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Uniqueness in the Cauchy problem for partial differential equations with multiple characteristic roots

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1. Introduction

We are concerned with the uniqueness theorem in the Cauchy problem for the following type of partial differential equations:

$$Pu\equiv\partial_t^m u+\sum_{\substack{|\alpha|+j\leq m\\ j\leq m-1}} a_{\alpha,j}(x,t)\partial_x^\alpha\partial_t^j u=0,\ (x\in \boldsymbol{R}^l).$$

Here we assume $a_{\alpha,j}(x,t)$ are sufficiently smooth functions. In the case where the characteristic roots are simple and the coefficients $a_{\alpha,j}(x,t)$ ($|\alpha|+j=m$) are all real, A. P. Calderón [1] proved the uniqueness theorem in 1958. When (x, t) is two-dimensional, T. Carleman [2] obtained the same result as early as 1939. S. Mizohata [11] proved the uniqueness in the case of elliptic type of order 4 with smooth characteristic roots. Many authors have studied the uniqueness in case of at most double smooth characteristic roots ([4], [9], etc.) Then a study for elliptic operators with triple characteristic roots, was made by K. Watanabe [15] under the assumption that the multiplicity of the characteristic roots are constant. Let us consider the following type of operator:

$$P=P_p(x,t;\partial_x,\partial_t)^{m}\,{}^{1)}+P_{mp-1}(x,t;\partial_x,\partial_t)+R(x,t;\partial_x,\partial_t),$$

1) We start assumig that the principal part of $\boldsymbol{P}$ is $(\boldsymbol{P}_p)^m$, but if $\boldsymbol{P}_{mp}$, differential operator of order mp, has the p characteristic roots with constant multiplicity m we can have a differential operator $\boldsymbol{P}_p$ of order p with simple characteristic roots such that $\boldsymbol{P}_{mp}=(\boldsymbol{P}_p)^m$ modulo order $mp-1$. Moreover if $m\geq 3$, $\boldsymbol{P}_{mp}-\boldsymbol{P}_p^m$ does not affect the conditions (A), (B_1) and (B_2), and if $m=2$

$$[\tfrac{1}{2}(\sum_{k=1}^{l}\partial_{\xi_k}\boldsymbol{P}'_{mp}\partial_{x_k}\lambda_j+\partial_\tau\boldsymbol{P}'_{mp}\partial_t\lambda_j)+\boldsymbol{P}_{mp-1}]|_{\tau=\lambda_j}$$

serves our need instead of $\boldsymbol{P}_{mp-1}|_{\tau=\lambda_j}$ in the conditions (A), (B_1) and (B_2) where $\boldsymbol{P}'_{mp}=\partial_\tau\boldsymbol{P}_{mp}$ (see S. Mizohata-Y. Ohya [13]).

where $m\geq 2$ and $p\geq 1$. Here we assume that, 1) P_p is a homogeneous partial differential operator of order p with real coefficients continuously differentiable up to order $l+\max\{mp, 6\}$. Moreover its characteristic roots $\{\lambda_j(x, t; \xi)\}_{1\leq j\leq p}$ of $P_p(x, t; \xi, \lambda)=0$ are distinct for all real $\xi(\neq 0)$, 2) P_{mp-1} is a homogeneous partial differential operator of order $mp-1$ with real coefficients belonging to $C^{l+\max\{mp-1,5\}}$, 3) R is a partial differential operator of order at most $mp-2$, with bounded measurable coefficients.

Let $\{\lambda_j(x, t; \xi)\}_{1\leq j\leq p}$ be the characteristic roots of P_p. We introduce the following conditions.

$$(A)\quad P_{mp-1}(0, 0; \xi, \tau)|_{\tau=\lambda_j(0,0;\xi)}\neq 0$$
$$\text{for all } \xi\in \boldsymbol{R}^l-\{0\},\ (1\leq j\leq p).$$

$$(B_1)\quad P_{mp-1}(x, t; \xi, \tau)|_{\tau=\lambda_j(x,t;\xi)}\equiv 0$$
$$\text{for all } (x, t, \xi)\in U\times(\boldsymbol{R}^l-\{0\}),\ (1\leq j\leq p),$$

U being a neighbourhood of the origin.

$$(B_2)\quad (B_1) \text{ and } \partial_\tau P_{mp-1}(0, 0; \xi, \tau)|_{\tau=\lambda_j(0,0;\xi)}\neq 0$$
$$\text{for all } \xi\in \boldsymbol{R}^l-\{0\},\ (1\leq j\leq p).$$

Then our result is the following.

Theorem 1.[2] *If $m=2$ and all λ_j satisfy the condition (A) or (B_1), or if $m\geq 3$ and all λ_j satisfy the condition (A) or (B_2), there exists a neighbourhood Ω of the origin such that the solution $u(x, t)\in C^{mp}$ of*

$$\begin{cases} Pu=0 \\ \partial_t^j u|_{t=0}=0 \quad (0\leq j\leq mp-1) \end{cases}$$

vanishes identically there.

Now we give some comments to the above new type conditions. When we do not assume the above condition (A), (B_1) or (B_2), the following examples show that we should assume another kind of conditions in order to obtain the uniqueness theorem. First, we give

[2] We stated this theorem with a short proof in [10].

three examples of elliptic type. Example 1 is in case of $l=1$, that is, $(x, t) \in \boldsymbol{R}^2$.

Example 1. (A. Pliś [14]). Let us consider the following operator:

$$P=(\partial_t-i\partial_x)^m+t^k(i\partial_x)^n+(i\partial_x)^{n-1}.$$

If $(m+3/2)<n\leq m$, $k>(m-1)/(2n-m-3)$, there exist $f(x, t)\in C^\infty$ and $u(x, t)\in C^\infty$ satisfying $(P+f)u=0$, $u\equiv 0$ $(t\leq 0)$, where u never vanishes in any neighbourhood of the origin.

Let us remark that if $m\geq 6$, we can take $n=m-1$, and if $m\geq 8$, $n=m-2$. In each case, neither the condition (A) nor (B_1) is satisfied.

Example 1 implies the following example in a higher dimensional space.

Example 2. Let $l\geq 1$, $m\geq 6$, and $(m+3)/2<n\leq m-1$, $k>(m-1)/(2n-m-3)$, Δ be the Laplacian in $\boldsymbol{R}_x^l\times\boldsymbol{R}_t^1$. There is an operator Q of order at most $2m-2$ with C^∞-coefficients and $u(x, t)=u(x_1, t)\in C^\infty$ satisfying

$$\begin{cases} [\Delta^m+P_{2m-1}+t^k(\partial_t+i\partial_{x_1})^m(i\partial_{x_1})^n+Q]u=0, \\ u\equiv 0 \quad (t\leq 0), \end{cases}$$

where P_{2m-1} is an arbitrary operator of order $2m-1$ containing only $\partial_{x_2}, \cdots, \partial_{x_l}$, and $u(x, t)$ never vanishes in any neighbourhood of the origin.

Note that the term of order $2m-1$ at the origin is nothing but $P_{2m-1}(0, 0; \partial_{x_2}, \cdots, \partial_{x_l})$. Since P_{2m-1} vanishes for $(\xi_1, \xi_2, \cdots, \xi_l)=(1, 0, \cdots, 0)$, the condition (A) is never satisfied. Moreover, for such P_{2m-1}, the condition (B_1) implies $P_{2m-1}\equiv 0$. Therefore (B_2) is not satisfied, too.

In particular if $m=8$ we can take $n=6$ and $P_{15}\equiv 0$. This means that for the operator of the form:

$$\Delta^8+(\text{l.o.t. of order at most } 14),$$

we cannot obtain in general the uniqueness theorem, although the homogeneous term of order 15 vanishes identically.

If we do not require sufficient differentiability of the coefficients of l.o.t., we have the following example:

Example 3.[3] (P. J. Cohen [3]). For arbitrary partial differential opetator of order m with constant coefficients $P(\partial_{x_1}, \cdots, \partial_{x_l}, \partial_t)$ and with a non-real characteristic root of multiple order r and arbitrary positive integer p, there exist a C^s-function $a(x, t)$ and a C^∞-function $u(x, t)=u(x_1, t)$ satisfying $a\equiv 0$ when $t\leq 0$ and

$$\begin{cases} Pu+P_{m-1}u+a(x, t)\partial_{x_1}^{m-p}u=0, \\ u=0 \quad (t\leq 0), \end{cases}$$

where $s<r-2p$ and P_{m-1} is an arbitrary operator of order $m-1$ containing only $\partial_{x_2}, \cdots, \partial_{x_l}$, and $u(x, t)$ never vanishes in any neighbourhood of the origin.

In the above example, if we want to take $s\geq 0$ and $p\geq 1$, r must be larger than 2 and neither the condition (A) nor (B_2) is satisfied as well by the same reason in Example 2.

Particularly, for $r=5$ we can take $p=2$ and $s<1$. Then for the operator with Δ^5 as the principal part, we cannot obtain in general the uniqueness theorem, although the homogeneous term of order 9 vanishes identically and the l.o.t. of order at most 8 have Hölder-continuous coefficients.

Next, we give an example of hyperbolic type.

Example 4. (L. Hörmander [5], see also P. J. Cohen [3]) Let $l\geq 1$, $r\geq 2$. There exist functions $a(x, t)$ and $u(x, t)=u(x_1, t)\in C^\infty$ satisfying $a\equiv 0$ when $t\leq 0$, and

$$\begin{cases} \partial_t^r u+P_{r-1}u+a(x, t)\partial_{x_1}u=0, \\ u\equiv 0 \quad (t\leq 0), \end{cases}$$

where P_{r-1} is an arbitrary operator of order $r-1$ containing only $\partial_{x_2}, \cdots, \partial_{x_l}$, and $u(x, t)$ never vanishes identically in any neighbourhood of the origin.

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[3] In [3], P. J. Cohen asserts that s can be taken smaller than $r-p-1$ but it seems to us that he proved only that $s<r-2p$.

