J. Math. Kyoto Univ. (JMKYAZ)
15-3 (1975) 619-646

On a generalization of complete intersections

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(Received Sep. 11, 1974)

§ 0. Introduction.

The purpose of this article is to introduce a family of varieties (called weighted complete intersections in § 3) which are quite similar to complete intersections. The idea is simply to embed a variety in the Proj of a graded polynomial ring (denoted by $\boldsymbol{Q}(e)$ in § 1) whose generators are not necessarily of degree 1. The main point is to find a good open set of $\boldsymbol{Q}(e)$ (denoted by $\boldsymbol{P}(e)$ and called a weak projective space in § 1) in which the above-mentioned variety should be contained, noting that $\boldsymbol{Q}(e)$ itself does not meet our requirements (Theorem 1. 7). Some geometric properties of weak projective spaces are studied in § 2, which are similar to those of projective spaces and used in the study of weighted complete intersections.

Weighted complete intersections have several properties similar to those of complete intersections and, on the other hand, they have the following characteristic property: A non-singular projective variety is a weighted complete intersection if it contains a weighted complete intersection of dimension ≥ 3 as an ample divisor (Corollary 3. 8). It should be noted that the family of complete intersections never has such a property. In fact, this is our motive of introduction of weighted complete intersections.

§ 4 includes some results on the deformation of weighted completed intersections, among which Example 4. 3 shows that some of the weighted complete intersections can be obtained by deforming hypersurfaces.

The author expresses his hearty thanks to his teachers, Professor M.Nagata, Professor H.Sumihiro, Professor M.Maruyama, Professor H.Tango and Professor A.Fujiki, and to his colleagues who encouraged and gave him useful suggestions.

§ 1. The definition of weak projective spaces.

In this section, we are going to study some schemes as a generalization of projective spaces.

As for the notation and conventions, we chiefly use those of [5] and, as usual, the ring of integers is denoted by $\boldsymbol{Z}$.

Definition 1.1. Let $n, e_0, \cdots, e_n$ be positive integers. We set $d=\text{g.c.d.}\{e_i | 0 \leq i \leq n\}$ and $m=\text{l.c.m.}\{e_i | 0 \leq i \leq n\}$. On the other hand, for each prime p, we consider $\#\{i | 0 \leq i \leq n, p \nmid e_i\}$. We denote by $r(e_0, \cdots, e_n)$, or simply by $r(e)$, the minimum of these numbers.

$\boldsymbol{Q}(e_0, \cdots, e_n)$, or simply $\boldsymbol{Q}(e)$, is the scheme Proj $(\boldsymbol{Z}[X_0, \cdots, X_n])$, where the gradation of $\boldsymbol{Z}[X_0, \cdots, X_n]$ is defined as follows;

$$\deg X_i = e_i \quad (0 \leq i \leq n), \quad \text{and} \quad \deg a = 0 \quad (a \in \boldsymbol{Z}).$$

For a positive integer k, S_k is the closed subset of $\boldsymbol{Q}(e)$, defined by the ideal generated by $\{X_i | k \nmid e_i\}$.

For an integer a, $\mathcal{O}_{\boldsymbol{Q}(e)}(a)$, or simply $\mathcal{O}_{\boldsymbol{Q}}(a)$, is the coherent $\mathcal{O}_{\boldsymbol{Q}(e)}$-module corresponding to the homogeneous $\boldsymbol{Z}[X_0, \cdots, X_n]$-module $\boldsymbol{Z}[X_0, \cdots, X_n](a)$ (for detail cf. [5]).

It must be noted that $\mathcal{O}_{\boldsymbol{Q}}(a)$ is not necessarily invertible as is seen in Theorem 1.7. But at least we have:

Proposition 1.1. *With the notation of Definition 1.1,* $\mathcal{O}_{\boldsymbol{Q}}(1)$ *is locally free on* $\boldsymbol{Q}(e) - \bigcup_{1<k} S_k$, *and the following three conditions are equivalent to each other: 1)* $\mathcal{O}_{\boldsymbol{Q}}(1) \neq 0$, *2)* $d=1$, *3)* $r(e) > 0$.

Remark 1.2. As is easily seen, we do not have to take account of so many integers k's in the first part of Proposition 1.1, to be precise: $\bigcup_{1<k} S_k = \bigcup_{\substack{k|m \\ k:\text{prime}}} S_k$.

For the proof of Proposition 1.1, we need some lemmas.

Lemma 1.3. *With the notation of Definition 1.1, $\mathcal{O}_Q(m)$ is an ample invertible sheaf, and for arbitrary integers a, b, the natural $\mathcal{O}_Q$-homomorphism $\mathcal{O}_Q(a)\otimes\mathcal{O}_Q(m)^{\otimes b}\to\mathcal{O}_Q(a+bm)$ is an isomorphism.*

The proof of this lemma is found in [5].

Lemma 1.4. *With the notation of Definition 1.1, we have*

$$\bigcap_{0\leq a} V_+(\boldsymbol{Z}[X_0, \cdots, X_n]_{am+1}) = \bigcup_{1<k} S_k .$$

Remark 1.5. As in [5], for a subset M of a graded ring R, $V_+(M)=\{P\in\operatorname{Proj} R \mid P\supset M\}$, $D_+(M)=\{P\in\operatorname{Proj} R \mid P\not\supset M\}$ and $R_a=$ (the homogeneous part of degree a of R).

Proof of Lemma 1.4. Let T denote $\boldsymbol{Z}[X_0, \cdots, X_n]$. Assume, for a homogeneous prime P of T, $P\notin\bigcup_{1<k}S_k$. Then for every prime number q with $q|m$, there exists a homogeneous element F_q of T with $F_q\notin P$ and $q\nmid\deg F_q$. By the last property there exists a positive integer a_q for each q, such that $\sum_{\substack{q|m\\ q:\text{prime}}} a_q \deg F_q=1+am$, for some positive integer a, hence $\prod_q F_q^{a_q}\in T_{am+1}-P$. Conversely, assume $P\in S_k$ for a prime number k with $k|m$ (cf. Remark 1.2). Then it follows that $P\supset\boldsymbol{Z}[X_0, \cdots, X_n]_{am+1}$ for every non-negative integer a, because no monomial of $\boldsymbol{Z}[X_0, \cdots, X_n]_{am+1}$ can be expressed as a product of X_i's such that $k|e_i$. q.e.d.

Proof of Proposition 1.1. The second part is trivial. As for the first part, let T denote $\boldsymbol{Z}[X_0, \cdots, X_n]$. Assume, for a homogeneous prime P of T, $P\notin\bigcup_{1<k}S_k$. Then, by Lemma 1.4, there exist a non-negative integer a and a homogeneous element F of T with $F\in T_{am+1}-P$. Then $\mathcal{O}_Q(am+1)$ is free at P, because $P\in D_+(F)$ and we have an equality of $\widetilde{T[1/F]_0}$-modules;

$$\mathcal{O}_Q(am+1)|_{D_+(F)}=\widetilde{T\left[\frac{1}{F}\right]}_{am+1}=\widetilde{T\left[\frac{1}{F}\right]}_0 F .$$

Hence $\mathcal{O}_{Q}(1)$ is free at P, by Lemma 1.3. q.e.d.

In view of Proposition 1.1, we are led to:

Definition 1.2. With the notation of Definition 1.1, $\boldsymbol{P}(e_0, \cdots, e_n)$, or simply $\boldsymbol{P}(e)$, is the open subscheme of $\boldsymbol{Q}(e)$, $\boldsymbol{Q}(e)-\bigcup_{1<k} S_k$. We call the scheme $\boldsymbol{P}(e)$ a weak projective space of size $(e_0, \cdots, e_n)$, or simply of size (e), and define $\mathcal{O}_{\boldsymbol{P}}(a)=\mathcal{O}_{\boldsymbol{Q}}(a)|_{\boldsymbol{P}(e)}$ for every integer a.

It must be noted that $\boldsymbol{P}(e)\neq\phi$ if and only if $d=1$ namely $r(e)>0$. In $\boldsymbol{P}(e)$, we have a result much simpler than Lemma 1.3.

Lemma 1.6. *With the notation of Definition 1.2, we have a natural isomorphism* $\mathcal{O}_{\boldsymbol{P}}(1)^{\otimes a}\xrightarrow{\sim}\mathcal{O}_{\boldsymbol{P}}(a)$, *for every integer* a.

Proof. It suffices to prove that the natural homomorphism $\mathcal{O}_{\boldsymbol{P}}(a)\otimes\mathcal{O}_{\boldsymbol{P}}(b)\to\mathcal{O}_{\boldsymbol{P}}(a+b)$ is an isomorphism for arbitrary integers a, b. This is induced by the natural T-module homomorphism $T(a)\otimes T(b)\to T(a+b)$, where T denotes $\boldsymbol{Z}[X_0, \cdots, X_n]$. For every homogeneous prime P of T with $P\in\boldsymbol{P}(e)$, there exist an integer i, a positive integer a and an element F of T with $0\leq i\leq n$, $X_i\notin P$ and $F\in T_{am+1}-P$. Therefore we have a commutative diagram;

$$\begin{array}{ccc}
\mathcal{O}_{\boldsymbol{P}}(a)\otimes\mathcal{O}_{\boldsymbol{P}}(b)|_{D_+(FX_i)} & \longrightarrow & \mathcal{O}_{\boldsymbol{P}}(a+b)|_{D_+(FX_i)} \\
\| & & \| \\
T\left[\frac{1}{FX_i}\right]_a \otimes_{T[1/FX_i]_0} T\left[\frac{1}{FX_i}\right]_b & \longrightarrow & T\left[\frac{1}{FX_i}\right]_{a+b} \\
\| & & \| \\
T\left[\frac{1}{FX_i}\right]_0\left(\frac{F}{X_i^{c_i}}\right)^a \otimes T\left[\frac{1}{FX_i}\right]_0\left(\frac{F}{X_i^{c_i}}\right)^b & \xrightarrow{\ \alpha\ } & T\left[\frac{1}{FX_i}\right]_0\left(\frac{F}{X_i^{c_i}}\right)^{a+b},
\end{array}$$

where $c_i=am/e_i$. The last homomorphism α is obviously an isomorphism. q.e.d.

Now $\boldsymbol{P}(e)$ can be characterized as an open set of $\boldsymbol{Q}(e)$ as follows.

