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Cohomology of classifying spaces of certain associative H-spaces

By

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§ 1. Introduction

Let p be a prime and $\mathcal{A}_p$ be the Steenrod algebra mod p. Let G be a compact, connected Lie group.

We define a set $\{G:p\}$ by

$$\{G:p\} = \{X;\text{ compactly generated, associative } H\text{-space such that } H^*(X;\boldsymbol{Z}_p) \equiv H^*(G;\boldsymbol{Z}_p) \text{ as algebra over } \mathcal{A}_p\}.$$

Remark that in the above definition we do not require existence of a map $f: G \to X$ inducing the isomorphism of cohomology mod p.

According to Dold-Lashof [6] or Milgram [10] an associative H-space X has a classifying space BX which is constructed by making use of the multiplication on X.

When $H_*(G;\boldsymbol{Z})$ is p-torsion free, the Borel's theorem states that the ring structure of $H^*(BX;\boldsymbol{Z}_p)$ and the Hopf algebra structure of $H^*(X;\boldsymbol{Z}_p)$ are isomorphic to those of G for all $X \in \{G:p\}$.

However this does not hold in general. Actually, as Baum-Browder have shown in [2], there exist compact Lie groups which are homeomorphic and which have different diagonal maps in cohomology. Meanwhile, Theorem 9.3 of [2] says that if both X and Y are compact, simple Lie groups and if X is homotopy equivalent to Y, then X is isomorphic to Y as Lie groups. Thus it is natural to ask if the following is true:

Statement: *Let G be a compact, connected, simple Lie group* *such that* $H_*(G; \boldsymbol{Z})$ *has p-torsion. For any* $X, Y \in \{G: p\}$,

(1) $H(X; \boldsymbol{Z}_p) \cong H(Y; \boldsymbol{Z}_p)$ *as Hopf algebra,*

(2) $H^*(BX; \boldsymbol{Z}_p) \cong H^*(BY; \boldsymbol{Z}_p)$.

Let F_4 be the compact, 1-connected, simple, exceptional Lie group of rank 4. Let $PU(3) = SU(3)/\Gamma_3$ the quotient of $SU(3)$ by the center Γ_3. As is well known, the integral homology groups of both groups have 3-torsion.

In this paper we show that for $G = F_4$ and $PU(3)$ the above statement is true. In fact (1) is easily checked to be true for these cases. This will be observed in § 2. In § 3 we calculated Cotor ${}^A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ with $A = H^*(X; \boldsymbol{Z}_3)$ for $X \in \{F_4: 3\}$ and $\{PU(3): 3\}$. The section 4 will be used to show the Eilenberg-Moore spectral sequence with $\boldsymbol{Z}_3$-coefficient collapses for X. Our main results are

Theorem A. *For any* $X \in \{F_4: 3\}$, *we have as module:*

$$H^*(BX; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[y_4, y_8, y_9, y_{20}, y_{21}, y_{25}, y_{26}, y_{36}, y_{48}]/R.$$

For R see Theorem 4.7 of § 4.

Theorem B. *For any* $Y \in \{PU(3): 3\}$, *we have as algebra:*

$$H^*(BY; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[y_2, y_3, y_7, y_8, y_{12}]/R,$$

where R is an ideal generated by $y_2y_3, y_3^2, y_2y_7, y_7^2, y_2y_8 + y_3y_7$.

§ 2. Non-primitivity.

Let $X \in \{F_4: 3\}$. By definition and by [3] we have

$$H^*(X; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}) \quad \text{with} \quad \deg x_i = i, \tag{2.1}$$

where $x_7 = \mathcal{P}^1 x_3$, $x_8 = \beta x_7$ *and* $x_{15} = \mathcal{P}^1 x_{11}$.

Let ϕ be the diagonal map in $H^*(X; \boldsymbol{Z}_3)$ induced from the multiplication of X and let $\bar{\phi}$ be the reduced one.