2. Preliminaries and outline of the proof of Theorem 1.

2.1. Notations.

Throughout this paper, we use the following notations. $\boldsymbol{R}_x^l$ denotes the Euclidian space of l dimensions; $\boldsymbol{R}_\xi^l$ denotes its dual space. For $\xi \in R_\xi^l$, we denote $\xi' = \xi/|\xi|$, and $\boldsymbol{S}_\xi^{l-1}$ denotes the unit sphere in $\boldsymbol{R}_\xi^l$ with its centre at the origin. ∂_t, ∂_x denote $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ respectively. Furthermore D_t, D_x denote $(1/i)\partial_t$, $(1/i)\partial_x$. "ψ.d.o." means pseudo-differential operator, and by saying that an operator T is of order p, we mean that T is a continuous map from $\mathcal{D}_{L^2}^{s+p}$ into $\mathcal{D}_{L^2}^s$.

For a differential operator $P(x, t; \partial_x, \partial_t)$, we denote its principal part by $P^0(x, t; \partial_x, \partial_t)$, and "l.o.t." means lower order terms. For $u(x) \in \mathcal{D}_{L^2}^j$, we define its norm by

$$||u||_j = ||(|\xi|+1)^j \hat{u}(\xi)||_{L^2}.$$

Finally, we use the following notations.

$$C_{(x,t)}^m \times C_\xi^\infty = \{a(x, t; \xi); a(x, t; \xi) \in C_{(x,t)}^m, a(x, t; \xi) \in C_\xi^\infty, \xi \neq 0\}.$$

Let E be a topological vector space, we write $f(t) \in \mathcal{E}_t^m(E)$, when f is a function m-times continuously differentiable in t with values in E.

2.2. Properties of pseudo-differential operators.

In this paper we use frequently pseudo-differential operators. Although its fundamental properties are fairly known, in order to make clear the required order of differentiability of symbol $a(x; \xi)$ in x we shall enumerate the following lemmas. We refer the readers, for example, to Kohn-Nirenberg [7]. $\mathfrak{H}_\lambda^p$ is the class of functions $a(x; \xi) \in C_x^p \times C^\infty$ satisfying the following three conditions:

1. $a(x; \xi)$ is homogeneous of degree λ in ξ.

2. there exists a limit $a(\infty; \xi)$ uniformly in ξ', when $|x| \to \infty$.

3. $|a(x; \xi') - a(\infty; \xi')| \leq \frac{\text{const.}(q)}{(1+|x|)^q}$ for all $q \in \boldsymbol{R}$.

Modification of $a(x; \xi)$. We often use the following modification. Let

$\theta(s)$ be a C^∞-function satisfying

$$\theta(s)=\begin{cases}1, & 0\leq s\leq R,\\ 0, & s\geq R+1,\end{cases}\qquad 0\leq\theta(s)\leq 1.$$

We define

$$\zeta(\xi)=(1-\theta(|\xi|))\cdot\xi+\theta(|\xi|)\cdot R\cdot\frac{\xi}{|\xi|}.$$

And finally we put

$$\tilde{a}(x;\xi)=a(x;\zeta(\xi)).$$

For $a(x;\xi)$ we define

$$a_0(x;\xi)=a(x;\zeta(\xi))-a(\infty;\zeta(\xi))\quad\text{and}\quad\tilde{a}(\infty;\xi)=a(\infty;\zeta(\xi)).$$

Therefore, for $a(x;\xi)$, $a_0(x;\xi)=\tilde{a}(x;\xi)-\tilde{a}(\infty;\xi)$.

Then, ψ.d.o. A and $\mathcal{A}$ associated with $a(x;\xi)$ are defined by

Definition 2.1.

1) $$Au=\left(\frac{1}{2\pi}\right)^l\int e^{ix\cdot\xi}\tilde{a}(x;\xi)\hat{u}(\xi)d\xi.$$

2) $$\widehat{\mathcal{A}u}=\int e^{-ix\cdot\xi}\tilde{a}(x;\xi)u(x)dx.$$

These can be written as

$$\widehat{Au}=\tilde{a}(\infty;\xi)\hat{u}(\xi)+\int\hat{a}_0(\xi-\eta;\eta)\hat{u}(\eta)d\eta,$$

where $\hat{a}_0(\tau;\xi)=\int e^{-ix\cdot\tau}a_0(x;\xi)dx$, and

$$\widehat{\mathcal{A}u}=\tilde{a}(\infty;\xi)\hat{u}(\xi)+\int\hat{a}_0(\xi-\eta;\xi)\hat{u}(\eta)d\eta.$$

Then,

Lemma 2.1.[4] *Let $a(x;\xi)\in\mathfrak{H}_\lambda{}^p$. Then, if $p>l+|s|$, A is continuous operator from $\mathcal{D}_{L^2}^{s+\lambda}$ into $\mathcal{D}_{L^2}^{s}$; if $p>l+|s+\lambda|$, then $\mathcal{A}$ is continuous operator from $\mathcal{D}_{L^2}^{s+\lambda}$ into $\mathcal{D}_{L^2}^{s}$.*

[4] From Lemma 2.1 to 2.4, we need the differentiabilities up to only some orders in ξ.

Next,

Lemma 2.2. *Suppose $a(x;\xi)\in\mathfrak{H}_\lambda{}^p$, and that $p>l+|s|$. Then it holds*

$$(Au, v)=(u, \mathcal{\bar{A}}v),$$

for all $u\in\mathcal{D}_{L^2}^{s+\lambda}$ and $v\in\mathcal{D}_{L^2}^{-s}$, where the symbol of $\mathcal{\bar{A}}$ is $\overline{a(x;\xi)}$.

Note. The above lemma shows $A^*=\mathcal{\bar{A}}$ in the above sense.

Now, we give two definitions.

Definition 2.2. Let A and B be two ψ.d.o.'s whose symbols are $a(x;\xi)$ and $b(x;\xi)$ respectively. We define a new ψ.d.o. $A\circ B$ by

$$A\circ B=\left(\frac{1}{2\pi}\right)^l\int e^{ix\cdot\xi}\tilde{a}(x;\xi)\tilde{b}(x;\xi)\hat{u}(\xi)d\xi.$$

Definition 2.3. Let A be a ψ.d.o. whose symbol is $a(x;\xi)$. Then,

$$A_{(\alpha)}^{(\beta)}u=\left(\frac{1}{2\pi}\right)^l\int e^{ix\cdot\xi}(D_x{}^\alpha\partial_\xi{}^\beta\tilde{a}(x;\xi))\hat{u}(\xi)d\xi.$$

Note. $A\circ B$ can be expressed as

$$\begin{aligned}\widehat{A\circ Bu}=&\tilde{a}(\infty;\xi)\tilde{b}(\infty;\xi)\hat{u}(\xi)+\int\tilde{a}(\infty;\eta)\hat{b}_0(\xi-\eta;\eta)\hat{u}(\eta)d\eta\\&+\int\hat{a}_0(\xi-\eta;\eta)\tilde{b}(\infty;\eta)\hat{u}(\eta)d\eta\\&+\int\hat{a}_0(\xi-\tau;\eta)\hat{b}_0(\tau-\eta;\eta)\hat{u}(\eta)d\tau d\eta.\end{aligned}$$

Now we obtain the following asymptotic formula.

Lemma 2.3. *Let $a(x;\xi)\in\mathfrak{H}_\lambda{}^p$, $b(x;\xi)\in\mathfrak{H}_\sigma{}^p$. If $p>l+|\lambda-\rho|+|s|+\rho$, AB can be expressed as*

$$AB=\sum_{|\alpha|\leq\rho-1}\frac{1}{\alpha!}A^{(\alpha)}\circ B_{(\alpha)}+T,$$

where T is continuous from $\mathcal{D}_{L^2}^{s\gamma++\sigma-\rho}$ to $\mathcal{D}_{L^2}^{s}$.

Corollary of Lemma 2. 3. *If $p>l+\max\{|\lambda-1|, |\sigma-1|\}+|s|+1$, $[A, B]=AB-BA$ is continuous from $\mathcal{D}_{L^2}^{s+\lambda+\sigma-1}$ to $\mathcal{D}_{L^2}^{s}$.*

Lemma 2. 4. *Let $a(x;\xi)\in\mathfrak{H}_\lambda{}^p$. If $p>l+\max\{|s|, |s+\lambda-\rho|\}+\rho$, we have the following expression:*

$$\mathcal{A}=\sum_{|\alpha|\leq\rho-1}\frac{1}{\alpha!}A_{(\alpha)}^{(\alpha)}+T,$$

where T is continuous from $\mathcal{D}_{L^2}^{s+\lambda-\rho}$ to $\mathcal{D}_{L^2}^{s}$.

In this paper we are mainly concerned with the symbol

$$a(x;\xi)=a_0(x;\xi)+a_1(x;\xi)|\xi|^{-(1/m)}+a_2(x;\xi)|\xi|^{-(2/m)}+\cdots$$

where $a_j(x;\xi)\in\mathfrak{H}_0{}^\sigma\,(0\leq j<\infty)$ and $\sigma>0$.

Put $M_j=\sum_{|\nu|\leq 2l}\sup|\partial_\xi{}^\nu a_j(x;\xi)|$ where supremum is taken over $x\in\boldsymbol{R}^l, \xi\in\boldsymbol{S}^{l-1}$, we have the following lemmas.

Lemma 2. 5. (*T. Kano* [6]). *Suppose that the above sequence $\{M_j\}$ satisfies the following condition: there exists $\varepsilon(>0)$ such that for all $\theta\in(0,\varepsilon)$, $\sum_{j=0}^{\infty}M_j\theta^{j/m}<\infty$. We choose R in paragraph 2. 2 larger than $1/\varepsilon$, then*

$$||Au||\leq\sum_{j=0}^{\infty}M_jR^{-(j/m)}||u||$$

for all $u\in L^2$ satisfying $\operatorname{supp}[\hat{u}(\xi)]\subset\{\xi;|\xi|>R+1\}$.