Theorem 1.7. *With the notation of Definition 1.2, $\boldsymbol{P}(e)$ is the largest among open subsets U's of $\boldsymbol{Q}(e)$ with the following two properties,*

(1) *$\mathcal{O}_Q(1)|_U$ is an invertible sheaf on U,*

(2) *for every positive integer a, we have a natural isomorphism;*

$$(\mathcal{O}_Q(1)|_U)^{\otimes a} \xrightarrow{\sim} \mathcal{O}_Q(a)|_U.$$

Furthermore if $r(e)>1$, $\boldsymbol{P}(e)$ is the largest among U's having the property (1).

Proof. Let U be an open set with the properties (1) and (2), and x a closed point of U. By Lemma 1.3, the natural map $H^0(\boldsymbol{Q}(e), \mathcal{O}_Q(am+1))\otimes k(x) \to \mathcal{O}_Q(am+1)\otimes k(x)$ is surjective for a sufficiently large a, where $k(x)$ denotes the residue field of x. As will be seen in Remark 2.2, there are isomorphisms $\boldsymbol{Z}[X_0, \cdots, X_n]_{am+1} \xrightarrow{\sim} H^0(\mathcal{O}_Q(e), \mathcal{O}_Q(am+1))$ $(a\in\boldsymbol{Z})$. Therefore, by the property (1), there are a positive integer a and an element F of $\boldsymbol{Z}[X_0, \cdots X_n]_{am+1}$ with $\mathcal{O}_{Q,x}F = \mathcal{O}_{Q,x}(am+1)$. Hence, by the property (2), $\mathcal{O}_{Q,x}F^m = \mathcal{O}_{Q,x}(m(am+1))$. On the other hand, there is an element G of $\boldsymbol{Z}[X_0, \cdots, X_n]_m$ such that $G\notin P$, where P denotes the homogeneous prime ideal corresponding to the point x. Since $\mathcal{O}_{Q,x}G^{am+1} = \mathcal{O}_{Q,x}(m(am+1))$ by the properties (1) and (2), we have $G^{am+1}/F^m \in \mathcal{O}^*_{Q,x}$, consequently $F^m = G^{am+1}\times(G^{am+1}/F^m)^{-1} \notin P$,[1] namely $F\in \boldsymbol{Z}[X_0, \cdots, X_n]_{am+1} - P$. Hence $x\in\boldsymbol{P}(e)$ by Lemma 1.4. In view of Proposition 1.1 and Lemma 1.6, this completes the proof of the first part.

To prove the second part, let U be an open set with the property (1). Then $U-\boldsymbol{P}(e)$ is a closed set of codimension 1 of U. To prove this assertion, first note that the property (2) is equivalent to

$$(\mathcal{O}_Q(1)|_U)^{\otimes m} \xrightarrow{\sim} \mathcal{O}_Q(m)|_U \quad \text{(cf. Lemma 1.3)}.$$

Therefore $U-\boldsymbol{P}(e)$ is the closed set defined by the section on U of

[1] It may be better to rewrite this part as follows. Put $S=\boldsymbol{Z}[X_0, \cdots, X_n]$ and $M=\{H|$ H: a homogeneous element of S, $H\notin P\}$. Then noting that $G^{am+1}/F^m \in \mathcal{O}^*_{Q,x} = (M^{-1}S)_0 \subset M^{-1}S$ and G is a unit in $M^{-1}S$, we see that F is a unit in $M^{-1}S$. Hence $F\notin P$.

the invertible sheaf $(\mathcal{O}_Q(m)|_U)\otimes(\mathcal{O}_Q(1)|_U)^{\otimes(-m)}$ corresponding to the natural homomorphism $(\mathcal{O}_Q(1)|_U)^{\otimes m}\to\mathcal{O}_Q(m)|_U$. This proves the above assertion. Now by the definition of $\boldsymbol{P}(e)$, the codimension of $\boldsymbol{Q}(e)-\boldsymbol{P}(e)$ in $\boldsymbol{Q}(e)$ is equal to $r(e)$. Hence, if $r(e)>1$, $U-\boldsymbol{P}(e)=\phi$ namely $U\subset\boldsymbol{P}(e)$. q.e.d.

It would be worth while to mention that we do not have to restrict the base ring. In fact:

Remark 1.8. For a commutative ring K with 1, $\boldsymbol{Q}(e)\times\operatorname{Spec}K$, $S_k\times\operatorname{Spec}K$, $\boldsymbol{P}(e)\times\operatorname{Spec}K$ enjoy the same properties as asserted in Proposition 1.1, Lemmas 1.3, 1.4, 1.6, and Theorem 1.7. These are proved by the theory of base change.

§ 2. Some properties of weak projective spaces.

In this section we fix a field K, and in view of Remark 1.8, we use the notation of Definitions 1.1 and 1.2 except that $\boldsymbol{Q}(e)\times\operatorname{Spec}K$, $S_k\times\operatorname{Spec}K$, $\boldsymbol{P}(e)\times\operatorname{Spec}K$ are used instead of $\boldsymbol{Q}(e), S_k, \boldsymbol{P}(e)$ respectively. Namely from now on, we define: $\boldsymbol{Q}(e)=\operatorname{Proj}(K[X_0,\cdots,X_n])$, $S_k=V_+(\{X_i|k\nmid e_i\})$ and $\boldsymbol{P}(e)=\boldsymbol{Q}(e)-\bigcup_{1<k}S_k$.

The purpose of this section is to prove some properties which will be used later. First we prove that $r(e)$ is a topological invariant of the variety $\boldsymbol{P}(e)$.

Proposition 2.1. (1) $r(e)=0$ *if and only if* $\boldsymbol{P}(e)=\phi$.

(2) $r(e)=1$ *if and only if* $\boldsymbol{P}(e)$ *is quasi-affine and non-empty.*

(3) *If* $r(e)\geq 1$, *then* $\boldsymbol{P}(e)$ *contains a complete subscheme* X *of dimension* $r(e)-1$, *which is defined on* $\boldsymbol{Q}(e)$ *by* $(n-r(e)+1)$ *elements* $F_0,\cdots,F_{n-r(e)}$ *of* $K[X_0,\cdots,X_n]_{am}$ *for some positive integer* a. *On the other hand,* $\boldsymbol{P}(e)$ *can not contain any complete subschemes of dimension* $\geq r(e)$.

Proof. (1) is already proposition 1.1. (2) and (3) follow immediately from the following three facts:

(i) The codimension of $\bigcup_{1<k}S_k$ in $\boldsymbol{Q}(e)$ is equal to $r(e)$.

(ii) S_k is the set-theoretically complete intersection $\boldsymbol{Q}(e)$ defined by the global sections X_i^{m/e_i} $(k\nmid e_i)$ of the ample invertible sheaf $\mathcal{O}_{\boldsymbol{Q}}(m)$ (cf. Lemma 1.3).

(iii) There are natural isomorphisms $K[X_0, \cdots, X_n]_{am} \xrightarrow{\sim} H^0(\boldsymbol{Q}(e), \mathcal{O}_{\boldsymbol{Q}}(am))$ $(a \in \boldsymbol{Z})$ (cf. Remark 2.2). q.e.d.

The following remark on cohomology are used in Proposition 2.1 and will be used later.

Remark 2.2. Let R be a commutative graded noetherian ring with 1 and with non-negative degrees, and assume that R is generated by homogeneous elements $f_0, \cdots, f_N$ of R_+ as an R_0-algebra. Then, by [6, § 2], for an arbitrary graded module M we have an exact sequence

$$0 \to H^0((f); M) \to M \to H^0(X, \widetilde{M}(*)) \to H^1((f); M) \to 0$$

and isomorphisms $H^i(X, \widetilde{M}(*)) \xrightarrow{\sim} H^{i+1}((f); M)$ $(i \in \boldsymbol{Z}, i \geq 1)$, where $X = \operatorname{Proj} R$ (for detail, cf. [6, § 2]). In particular if the ring R is Cohen-Macaulay of dimension $n+1$ $(n \geq 1)$, then $H^i((f); R) = 0$ $(i \in \boldsymbol{Z}, i < n+1)$, hence;

$$R_i \xrightarrow{\sim} H^0(X, \mathcal{O}_X(i)) \quad (i \in \boldsymbol{Z}),$$

$$H^j(X, \mathcal{O}_X(i)) = 0 \quad (i, j \in \boldsymbol{Z}, 0 < j < n).$$

The followings are simple generalizations of some properties of projective spaces to the case of weak projective spaces.

Proposition 2.3. (1) *Set* $\boldsymbol{A}(e) = \operatorname{Spec} K[X_0, \cdots, X_n] - \bigcup_{1<k} V(\{X_i | k \nmid e_i\})$. *Then there is a natural morphism* $\pi: \boldsymbol{A}(e) \to \boldsymbol{P}(e)$, *and this is a* $\boldsymbol{G}_m$*-bundle.*

(2) $\mathcal{O}_{\boldsymbol{P}}(1)$ *generates* $\operatorname{Pic} \boldsymbol{P}(e)$ *in general, and* $\operatorname{Pic} \boldsymbol{P}(e) \simeq \boldsymbol{Z}$ *if* $r(e) > 1$.[2)]

(3) *If* $r(e) > 1$, *we have* $K_{\boldsymbol{P}} = \mathcal{O}_{\boldsymbol{P}}(-\sum_{j=0}^n e_i)$, *where* $K_{\boldsymbol{P}}$ *denotes the canonical sheaf of* $\boldsymbol{P}(e)$.[3)]

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2) In case $r(e) = 1$, it holds that $\operatorname{Pic} \boldsymbol{P}(e) \simeq \boldsymbol{Z}/t\boldsymbol{Z}$ with
$t = \text{g.c.d.}\{e_i | \text{g.c.d.}\{e_0, \cdots, \check{e}_i, \cdots, e_n\} > 1\}$.

3) This assertion holds even if $r(e) = 1$.