Proposition 2.2. *In (2.1),*

$$\bar{\phi}(x_i) \quad for \quad i=3,7,8,$$

$$\bar{\phi}(x_j)=x_8 \otimes x_{j-8} \quad for \quad j=11,15.$$

Proof. Clearly x_3, x_7 and x_8 are primitive. Suppose x_{11} primitive. Then x_{15} would also be primitive by naturality. Hence $H^*(X; \boldsymbol{Z}_3)$ would be primitively generated. This contradicts to Theorem 1 of [4] (cf. the footnote in p. 319). Therefore x_{11} is not primitive. The only possible form is:

$$\bar{\phi}(x_{11})=x_8 \otimes x_3.$$

Applying $\mathscr{P}^1$ we obtain

$$\bar{\phi}(x_{15})=x_8 \otimes x_7. \qquad \text{q.e.d.}$$

Let $Y \in \{PU(3):3\}$. Then by definition and by [3] we have

(2.3) $H^*(Y; Z_3) \cong \boldsymbol{Z}_3[x_2]/(x_2^2) \otimes \Lambda(x_1, x_3)$ *with* $\deg x_i = i$, *where* $x_2 = \beta x_1$.

Similarly one obtains

Proposition 2.4. *In (2.3)*

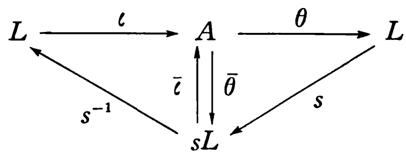
$$\bar{\phi}(x_i)=0 \quad for \quad i=1,2; \ \bar{\phi}(x_3)=x_2 \otimes x_1.$$

§ 3. The twisted tensor product.

In this section all algebras are graded [11]. We recall a construction of the twisted tensor product due to Brown (see [5], [7] or [15]).

Let A be an augmented coalgebra over a commutative field K with an augmentation $\eta: K \to A$ and the diagonal map ϕ. So we may consider $A = K \otimes J(A)$, where $J(A) \cong \operatorname{Coker} \eta$. Let L be a K-submodule of $J(A)$ and $\iota: L \to A$ be the inclusion and $\theta: A \to L$ a map such that $\theta \circ \iota = 1_L$.

Let $s: L \to sL$ be a suspension. Define $\bar{\theta}: A \to sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota}: sL \to A$ by $\bar{\iota} = \iota \circ s^{-1}$. Construct the tensor algebra $T(sL)$ and denote by by ψ the product in $T(sL)$. Let I be the ideal of $T(sL)$ generat-



ed by $\mathrm{Im}(\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi)\circ(\mathrm{Ker}\,\bar{\theta})$. Put $\bar{X}=T(sL)/I$. Then the map $\bar{\theta}: A\to sL$ induces a map $A\to\bar{X}$ which is again denoted by $\bar{\theta}$.

We define a map

$$d=-\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi\circ\bar{\iota}: sL\to T(sL)$$

and extend it naturally over $T(sL)$. Since $d\circ\phi=\psi\circ(d\otimes 1+1\otimes d)$ holds, we deduce $d(I)\subset I$. So d induces a map $\bar{X}\to\bar{X}$, which is again denoted by $d: \bar{X}\to\bar{X}$. Then it is easy to see $d\circ d=0$. This shows that $\bar{X}$ is a differential algebra over K.

Since the relation

$$d\circ\bar{\theta}+\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi=0$$

holds, we now can construct the twisted tensor product $W=A\otimes\bar{X}$ with respect to $\bar{\theta}$. That is, $W=A\otimes\bar{X}$ is an A-comodule with the differential operator

$$\bar{d}=1\otimes d+(1\otimes\psi)\circ\bar{\theta}\otimes 1)\circ(\phi\otimes 1).$$

We now apply this to calculate $\mathrm{Cotor}^{A}(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$, where $A=H^*(X; \boldsymbol{Z}_3)$ for $X\in\{F_4: 3)$ or $\{PU(3): 3\}$.

Let $X\in\{F_4; 3\}$. Then by (2.1) and Proposition 2.2

$$H^*(X; \boldsymbol{Z}_3)=\boldsymbol{Z}_3[x_8]/(x_8^3)\otimes\Lambda(x_3, x_7, x_{11}, x_{15}), \tag{3.1}$$

where $\bar{\phi}(x_i)=0$ for $i=3, 7, 8$ and $\bar{\phi}(x_j)=x_8\otimes x_{j-8}$ for $j=11, 15$.

