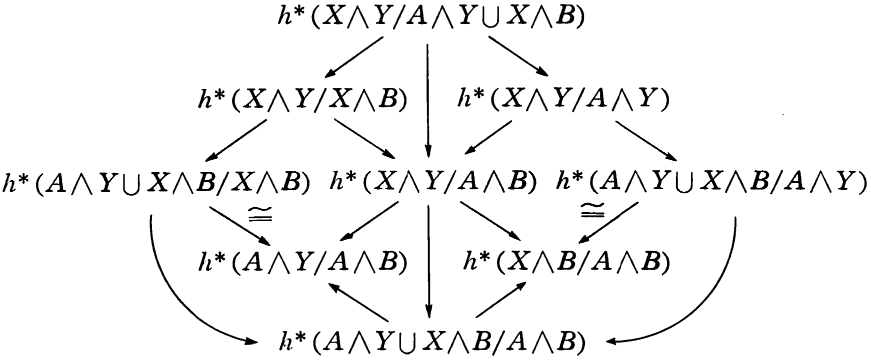
5-4. Under the same conditions of 5-3, the following diagram commutes up to sign $(-1)^{t-s}$;

$$\begin{array}{ccc} Ext_A^{s,t}(N, A)\otimes Ext_A^{s',t'}(M, A) & \xrightarrow{U} & Ext_A^{s+s',t+t'}(N\underset{A}{\otimes}M, A) \\ \big\downarrow{\scriptstyle id\otimes\Delta} & & \big\downarrow{\scriptstyle \Delta''} \\ Ext_A^{s,t}(N, A)\otimes Ext_A^{s'+1,t'}(M', A) & \xrightarrow{U} & Ext_A^{s+s'+1,t+t'}(N\underset{A}{\otimes}M', A), \end{array}$$

where Δ'' is the connecting homomorphism induced by the short exact sequence; $0\to N\underset{A}{\otimes}M\to N\underset{A}{\otimes}M''\to N\underset{A}{\otimes}M'\to 0$.

Secondly we summarize the results of the reduced multiplicative cohomology theory $h^*(\ \ )$. Our reference is [2].

5-5. Let (X, A) and (Y, B) be a pair of CW-spectra with base-point. We denote the smash product of X and Y by $X\wedge Y$. The following diagram commutes;



In the above diagram, all straight sequences are exact. The above diagram displays $h^*(A\wedge Y\cup X\wedge B/A\wedge B)$ as the direct sum $h^*(A\wedge Y/A\wedge B)\oplus h^*(X\wedge B/A\wedge B)$. It is easily proved that if the sequences; $h^*(X\wedge Y/A\wedge Y\cup X\wedge B)\to h^*(X\wedge Y/X\wedge B)\to h^*(A\wedge Y\cup X\wedge B/X\wedge B)$ and $h^*(X\wedge Y/A\wedge Y\cup X\wedge B)\to h^*(X\wedge Y/A\wedge Y)\to h^*(A\wedge Y\cup X\wedge B/A\wedge Y)$ are short exact sequences, then so are the other exact sequences in the above diagram.

Thirdly we shall apply the results 5-1~5 to the case $h^*(\ )=BP^*(\ )$, $A=A^{BP}$, and $\Lambda=\pi_*(BP)$. From Theorem 1 in § 1, A^{BP} is clearly Λ-projective.

Now we shall construct a pairing $\prod_r: E_r^{u,s,t}\otimes E_r^{u',s',t'}\to E_r^{u+u',s+s',t+t'}$. First we construct a pairing in the E_1-terms. We consider the skeletal filtration $\{K^r\}$ as Lemma 2-1 or Remark 2 in § 2. Then it is obvious that

i) $BP^*(K^u/K^v)$ is Λ-free for any $u\geqq v$,

ii) the Künneth formula holds for $u\geqq v, u'\geqq v'$:

$$\kappa:\ BP^*(K^u/K^v)\otimes BP^*(K^{u'}/K^{v'}) \to BP^*(K^u\wedge K^{u'}/K^u\wedge K^{v'}\cup K^v\wedge K^{u'}),$$

where $\wedge$ is the smash product.

By i) and ii) we can define $\prod_1$ as the composition $(\mu^*)\#\circ(\kappa^{-1})\#\circ U$:

$$Ext_{A^{BP}}^{s,t+u}(BP^*(K^u/K^{u-1}),\Lambda)\otimes Ext_{A^{BP}}^{s',t'+u'}(BP^*(K^{u'}/K^{u'-1}),\Lambda)\xrightarrow{U}$$

$$Ext_{A^{BP}}^{s+s',t+t'+u+u'}(BP^*(K^u/K^{u-1})\underset{\Lambda}{\otimes}BP^*(K^{u'}/K^{u'-1}),\Lambda)\xrightarrow{(\kappa^{-1})\#}$$

$$Ext_{A^{BP}}^{s+s',t+t'+u+u'}(BP^*(K^u\wedge K^{u'}/K^{u-1}\wedge K^{u'}\cup K^u\wedge K^{u'-1}),\Lambda)\xrightarrow{(\mu^*)\#}$$

$$Ext_{A^{BP}}^{s+s',t+t'+u+u'}(BP^*(K^{u+u'}/K^{u+u'-1}),\Lambda),$$

where $\mu: K^u\wedge K^{u'}/K^{u-1}\wedge K^{u'}\cup K^u\wedge K^{u'-1}\to K^{u+u'}/K^{u+u'-1}$ is the structure map of the ring spectrum K.

Now Theorem 5-1 follows from the standard arguments. So we omit the proof.

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