Proof. (1) π is given as follows: For every homogeneous element F of $K[X_0, \cdots, X_n]$ with $D_+(F) \subset \boldsymbol{P}(e)$, the morphism $\boldsymbol{A}(e)_F \to \boldsymbol{P}(e)_F$ is induced by the natural K-algebra homomorphism; $K[X_0, \cdots, X_n, 1/F]_0 \to K[X_0, \cdots, X_n, 1/F]$. These morphisms are obviously patched together and we obtain π. Next we consider an open set $D_+(FG)$ of $\boldsymbol{Q}(e)$, for a positive integer a and two homogeneous elements F and G of $K[X_0, \cdots, X_n]$ with $\deg F = am$ and $\deg G = am+1$. Then, by Lemmas 1. 3, 1. 4 and Remark 1. 8, $\boldsymbol{P}(e)$ is covered by such open sets $D(FG)$'s. The morphism $\pi|_{D(FG)}$ is induced by the natutural K-algebra homomorphism $K[X_0, \cdots, X_n, 1/F, 1/G]_0 \to K[X_0, \cdots, X_n, 1/F, 1/G]$. Now putting $H = G/F$, we have;

$$K\left[X_0, \cdots, X_n, \frac{1}{F}, \frac{1}{G}\right]_0$$

$$= K[X_0/H^{e_0}, \cdots, X_n/H^{e_n}, H^{am}/F, H^{am+1}/G],$$

$$K\left[X_0, \cdots, X_n, \frac{1}{F}, \frac{1}{G}\right]$$

$$= K[X_0/H^{e_0}, \cdots, X_n/H^{e_n}, \cdots, H^{am}/F, H^{am+1}/G][H, 1/H].$$

These mean that $D(FG) \simeq D_+(FG) \times \boldsymbol{G}_m$. It is immediate to check that these are patched together and we have a $\boldsymbol{G}_m$-bundle.

(2) Since $\mathcal{O}_{\boldsymbol{P}}(1)$ is an ample invertible sheaf and $P(e)$ is smooth by (1), it follows that $\mathcal{O}_{\boldsymbol{P}}(1)$ generates Pic $\boldsymbol{P}(e)$, provided that it is proved that for every subvariety D of codimension 1 of $\boldsymbol{P}(e)$, there is a homogeneous prime element F of $K[X_0, \cdots, X_n]$ such that $\operatorname{Supp} D = V_+(F)$.

Now assume that D is a subvariety of codimension 1 of $\boldsymbol{P}(e)$, then $\pi^{-1}(D)$ is a homogeneous subvarity of codimension 1 of $\boldsymbol{A}(e)$. Since $\boldsymbol{A}(e)$ is an open set of the affine space $\boldsymbol{A}^{n+1}$, there exists a homogeneous prime element F of $K[X_0, \cdots, X_n]$ such that $\operatorname{Supp} \pi^{-1}(D) = V(F)$. This means $\operatorname{Supp} D = V_+(F)$.

If $r(e) > 1$, then $\mathcal{O}_{\boldsymbol{P}}(a)$ is not trivial for any positive integer a; this follows from the fact that $\mathcal{O}_{\boldsymbol{P}}(1) \otimes \mathcal{O}_C = O_C(1)$ is ample on C, where C is a complete subscheme of dimension ≥ 1 given in Proposition 2. 1, (3).

(3) By (1) and (2), we can define an integer $s = s(e_0, \cdots, e_n)$ with $K_{\boldsymbol{P}(e)} = \mathcal{O}_{\boldsymbol{P}(d)}(-s)$, in the case $r(e) > 1$. If $r(e_0, \cdots, e_{n-1}) > 1$, we obtain

$$(*) \qquad s(e_0, \cdots, e_n) - e_n = s(e_0, \cdots, e_{n-1})$$

by embedding $\boldsymbol{P}(e_0, \cdots, e_{n-1})$ in $\boldsymbol{P}(e_0, \cdots, e_n)$ as the closed subscheme defined by X_n and applying the adjunction formula

$$K_{\boldsymbol{P}(e_0, \cdots, e_n)}(X_n) \otimes \mathcal{O}_{\boldsymbol{P}(e_0, \cdots, e_{n-1})} = K_{\boldsymbol{P}(e_0, \cdots, e_{n-1})}.^{4)}$$

If $r(e_0, \cdots, e_n) > 1$, by $(*)$, we have

$$s(e_0, \cdots, e_n) = -2 + s(1, 1, e_0, \cdots, e_n),$$

$$s(1, 1, e_0, \cdots, e_n) - \sum_{j=0}^{n} e_j = s(1, 1).$$

As is well known, $s(1, 1) = 2$, hence $s(e_0, \cdots, e_n) = \sum_{j=0}^{n} e_j$.

This completes the proposition 2. 3.

Remark 2. 4. More precisely than Proposition 2. 3, (3), we have an exact sequence;

$$0 \to \mathcal{O}_{\boldsymbol{P}} \to \bigoplus_{i=0}^{n} \mathcal{O}_{\boldsymbol{P}}(e_i) \to T_{\boldsymbol{P}} \to 0,$$

but, in this paper, Proposition 2. 3, (3) is sufficient for our use.

§ 3. The definition and some properties of weighted complete intersections.

The aim of this section is not only to introduce the notion of a "weighted complete intersection", but also to give some evidence that the notion of a weighted complete intersection is a natural generalization of the one of a complete intersection.

We use the notation of § 2.

Definition 3. 1. With the notation of Definitions 1. 1 and 1. 2, let c, $a_1, \cdots, a_c$ be positive integers. We consider $\mathrm{Proj}(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$, for arbitrary homogeneous elements $F_1, \cdots, F_c$ of the graded ring $K[X_0, \cdots, X_n]$ (given in Definition 1. 1) with $\deg F_j = a_j$ $(1 \leq j \leq c)$, satisfying the following two conditions:

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[4] We have used the equality

$$\mathcal{O}_{\boldsymbol{P}(e_0, \cdots, e_n)}(1) \otimes \mathcal{O}_{\boldsymbol{P}(e_0, \cdots, e_{n-1})} = \mathcal{O}_{\boldsymbol{P}(e_0, \cdots, e_{n-1})}(1)$$

which is non-trivial but is easy to prove.

(1) $(F_1, \cdots, F_c)$ is a regular sequence of $K[X_0, \cdots, X_n]$.

(2) $V_+(F_1, \cdots, F_c) \cap \bigcup_{1<k} S_k = \phi$.

Then an algebraic K-scheme X is called a weighted complete intersection of $\boldsymbol{P}(e)$ of type $(a_1, \cdots, a_c)$, if X is isomorphic to such a K-scheme $\mathrm{Proj}(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$.

In this case, for an arbitrary integer a, we denote by $\mathcal{O}_X(a)$ the invertible sheaf on X induced by $\mathcal{O}_{\boldsymbol{P}(e)}(a)$.

Remark 3.1. By Proposition 2.1, (3), we have $\dim X = n-c < r(e)$, namely $c > n-r(e)$. Then if $\dim X > 0$ (resp. ≥ 0) it follows necessarily that $r(e) > 1$ (resp. ≥ 1).

As for the degree of $\mathcal{O}_X(1)$ (see Definition 3.1), it is calculated by using the result of the appendix (Corollary A.2):

Proposition 3.2. *If X is a weighted complete intersection of type $(a_1, \cdots, a_c)$ of $\boldsymbol{P}(e)$, then $\mathcal{O}_X(1)$ is an ample invertible sheaf and we have an isomorphism $\mathcal{O}_X(1)^{\otimes a} \xrightarrow{\sim} \mathcal{O}_X(a)$ for every integer a. Furthermore $(\mathcal{O}_X(1)^{n-c}) = \prod_{j=1}^{c} a_j / \prod_{i=0}^{n} e_i$.*[5]

Proof. The first part follows immediately from Lemmas 1.3 and 1.6. As for the second part, we have $h^0(X, \mathcal{O}_X(u)) = \Delta_{a_1} \cdots \Delta_{a_c} H(e_0, \cdots, e_n; u)$ for sufficiently large u, because $F_1, \cdots, F_c$ is a regular sequence of $K[X_0, \cdots, X_n]$ (the definition of $H(e; u)$ is found in Definition A.1, and the proof of this assertion is similar to the one of Theorem A.1, (2)). Since $r(e) > \dim X \geq 0$ by Remark 3.1, the coefficient of $u^{n-c}/(n-c)!$ in $\Delta_{a_1} \cdots \Delta_{a_c} H(e_0, \cdots, e_n; u)$ is $\prod_{j=1}^{c} a_j / \prod_{i=0}^{n} e_i$ by Corollary A.2. q.e.d.

As for the cohomology groups of $\mathcal{O}_X(a)$ $(a \in \boldsymbol{Z})$, we have the following result.

Proposition 3.3. *Let X be a weighted complete intersection of dimension ≥ 1 of $\boldsymbol{P}(e)$ of type $(a_1, \cdots, a_c)$. Then we have:*

[5] $(\mathcal{O}_X(1)^{n-c})$ is the intersection number of $(n-c)$ invertible sheaves $\mathcal{O}_X(1), \cdots, \mathcal{O}_X(1)$ on X.

$$(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))_a \tilde{\to} H^0(X, \mathcal{O}_X(a)) \quad (a \in \boldsymbol{Z}),$$

$$H^j(X, \mathcal{O}_X(a)) = 0 \quad (a, j \in \boldsymbol{Z}, 0 < j < n - c = \dim X),$$

$$\omega_X = \mathcal{O}_X(\sum_{j=1}^{c} a_j - \sum_{i=0}^{n} e_i),$$

where ω_X *denotes the dualyzing sheaf of* X.

Proof. The first and the second equalities follow immediately from Remark 2. 2.

As for the last one, first note that $\omega_X = Ext^c_{\mathcal{O}_{\boldsymbol{P}}}(\mathcal{O}_X, \mathcal{O}_{\boldsymbol{P}}(-\sum_{i=0}^n e_i))$ (see Proposition 2. 3, (3)). Since $(F_1, \cdots F_c)$ is a regular sequence of $K[X_0, \cdots, X_n]$ by the definition of X, the Koszul complex $\otimes_{j=1}^c K_j$ is a resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\boldsymbol{P}}$-module; K_j is a complex of $\mathcal{O}_{\boldsymbol{P}}$-modules with

$$(K_j)_b = \begin{cases} \mathcal{O}_{\boldsymbol{P}} & \text{if } b=0, \\ \mathcal{O}_{\boldsymbol{P}}(-a_j) & \text{if } b=1, \\ 0 & \text{if } b \in \boldsymbol{Z} \text{ and } b \neq 0, 1, \end{cases}$$

$$(d_j)_1 : \mathcal{O}_{\boldsymbol{P}}(-a_j) \to \mathcal{O}_{\boldsymbol{P}} \text{ is a multiplication by } F_j,$$

(for detail, see [6]). This proves that $\omega_X = \mathcal{O}_X(\sum_{j=1}^c a_j - \sum_{i=0}^n e_i)$.
q.e.d.

The following example is the simplest one which is non-trivial.