Put $A=H^*(X; \boldsymbol{Z}_3)$. Take L to be a $\boldsymbol{Z}_3$-submodule of A generated by $\{x_3, x_7, x_8, x_8^2, x_{11}, x_{15}\}$ and θ to be the projection $A\to L$. We name the corresponding elements under the suspension s as $sL=\{a_4, a_8, a_9, c_{17}, b_{12}, b_{16}\}$ respectively. Then

$$\bar{X}=\boldsymbol{Z}_3\{a_4, a_8, a_9, b_{12}, b_{16}, c_{17}\}/I,$$

which is a quotient of $T(sL)$ by the ideal I generated by

$$[a_4, a_8],\ [a_4, a_9],\ [a_4, c_{17}],\ [a_4, b_{12}],\ [a_4, b_{16}] \tag{3.2}$$

$$[a_8, a_9],\ [a_8, c_{17}],\ [a_8, b_{12}],\ [a_8, b_{16}]$$

$$[a_9, b_{12}]+c_{17}a_4,\ \ [a_9, b_{16}]+c_{17}a_8,$$

$$[b_{12}, b_{16}],\ [c_{17}, b_{12}],\ [c_{17}, b_{16}],$$

where $[x, y]=xy-(-1)^{*}yx$ *with* $*=\deg x\cdot\deg y$.

We define weight in $W=A\otimes\bar{X}$ as follows:

(3.3)

A:	x_3	x_7	x_8	x_8^2	x_{11}	x_{15}
$\bar{X}$:	a_4	a_8	a_9	c_{17}	b_{12}	b_{16}
weight:	0	0	1	2	2	2

The weight of a monomial is a sum of the weight of each element. Define filtration $F_r=\{x \mid \text{weight}\, x\leq r\}$.

The differential operator $\bar{d}$ on A is given by

(3.4)
$$\bar{d}x_i=a_{i+1} \quad \textit{for} \quad i=3, 7, 8$$
$$\bar{d}x_8^2=c_{17}-x_8a_9$$
$$\bar{d}x_j=b_{j+1}+x_8a_{j-7} \quad \textit{for} \quad j=11, 15.$$

Put $E_0W=\sum_i F_i/F_{i-1}$. Then

$$E_0W\cong\Lambda(x_3, x_7, x_{11}, x_{15})\otimes \boldsymbol{Z}_3[a_4, a_8, b_{12}, b_{16}]\otimes C(Q(x_8)),$$

where $C(Q(x_8))$ is the cobar construction of $\boldsymbol{Z}_3[x_8]/(x_8^3)$. It follows from (3.4) that E_0W is acyclic and hence W is acyclic. Thus W is an injective resolution of $\boldsymbol{Z}_3$ over $H^*(X; \boldsymbol{Z}_3)$. Therefore by definition

$$H(\bar{X}; d)=\operatorname{Ker} d/\operatorname{Im} d=\operatorname{Cotor}^A(\boldsymbol{Z}_3, \boldsymbol{Z}_3).$$

Remark that the differential operator d on $\bar{X}$ is given by

(3.5)
$$da_i=0 \quad \textit{for} \quad i=4, 8, 9;$$
$$dc_{17}=a_9^2;$$
$$db_j=-a_9a_{j-8} \quad \textit{for} \quad j=12, 16.$$

Then it is easy to see that the following set is a system of generators of $\operatorname{Cotor}^A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$:

(3.6) $$\{y_4, y_8, y_9, y_{20}, y_{21}, y_{25}, y_{26}, y_{36}, y_{48}\},$$

where $y_i=\{a_i\}$ for $i=4, 8, 9$, $y_{20}=\{a_8b_{12}-a_4b_{16}\}$, $y_{26}=\{[a_9, c_{17}]\}$, $y_{21}=\{a_9b_{12}-c_{17}a_4\}$, $y_{25}=\{a_9b_{16}-c_{17}a_8\}$, $y_{36}=\{b_{12}^3\}$ and $y_{48}=\{b_{16}^3\}$.

Remark 3.7. y_i for $i=20, 21, 25, 26$ are represented by Massey products $\langle a_4, a_9, a_8\rangle$, $\langle a_9, a_9, a_4\rangle$, $\langle a_9, a_9, a_8\rangle$ and $\langle a_9, a_9, a_9\rangle$ respectively ([9]).