Finally we need the following

Lemma 2. 6. (*T. Kano* [6]). *Put $\inf|a_0(x;\xi)|=\delta>0$, where infimum is taken over $x\in\boldsymbol{R}^l, \xi\in\boldsymbol{S}^{l-1}$. Then, for all $u(x)\in L^2$ satisfying $\operatorname{supp}[\hat{u}(\xi)]\subset\{\xi;|\xi|>R+1\}$, we have the following estimate*

$$||Au||\geq\Big(\frac{\delta}{2}-c_1R^{-1}-c_2\sum_{j=1}^{\infty}M_jR^{-(j/m)}\Big)||u||.$$

2. 3. Outline of the proof of Theorem 1.

Under the condition (B_1) or (B_2), we can easily obtain the theorem by applying the result under the condition (A). Thus we give the outline of the proof of the theorem under the condition (A).

Reduction to a system of first order

We put $u\equiv 0$ when $t\leq 0$, then u remains as a solution of $Pu=0$. When we perform a Holmgren's transformation, all the conditions in the theorem are invariant in a neighbourhood of the origin. Moreover, modifying the coefficients out of the neighbourhood of the origin, we can assume

$$|P_{mp-1}(x, t; \xi, \tau)|_{\tau=\lambda_j(x,t;\xi)}|\geq\delta_0|\xi|^{mp-1},$$

where δ_0 is a positive constant (paragraph 3. 1).

Next, in paragraph 3. 2 we reduce the equation to a system of first order regarding $(P_p)^m+P_{mp-1}$ as the principal part, in the same way as S. Mizohata-Y. Ohya [13], then we have[5)]

$$D_tU=HU+BU+GU+U_0,$$

where D_t-H is the principal part of the new equation. Then the characteristic roots of $\det(\mu I-H(x, t; \xi))=0$ can be expanded with rsepect to $|\xi|^{-(1/m)}$ in the sense of Puiseux by virtue of the condition (A) and they are distinct (paragraph 3. 3). Let λ_i be real when $1\leq i\leq p_1$, and non-real when $p_1+1\leq i\leq p$.

Lemma 3. 3. *The characteristic roots* $\{\mu_i^{(j)}\}_{\substack{1\leq i\leq p\\ 1\leq j\leq m}}$ *are expanded in the following manner,*

$$\mu_i^{(j)}(x, t; \xi)=\lambda_i(x, t; \xi)+\sum_{k=1}^{\infty}\nu_{i,k}^{(j)}(x, t; \xi)|\xi|^{1-(k/m)},$$

$$(\nu_{i,1}^{(j)})^m=\sqrt{-1}P_{mp-1}(x, t; \xi, \tau)|_{\tau=\lambda_i(x,t;\xi)}/\prod_{k\neq i}(\lambda_i(x, t; \xi)-\lambda_k(x, t; \xi))^m,$$

for $1\leq i\leq p, 1\leq j\leq m$, *and where* $\nu_{i,k}^{(j)}$ *are homogeneous of order 0 with respect to* ξ *and belong to* $C_{(x,t)}^{l+5}\times C_\xi^\infty$.

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5) In the case when $l=2$, we encounter the difficulty. That is, in this case, in general we cannot expect $\lambda_i(x, t; \xi)$ to be one-valued on the unit circle S_ξ^1. Moreover, even if $\lambda_i(x, t; \xi)$ is one-valued, we cannot expect $\mu_i^{(j)}(x, t; \xi)$ to be one-valued in general. On this subject, see A.5 in Appendices.

Then the imaginary part of $\nu_{i,1}^{(j)}$ never vanishes $(1\leq i\leq p_1)$.

Now, in paragraph 3. 4 let us construct the diagonalizator $\mathscr{N}(x, t; \xi)$ of $H(x, t; \xi)$. Let us put $\mathscr{N}(x, t; \xi)=(n_{ij}(x, t; \xi))$, then we have

$$n_{ij}\equiv \prod_{k=j-p[j/p]+1}^{p} (\mu_r^{(s)}-\lambda_k)\{\nu_{r,1}^{(s)}\prod_{k\neq r}(\mu_r^{(s)}-\lambda_k)\}^{m-[j/p]-1} \quad \text{mod. l.o.t.},$$

where $r=i-p[(i-1)/p]$, $s=[(i-1)/p]+1$.

Because $\mu_i^{(j)}$ is not homogeneous, $\mathscr{N}(x, t; \xi)$ degenerates near the point at infinity. So the operator with the symbol $\mathscr{M}=\mathscr{N}^{-1}$ is not bounded, but by the detailed consideration we can see that the order of $m_{ij}(x, t; D_x)$, the (i, j)-element of $\mathscr{M}$, is at most $1-(1/m)([(i-1)/p]+1)$.

The above fact gives us $||\mathscr{N}U||\geq\text{const.}||(\Lambda+1)^{-1+(1/m)}U||$ if we restrict h sufficiently small.

Energy with a weight function

Now, operating $\mathscr{N}$ to $PU=0$, we have

$$\mathscr{N}PU\equiv D_t\mathscr{N}U-\mathscr{D}\mathscr{N}U-\mathscr{N}_t'U-(\mathscr{N}H-\mathscr{D}\mathscr{N})U$$
$$-\mathscr{N}BU-\mathscr{N}GU-\mathscr{N}U_0=0,$$

where $\mathscr{D}$ is a diagonal matrix whose diagonal elements are $\mu_i^{(j)}$. In paragraph 4. 2 and 4. 3 let us estimate the energy of $\mathscr{N}PU$ with a weight function $\varphi_n(t)=(t+h)^{-n}$, namely $E_n=\int_0^h\varphi_n^2(t)||\mathscr{N}PU||^2dt$. Concerning the four terms $\mathscr{N}_t'U$, $(\mathscr{N}H-\mathscr{D}\mathscr{N})U$, $\mathscr{N}B$, and $\mathscr{N}G$ in paragraph 4. 4, we have

$$||\mathscr{N}_t'U||\leq\text{const.}(||\mathscr{N}U||+||(\Lambda+1)^{-1}U||),$$
$$||(\mathscr{N}H-\mathscr{D}\mathscr{N})U||\leq\text{const.}(||\mathscr{N}U||+||(\Lambda+1)^{-1}U||),$$
$$||\mathscr{N}BU||\leq\text{const.}(||\mathscr{N}U||+||(\Lambda+1)^{-1}U||),$$
$$||\mathscr{N}GU||\leq\text{const.}(||\mathscr{N}U||+||(\Lambda+1)^{-1}U||).$$

Then a slight modification of the Calderón's argument in [1] (see also S. Mizohata [12]) gives the following proposition.

Proposition 1. *There exists a sufficiently small h depending only on P such that for sufficiently large n we have*

$$E_n = \int_0^h \varphi_n^2(t)\,||\mathscr{N}PU(t)||^2 dt$$

$$\geq \frac{c_1}{n}\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||\partial_t^{mp-j}u(t)||_{j-1}^2 dt$$

$$+\frac{c_2 n}{h^2}\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1+(1/m)}\partial_t^{mp-j}u(t)||_{j-1}^2 dt.$$

In this estimate, $||\cdot||_{j-1}$ *means* $\mathscr{D}_{L^2}^{j-1}(\boldsymbol{R}_x^l)$*-norm, and the constants* c_1, c_2 *are independent of* u, h *and* n.

On the other hand since $\mathscr{N}PU=0$, we have $E_n=0$. This implies, by virtue of the above inequality, that $u\equiv 0$ in a n.b.d. of the origin. So we have Theorem 1 under the condition (A).

3. Reduction to a system of first order.

From now on we start out to prove Theorem 1. First we consider the theorem under the condition (A).

3. 1. Modification of coefficients and solution.

First we put $u(x, t)\equiv 0$ for $t\leq 0$. So $u(x, t)$ is still m-times continuously differentiable in $\boldsymbol{R}^{l+1}$ because of $\partial_t^j u(x, 0)=0\,(0\leq j\leq mp-1)$ and $Pu=0$. Let us transform the coordinates, $\tilde{x}_j=x_j$, $\tilde{t}=t+x_1^2+x_2^2+\cdots+x_l^2$ (Holmgren's transformation). Then $u(x, t)\equiv 0$ if $\tilde{t}\leq|\tilde{x}|^2$. Regarding $\tilde{t}$ as a parameter, we have $\mathrm{supp}[u(\tilde{x}, \tilde{t})]\subset\{\tilde{x};\,|\tilde{x}|^2\leq h\}$ for $0\leq\tilde{t}\leq h$. By $\tilde{P}_p$ we mean the normalized homogeneous principal part of the transformed operator. Namely the symbol of $\tilde{P}_p$ is defined by

$$P_p(\tilde{x}, \tilde{t}-\sum_{j=1}^{l}\tilde{x}_j^2;\,\xi+2\tilde{x}\tau, \tau)/P_p(\tilde{x}, \tilde{t}-\sum_{j=1}^{l}\tilde{x}_j^2;\,2\tilde{x}, 1).$$

And similarly we shall use the same convention for general differential operators. Thus $\tilde{P}_p$ satisfies the same condition as P_p stated in Theorem 1 in a n.b.d. of the origin. Moreover it holds $\tilde{P}_p=P_p$, $\tilde{P}_{mp-1}=P_{mp-1}$ mod. $\prod_{j=1}^{p}(D_t-\lambda_j(0, 0; D_x))$, and $\tilde{\lambda}_j=\lambda_j$ at the origin, thus

$$\tilde{P}_{mp-1}(0, 0;\,\xi, \tau)\,|_{\tau=\lambda_j(0,0;\xi)}=P_{mp-1}(0, 0;\,\xi, \tau)\,|_{\tau=\lambda_j(0,0;\xi)}\neq 0,\ (\xi\in\boldsymbol{R}^l-\{0\}).$$