Example 3. 4. Put $p = \mathrm{char}\, K$. Assume that $n, a, e_0, \cdots, e_n$ are positive integers such that arbitrary two of $p, a, e_0, \cdots, e_n$ are relatively prime to each other. We define $X = \mathrm{Proj}(K[X_0, \cdots, X_n]/(F))$, with $\deg X_i = e_i$ $(0 \leq i \leq n)$, $m = \prod_{i=0}^n e_i$ and $F = \sum_{i=0}^n X_i^{am/e_i}$. Then X enjoys the following four properties:

(1) X is a smooth projective variety of dimension $n-1$.

(2) If $n \geq 4$ i.e. $\dim X \geq 3$, then $\mathrm{Pic}\, X$ is isomorphic to $\boldsymbol{Z}$ and is generated by $\mathcal{O}_X(1)$.

(3) $\mathcal{O}_X(1)$ is an ample invertible sheaf of degree a, and $K_X = \mathcal{O}_X(am - \sum_{i=0}^n e_i)$.

(4) If $e_i > 1$ for every i, then $\mathcal{O}_X(1)$ has no global sections.

On the other hand, assume $e_0=\cdots e_{n-1}=1$. Then $\mathcal{O}_X(1)$ is generated by global sections $X_0, \cdots X_{n-1}$, which define a morphism of X to $\boldsymbol{P}_{n-1}$. This morphism makes X an a-sheeted cyclic branched covering of $\boldsymbol{P}_{n-1}$ with branch locus a smooth hypersurface of degree am.

Proof. (3) is proved in Propositions 3.2 and 3.3, and (4) follows immediately from Proposition 3.3. (2) will be proved in Theorem 3.6. As for the smoothness of X, it suffices to prove that the variety Spec $(K[X_0, \cdots, X_n]/(F)) - V(X_0, \cdots, X_n)$ is smooth (cf. Proposition 2.3, (1)). This is an immediate consequence of the Jacobian Criterion.

q.e.d.

It would be worth while to mention that the variety X given in Example 3.4 is obtained as a quotient of a smooth hypersurface $\tilde{X}$ of $\boldsymbol{P}_n$ by a finite group $\boldsymbol{Z}/m\boldsymbol{Z}$. To be precise, put $\boldsymbol{Z}/m\boldsymbol{Z}=\operatorname{Spec} K[T]/(T^m-1)$ and $X=\operatorname{Proj}(K[Y_0, \cdots, Y_n]/(G))$; $G=\sum_{i=0}^n Y_i^{am}$, and $K[Y_0, \cdots, Y_n]$ is the graded ring defined by $\deg Y_i=1$ $(0\leq i\leq n)$ and $\deg r=0$ $(r\in K)$. Define a dual action σ^*: $K[Y_0, \cdots, Y_n]/(G) \to K[T]/(T^m-1)\otimes K[Y_0, \cdots, Y_n]/(G)$ by $\sigma^* Y_i=T^{m/e_i}\otimes Y_i$ $(0\leq i\leq n)$. This induces an action $\sigma: \boldsymbol{Z}/m\boldsymbol{Z}\times\tilde{X}\to\tilde{X}$. Then X is a quotient of $\tilde{X}$ by the group $\boldsymbol{Z}/m\boldsymbol{Z}$.

But, even in the case char $K=0$, not all of the smooth weighted complete intersections are obtained as quotients of smooth complete intersections by finite groups in such a way.

Example 3.5. Assume that char $K\neq 2$. Then $X=\operatorname{Proj}(K[X_0, X_1, Y]/(F))$ is a smooth curve over K, where $\deg X_0=\deg X_1=1$, $\deg Y=2$ and $F=Y^2+2(X_0^2+X_1^2)Y+2X_0^2X_1^2$. On the other hand, $X=\operatorname{Proj}(K[X_0, X_1, X_2]/(G))$ has a singular point $(0:1:0)$, where $\deg X_i=1$ $(i=0, 1, 2)$ and $G=X_2^4+2(X_0^2+X_1^2)X_2^2+2X_0^2X_1^2$.

The following theorem gives some evidence of the naturality of the notion "complete intersection".

Theorem 3.6. *Let X be a projective K-scheme with an ample effective Cartier divisor Y. Assume Y is, as a K-scheme, a weighted complete intersection of dimension ≥ 2 of type $(a_1, \cdots, a_c)$ of*

$P(e_0, \cdots, e_n)$. *Assume furthermore the following three conditions:*

(1) *For every closed point x of X*, depth $\mathcal{O}_{X,x} \geq 2$.

(2) *There exists an invertible sheaf L on X such that $L \otimes \mathcal{O}_Y \simeq \mathcal{O}_Y(1)$, where $\mathcal{O}_Y(1)$ is the invertible sheaf given in Definition 3. 3.*

(3) *There exists a positive integer a such that $\mathcal{O}_X(Y) \otimes \mathcal{O}_Y \simeq \mathcal{O}_Y(a)$.*

Then X is a weighted complete intersection of type $(a_1, \cdots, a_c)$ of $\boldsymbol{P}(e_0, \cdots, e_n, a)$, and $L \simeq \mathcal{O}_X(1)$.

Proof. We prove the theorem is several steps.

Step 1. $\mathcal{O}_X(Y) \simeq L^{\otimes a}$, in particular L is ample. And we have

$$H^1(X, L^{\otimes i}) = 0 \quad \text{and} \quad H^0(X, L^{\otimes(-j)}) = 0 \quad (i \in \boldsymbol{Z}, j \in \boldsymbol{Z}, j > 0).$$

Proof of Step 1. We obtain an injection $\operatorname{Pic} X \hookrightarrow \operatorname{Pic} Y$, using the assumption (1), $\dim Y \geq 2$ and $H^1(Y, \mathcal{O}_Y(i)) = 0$ $(i \in \boldsymbol{Z})$ (see Proposition 3. 3 in this paper and Corollary 3. 6 in [8, Exposé XII]). In particular we have $\mathcal{O}_X(Y) \simeq L^{\otimes a}$. Hence we have an exact sequence for every integer i,

$$0 \to L^{\otimes(i-a)} \to L^{\otimes i} \to \mathcal{O}_Y(i) \to 0.$$

By the equalities $H^1(Y, \mathcal{O}_Y(i)) = 0$ $(i \in \boldsymbol{Z})$, we obtain surjections $H^1(X, L^{\otimes(i-aj)}) \to H^1(X, L^{\otimes i})$ $(i, j \in \boldsymbol{Z}, j \geq 0)$. On the other hand, by the assumption (1) and Corollary 1. 4 in [8, Exposé XII], $H^1(X, L^{\otimes(-j)}) = 0$ for sufficiently large j. Hence we obtain $H^1(X, L^{\otimes i}) = 0$ for every integer i. By a similar method, the last assertion can be proved.

Step 2. There exists an element Φ of $H^0(X, L^{\otimes a})$ such that we have a naturally induced isomorphism of graded rings

$$\bigoplus_{i \in \boldsymbol{Z}} H^0(X, L^{\otimes i}) / \Phi \bigoplus_{i \in \boldsymbol{Z}} H^0(X, L^{\otimes i}) \xrightarrow{\sim} \bigoplus_{j \in \boldsymbol{Z}} H^0(Y, \mathcal{O}_Y(j))$$

where the graded ring structures are the naturally induced ones.

Proof of Step 2. By Step 1, we have an exact sequence

$$0 \to L^{\otimes(i-a)} \to L^{\otimes i} \to \mathcal{O}_Y(i) \to 0$$

for every integer i. Therefore, again by Step 1, we have an exact sequence for every integer i,

$(*)_i \qquad 0 \to H^0(X, L^{\otimes(i-a)}) \to H^0(X, L^{\otimes i}) \to H^0(Y, \mathcal{O}_Y(i)) \to 0.$

By $(*)_0$ and Step 1, we have $H^0(X, \mathcal{O}_X) = K$. In particular, we have the following exact sequence

$(*)_a \qquad 0 \to K \to H^0(X, L^{\otimes a}) \to H^0(Y, \mathcal{O}_Y(a)) \to 0.$

Let Φ be the image of 1 by the map $K \to H^0(X, L^{\otimes a})$. Then $(*)_i$ becomes

$$0 \to H^0(X, L^{\otimes(i-a)}) \xrightarrow{\times\Phi} H^0(X, L^{\otimes i}) \to H^0(Y, \mathcal{O}_Y(i)) \to 0.$$

This proves the assertion of Step 2.

Step 3. In view of Proposition 3.3, assume that we have a graded K-algebra isomorphism

$$K[X_0, \cdots, X_n]/(F_1, \cdots, F_c) \to \bigoplus_{j \in \mathbf{Z}} H^0(Y, \mathcal{O}_Y(i))$$

where $\deg X_i = e_i$, $\deg F_j = a_j$ $(0 \leq i \leq n, 1 \leq j \leq c)$ and $\dim Y = n - c$. We define K-algebra homomorphisms

$$\alpha: K[\bar{X}_0, \cdots, \bar{X}_n, Z] \to K[X_0, \cdots, X_n],$$
$$\beta: K[\bar{X}_0, \cdots, \bar{X}_n, Z] \to \bigoplus_{i \in \mathbf{Z}} H^0(X, L^{\otimes i}),$$

with $\deg \bar{X}_i = e_i$ $(0 \leq i \leq n)$, $\deg Z = a$, $\alpha \bar{X}_i = X_i$ $(0 \leq i \leq n)$, $\alpha Z = 0$, and $\beta Z = \Phi$, such that the following diagram is commutative:

$$\begin{array}{ccc} K[\bar{X}_0, \cdots, \bar{X}_n, Z] & \xrightarrow{\beta} & \bigoplus_{i \in \mathbf{Z}} H^0(X, L^{\otimes i}) \\ \alpha \downarrow & & \gamma \downarrow \text{nat.} \\ K[X_0, \cdots, X_n] & \underset{\text{nat.}}{\xrightarrow{\delta}} & \bigoplus_{i \in \mathbf{Z}} H^0(Y, \mathcal{O}_Y(i)). \end{array}$$

Then β is surjective. This follows immediately by applying Nakayama's Lemma to

$$\bigoplus_{i \in \mathbf{Z}} H^0(X, L^{\otimes i}) = \operatorname{Im} \beta + \Phi \bigoplus_{i \in \mathbf{Z}} H^0(Y, L^{\otimes i}).$$

Step 4. There exist homogeneous elements $\bar{F}_1, \cdots, \bar{F}_c$ of $\operatorname{Ker} \beta$ with $\deg \bar{F}_j = a_j$ and $\alpha \bar{F}_j = F_j$ $(1 \leq j \leq c)$, such that $\bar{F}_1, \cdots, \bar{F}_c$ generate $\operatorname{Ker} \beta$.