By a routine calculation we obtain

Theorem 3.8. *For any $X\in\{F_4\, 3\}$ we have as module*

$$\mathrm{Cotor}^{H^*(X;\boldsymbol{Z}_3)}(\boldsymbol{Z}_3, \boldsymbol{Z}_3)\cong\boldsymbol{Z}_3[y_4, y_8, y_9, y_{20}, y_{21}, y_{25}, y_{26}, y_{36}, y_{48}]/R,$$

where R is an ideal generated by

$$y_4y_9,\ y_8y_9,\ y_9^2,\ y_4y_{21},\ y_8y_{25},\ y_4y_{25}+y_8y_{21},$$

$$y_{20}y_{21},\ y_{20}y_{25},\ y_{21}^2,\ y_{25}^2,\ y_9y_{20}-y_4y_{25}+y_8y_{21},$$

$$y_{20}^3-y_4^3y_{48}+y_8^3y_{36},\ y_{26}y_4+y_{21}y_9,\ y_{26}y_8+y_{25}y_9,$$

$$y_{26}y_{20}-y_{21}y_{25}.$$

Let $Y\in\{PU(3):3\}$. Then by (2.3) and Proposition 2.4

(3.9) $$H^*(Y;\boldsymbol{Z}_3)\cong\boldsymbol{Z}_3[x_2]/(x_2^3)\otimes\Lambda(x_1, x_2),$$

where $\bar{\phi}(x_1)=\bar{\phi}(x_2)=0$ and $\bar{\phi}(x_3)=x_2\otimes x_1$. For simplicity we put $A=H^*(Y;\boldsymbol{Z}_3)$. Take a $\boldsymbol{Z}_3$-subspace $L=\{x_1, x_2, x_2^2, x_3\}$ and name the corresponding elements in sL as $sL=\{a_2, a_3, c_5, b_4\}$ respectively. Then $\bar{X}=\boldsymbol{Z}_3\{a_2, a_3, c_5, b_4\}/I$, where I is generated by

(3.10) $$[a_2, a_3], [a_2, c_5], [a_2, b_4], [c_5, b_4] \quad \textit{and} \quad [a_3, b_4]+c_5a_2.$$

Similarly as before we can prove that $W=A\otimes\bar{X}$ is acyclic. So by definition

$$H(\bar{X};d)=\mathrm{Cotor}^{A}(\boldsymbol{Z}_3, \boldsymbol{Z}_3) \quad \text{with} \quad A=H^*(Y;\boldsymbol{Z}_3)$$

where the differential d on $\bar{X}$ is explicitly given as follows:

(3.11) $$da_2=da_3=0,\ db_4=-a_2a_3 \quad \textit{and} \quad dc_5=a_3^2.$$

Easy calculation yields

Theorem 3.12. *For any* $Y\in\{PU(3):3\}$ *we have as module,*

$$\mathrm{Cotor}^{H^*(Y;Z_3)}(Z_3, Z_3)\cong Z_3[y_2, y_3, y_7, y_8, y_{12}]/R,$$

where R *is an ideal generated by*

$$y_2y_3,\ y_2y_7,\ y_3^2,\ y_7^2,\ y_2y_8+y_3y_7 \tag{3.13}$$

Remark 3.14. In the above theorem the generators are: $y_i=\{a_i\}$ for $i=2, 3$, $y_7=\{a_3b_4-a_2c_5\}$, $y_8=\{[a_3, c_5]\}$ and $y_{12}=\{b_4^3\}$.

§ 4. The Eilenberg-Moore spectral sequence

In this section we use the Eilenbery-Moore spectral sequence with Z_3-coefficient $\{E_r(X), d_r\}$ for an associative H-space X such that $E_2(X)=\mathrm{Cotor}^{H^*(X;Z_3)}(Z_3, Z_3)$ and $E_\infty(X)\cong \mathcal{G}rH^*(BX; Z_3)$. (For construction and properties see [12] and [13]).

Let $X\in\{F_4: 4\}$. Put $A=H^*(X; Z_3)$.