From these considerations we see that the assumptions are invariant by Holmgren's transformation. From now on we write $x, t, P_p, P_{mp-1}, \lambda_j$ instead of $\tilde{x}, \tilde{t}, \tilde{P}_p, \tilde{P}_{mp-1}, \tilde{\lambda}_j$ respectively. From

$$P_{mp-1}(0, 0; \xi, \tau)|_{\tau=\lambda_j(0,0;\xi)} \neq 0, \quad \xi \in \boldsymbol{R}^l - \{0\} \quad (1 \leq j \leq p),$$

it follows that there exists a $(l+1)$-dimensional cube n.b.d. U_0 of the origin such that

$$P_{mp-1}(x, t; \xi, \tau)|_{\tau=\lambda_j(x,t;\xi)} \neq 0 \quad (x, t, \xi) \in U_0 \times (\boldsymbol{R}_\xi^l - \{0\}),$$

where $U_0 = U_{0,x} \times [0, h]$. Because of the compactness of $\frac{1}{2} U_0 \times \boldsymbol{S}_\xi^{l-1}$ and the homogeneity of $P_{mp-1}|_{\tau=\lambda_j}$ in ξ, there exists $\delta_0 > 0$ such that

$$(1) \quad |P_{mp-1}(x, t; \xi, \tau)|_{\tau=\lambda_j(x,t;\xi)}| \geq \delta_0 |\xi|^{mp-1}, \quad (x, t, \xi) \in \tfrac{1}{2} U_0 \times (\boldsymbol{R}_\xi^l - \{0\}).$$

Now we change all the coefficients appearing in P by replacing $\eta(x/r)$ instead of x, where $\eta(x)$ is defined by $\eta(x) = \theta(|x|)(x/|x|)$, $\theta(s) \in C^\infty(\boldsymbol{R}_+^1)$ and

$$\theta(s) = \begin{cases} s & 0 \leq s \leq \frac{1}{2}, \\ 0 & s \geq 1, \end{cases} \quad 0 \leq \theta(s) \leq 1. \quad \text{After this modification}$$

we can make small the oscillations of the coefficients and those of their derivatives in ξ up to order $2l$ by taking r small. Since for sufficiently small r we have $\{|x| \leq r\} \subset \frac{1}{2} U_{0,x}$. Then, if we restrict t to $0 \leq t \leq h'$ where $h' \leq (r^2/16)$, we have $\text{supp}[u] \subset \frac{1}{4} U_{0,x}$. Accordingly since the coefficients are not modified on $\frac{1}{4} U_{0,x} \times [0, h']$, $u(x, t)$ remains the solution of $Pu = 0$ and moreover $|P_{mp-1}(\eta(x/r), t; \xi, \tau)|_{\tau=\lambda_j}| \geq \delta_0 |\xi|^{mp-1}$ in $\boldsymbol{R}_x^l \times [0, h'] \times (\boldsymbol{R}_\xi^l - \{0\})$. We write h instead of h', and denote the modified operators by the same notation.

We note that in this modification we can assume that the oscilations of $\partial_\xi^\nu \lambda_j(x, t; \xi)$ $(|\nu| \leq 2l)$ are sufficiently small on $\boldsymbol{S}_\xi^{l-1}$.

3.2. Transformation of the base.

From paragraph 3.2 to paragraph 3.4 we use the same method as in Mizohata-Ohya [13]. We assume that λ_j is real when $1 \leq j \leq p_1$, and that λ_j is non-real when $p_1 + 1 \leq j \leq p$.

Put

$$\lambda_j(x, t; \xi) = \lambda_{j-p[(j-1)/p]}(x, t; \xi)^{6)} \qquad 1\leq j\leq mp,$$

$$\partial_j = D_t - \lambda_j(x, t; D_x) \qquad 1\leq j\leq mp,$$

$$\partial_0 = 1,$$

$$\Pi_0 = \partial_{mp}\partial_{mp-1}\cdots\partial_1,$$

$$i^{mp-1}\Pi_1 = i^{mp}(P_p(x, t; D_x, D_t))^m + i^{mp-1}P_{mp-1}(x, t; D_x, D_t) - i^{mp}\Pi_0.$$

Lemma 3.1. 1) *For all $j\geq 0$ there exist $a_i(x, t; D_x)$, ψ.d.o.'s of order i $(0\leq i\leq j$ and $a_0(x, t; D_x)=1)$ such that*

$$\partial_j\partial_{j-1}\cdots\partial_1 = \sum_{i=0}^{j} a_i(x, t; D_x)D_t^{j-i} + T_1,$$

where $T_1=\sum_{i=0}^{j-1}c_i(x, t; D_x)D_t^{j-i-1}$, order$(c_i)\leq i$.

2) *Conversely for all $j\geq 0$ there exist $b_i(x, t; D_x)$, ψ.d.o.'s of order i $(0\leq i\leq j$ and $b_0(x, t; D_x)=1)$ such that*

$$D_t^j = \sum_{i=0}^{j} b_i(x, t; D_x)\partial_{j-i}\partial_{j-i-1}\cdots\partial_0 + T_2,$$

where $T_2=\sum_{i=0}^{j-1}d_i(x, t; D_x)\partial_{j-i-1}\partial_{j-i-2}\cdots\partial_0$, order*order*$(d_i)\leq i$.

Proof. This is easily proved by induction on j. Q.E.D.

Corollary 3.1. *Every partial differential operator of order k has the expression*

$$\sum_{i=0}^{k} c_i(x, t; D_x)\partial_{k-i}\partial_{k-i-1}\cdots\partial_0 + T,$$

where $c_i(x, t; D_x)$ are ψ.d.o.'s of order i, and T is an operator of order at most $k-1$ for x and t. (More precisely, T, has the following form $T=\sum_{i=0}^{k-1}e_i(x, t; D_x)\partial_{k-i-1}\partial_{k-i-2}\cdots\partial_0$, order$(e_i)\leq i$.)

Proof. For $|\alpha|+j=k$,

$$\partial_x^\alpha\partial_t^j = a(D_x)\partial_t^j$$

[6] $\lambda_{ip+j}=\lambda_j$ for $1\leq j\leq p$ $(0\leq i\leq m-1)$. Then when $1\leq j\leq p_1$, λ_{ip+j} is real and when $p_1+1\leq j\leq p$, λ_{ip+j} is non-real.

$$= a(D_x) \sum_{i=0}^{j} b_i(x, t; D_x) \partial_{j-i} \partial_{j-i-1} \cdots \partial_0 + T'$$

$$= \sum_{i=0}^{j} (a \circ b_i)(x, t; D_x) \partial_{j-i} \partial_{j-i-1} \cdots \partial_0 + T,$$

where $a(x, t; D_x)$ and $b_i(x, t; D_x)$ are ψ.d.o.'s of order $|\alpha|$ and i, respectively, and T and T' are operators of order at most $k-1$ for x, t.

Q.E.D.

Corollary 3.2. *For any non-positive s there exist positive constants c_1 and c_2 such that*

$$c_1 \sum_{i=0}^{k} ||(\Lambda+1)^i D_t^{k-i} u||_s \leq \sum_{i=0}^{k} ||(\Lambda+1)^i \partial_{k-i} \cdots \partial_0 u||_s$$

$$\leq c_2 \sum_{i=0}^{k} ||(\Lambda+1)^i D_t^{k-i} u||_s,$$

for any $u(x, t) \in \bigcap_{j=0}^{k} \mathcal{E}_t^{k-j}(\mathcal{D}_{L^2}^j)$, $(k \geq 0)$.

Proof. This follows immediately from Lemma 3.1. Q.E.D.

In view of Corollary 3.1, we have the following expression,

$$\Pi_1 = a_{mp-1}(x, t; D_x) + a_{mp-2}(x, t; D_x) \partial_1 + \cdots$$

$$+ a_{mp-i}(x, t; D_x) \partial_{i-1} \cdots \partial_1 + \cdots + a_0 \partial_{mp-1} \cdots \partial_1 + T,$$

where $a_j(x, t; D_x)$ are homogeneous ψ.d.o.'s of order j, and T is an operator of order at most $mp-2$ for x, t. To make clear our essential assumption stated in Theorem 1, we introduce

Definition 3.1.

$$L_j(x, t; \xi) = \Pi_1^0(x, t; \xi, \lambda_j(x, t; \xi))$$

$$= a_{mp-1} + a_{mp-2}(\lambda_{j'} - \lambda_1) + \cdots$$

$$+ a_{mp-j'}(\lambda_{j'} - \lambda_{j'-1})(\lambda_{j'} - \lambda_{j'-2}) \cdots (\lambda_{j'} - \lambda_1),$$

where $j' = j - p[(j-1)/p]$.

Lemma 3.2.

$$L_j(x, t; \xi) = P_{mp-1}(x, t; \xi, \tau)|_{\tau = \lambda_j(x, t; \xi)}.$$

Proof. Put

$$\prod_{1\leq j\leq p}^{o} \partial_j = \partial_p \circ \partial_{p-1} \circ \cdots \circ \partial_1.$$

Let us denote by $\sigma_{mp-1}(A)$ the symbol of the homogeneous part of order $mp-1$ of a ψ.d.o. A. Then it sufficies to prove

$$\sigma_{mp-1}((P_p)^m - \Pi_o) \equiv 0 \quad \text{mod.} \prod_{j=1}^{p} (\tau - \lambda_j).$$

$$\begin{aligned}\sigma_{mp-1}(\Pi_o) &= \sigma_{mp-1}((\partial_p \cdots \partial_1)^m) \\ &\equiv \sigma_{mp-1}((\prod_{1\leq j\leq p}^{o} \partial_j)^{m-1}(\partial_p \cdots \partial_1)) \quad \text{mod.} \prod_{j=1}^{p} (\tau - \lambda_j) \\ &\equiv \sigma_{mp-1}((\prod^{o} \partial_j)^{m-1}(\prod^{o} \partial_j)) \quad \text{mod.} \prod_{j=1}^{p} (\tau - \lambda_j) \\ &= \sigma_{mp-1}(P_p)^m. \end{aligned}$$

Q.E.D.