Proof of Step 4. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & K[\bar{X}_0, \cdots, \bar{X}_n, Z] & \xrightarrow{\times Z} & K[\bar{X}_0, \cdots, \bar{X}_n, Z] & \xrightarrow{\alpha} & K[X_0, \cdots X_n] & \to 0 \\ & \downarrow \beta & & \downarrow \beta & & \downarrow \delta & \\ 0 \to & \bigoplus_{i \in \mathbf{Z}} H^0(X, L^{\otimes i}) & \xrightarrow{\times \Phi} & \bigoplus_{i \in \mathbf{Z}} H^0(X, L^{\otimes i}) & \xrightarrow{r} & \bigoplus_{i \in \mathbf{Z}} H^0(Y, \mathcal{O}_Y(i)) & \to 0 \end{array}$$

where the rows are exact and the vertical arrows are surjective. Hence we have an exact sequence

$$0 \to \operatorname{Ker} \beta \xrightarrow{\times Z} \operatorname{Ker} \beta \xrightarrow{\alpha} \operatorname{Ker} \delta \to 0.$$

In view of this exact sequence, we can take elements $\bar{F}_1, \cdots, \bar{F}_c$ of $\operatorname{Ker} \beta$ such that $\bar{F}_j \in (\operatorname{Ker} \beta)_{a_j}$ and $\alpha \bar{F}_j = F_j$ $(1 \leq j \leq c)$. Then $\bar{F}_1, \cdots, \bar{F}_c$ generate $\operatorname{Ker} \beta$; this can be proved by applying Nakayama's Lemma to $K[\bar{X}_0, \cdots, \bar{X}_n, Z](\bar{F}_1, \cdots, \bar{F}_c) + Z \operatorname{Ker} \beta = \operatorname{Ker} \beta$. This proves Step 4.

Step 5. X *is a weighted complete intersection of* $P(e_0, \cdots, e_n, a)$ *of type* $(a_1, \cdots, a_c)$.

Proof of Step 5. Since L is an ample invertible sheaf on X, there exists a positive integer b such that $H^0(X, L^{\otimes(bm+1)})$ generates $\bigoplus_{i \in \mathbf{Z}} H^0(X, L^{\otimes(bm+1)i})$ (cf. Theorem 3 in page 45 of [11]), where $m = \text{l.c.m.}\{e_0, \cdots, e_n, a\}$. By considering the homogeneous part of degree $m(bm+1)$, we see that

$$(K[\bar{X}_0, \cdots, \bar{X}_n, Z]_m)^{bm+1} \subset (K[\bar{X}_0, \cdots, \bar{X}_n, Z]_{bm+1})^m + (\text{the ideal generated by } \bar{F}_1, \cdots, \bar{F}_c).$$

Consequently $V_+(\bar{F}_1, \cdots, \bar{F}_c) \cap V_+(K[\bar{X}_0, \cdots, \bar{X}_n, Z]_{bm+1}) = \phi$. In view of Lemma 1.4 and the assumption $\dim X = n+1-c$, this proves Step 5.

Step 6. $L \simeq \mathcal{O}_X(1)$.

This is obvious, because it is proved in Step 1 that the natural map $\operatorname{Pic} X \to \operatorname{Pic} Y$ is an injection. This completes the proof of Theorem 3.6.

In some cases, the assumptions in Theorem 3.6 can be simplified. In this simplification the following theorem plays an essential role.

Theorem 3.7. *Let* X *be a weighted complete intersection of* *dimension* ≥ 3. *Then* Pic $X \simeq \boldsymbol{Z}$, *and* $\mathcal{O}_X(1)$ *generates* Pic X.

Proof. Assume, with the notation of Definition 3.1, $X=\text{Proj}(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$. Let $K[Y_0, \cdots Y_n]$ be a graded polynomial ring with $\deg Y_i = 1$ $(0 \leq i \leq n)$. We define a graded K-algebra homomorphism $\Phi: K[X_0, \cdots, X_n] \to K[Y_0, \cdots, Y_n]$ by $\Phi X_i = Y_i^{e_i}$, and put $G_j = \Phi F_j$ $(1 \leq j \leq c)$. Then Φ induces a graded ring homomorphism

$$K[X_0, \cdots, X_n]/(F_1, \cdots, F_c) \to K[Y_0, \cdots, Y_n]/(G_1, \cdots, G_c).$$

Thus we obtain a morphism of K-schemes $\varphi: \tilde{X} \to X$ with $\varphi^* \mathcal{O}_X(1) = \mathcal{O}_{\tilde{X}}(1)$ (define $\tilde{X} = \text{Prroj}(K[Y_0, \cdots, Y_n]/(G_1, \cdots, G_c))$). On the other hand, it is easy to see that $K[Y_0, \cdots, Y_n]/(G_1, \cdots, G_c)$ has the following decomposition as a $K[X_0, \cdots, X_n]/(F_1, \cdots, F_c)$-module

$$K[Y_0, \cdots, Y_n]/(G_1, \cdots, G_c) \simeq \bigoplus_{\substack{0 \leq v_i < e_i \\ 0 \leq i \leq n \\ v_i,\, i:\ \text{integers}}} (K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))\, Y_0^{v_0} \cdots Y_n^{v_n}.$$

Consequently we have a similar decomposition of $\varphi_* \mathcal{O}_{\tilde{X}}$ as an $\mathcal{O}_X$-module

$$\varphi_* \mathcal{O}_{\tilde{X}} \simeq \bigoplus_{v} \mathcal{O}_X\Big(-\sum_{i=0}^{n} v_i\Big). \tag{1}$$

Now (1) implies that $\varphi^*: \text{Pic } X \to \text{Pic } \tilde{X}$ is an injection. In fact, if L is an invertible sheaf on X with $\varphi^* L \simeq \mathcal{O}_X$, we have

$$L \otimes \varphi_* \mathcal{O}_{\tilde{X}} \simeq \varphi_*(\varphi^* L) \simeq \varphi_* \mathcal{O}_{\tilde{X}}$$

by the projection formula. Then (1) implies

$$\bigoplus L\Big(-\sum_{i=0}^{n} v_i\Big) \simeq \bigoplus \mathcal{O}_X\Big(-\sum_{i=0}^{n} v_i\Big)$$

whereas such a decomposition is unique up to a permutation of the direct summands, by the Krull-Schmidt theorem stated in [1].[6] Hence we have $L \simeq \mathcal{O}_X(b)$ for some integer b. Here assume $b \neq 0$. Then L or $L^\vee$ is ample on X, consequently $\mathcal{O}_{\tilde{X}} = \varphi^* L = \varphi^* L^\vee$ is ample on $\tilde{X}$ (note that φ is a finite morphism). This is a contradiction, because $\tilde{X}$ is a proper K-scheme of dimension >0. Hence $L \simeq \mathcal{O}_X$,

[6] Note that the theorem stated in [1] is also applicable to an algebraic K-scheme X proper over K with $H^0(X, \mathcal{O}_X) = K$.

namely φ^* is an injection. On the other hand, by Corollary 3.7 in [8, Exposé XII], Pic $\tilde{X}$ is isomorphic to $\boldsymbol{Z}$ and is generated by $\mathcal{O}_{\tilde{X}}(1)$. This implies that $\varphi^*: \text{Pic } X \to \text{Pic } \tilde{X}$ is an isomorphism, because $\varphi^* \mathcal{O}_X(1) = \mathcal{O}_{\tilde{X}}(1)$. q.e.d.

Hence as a special case of Theorem 3.6, we have:

Corollary 3.8. *Let X be a locally factorial projective K-scheme with an ample effective divisor Y. Assume Y is a weighted complete intersection of type $(a_1, \cdots, a_c)$ of $\boldsymbol{P}(e_0, \cdots, e_n)$. If* $\dim Y \geq 3$, *all of the assumptions (1), (2) and (3) stated in Theorem 3.6 are satisfied. Hence X is a weighted complete intersection of type $(a_1, \cdots, a_c)$ of $\boldsymbol{P}(e_0, \cdots, e_n, a)$.*

Proof. This is an immediate consequece of Theorem 3.7 in this paper and Corollary 3.6 in [8, Exposé XII].

With the notation of Corollary 3.8, if we assume $\dim Y = 2$ instead of $\dim Y \geq 3$, then X need not be a weighted complete intersection.

Example 3.9. Let Y' be a smooth quadric surface in $\boldsymbol{P}_3$ such that $Y' \supset$ a line l (such a line exists if K is algebraically closed). We denote by $\pi: X \to \boldsymbol{P}_3$ the blowing-up of $\boldsymbol{P}_3$ along l, and by Y the proper transform of Y' with π. Then the variety X and its subvariety Y enjoy the following three properties;

(1) X is a smooth projective variety,

(2) Y is isomorphic to a smooth quadric surface in $\boldsymbol{P}_3$,

(3) Y is an ample divisor of X, whereas X is not a weighted complete intersection.

Proof. Verification of (1), (2) and the first part of (3) is immedite. As for the last assertion, it suffices to prove that $\text{Pic } X \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$ (cf. Theorem 3.5). On the other hand, this is an immediate consequence of the fact that X is a blowing-up of $\boldsymbol{P}_3$ along l (cf. [2]). q.e.d.

In view of the above example, the assumptions in the following proposition are reasonable (at least in the case char $K=0$).

Proposition 3.10. *Let X be a locally factorial projective K-scheme with an ample effective divisor Y. Assume* $\dim Y=2$ *and Y is a weighted complete intersection of type $(a_1, \cdots, a_c)$ of $\boldsymbol{P}(e_0, \cdots, e_n)$. If one of the following two conditions holds, then all of the assumptions (1), (2) and (3) in Theorem 3.6 are satisfied.*

1) $\operatorname{Pic} Y \simeq \boldsymbol{Z}$ *and* $\omega_Y \simeq \mathcal{O}_Y(b)$ *with b prime to* char K.

2) $\mathcal{O}_X(Y) \otimes \mathcal{O}_Y \simeq \mathcal{O}_Y(b')$ *with b' prime to* char K.

Hence if one of these two assumptions holds, then X is a weighted complete intersection.

Proof. This proposition follows from the fact that the cokernel of the natural map $\operatorname{Pic} X \to \operatorname{Pic} Y$ has no torsion prime to char K. This is proved in [12]. q.e.d.