Lemma 4.1. *There are no elements in* $\mathrm{Cotor}^A(Z_3, Z_3)$ *of degree 37 or 49.*

Proof. (1) Because of relations in Theorem 3.8 the element of degree 37 is of the form:

$$y_9f(y_{20}, y_{26})+y_{21}g(y_8, y_{26})+y_{25}h(y_4, y_{26})$$

with polynomials f, g and h. Here f cannot have degree 28. The possible form of g of degree 16 is a scalar multiple of y_8^2. Similarly $h=\alpha y_4^3$ with $\alpha\in Z_3$. However by relations in Theorem 3.8 we have: $y_{21}y_8^2=-y_{20}y_9y_8=0$ and $y_{25}y_4^3=-y_{20}y_9y_4^2=0$. So there are no elements of degree 37.

(2) The element of degree 49 is of the form:

$$y_9\bar{f}(y_{36}, y_{20}, y_{26})+y_{21}\bar{g}(y_8, y_{26})+y_{25}\bar{h}(y_4, y_{26}).$$

Here $\bar{f}$ is a scalar multiple of y_{20}^2. $\bar{g}=0$ because of dimensional reason and $\bar{h}$ is a scalar multiple of y_4^6. Since $y_9y_{20}^2=-y_{25}y_4y_{20}=0$

and since $y_{25}y_4^6=-y_{20}y_9y_4^5=0$, there are no elements of degree 49.

q.e.d.

Now we will show that the Eilenberg-Moore spectral sequence with $\boldsymbol{Z}_3$-coefficient collapses for X, where the E_2-term is given by Theorem 3.8.

For dimensional reason, x_3 and hence x_7 and x_8 of $H(X;\boldsymbol{Z}_3)$ are universally transgressive in the universal spectral sequence for X with $\boldsymbol{Z}_3$-coefficient. Then it is easy to see that the elements y_4, y_8 and y_9 of the E_2-term survive in the E_∞-term and represent the transgressive image of x_3, x_7 and x_8 respectively. Thus we obtain an isomorphism as glgebras:

(4.2) $H^*(BX;\boldsymbol{Z}_3)\cong\boldsymbol{Z}_3[y_4, y_8, y_9]/(y_4y_9, y_8y_9, y_9^2)$ for $*<20$.

Lemma 4.3. *In $H^*(BX;\boldsymbol{Z}_3)$ $\mathcal{P}^3y_8$ is not decomposable.*

Proof. In the universal spectral sequence of X the element $y_9\otimes x_8^2$ is not coboundary and $d_{17}(y_9\otimes x_8^2)=\beta\mathcal{P}^4y_9\otimes 1$ by the Kudo transgression theorem [8]. Then by the Adem relation $\beta\mathcal{P}^4y_9=\beta\mathcal{P}^4\beta y_8=\beta\mathcal{P}^1\beta\mathcal{P}^3y_8$, which is non-trivial. If $\mathcal{P}^3y^8$ is decomposable, say $\mathcal{P}^3y=f(y_4, y_8)$, then $\beta\mathcal{P}^3x_8=y_9h(y_4, y_8)=0$. This is a contradiction.

q.e.d.

Lemma 4.4. *The elements y_i for $i=20, 21, 25, 26$ are permanent cycles.*

Proof. The generators in $E_2(X)$ of degree 21, 25, 26 are unique, respectively. On the other hand there are non-trivial elements $\beta\mathcal{P}^3y_8$, $\mathcal{P}^1\beta\mathcal{P}^3y_8$, $\beta\mathcal{P}^1\beta\mathcal{P}^3y_8$ in $H^*(BX;\boldsymbol{Z}_3)$ of degree 21, 25, 26 respectively. So y_{21}, y_{25}, y_{26} are permanent cycles and represent these elements respectively. Similarly for y_{20}, which survives to $\mathcal{P}^3y_8$. q.e.d.

Lemma 4.5. *The elements y_{36} and y_{48} are permanent cycles.*

Proof. This follows from Lemma 4.1, since d_ry_{36} and d_ry_{48} are in degree 37 and 49 respectively. q.e.d.