Now let us reduce the equation to a system. Put

$$u_j = (\Lambda + 1)^{mp-j} \partial_{j-1} \cdots \partial_0 u \quad (1 \leq j \leq mp),$$

then

$$\begin{aligned} D_t u_j &= (\Lambda+1)^{mp-j} D_t \partial_{j-1} \cdots \partial_0 u \\ &= (\Lambda+1)^{mp-j} (\partial_j + \lambda_j) \partial_{j-1} \cdots \partial_0 u \\ &= (\Lambda+1) u_{j+1} + \lambda_j u_j + [(\Lambda+1)^{mp-j}, \lambda_j] \partial_{j-1} \cdots \partial_0 \\ &= (\Lambda+1) u_{j+1} + \lambda_j u_j + b_j u_j, \qquad (1 \leq j \leq mp-1), \end{aligned}$$

where $b_j(x, t; D_x) = [(\Lambda+1)^{mp-j}, \lambda_j](\Lambda+1)^{-(mp-j)}$. Denote $a_j'(x, t; \xi) = a_j(x, t; \xi')$, $\xi' = (\xi/|\xi|)$, then

$$\begin{aligned} \Pi_1^o u &= \sum_{j=1}^{mp} a'_{mp-j}(x, t; D_x) \Lambda^{mp-j} \partial_{j-1} \cdots \partial_0 u \\ &= \sum_{j=1}^{mp} a'_{mp-j}(x, t; D_x) u_j + \sum_{j=1}^{mp} g_j u_j, \end{aligned}$$

where $g_j(D_x) = a'_{mp-j}(x, t; D_x)\{\Lambda^{mp-j} - (\Lambda+1)^{mp-j}\}(\Lambda+1)^{-(mp-j)}$. We note that $\text{order}(g_j) \leq -1$. From the equation we have

$$D_t u_{mp} = \lambda_m u_{mp} + i \sum_{j=1}^{mp} a'_{mp-j} u_j + i \sum_{j=1}^{mp} g_j u_j + Tu + T'u,$$

where T' and T are the lower order terms of P and Π_1, respectively, and they are of order at most $mp-2$ for x, t. So we arrive at the following system

(2) $$D_t U = H(x, t; D_x) U + B(x, t; D_x) U + G(x, t; D_x) U + U_0,$$

where

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_{mp} \end{pmatrix}, \qquad U_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (T+T')u \end{pmatrix}.$$

$$H(x, t; D_x) = \begin{pmatrix} \lambda_1 & \Lambda & & & 0 \\ & \lambda_2 & \Lambda & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \Lambda \\ ia_{mp-1} & ia_{mp-2} & \cdots & \cdots & \lambda_{mp} + ia_0 \end{pmatrix},$$

$$\lambda_j = \lambda_j(x, t; D_x) = \lambda_{j-p[(j-1)/p]}(x, t; D_x).$$

$$B(x, t; D_x) = \begin{pmatrix} b_1 & 1 & & & 0 \\ & b_2 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & b_{mp} \end{pmatrix},$$

$$b_j = b_j(x, t; D_x) = [(\Lambda+1)^{mp-j}, \lambda_j](\Lambda+1)^{-(mp-j)}, \quad (b_{mp}=0).$$

$$G(x, t; D_x) = \begin{pmatrix} & 0 & \\ ig_1 & ig_2 \cdots & ig_{mp} \end{pmatrix},$$

$$g_j(x, t; D_x) = a'_{mp-j}(x, t; D_x)\{\Lambda^{mp-j} - (\Lambda+1)^{mp-j}\}(\Lambda+1)^{-(mp-j)}.$$

3. 3. Puiseux's expansion of the eigenvalues of $H(x, t; \xi)$. In this paragraph we try to expand the eigenvalues of $H(x, t; \xi)$ in $|\xi|$ in a n.b.d. of the point at infinity.

$\mathrm{Det}(\mu I - H(x, t; \xi))$

$$
= \det \begin{pmatrix}
\mu-\lambda_1 & -|\xi| & & & & & & & \\
 & \mu-\lambda_2 & \ddots & & & & & 0 & \\
 & & \ddots & -|\xi| & & & & & \\
 & & & \mu-\lambda_p & -|\xi| & & & & \\
 & & & & \mu-\lambda_1 & \ddots & & & \\
 & 0 & & & & \ddots & & & \\
 & & & & & & \ddots & -|\xi| & \\
-ia'_{mp-1} & -ia'_{mp-2} & \cdots & -ia'_{(m-1)p} & -ia'_{(m-1)p-1} & \cdots & & \mu-\lambda_p-ia_0' &
\end{pmatrix}
$$

$$
= \sum_{j=1}^{p} (\mu-\lambda_j)^m - i \sum_{j=1}^{mp} a'_{mp-j}(x, t; \xi)|\xi|^{mp-j}(\mu-\lambda_{j-1})\cdots(\mu-\lambda_1).
$$

Now we show

Lemma 3.3. *The roots of* $\det(\mu I - H) = 0$ *can be expanded for* $|\xi| \geq {}^{\exists}R_1$ *in the following manner.*

$$
\mu_i^{(j)}(x, t; \xi) + \sum_{k=1}^{\infty} \nu_{i,k}^{(j)}(x, t; \xi)|\xi|^{1-(k/m)},
$$

where

$$
(\nu_{i,1}^{(j)})^m = \frac{\sqrt{-1}L_i(x, t; \xi')}{\prod_{k\neq i}(\lambda_i(x, t; \xi') - \lambda_k(x, t; \xi'))^m}
$$

$$
(1 \leq i \leq p,\ 1 \leq j \leq m)
$$

and $\nu_{i,k}^{(j)}(x, t; \xi) \in C_{(x,t)}^{l+5} \times C_{\xi}^{\infty}$.

Proof. Put $\mu = \lambda_i + \nu$, then $\det(\mu I - H) = 0$ becomes

$$
\nu^m \prod_{k\neq i} (\lambda_i - \lambda_k + \nu)^m
$$

$$
= i \sum_{j=1}^{mp} a'_{mp-j}|\xi|^{mp-j}(\lambda_i - \lambda_{j-1} + \nu)\cdots(\lambda_i - \lambda_1 + \nu).
$$

(the right side) $= \sqrt{-1}L_i(x, t; \xi) + b_{mp-2}(x, t; \xi)\nu + b_{mp-3}(x, t; \xi)\nu^2 + \cdots + b_0\nu^{mp-1}$, (the left side) $= \nu^m\{\prod_{k\neq i}(\lambda_i - \lambda_k)^m + c_{m(p-1)-1}(x, t; \xi)\nu + \cdots$

$+\nu^{m(p-1)}\}$, b_j and c_j being homogeneous of order j in ξ. Put $\nu' = \nu/|\xi|$, then

$$\nu'^m\{\prod_{k\neq i}(\lambda_i(x,t;\xi')-\lambda_k(x,t;\xi'))^m + c_{m(p-1)-1}(x,t;\xi')\nu' + \cdots + \nu'^{m(p-1)}\}$$

$$=\frac{1}{|\xi|}\{\sqrt{-1}L_i(x,t;\xi') + b_{mp-2}(x,t;\xi')\nu' + \cdots + b_0(x,t;\xi')\nu'^{mp-1}\}.$$

Because $\prod_{k\neq i}(\lambda_i(x,t;\xi')-\lambda_k(x,t;\xi'))^m$ does not vanish at $(x,t,\xi') \in \boldsymbol{R}_x^l \times [0,h] \times S_{\xi'}^{l-1}$, we can put for $|\nu'| \leq^{\exists} \varepsilon_0$

$$\psi(\nu';x,t;\xi') = \frac{\sqrt{-1}L_i' + \sum_{j=1}^{mp-1} b'_{mp-j-1}\nu'^j}{\prod_{k\neq i}(\lambda_i'-\lambda_k')^m + \sum_{1\leq j\leq m(p-1)} c'_{m(p-1)-j}\nu'^j},$$

where L_i', b_i', c_i', means that we replace ξ' instead of ξ. We verify easily that

$$\psi(\nu';x,t;\xi') \in C^{l+5}_{(x,t)}, \text{ for } |\nu'| \leq \varepsilon_0.$$

Let us remark that we can take ε_0 positive in view of our modification of the coefficients. On the other hand, since $|L_i'| \geq \delta_0$ (from (1) and Lemma 3.2), ψ does not vanish if ν' is sufficiently small. Therefore

$$(\psi(\nu';x,t;\xi'))^{1/m} \in C^{l+5}_{(x,t)} \times C^{\infty}_{\xi'}.$$

From $\nu'^m = (1/|\xi|)\psi(\nu';x,t;\xi')$, we have $\nu' = \omega^j \varepsilon^{1/m}(\psi(\nu';x,t;\xi'))^{1/m}$, $(0 \leq j \leq m-1)$, where $\varepsilon = 1/|\xi|$, and ω is a primitive m-th root of 1. So well-known Lagrange's formula shows

$$\nu' = \sum_{k=1}^{\infty} \nu_{i,k}(x,t;\xi')(\omega^j\varepsilon^{1/m})^k = \sum_{k=1}^{\infty} \nu_{i,k}(x,t;\xi')(\omega^j|\xi|^{-(1/m)})^k,$$

where

$$\nu_{i,k}(x,t;\xi') = \frac{1}{2k\pi i}\int_{|\zeta|=\eta} \frac{\psi(\zeta;x,t;\xi')^{k/m}}{\zeta^k}d\zeta.$$

Since $\psi(\zeta;x,t;\xi')$ is not zero if $\eta(>0)$ is sufficiently small, we have $\nu_{i,k}(x,t;\xi') \in C^{l+5}_{(x,t)} \times C^{\infty}_{\xi'}$. In particular

$$\nu_{i,1}(x,t;\xi') = \frac{1}{2\pi i}\int_{|\zeta|=\eta} \frac{\psi(\zeta;x,t;\xi')^{1/m}}{\zeta}d\zeta$$

$$=\psi(0; x, t; \xi')^{1/m}=\left(\frac{\sqrt{-1}L_i'}{\prod_{k\neq i}(\lambda_i'-\lambda_k')^m}\right)^{1/m}.$$

Finally we put $\nu_{i,k}^{(j)}=\omega^{(j-1)k}\nu_{i,k}$, $\mu_i^{(j)}=\lambda_i+\sum_{k=1}^{\infty}\nu_{i,k}^{(j)}|\xi|^{1-(k/m)}$ $(1\leq j\leq m)$.