For example, by Propositions 3.8 and 3.10, we have:

Corollary 3.11.[7] *Let X be a smooth projective K-variety of dimension n containing a closed K-subscheme $Y \simeq P_{n-1,K}$ as an ample divisor. Assume furthermore $n \geq 4$, or $n=3$ and* char $K \neq 3$. *Then $X \simeq \boldsymbol{P}_{n,K}$ and Y is contained in X as a hyperplane.*

§4. Related results.

In this section, we consider algebraic small deformations of weighted complete intersections.

Proposition 4.1. *Let $\pi: X \to S$ be a proper and flat morphism with $S=\operatorname{Spec} A$, where A is a local ring with residue field K and maximal ideal $\mathfrak{M}$. Assume the following two conditions:*

(1) *$X_K = X \otimes_A K$ is a weighted complete intersection of dimension ≥ 2 such that $X_K \simeq \operatorname{Proj}(K[X_0, \cdots, X_n]/(F_1, \cdots F_c))$, with the notation of Definition 3.1.*

(2) *There exists an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L} \otimes \mathcal{O}_{X_K}$*

[7] Professor H. Tango proved this result without any restriction except for $n \geq 3$ with a geometric argument. The case $n=2$ is treated in [4].

$\simeq\mathcal{O}_{X_K}(1)$. *Then there exist homogeneous elements* G_i *with* $\deg G_i=a_i$ $(1\leq i\leq c)$ *of* $A[X_0,\cdots,X_n]$[8] *enjoying the following three properties:*

i) $G_i \bmod \mathfrak{M}A[X_0,\cdots X_n]_{a_i}=F_i$ $(1\leq i\leq c)$.

ii) $X\simeq\mathrm{Proj}\,(A[X_0,\cdots,X_n]/(G_1,\cdots,G_c))$ *and the imbedding* $X_K\to X$ *is induced by the natural graded A-algebra homorphism*

$$A[X_0,\cdots,X_n]/(G_1,\cdots,G_c)\to K[X_0,\cdots X_n]/(F_1,\cdots,F_c).$$

iii) $\mathcal{L}\simeq\mathcal{O}_{\boldsymbol{P}(e)\times\mathrm{Spec}\,A}(1)\,|_X$.

The proof is similar to those of Theorem 3.6 in this paper and Lemma 1.5 in [9], therefore we omit it.

Similarly to Theorem 3.6, if $\dim X_K\geq 3$, the assumption 2) in Proposition 4.1 can be simplified a little.

Remark 4.2. Let $\pi: X\to S$ be a proper and flat morphism with $S=\mathrm{Spec}\,A$, where A is a local ring with residue field K and maximal ideal $\mathfrak{M}$. In addition to the assumption (1) of Proposition 4.1, assume $H^2(X_K,\mathcal{O}_{X_K})=0$. Furthermore assume that one of the following three conditions holds:

(a) A is a complete local ring.

(b) π is smooth or $\dim X_K\geq 3$, and π has section, i.e. there exists a morphism $\sigma: S\to X$ such that $\pi\circ\sigma=id_S$.

(c) π is smooth or $\dim X_K\geq 3$, and $\underset{i\in\boldsymbol{Z}}{\mathrm{g.c.d.}}\, i\, h^0(\mathcal{O}_{X_K}(i))=1$.[9]

Then the assumption (2) of Proposition 4.1 is necessarily satisfied, consequently the result of Proposition 4.1 holds.

Proof. Case 1. Assume that (a) holds. Then the existence of $\mathcal{L}$ in Proposition 4.1, (2) is an immediate consequence of Corollary 2.2 in [8, Exposé XI] and Corollary 5.10 in [8, Exposé XII].

Case 2. Assume that π is smooth or $\dim X_K\geq 3$. In this case, the method of our proof is to give, under some additional assumptions,

8) With the notation of Definition 3.1, the gradation of $A[X_0,\cdots,X_n]$ is defined by $\deg r=0$ $(r\in A)$, $\deg X_i=e_i$ $(0\leq i\leq n)$.

9) It is easy, in some cases, to check this condition. Indeed if X_K is a complete intersection of type $(a_1,\cdots,a_c)$ of P_n with $1<a_1<\cdots<a_c$, then $\underset{i\in\boldsymbol{Z}}{\mathrm{g.c.d.}}\, i\, h^0(X_K,\mathcal{O}_{X_K}(i))$ $=\mathrm{g.c.d.}\{n+1,a_1,\cdots,a_c\}$.

descent data of $\hat{\mathcal{L}}$ the invertible sheaf on $X\otimes_A\hat{A}$ which is given by the result of Case 1 ($\hat{A}$ denotes the completion of A). Let us assume that B is a commutative ring $\ni 1$ and

$$A \xrightarrow{{}^t p} \tilde{A} \underset{{}^t\beta}{\overset{{}^t\alpha}{\rightrightarrows}} B$$

are ring homomorphisms such that ${}^t\alpha\circ{}^t p={}^t\beta\circ{}^t p$. Then we consider the following commutative diagram of naturally induced morphisms:

$$\begin{array}{ccccc} X & \xleftarrow{\bar{p}} & X_{\hat{A}} & \underset{\bar{\beta}}{\overset{\bar{\alpha}}{\leftleftarrows}} & X_B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\hat{A}}} & & \downarrow{\scriptstyle \pi_B} \\ \operatorname{Spec} A & \xleftarrow{p} & \operatorname{Spec}\hat{A} & \underset{\beta}{\overset{\alpha}{\leftleftarrows}} & \operatorname{Spec} B \end{array}$$

where $X_{\hat{A}}$ (resp. X_B) denotes $X\otimes_A\hat{A}$ (resp. $X\otimes_A B$). With the notation of the above diagram, we put $\mathcal{M}=\bar{\beta}^*\hat{\mathcal{L}}\otimes(\bar{\alpha}^*\hat{\mathcal{L}})^{\otimes(-1)}$, where $\hat{\mathcal{L}}$ is the invertible sheaf on $X_{\hat{A}}$ obtained by the result of Case 1.

Then we claim: $\mathcal{N}=\pi_{B*}\mathcal{M}$ is an invertible sheaf on $\operatorname{Spec} B$, and the natural map $\pi_B{}^*\mathcal{N}\to\mathcal{M}$ is an isomorphism.

Proof of the claim: Let A' and B' denote noetherian subrings of $\hat{A}$ and B with the following three properties; A' is a local A-algebra dominated by $\hat{A}$, we have the following commutative diagram of naturally induced morphisms

$$\begin{array}{ccccc} \operatorname{Spec} A & \xleftarrow{p} & \operatorname{Spec}\hat{A} & \underset{\beta}{\overset{\alpha}{\leftleftarrows}} & \operatorname{Spec} B \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \operatorname{Spec} A & \xleftarrow{p'} & \operatorname{Spec} A' & \underset{\beta'}{\overset{\alpha'}{\leftleftarrows}} & \operatorname{Spec} B' \end{array}$$

and furthermore there exists an invertible sheaf $\mathcal{L}'$ on $X_{A'}=X\otimes_A A'$ such that $\bar{f}^*\mathcal{L}'=\hat{\mathcal{L}}$ with $\bar{f}=f\otimes_{\operatorname{Spec} A'}X_{A'}: X_{\hat{A}}\to X_{A'}$. If we put $\mathcal{M}'=\bar{\beta}'^*\mathcal{L}'\otimes(\bar{\alpha}'^*\mathcal{L}')^{\otimes(-1)}$ (the definitions of $\bar{\alpha}'$ and $\bar{\beta}'$ are similar to those of $\bar{\alpha}$ and $\bar{\beta}$), we have:

$$(*)\qquad \mathcal{M}'|_{X_{k(x)}}\simeq\mathcal{O}_{X_{k(x)}},\ \text{with } X_{k(x)}=X\otimes_A k(x)\quad (x\in\operatorname{Spec} B').$$

Proof of (*). First note that Proposition 4.1 is applicable to the A'-scheme $X_{A'}$. Consequently $X_{k(x)}$ is a weighted complete intersection of type $(a_1,\cdots,a_c)$ of $P(e_0,\cdots,e_n)$. So if $\dim X_K=2$ and π is smooth, there exists a non-zero integer a such that

$$(\bar{\alpha}'^*\mathcal{L}'|_{X_{k(x)}})^{\otimes a}\simeq(\bar{\beta}'^*\mathcal{L}'|_{X_{k(x)}})^{\otimes a}\simeq K_{X_{k(x)}}$$

(a is non-zero by the assumption $H^2(X_K, \mathcal{O}_{X_K})=H^0(X_K, K_{X_K})=0$). This proves (*) because Pic $X_{k(x)}$ is torsion free.[10] If dim $X_K\geq 3$, both $\bar{\alpha}'^*\mathcal{L}'|_{X_{k(x)}}$ and $\bar{\beta}'^*\mathcal{L}'|_{X_{k(x)}}$ are ample generators of Pic $X_{k(x)}\simeq \boldsymbol{Z}$ (cf. Theorem 3.7). This proves (*), hence the proof of (*) is completed.

Proof of the claim (continued). Now we have $H^1(X_{k(x)}, \mathcal{O}_{X_{k(x)}})=0$ ($x\in \mathrm{Spec}\, B'$) from the proof of (*) (cf. Proposition 3.3). Therefore by applying the results of [6, § 7], we see that $\pi_{B'*}\mathcal{M}'$ is a locally free sheaf and $\pi_{B'*}$ commutes with base change.[11] Noting that $(\pi_{B'*}\mathcal{M}')\otimes k(x)\tilde{\rightarrow} H^0(X_{k(x)}, \mathcal{M}'|_{X_{k(x)}})\simeq k(x)$ for every point x of Spec B', we obtain the following two results;

(**) $\qquad \pi_{B'*}\mathcal{M}'$ is an invertible sheaf,

(***) $\qquad \pi_{B*}\mathcal{M}=g^*(\pi_{B'*}\mathcal{M}')$.

Combining (**) and (***), we have first part of the claim. The combination of (*) and (**) implies that the natural homomorphism $\pi_{B'}{}^*\pi_{B'*}\mathcal{M}'\rightarrow\mathcal{M}'$ is an isomorphism, then (***) implies the second part of the claim. This completes the proof of the claim.