Thus we have shown

Theorem 4.6. *For any $X \in \{F_4 : 3\}$, the Eilenberg-Moore spectral sequence with $\boldsymbol{Z}_3$-coefficient collapses.*

As an immediate corollary

Theorem 4.7. *For any $X \in \{F_4 : 3\}$ we have as module*

$$H^*(BX; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[y_4, y_8, y_9, y_{20}, y_{21}, y_{25}, y_{26}, y_{36}, y_{48}]/R,$$

where R is an ideal generated by y_4y_9, y_8y_9, y_9^2, y_4y_{21}, y_8y_{25}, $y_4y_{25}+y_8y_{21}$, $y_{20}y_{21}$, $y_{20}y_{25}$, y_{21}^2, y_{25}^2, $y_9y_{20}-y_4y_{25}+y_8y_{21}$, $y_{20}^3-y_4^3y_{48}+y_8^3y_{36}$, $y_{26}y_4+y_{21}y_9$, $y_{26}y_8+y_{25}y_9$, $y_{26}y_{20}-y_{21}y_{25}$. *Furthermore, up to non-zero multiple,* $y_8=\mathcal{P}^1y_4$, $y_9=\beta y_8$, $y_{20}=\mathcal{P}^3y_8$, $y_{21}=\beta y_{20}$, $y_{25}=\mathcal{P}^1y_{21}$ *and* $y_{26}=\beta y_{25}$.

Remark 4.8. This theorem gives the module structure of $H(BF_4; \boldsymbol{Z}_3)$. The algebra structure of $H^*(BF_4; \boldsymbol{Z}_3)$ was already obtained in [16], in which Toda used the fibering $\prod \to \mathrm{BSpin}(9) \to BF_4$ and made elaborate calculations of invariant forms under the Weyl group of F_4.

Now we turn to the case $\{PU(3) : 3\}$.

Take $Y \in \{PU(3) : 3\}$. Put $A = H^*(Y; \boldsymbol{Z}_3)$. Then it is easy to obtain an isomorphism as algebras:

$$(4 \cdot 9) \qquad H^*(BY; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[y_2, y_3]/(y_2y_3, y_3^2) \quad for \quad * < 7.$$

We show

Theorem 4.10. *For any $Y \in \{PU(3) : 3\}$ the Eilenberg-Moore spectral sequence with $\boldsymbol{Z}_3$-coefficient collapses.*

Proof. (1) The elements y_2, y_3 of $\mathrm{Cotor}^A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ are permanent cycles by (4.9).

(2) In the universal spectral sequence for Y we use the Kudo theorem [8] and obtain a non-trivial element $\beta\mathcal{P}^1y_3 \in H^8(BY; \boldsymbol{Z}_3)$. Hence $H^7(BY; \boldsymbol{Z}_3) \neq 0$. There is only one element y_7 in $\mathrm{Cotor}^A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ of degree 7. So y_7 is a permanent cycle and represent $\mathcal{P}^1y_3$.

(3) By easy observation one can see that there are no elements in $\mathrm{Cotor}^{A}(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ of degree 9 or 13. So y_8 and y_{12} of $\mathrm{Cotor}^{A}(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ are permanent cycles, since $d_r y_8$ and $d_r y_{12}$ are of degree 9 and 13 respectively. q.e.d.

Remark that $y_2 y_3 = 0$ is a relation as algebra. By applying $\mathcal{P}^1$ we have $y_2 y_7 = \mathcal{P}^1(y_2 y_3) = 0$. Further applying β on $y_2 y_7 = 0$ we obtain $y_3 y_7 + y_2 y_8 = 0$. Since $y_3^2 = y_7^2 = 0$ are relations as algebra, we therefore have proved:

Theorem 4.11. *For any* $Y \in \{PU(3):3\}$, *we have as algebra*

$$H^*(BY; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[y_2, y_3, y_7, y_8, y_{12}]/R,$$

where R is an ideal generated by $y_2 y_3$, y_3^2, $y_2 y_7$, y_7^2, $y_2 y_8 + y_3 y_7$. *Further,* $y_3 = \beta y_2$, $y_7 = \mathcal{P}^1 y_3$, $y_8 = \beta y_7$.

Remark 4.12. In particular, this theorem gives the algebra structure of $H^*(BPU(3); \boldsymbol{Z}_3)$.

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