Q.E.D.

Remark. 1) If we put

(3) $$M_k=\max_{1\leq i\leq p}\sum_{|\alpha|\leq 2l}\sup|(\partial_\xi^{\alpha})\nu_{i,k}(x, t; \xi)|,$$

where sup is taken over $(x, t; \xi)\in \boldsymbol{R}_x^l\times[0, h]\times \boldsymbol{S}_\xi^{l-1}$, we have easily

$$\sum_{k=1}^{\infty}M_k\theta^{k/m}<\infty \text{ for } \theta\in(0, {}^{\exists}\varepsilon_1),\ \varepsilon_1>0.$$

(From our modification of the coefficients, we can take $\varepsilon_1>0$. See T. Kano [6].) 2) Since we are concerned only with real coefficients, if $1\leq i\leq p_1$ $L_i(x, t; \xi)$ and $\prod_{k\neq i}(\lambda_i-\lambda_k)^m$ are real, then $\nu_{i,1}^{(j)}=\omega^{j-1}\times(\sqrt{-1}L_i/\prod_{k\neq i}(\lambda_i-\lambda_k)^m)^{1/m}$ are non-real.

3.4. Diagonalizator $\mathcal{N}(x, t; \xi)$ of $H_0(x, t; \xi)$.

In this paragraph we aim at the construction of the diagonalizator of $H_0(x, t; \xi)=(H(x, t; \xi)/|\xi|)$, that is, $\mathcal{N}H_o=\mathcal{D}_o\mathcal{N}$,

$$\mathcal{D}_o=\begin{pmatrix} \mu_1^{(1)'} & & & & & & 0 \\ & \ddots & & & & & \\ & & \mu_p^{(1)'} & & & & \\ & & & \mu_1^{(2)'} & & & \\ & & & & \ddots & & \\ & & & & & \mu_p^{(2)'} & \\ 0 & & & & & & \ddots \\ & & & & & & & \mu_p^{(m)'} \end{pmatrix},$$

near the point at infinity for ξ.

From $|\nu_{i,1}^{(j)}(x, t; \xi')|\geq{}^{\exists}\delta\geq 0$, it follows that the eigenvalues of H_o are distinct near the point at infinity, so H_o is diagonalizable. Since the row-vectors of the diagonalizator of H_o are the left eigen-vectors of H_o, so we investigate the left eigenvectors of H_o.

Let us Δ_{ij} be (i, j)-cofactor of $((\mu_r^{(s)}/|\xi|)I-H_o)$, then $(\Delta_{1k}, \cdots, \Delta_{mpk})$ is an eigenvector (possibly 0-vector) corresponding to the eigenvalue $\mu_r^{(s)}/|\xi|$. For general μ,

$$\mu I-H_0=\begin{pmatrix} \mu-\lambda_1'(x,t;\xi) & -1 & & & 0 \\ & \mu-\lambda_2' & \ddots & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & -1 \\ -d_1 & \cdots & \cdots & \cdots & \mu-\lambda_p'-d_{mp} \end{pmatrix},$$

where $\text{degree}(d_j)\leq -1$ $(1\leq j\leq mp)$. Especially $(\varDelta_{11},\cdots,\varDelta_{mp\,1})$ is non-zero vector because $\varDelta_{mp\,1}=1$ for any μ, so it is a global eigenvector with eigenvalue μ.

Let us express explicitly $\varDelta_{j1}$ modulo order -1. A simple observation shows that

$$\varDelta_{j1}=(\mu-\lambda'_{j+1-p[j/p]})\varDelta_{j+1\,1}-d_{j+1}\equiv\prod_{k=j+1}^{mp}(\mu-\lambda_k') \quad \text{mod. order } -1.$$

when we replace $\mu_r^{(s)\prime}$ instead of μ $(\mu_r^{(s)\prime}=\mu_r^{(s)}/|\xi|)$, we have $\varDelta_{j,1}^{(r,s)}=\prod_{k=j+1}^{mp}(\mu_r^{(s)\prime}-\lambda_k')$ mod. order -1. By the relation $\lambda_{ip+j}=\lambda_j$ and the expansion formula of $\mu_r^{(s)}$, we have if $j-p[j/p]+1>r$,

$$\varDelta_{j,1}^{(r,s)}\equiv\omega^{(s-1)(m-[j/p]-1)}\prod_{k=j-p[j/p]+1}^{p}(\lambda_r'-\lambda_k')\{\prod_{k\neq r}(\lambda_r'-\lambda_k')\nu_{r,1}^{(1)\prime}\}^{m-[j/p]-1},$$

and if $j-p[j/p]+1\leq r$,

$$\varDelta_{j,1}^{(r,s)}\equiv\omega^{(s-1)(m-[j/p])}\prod_{\substack{k=j-p[j/p]+1\\ k\neq r}}^{p}(\lambda_r'-\lambda_k')\prod_{k\neq r}(\lambda_r'-\lambda_k')^{m-[j/p]-1}(\nu_{r,1}^{(1)\prime})^{m-[j/p]}.$$

We understand the above formula in the following sense: If the right-hand side is of degree>-1, then the principal part of $\varDelta_{j,1}^{(r,s)}$ is given by the right-hand side; if the right-hand side is of degree -1, then the above formula means merely $\text{degree}(\varDelta_{j,1}^{(r,s)})=-1$.

From the above relation, it follows that

$$\text{degree}(\varDelta_{j,1}^{(r,s)})\leq\frac{-1}{m}\left(m-\left[\frac{j-r}{p}\right]-1\right)=\frac{1}{m}\left(\left[\frac{j-r}{p}\right]+1\right)-1,$$

and if $[(j-r)/p]+1>0$, it is of true degree.

Let us put $\mathscr{N}(x,t;\xi)=(n_{ij}(x,t;\xi))$ where

$$(4)\qquad n_{ij}=\varDelta_{j,1}^{(r,s)},\ r=i-p\left[\frac{i-1}{p}\right],\ s=\left[\frac{i-1}{p}\right]+1\ \ (\text{i.e. } i=(s-1)p+r).$$

More precisely, we denote by $\varDelta^{(r,s)}$ the row-vector

$$\Delta^{(r,s)}=(\Delta_{1,1}^{(r,s)},\cdots,\Delta_{mp,1}^{(r,s)})\quad(1\leq r\leq p,\ 1\leq s\leq m),$$

then $\mathcal{N}$ is composed of these mp row-vectors arranged with the following order

$$\Delta^{(1,1)},\ \Delta^{(2,1)},\cdots,\ \Delta^{(p,1)},\ \Delta^{(1,2)},\cdots\ \Delta^{(p,2)},\cdots\ \Delta^{(p,m)}.$$

Obviously it follows that $\text{degree}(n_{ij})\leq(1/m)([(j-r)/p]+1)-1$, and moreover if $[(j-r)/p]+1>0$, the equality holds, where $r=i-p[(i-1)/p]$, $s=[(i-1)/p]+1$.

Therefore we have the following

Lemma 3.4. Det $\mathcal{N}(x,t;\xi)$ *is of true degree* $-p(m-1)/2$ *in* $|\xi|\geq R$, $R>(1/\varepsilon_1)$.

Proof. We decompose $\mathcal{N}$ into m^2 blocks $(n_{ij})_{\substack{rp+1\leq i\leq(r+1)p\\ sp+1\leq j\leq(s+1)p}}$, $0\leq r$, $s\leq m-1$. The principal part of det $\mathcal{N}$ is equal to that of the matrix A_m obtained by taking only the diagonal elements in each block. So now we prove that it is of true degree $-p(m-1)/2$. From now on we consider $A_m=(a_{ij})$, $a_{ij}=\begin{cases} n_{ij} & i=j \bmod. p\\ 0 & i\neq j \bmod. p.\end{cases}$ The j-th column-vector of A_m has the following common term

$$\{\nu_{r,1}^{(1)}\prod_{k=1}^{r-1}(\lambda_r'-\lambda_k')\}^{m-[j/p]-1}(\prod_{k=r+1}^{p}(\lambda_r'-\lambda_k'))^{m-[j/p]},\quad\left(r=j-p\left[\frac{j-1}{p}\right]\right),$$

so that

$$\det A_m=(-1)^{p(p-1)m(m+1)/4}(\prod_{r=1}^{p}\nu_{r,1}^{(1)})^{m(m-1)/2}\prod_{1\leq r<s\leq p}(\lambda_s'-\lambda_r')^{m^2}\det B_m,$$

where B_m is a constant matrix, then $\text{degree}(\det A_m)=-p(m-1)/2$ if $\det B_m\neq 0$.

Finally, we show that $\det B_m\neq 0$. Put $B_s=((i,j)\text{-block})_{1\leq i,j\leq s}$, where (i,j)-block is $\omega^{(i-1)(s-j)}I$ and I is the unit matrix of order p. Now, let us show $\det B_s=\omega^{p(s-1)(s-2)/2}\prod_{i=1}^{s-1}(1-\omega^i)^p\det B_{s-1}$. In B_s let us subtract the $(j+p)$-th row-vector from the j-th row-vector, then

$$(i,s)\text{-block}=\begin{cases}0 & (1\leq i\leq s-1),\\ I & (i=s),\end{cases}$$

$$(i,j)\text{-block}=\omega^{(i-1)(s-j)}(1-\omega^{s-j})I\quad(1\leq i,j\leq s-1).$$

Therefore $\det B_s = \det C_{s-1}$, where C_{s-1} is a matrix of order $(s-1)p$ with (i,j)-block $=\omega^{(i-1)(s-j)}(1-\omega^{s-j})I$. The i-th row-vector and the j-th column-vector of C_{s-1} have common factors $\omega^{[j-1/p]-1}$ and $1-\omega^{s-[j-1/p]-1}$, respectively. Thus

$$\det B_s = \det C_{s-1} = \omega^{p(s-1)(s-2)/2} \prod_{i=1}^{s-1} (1-\omega^i)^p \det B_{s-1}.$$

Starting from $\det B_1 = 1$, step by step we have $\det B_m \neq 0$. Q.E.D.