Case 2.1. Assume that (b) holds. Bp the equality

$$\begin{aligned}\mathcal{N}\simeq(\pi_B\circ\sigma_B)^*\mathcal{N}\tilde{\rightarrow}\sigma_B{}^*\mathcal{M}&=(\bar{\beta}\circ\sigma_B)^*\hat{\mathcal{L}}\otimes\{(\bar{\alpha}\circ\sigma_B)^*\hat{\mathcal{L}}\}^{\otimes(-1)}\\&=(\sigma_{\hat{A}}\circ\beta)^*\hat{\mathcal{L}}\otimes\{(\sigma_{\hat{A}}\circ\alpha)^*\hat{\mathcal{L}}\}^{\otimes(-1)}\\&=\beta^*(\sigma_{\hat{A}}{}^*\hat{\mathcal{L}})\otimes\alpha^*(\sigma_{\hat{A}}{}^*\hat{\mathcal{L}})^{\otimes(-1)},\end{aligned}$$

we have a natural isomorphism of B-modules;

$$\Phi_{\beta,\alpha}: \mathrm{Hom}_{X_B}(\bar{\alpha}^*\hat{\mathcal{L}}, \bar{\beta}^*\hat{\mathcal{L}})\rightarrow \mathrm{Hom}_B(\alpha^*\hat{\mathcal{D}}, \beta^*\hat{\mathcal{D}})$$

where $\hat{\mathcal{D}}$ denotes the invertible sheaf $\sigma_{\hat{A}}{}^*\hat{\mathcal{L}}$ on Spec $\hat{A}$.

To be precise, $\Phi_{\beta,\alpha}$'s have the following properties:

(i) If $\alpha=\beta$, $\Phi_{\alpha,\alpha}(id_{\bar{\alpha}^*\hat{\mathcal{L}}})=id_{\alpha^*\hat{\mathcal{D}}}$.

(ii) For three morphisms α, β, γ: Spec $B\rightarrow$ Spec $\hat{A}$ with $p\circ\alpha=p\circ\beta$

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[10] First we have an injection Pic $X_{k(x)}\hookrightarrow$ Pic $X_{\overline{k(x)}}$, since $X_{k(x)}$ is projective. Next by the assumptions $H^1(X_{k(x)}, \mathcal{O}_{X_{k(x)}})=0$ and that $K_{X_{k(x)}}^{\vee}$ is ample, $X_{\overline{k(x)}}$ is a rational surface by the theorem of Castelnuovo-Zariski. This proves that Pic $X_{k(x)}$ is torsion free.

[11] This means that, for every morphism q: Spec $C\rightarrow$ Spec B, the natural map $q^*\pi_{B'*}\mathcal{M}'\rightarrow\pi_{C*}\bar{q}^*\mathcal{M}'$ is an isomorphism, where $\bar{q}=q\times_{\mathrm{Spec}\,B'}X_{B'}$, $\pi_C=\pi\times_{\mathrm{Spec}\,A}\mathrm{Spec}\, C$.

$=p\circ\gamma$, the following diagram commutes;

$$\begin{array}{ccccc}
\mathrm{Hom}_{X_B}(\bar{\alpha}^*\hat{\mathcal{L}}, \bar{\beta}^*\hat{\mathcal{L}}) & \times & \mathrm{Hom}_{X_B}(\bar{\beta}^*\hat{\mathcal{L}}, \bar{\gamma}^*\hat{\mathcal{L}}) & \longrightarrow & \mathrm{Hom}_{X_B}(\bar{\alpha}^*\hat{\mathcal{L}}, \bar{\gamma}^*\hat{\mathcal{L}}) \\
\downarrow{\scriptstyle \Phi_{\beta,\alpha}} & & \downarrow{\scriptstyle \Phi_{\gamma,\beta}} & & \downarrow{\scriptstyle \Phi_{\gamma,\alpha}} \\
\mathrm{Hom}_B(\alpha^*\hat{\mathscr{D}}, \beta^*\hat{\mathscr{D}}) & \times & \mathrm{Hom}_B(\beta^*\hat{\mathscr{D}}, \gamma^*\hat{\mathscr{D}}) & \longrightarrow & \mathrm{Hom}_B(\alpha^*\hat{\mathscr{D}}, \gamma^*\hat{\mathscr{D}}).
\end{array}$$

(iii) For a scheme $\mathrm{Spec}\, C$ and a morphism $q: \mathrm{Spec}\, C \to \mathrm{Spec}\, B$, we have

$$q^*\Phi_{\beta,\alpha} = \Phi_{\beta\circ q, \alpha\circ q}.$$

Due to these properties (i), (ii) and (iii), giving descent data of $\hat{\mathcal{L}}$ with respect to $\bar{p}$ is equivalent to giving those of $\sigma_{\hat{A}}^*\hat{\mathcal{L}}$ with respect to p. Now since $\sigma_{\hat{A}}^*\hat{\mathcal{L}}$ is isomorphic to $\hat{A}$ as an $\hat{A}$-module, it is immediate by the theory of flat descent in [7, Exposé VIII] to check the existence of an invertible sheaf $\mathcal{L}$ on X such that $\bar{p}^*\mathcal{L} \simeq \hat{\mathcal{L}}$.

Case 2.2. Assume that (c) holds. Take integers a_i's ($i \in \boldsymbol{Z}$) almost all of which are zero, satisfying the following equality

$$\sum_{i\in\boldsymbol{Z}} a_i \cdot i \cdot h^0(X_K, \mathcal{O}_{X_K}(i)) = 1.$$

By the result of Case 2, we have a natural $\mathcal{O}_{X_B}$-isomorphism

$$\bar{\beta}^*\hat{\mathcal{L}} \cong (\pi_B{}^*\mathcal{N}) \otimes \bar{\alpha}^*\hat{\mathcal{L}}.$$

Note that since $H^1(X_{(\hat{A}/\mathfrak{M}\hat{A})}, \hat{\mathcal{L}}^{\otimes i}|_{X_{(\hat{A}/\mathfrak{M}\hat{A})}}) = H^1(X_K, \mathcal{O}_{X_K}(i)) = 0$, again by the results of [6, § 7], we have the following results (1) and (2) for every integer i,

(1) $\pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i}$ is a free $\hat{A}$-module of rank $h^0(X_K, \mathcal{O}_{X_K}(i))$,

(2) we have natural B-module isomorphisms,

$$\alpha^*(\pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i}) \xrightarrow{\sim} \pi_{B*}\bar{\alpha}^*\hat{\mathcal{L}}^{\otimes i} \quad \text{and} \quad \beta^*(\pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i}) \xrightarrow{\sim} \pi_{B*}\bar{\beta}^*\hat{\mathcal{L}}^{\otimes i}$$

(we identify the terms on both sides by these isomorphisms).

For every integer i, we have an equality of B-modules;

$$\begin{aligned} \beta^*(\pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i} &= \pi_{B*}\bar{\beta}^*\hat{\mathcal{L}}^{\otimes i} = \mathcal{N}^{\otimes i} \otimes \pi_{B*}\bar{\alpha}^*\hat{\mathcal{L}}^{\otimes i} \\ &= \mathcal{N}^{\otimes i} \otimes \alpha^*(\pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i}). \end{aligned}$$

By taking the determinant of this equality, we have

$$\beta^*(\det \pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i}) = \mathcal{N}^{\otimes b(i)} \otimes \alpha^*(\det \pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i})$$

where $b(i)$ denotes $i\cdot h^0(X_K, \mathcal{O}_{X_K}(i))$ $(i\in \boldsymbol{Z})$. Since $\sum_i a_i\cdot b(i)=1$, we have an equality of B-modules

$$\beta^*\hat{\mathcal{E}}=\mathcal{N}\otimes\alpha^*\hat{\mathcal{E}} \quad \text{with} \quad \hat{\mathcal{E}}=\bigotimes_{i\in \boldsymbol{Z}}(\det \pi_{\hat{A}*}\hat{\mathcal{L}}^{\otimes i})^{\otimes a_i}.$$

Since $\hat{\mathcal{E}}$ is an invertible sheaf on Spec $\hat{A}$, we have a natural isomorphism of B-modules;

$$\Psi_{\beta,\alpha}\colon \mathrm{Hom}_{X_B}(\bar{\alpha}^*\hat{\mathcal{L}}, \bar{\beta}\hat{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}_B(\alpha^*\hat{\mathcal{E}}, \beta^*\hat{\mathcal{E}}).$$

It is easily checked that these $\Psi_{\beta,\alpha}$'s enjoy the three properties similar to those of $\Phi_{\beta,\alpha}$'s stated in Case 2. 1. Hence the existence of $\mathcal{L}$ stated in the assumption (2) of Proposition 4. 1 is proved. q.e.d.

As a simple example of deformation of weighted complete intersections, we have the following.

Example 4. 3. Let A be a discrete valuation ring with uniformizant π and residue field K. Take positive integers a, b, n with $a>1, b>1$, and $n>1$. Take two homogeneous elements F_1 and F_2 of $A[X_0, \cdots, X_n]$ with $\deg F_1=a$, $\deg F_2=ab$.[12] Assume that Proj $(K[X_0, \cdots, X_n]/(\bar{F}_1, \bar{F}_2))$ is a smooth variety over K, where $\bar{F}_i$ denotes the image of F_i by the natural map $A[X_0, \cdots, X_n]\to K[X_0, \cdots, X_n]$. Then $X=\mathrm{Proj}\,(A[X_0, \cdots, X_n, Y]/(\pi Y-F_1, Y^b+F_2))$[13] enjoys following properties:

(1) The natural map $\pi\colon X\to \mathrm{Spec}\, A$ is smooth and projective.

(2) If L denotes the quotient field of A, then $X_L=X\otimes_A L$ is a smooth hypersurface of degree ab of $\boldsymbol{P}_n$ defined over L.

(3) $X_K=X\otimes_A K$ is not a hypersurface of degree ab of $\boldsymbol{P}_n$ but a weighted complete intersection of type (a, ab) of $\boldsymbol{P}(\underbrace{1, \cdots, 1}_{n+1}, a)$.

To understand the meaning of this example, we consider smooth specializations of weighted complete intersections.

Definition 4. 1. We consider a proper and smooth morphism of schemes $\pi\colon X\to S$ with the following two properties:

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[12] The gradation of $A[X_0, \cdots, X_n]$ is defined by $\deg r=(r\in A)$ and $\deg X_i=1\,(0\leq i\leq n)$.
[13] The gradation of $A[X_0, \cdots, X_n, Y]$ is defined by that of $A[X_0, \cdots, X_n]$ and $\deg Y=a$.

(1) $S=\operatorname{Spec} A$, where A is a local domain with residue field K and quotient field L.

(2) The L-scheme $X_L=X\otimes_A L$ is a weighted complete intersection of type $(a_1,\cdots,a_c)$ of $\boldsymbol{P}(e)$.