We modify $\mathcal{N}(x,t;\xi)$ to $\mathcal{N}'(x,t;\zeta(\xi))$ in the same way as paragraph 2.2 in taking account of Remark of Lemma 3.3. Hereafter we denote $\mathcal{N}(x, t; \zeta(\xi))$ simply by $\mathcal{N}(x, t; \xi)$. Put $\mathcal{M}(x, t; \xi) = (m_{ij}(x,t;\xi)) = \mathcal{N}^{-1}(x,t;\xi)$, then we see that $\mathcal{D}(x,t;\xi) = \mathcal{N}(x,t;\xi) H(x,t;\xi)\mathcal{M}(x,t;\xi)$ is a diagonal matrix, namely,

$$\mathcal{D} = \begin{pmatrix} \mu_1^{(1)}(x,t;\xi) & & & & & & & 0 \\ & \mu_2^{(1)}(x,t;\xi) & & & & & & \\ & & \ddots & & & & & \\ & & & \mu_p^{(1)} & & & & \\ & & & & \mu_1^{(2)} & & & \\ & & & & & \mu_2^{(2)} & & \\ & 0 & & & & & \ddots & \\ & & & & & & & \mu_p^{(m)} \end{pmatrix},$$

when $|\xi| > R+1$. Then we have from Lemma 3.4 and (4),

$$\text{degree}\,(m_{ij}(x,t;\xi)) \leq 1 - \frac{1}{m}\left(\left[\frac{i-1}{p}\right] + 1\right).$$

4. Proof of Theorem 1 under the condition (A).

4.1. Energy inequality with a weight function.

From now on we denote several different constants indepent of u, h, and n simply by c.

We operate $\mathcal{N}(x,t;D_x)$ to our system and denote it $\tilde{P}U$. Put $V = \mathcal{N}U$, then we have

(5) $\tilde{P}U \equiv D_t V - \mathcal{D}V - \mathcal{N}_t' U - (\mathcal{N}H - \mathcal{D}\mathcal{N})U - \mathcal{N}BU - \mathcal{N}GU - \mathcal{N}U_0.$

We use $\varphi_n(t) = (t+h)^{-n}$ as the weight function and consider the energy

$$E_n = \int_0^h \varphi_n^2(t) \| \tilde{P}U(t) \|^2_{L_x^2} dt,$$

where $\|V\|^2_{L_x^2} = \sum_{j=1}^{mp} \|v_j\|^2_{L_x^2}$ $\left(V = \begin{pmatrix} v_1 \\ \vdots \\ v_{mp} \end{pmatrix} \right)$, then $E_n = 0$ for all n because of $\tilde{P}U = 0$. However we have the following proposition.

Proposition 1. *There exists a sufficiently small h depending only on P such that for sufficient large n we have*

$$E_n = \int_0^h \varphi_n^2(t) \| \tilde{P}U(t) \|^2 dt \geq \frac{c_1}{n} \sum_{j=1}^{mp} \int_0^h \varphi_n^2(t) \| \partial_t^{mp-j} u(t) \|^2_{j-1} dt$$

$$+ \frac{c_2 n}{h^2} \sum_{j=1}^{mp} \int_0^h \varphi_n^2(t) \| (\Lambda+1)^{-1+(1/m)} \partial_t^{mp-j} u(t) \|^2_{j-1} dt.$$

where $\| \cdot \|_{j-1}$ means $\mathcal{D}^{j-1}_{L^2}(\boldsymbol{R}_x^l)$-norm, and the constants c_1, c_2 are independent of u, h and n.

Before proving Proposition 1, we consider the estimate $\mathcal{N}U$. For this purpose we prove

Lemma 4.1. *For all pair (s, s'), $0 \leq s \leq s'$, there exists a constant $c(s, s', l)$, such that, for all $u \in \mathcal{D}^{s'}_{L^2}(|x| \leq d)$, it holds*

$$\|u\|_s \leq c d^{s'-s} \|u\|_{s'},$$

where d is an arbitrary positive number.

Proof. This lemma is a result of H. Kumanogo-M. Nagase [8]. Here we present a short proof. Since the result is trivial when $d \geq 1$, we assume $d < 1$. Let us put $x' = (\delta/d)x$, $\xi' = (d/\delta)\xi$, $v(x') = u((d/\delta)x')$, then we have $dx = (d/\delta)^l dx'$, $d\xi = (\delta/d)^l d\xi'$, $u(x) = v(x')$ and $\mathrm{supp}[v] \subset B_\delta$ where B_δ is the ball with radius δ in $\boldsymbol{R}^l$. Since $\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx = (d/\delta)^l \int e^{-ix'\cdot\xi'} v(x') dx' = (d/\delta)^l \hat{v}(\xi')$,

$$\|\Lambda^s u\|^2 = \int |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = \left(\frac{d}{\delta}\right)^{l-2s} \int |\xi'|^{2s} |\hat{v}(\xi')|^2 d\xi'.$$

For the estimate of the right side, we prove the following lemma.

Lemma. *If we fix δ sufficiently small, we have the following property for any $v(x)\in\mathcal{D}(B_\delta)$:*

$$\int_{|\xi|<1}|\hat{v}(\xi)|^2d\xi\leq\int_{|\xi|\geq1}|\hat{v}(\xi)|^2d\xi.$$

Proof. Since $|\hat{v}(\xi)|\leq\int|v(x)|dx\leq\{\mathrm{vol}(B_\delta)\int|v(x)|^2dx\}^{1/2}=\sqrt{\mathrm{vol}(B_\delta)}\,||v||$, we have $\int_{|\xi|<1}|\hat{v}(\xi)|^2d\xi\leq\mathrm{vol}(B_\delta)\,\mathrm{vol}(B_1)\,||v||^2$. Let us fix δ in such a way that $\mathrm{vol}(B_\delta)\cdot\mathrm{vol}(B_1)\leq\frac{1}{2}$, then

$$\int_{|\xi|<1}|\hat{v}|^2d\xi\leq\frac{1}{2}\int_{R^l}|\hat{v}|^2d\xi.$$

Therefore $\int_{|\xi|\geq1}|\hat{v}|^2d\xi=\int_{R^l}|\hat{v}|^2d\xi-\int_{|\xi|<1}|\hat{v}|^2d\xi\geq\frac{1}{2}\int_{R^l}|\hat{v}|^2d\xi$, that is,

$$\int_{|\xi|<1}|\hat{v}|^2d\xi\leq\frac{1}{2}\int_{R^l}|\hat{v}|^2d\xi\leq\int_{|\xi|\geq1}|\hat{v}|^2d\xi.\qquad\text{Q.E.D.}$$

In virtue of the above lemma $||\Lambda^su||^2$ is estimated as follows

$$\left(\frac{d}{\delta}\right)^{l-2s}\int_{|\xi'|\geq1}|\xi'|^{2s}|v(\xi')|^2d\xi'\leq\left(\frac{d}{\delta}\right)^{2s'-2s}\left(\frac{d}{\delta}\right)^{l-2s'}$$

$$\times\int_{|\xi|\geq1}|\xi'|^{2s'}|v(\xi')|^2d\xi'\leq\left(\frac{d}{\delta}\right)^{2(s'-s)}||\Lambda^{s'}u||^2.$$

Then we have

$$||\Lambda^su||\leq2\left(\frac{d}{\delta}\right)^{s'-s}||\Lambda^{s'}u||\leq2\delta^{-s'}d^{s'-s}||\Lambda^{s'}u||,$$

$$||u||\leq2\left(\frac{d}{\delta}\right)^{s'}||\Lambda^{s'}u||\leq2\delta^{-s'}d^{s'-s}||\Lambda^{s'}u||,$$

that is,

$$||u||_s\leq c(s)\,(||\Lambda^su||+||u||)\leq c(s)\,4\delta^{-s'}d^{s'-s}||\Lambda^{s'}u||$$

$$\leq\mathrm{Const.}(s,s',l)\,d^{s'-s}||u||_{s'}.$$

Since $\mathcal{D}$ is dense in $\mathcal{D}^{s'}_{L^2}$ under the topology of $\mathcal{D}^{s'}_{L^2}$, so the above result remains true for all $u\in\mathcal{D}^{s'}_{L^2}(|x|\leq d)$. Q.E.D.

Remark. When $s'\geq 0>s$, the above proof implies the following inequality,

$$||u||_s\leq c(s, s', l)d^\rho||u||_{s'} \quad \text{for all} \quad u\in\mathcal{D}_{L^2}^{s'}(|x|\leq d),$$

where $\rho=s'+(-ls)/(l-2s)$.

More generally, when $s'\geq -l/2$ and $s'>s$, for each $\varepsilon>0$, there exists a sufficiently small $d>0$ such that

$$(*) \qquad ||u||_s\leq\varepsilon||u||_{s'} \quad \text{for all} \quad u\in H_{B_d}^{s'}(\boldsymbol{R}^l),$$

(see F. Trèves: Linear partial differential equations with constant coefficients (Goadon and Breach, New York, 1966) Theorem 0.41).

But, when $-l/2>s'>s$, considering Dirac's δ-function, it seems that the inequality like (*) is not correct for arbitrary small $\varepsilon>0$.

Here, we need only the case where $s'>s\geq 0$.

In virtue of the above lemma, we have

Lemma 4.2. *For sufficiently small $h_0>0$ depending only on P, and $k\geq 0$, we have*

$$||(\Lambda+1)^k\mathcal{N}U(t)||\geq c\sum_{j=1}^{mp}||(\Lambda+1)^{k-1+(1/m)}\partial_t^{mp-j}u(t)||_{j-1} \quad 0\leq t\leq h_0,$$

where c is a constant independent of u and t.