Then $X_K=X\otimes_A K$ is, by definition, a smooth K-variety. Now a smooth K-variety Y is called a smooth specialization of weighted complete intersections of type $(a_1,\cdots,a_c)$ of $\boldsymbol{P}(e)$ if Y is isomorphic to such X_K.

Now assume $\operatorname{char} K=0$ and $K=\bar{K}$. Then it is proved that smooth specializations of smooth hypersurfaces of degree ≤ 3 and dimension ≥ 3 are smooth hypersurfaces of the same degree (cf. [3] and [10]). Example 4.3 shows that, if d is a composite number, the family of smooth hypersurfaces of degree d is not closed under smooth specialization. On the other hand, it is proved that the family of weighted complete intersections of type $(2,4)$ of $\boldsymbol{P}(\underbrace{1,\cdots,1}_{n+2},2)$ with $n\geq 3$ (note that every smooth hypersurface of degree 4 of $\boldsymbol{P}_{n+1}$ is isomorphic to some member of this family) is closed under smooth specialization (cf. [3]). The author knows no further results about smooth hypersurfaces of degree ≥ 5 and dimension ≥ 3.

Appendix. On some Hilbert functions.

The purpose of this appendix is to study some Hilbert functions and determine the leading coefficients of them which are used in § 3.

Throughout this appendix, K denotes a field.

Definition A.1. Assume that n is a non-negative integer and $e_0,\cdots,e_n$ are positive integers. Then as in Definition 1.1, $m=\text{l.c.m.}\{e_i|0\leq i\leq n\}$ and $r(e)=\min_{p:\text{prime}}\sharp\{i|0\leq i\leq n,\ p\nmid e_i\}$. Let $K[X_0,\cdots,X_n]$ be the graded ring defined by $\deg X_i=e_i$ $(0\leq i\leq n)$ and $\deg r=0$ $(r\in K)$. Then we define:

$$H(e_0,\cdots,e_n;u)=\operatorname{length}_K K[X_0,\cdots,X_n]_u\quad(u\in\boldsymbol{Z}).$$

For an integer b and a function $f:\boldsymbol{Z}\to\boldsymbol{Z}$, $\Delta_b f$ is the function of $\boldsymbol{Z}$ to $\boldsymbol{Z}$ defined as follow:

$$(\Delta_b f)(x)=f(x)-f(x-b)\quad(x\in\boldsymbol{Z}).$$

The main result of this appendix is the assertion (2) of the following theorem, which shows that $r(e)$ tells us how far the function $H(e; u)$ is from being a polynomial.

Theorem A. 1. (1) *With the notation of Definition A. 1, take an arbitrary integer i. For every integer u with* $u \equiv i$ *(modulo m), we consider the function* $H(e_0, \cdots, e_n; u)$. *Then, for sufficiently large u,* $H(e_0, \cdots, e_n; u)$ *is a polynomial in u of degree* $\leq n$ *(degree n if* $i=0$*). We denote this polynomial by* $H_i(e_0, \cdots, e_n; u)$.

(2) *With the notation of* (1), *we have*

$$\max_{0 \leq i, j \leq m-1} \deg(H_i(e_0, \cdots e_n; u) - H_j(e_0, \cdots, e_n; u)) = n - r(e),$$

where the degree of the polynomial 0 *is defined to be* -1.

Proof. (1) If $n=0$, it is immediate to check the assertion. Therefore assume that $n>0$. Then applying Remark 2. 2 to the $\mathcal{O}_{Q(e)}$-coherent sheaf $\mathcal{O}_Q(u)$ (see, for the notation, Definition 1. 1), we obtain;

$$H(e; u) = h^0(\boldsymbol{Q}(e), \mathcal{O}_Q(u)) \quad (u \in \boldsymbol{Z}).$$

Put $u = mv + i$ $(v \in \boldsymbol{Z})$, then $\mathcal{O}_Q(u) = \mathcal{O}_Q(i) \otimes \mathcal{O}_Q(m)^{\otimes v}$ and $\mathcal{O}_Q(m)$ is an ample invertible sheaf on the projective variety $\boldsymbol{Q}(e)$ (see Lemma 1. 3). Hence we have the following equality for sufficiently large v;

$$h^0(\boldsymbol{Q}(e), \mathcal{O}_Q(mv+i)) = \chi(\boldsymbol{Q}(e), \mathcal{O}_Q(i) \otimes \mathcal{O}_Q(m)^{\otimes v}).$$

On the other hand, by [6], we know that $\chi(\boldsymbol{Q}(e), \mathcal{O}_Q(i) \otimes \mathcal{O}_Q(m)^{\otimes v})$ is a polynomial in v of degree$=$dim Supp $\mathcal{O}_Q(i)$. This proves (1).

(2) For simplicity, we put $t(e) = \max_{i,j} \deg(H_i(e; u) - H_j(e; u))$. It is immediate to check that $t(e) = n - r(e)$ under the condition "$r(e) = 0$ or $n=0$". First we prove the inequality $t(e) \leq n - r(e)$. By Proposition 2. 1, (3), there exist a positive integer a and elements $F_0, \cdots, F_{n-r(e)}$ of $K[X_0, \cdots, X_n]_{am}$ satisfying the following two conditions;

(i) $(F_0, \cdots, F_{n-r(e)})$ is a regular sequence of $K[X_0, \cdots, X_n]$,

(ii) $X = \mathrm{Proj}(K[X_0, \cdots, X_n]/(F_0, \cdots, F_{n-r(e)}))$ is contained in $\boldsymbol{P}(e)$.

Then we have, by Proposition 3. 3,

(iii) $$\text{length}_K(K[X_0,\cdots,X_n]/(F_0,\cdots,F_{n-r(e)}))_u = h^0(X,\boldsymbol{Q}_X(u)) \quad (u\in\boldsymbol{Z}),$$

and, by the property (i),

(iv) $$\text{length}_K(K[X_0,\cdots,X_n]/(F_0,\cdots,F_{n-r(e)}))_u = \Delta_{am}^{n-r(e)+1}H(e;u) \quad (u\in\boldsymbol{Z}).$$

Since $\mathcal{O}_X(1)$ is an ample invertible sheaf on X, we obtain the following equality by (iii) and (iv):

(v) $$\Delta_{am}^{n-r(e)+1}H(e;u)=\chi(X,\mathcal{O}_X(u)) \quad (u\in Z, u\gg 0).$$

By assuming $u\equiv i$ (modulo m) in the equality (v), we have an equality of polynomials in u;

$$\Delta_{am}^{n-r(e)+1}H_i(e;u)=\chi(X,\mathcal{O}_X(u)).$$

This equality shows that for arbitrary integers i and j,

$$\Delta_{am}^{n-r(e)+1}(H_i(e;u)-H_j(e;u))=0.$$

Hence we have $\deg(H_i(e;u)-H_j(e;u))\leq n-r(e)$, from which the inequality $t(e)\leq n-r(e)$ follows.

Next we prove the reverse inequality $t(e)\geq n-r(e)$, by induction on n. As is previously seen, we may assume $n>0$ and $r(e)>0$. Then there exists an integer b with $0\leq b\leq n$ such that $r(e_0,\cdots,e_n)-1 =r(e_0,\cdots,\check{e}_b,\cdots,e_n)$ (we may assume $b=n$ by symmetry). On the other hand, we have an equality of polynomials in u;

(vi) $$H_i(e_0,\cdots,e_n;u)-H_{i-e_n}(e_0,\cdots,e_n;u-e_n)=H_i(e_0,\cdots,e_{n-1};u)$$

(note that we have the exact sequence

$$0\to K[X_0,\cdots,X_n]_{u-e_n}\xrightarrow{\times X_n}K[X_0,\cdots,X_n]_u\to(K[X_0,\cdots,X_n]/(X_n))_u\to 0$$
$$(K[X_0,\cdots,X_n]/(X_n))_u \cong K[X_0,\cdots,X_{n-1}]_u$$

for every integer u with $u\equiv i$ (modulo m)). Now by the equality (vi), we have

(vii) $$\begin{aligned}H_i(e_0,\cdots,e_{n-1};u)&-H_j(e_0,\cdots,e_{n-1};u)\\ &=\{H_i(e_0,\cdots,e_n;u)-H_j(e_0,\cdots,e_n;u)\}\\ &\quad-\{H_{i-e_n}(e_0,\cdots,e_n;u-e_n)-H_{j-e_n}(e_0,\cdots,e_n;u-e_n)\}.\end{aligned}$$

Here the induction assumption implies the existence of a pair of integers (i,j) such that the term on the left side of (vii) is a polynomial in u of degree $n-1-r(e_0,\cdots,e_{n-1})=n-r(e_0,\cdots,e_n)$. Therefore, for such (i,j), one of the { }'s on the right side of (vii) is a polynomial of degree $n-r(e_0,\cdots,e_n)$. This establishes the required inequality, hence the proof of Theorem A. 1 is complete.

Due to Theorem A. 1, we can speak of the coefficients of u^j, with $j>n-r(e)$, in $H(e;u)$. Here we determine the coefficient of u^n in the case $r(e)\geq 1$.

Corollary A. 2. *If we define rational numbers $c_0,\cdots,c_n$ depending on $e_0,\cdots,e_n$ and an integer i by*

$$H_i(e;u)=\sum_{j=0}^{n}c_j\frac{u^{n-j}}{(n-j)!},$$

then c_j is independent of the choice of i if $j<r(e)$. For instance, we have $c_0=1/\prod_{i=0}^{n}e_i$ if $r(e)\geq 1$.

Proof. The first part follows immediately from Theorem A. 1. In order to prove the second part, we denote the above-metioned c_0 by $c(e_0,\cdots,e_n)$, in the case $r(e)\geq 1$. Then we have

(1) $$e_n c(e_0,\cdots,e_n)=c(e_0,\cdots,e_{n-1}) \quad \text{if} \quad r(e_0,\cdots,e_n)\geq 2$$

by the equality (iv) stated in the proof of Theorem A. 1, (2). By (1), we obtain

(2) $$c(1,1,e_0,\cdots,e_n)=c(e_0,\cdots,e_n) \quad \text{if} \quad r(e_0,\cdots,e_n)\geq 1,$$

(3) $$\left(\prod_{i=0}^{n}e_i\right)c(1,1,e_0,\cdots,e_n)=c(1,1).$$

On the other hand, it is well known that $c(1,1)=1$. Hence, by (2) and (3), we have $c(e_0,\cdots,e_n)=1/\prod_{i=0}^{n}e_i$. q.e.d.

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