Proof.

$$||(\Lambda+1)^{k-1+(1/m)}U||=||(\Lambda+1)^{k-1+(1/m)}\mathcal{M}\circ\mathcal{N}U||$$

$$\leq||(\Lambda+1)^{k-1+(1/m)}\mathcal{M}\mathcal{N}U||+||(\Lambda+1)^{k-1+(1/m)}(\mathcal{M}\mathcal{N}-\mathcal{M}\circ\mathcal{N})U||.$$

Thus taking account of the fact that $\text{order}(n_{ij})\leq 0$, $n_{i\,mp}=1$ and $\text{order}(m_{ij})\leq 1-(1/m)$, we obtain the following inequality,

$$||(\Lambda+1)^{k-1+(1/m)}U||\leq c_0||(\Lambda+1)^k\mathcal{N}U||+c_1\sum_{j=1}^{mp-1}||(\Lambda+1)^{k-1}u_j||,$$

where c_0 and c_1 depend only on P. Therefore

$$||(\Lambda+1)^k\mathcal{N}U||\geq c_0'||(\Lambda+1)^{k-1+(1/m)}U||-c_1'\sum_{j=1}^{mp-1}||(\Lambda+1)^{k-1}u_j||.$$

Applying Corollary 3.2 to the right-hand side we have

$$\|(\Lambda+1)^k\mathscr{N}U\|\geq c_0''\sum_{j=1}^{mp}\|(\Lambda+1)^{k-1+(1/m)}\partial_t^{mp-j}u\|_{j-1}$$

$$-c_1''\sum_{j=2}^{mp}\|(\Lambda+1)^{k-1}\partial_t^{mp-j}u\|_{j-1}.$$

Since $u(x,t)$ belongs to $\mathscr{D}^{mp}(|x|\leq\sqrt{h})$ for $0\leq t\leq h$, Lemma 4.1 gives,

$$\|(\Lambda+1)^{k-1}\partial_t^{mp-j}u\|_{j-1}\leq c_j h^{1/2m}\|(\Lambda+1)^{k-1+(1/m)}\partial_t^{mp-j}u\|_{j-1},$$

where c_j depends only on P and j, $(2\leq j\leq mp)$.

Let us take h sufficiently small such that $c_1''\max_{2\leq j\leq mp}\{c_j\}h^{1/2m}\leq\frac{1}{2}c_0''$, then we have

$$\|(\Lambda+1)^k\mathscr{N}U\|\geq\frac{c_0''}{2}\sum_{j=1}^{mp}\|(\Lambda+1)^{k-1+(1/m)}\partial_t^{mp-j}u\|_{j-1}.\quad\text{Q.E.D.}$$

From now on, we try to estimate the energy $\int_0^h\varphi_n^2(t)\|\tilde{P}U(t)\|^2dt$. In view of the expression (5) of $\tilde{P}U$, introduced at the beginning of paragraph 4.1, we have

$$(6)\qquad \|\tilde{P}U\|\geq\tfrac{1}{2}\|D_tV-\mathscr{D}V\|^2-c(\|\mathscr{N}_t'U\|^2+\|(\mathscr{N}H-\mathscr{D}\mathscr{N})U\|^2$$
$$+\|\mathscr{N}BU\|^2+\|\mathscr{N}GU\|^2+\|\mathscr{N}U_0\|^2).$$

First we consider $\|D_tV-\mathscr{D}V\|^2$. For this purpose, first we define $\gamma_1(s)\in C^\infty(\boldsymbol{R}_+^1)$ such that $0\leq\gamma_1(s)\leq 1$, $\gamma_1(s)=1$ for $s\geq R_0+1$, and $\gamma_1(s)=0$ for $s\leq R_0$. In view of (3) at the end of paragraph 3.3, and also of Lemma 2.6, $R_0(>(1/\varepsilon_1))$ is taken sufficiently large in such a way that, denoting $\hat{V}_1(\xi)=\gamma_1(|\xi|)\hat{V}(\xi)$,

$$\|(\mathrm{Im}\,\mu_i^{(j)})V_1\|\geq\frac{\delta^0}{4}\|(\Lambda+1)^{1-(1/m)}V_1\|\quad(1\leq i\leq p_1),$$

$$\|(\mathrm{Im}\,\mu_i^{(j)})V_1\|\geq\frac{\delta^0}{4}\|(\Lambda+1)V_1\|\quad(p_1+1\leq i\leq p),$$

where $\delta^0=\min\{\inf_{\substack{1\leq i\leq p_1\\1\leq j\leq m}}|\mathrm{Im}\,\nu_{i,1}^{(j)}(x,t;\xi)|,\ \inf_{p_1+1\leq i\leq p}|\mathrm{Im}\,\lambda_i(x,t;\xi)|\}$ and inf is taken over $(x,t,\xi)\in\boldsymbol{R}_x^l\times[0,h]\times S_\xi^{l-1}$. Moreover, let $\gamma_2(s)=1-\gamma_1(s)$, and $\hat{V}_2(\xi)=\gamma_2(|\xi|)\hat{V}(\xi)$, then we have

$$V=\gamma_1V+\gamma_2V=V_1+V_2.$$

Lemma 4.3. *We have*

$$||D_tV-\mathcal{D}V||^2\geq c(||D_tV_1-\mathcal{D}V_1||^2+||D_tV_2-\mathcal{D}V_2||^2)-c'||V||^2,$$

where c, c' are depending only on P and R_o.

Proof.

$$||D_tV-\mathcal{D}V||^2=||D_tV_1-\mathcal{D}V_1||^2+||D_tV_2-\mathcal{D}V_1||^2$$
$$+2\,\mathrm{Re}[(D_t-\mathcal{D})V_1,\ (D_t-\mathcal{D})V_2].$$
$$[(D_t-\mathcal{D})\gamma_1V,\ (D_t-\mathcal{D})\gamma_2V]=[\gamma_1(D_t-\mathcal{D})V,\gamma_2(D_t-\mathcal{D})V]$$
$$+[\gamma_2[\gamma_1,\mathcal{D}]V,\ (D_t-\mathcal{D})V]+[(D_t-\mathcal{D})V,\gamma_1[\gamma_2,\mathcal{D}]V]$$
$$+[[\gamma_1,\mathcal{D}]V,\ [\gamma_2,\mathcal{D}]V],$$

where $\gamma_1, \gamma_2, [\gamma_1, \mathcal{D}]$ and $[\gamma_2, \mathcal{D}]$ are bounded operators in L^2, so we have

$$|[(D_t-\mathcal{D})V_1,\ (D_t-\mathcal{D})V_2]|\leq c_1||(D_t-\mathcal{D})V||^2+c_2||V||^2.$$

Then

$$||(D_t-\mathcal{D})V||^2\geq||(D_t-\mathcal{D})V_1||^2+||(D_t-\mathcal{D})V_2||^2$$
$$-2c_1||(D_t-\mathcal{D})V||^2-2c_2||V||^2,$$

i.e.

$$||(D_t-\mathcal{D})V||^2\geq c(||(D_t-\mathcal{D})V_1||^2+||(D_t-\mathcal{D})V_2||^2)-c'||V||^2.$$

Q.E.D.

4. 2. Estimate of $\int_0^h\varphi_n^2(t)\,||(D_t-\mathcal{D})V_1(t)||^2dt$.

In this and the next paragraphs we follow essentially the method of A. P. Calderón [1].

Let us decompose $\mu_t^{(j)}(x, t; \xi)$ into the positive order parts $\mu_{i,1}^{(j)}$ and the rest parts $\mu_{i,2}^{(j)}$. We denote the real and the imaginary parts of $-i\mu_{i,1}^{(j)}$ by $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ respectively. More explicitely,

$$\alpha_i^{(j)}(x, t; \xi)=\mathrm{Im}\{\lambda_i(x, t; \xi)+\sum_{k=1}^{m-1}\nu_{i,k}^{(j)}(x, t; \xi)|\xi|^{1-(k/m)}\},$$

$$\beta_i^{(j)}(x, t; \xi)=-\mathrm{Re}\{\lambda_i(x, t; \xi)+\sum_{k=1}^{m-1}\nu_{i,k}^{(j)}(x, t; \xi)|\xi|^{1-(k/m)}\}.$$

Recalling that $D_t=(1/i)\partial_t$ and the form of $\mathcal{D}$ in $|\xi|\geq R_o+1$, we have

$$\int_0^h \varphi_n^2(t)\,\|(D_t-\mathcal{D})V_1\|^2dt$$

$$=\sum_{\substack{1\leq i\leq p\\ 1\leq j\leq m}}\int_0^h \varphi_n^2(t)\,\|(\partial_t-\sqrt{-1}\,\mu_i^{(j)})\,v_{1,(j-1)p+i}\|^2dt$$

$$=\sum_{i,j}\int_0^h \varphi_n^2(t)\,\|\{(\partial_t+\alpha_i^{(j)}+\sqrt{-1}\,\beta_i^{(j)})-\sqrt{-1}\,\mu_{i,2}^{(j)}\}\,v_{1,(j-1)p+i}\|^2dt,$$

where $V_1={}^t(v_{1,1},\cdots,v_{1,mp})$. Now the last integral is greater than

$$\sum_{i,j}\Big\{\frac{1}{2}\int_0^h \varphi_n^2(t)\,\|(\partial_t+\alpha_i^{(j)}+\sqrt{-1}\,\beta_i^{(j)})\,v_{1,(j-1)p+i}\|^2dt$$

$$-c\int_0^h \varphi_n^2(t)\,\|\mu_{i,2}^{(j)}v_{1,(j-1)p+i}\|^2dt\Big\}$$

$$\geq\sum_{i,j}\Big\{\frac{1}{2}\int_0^h \varphi_n^2(t)\,\|(\partial_t+\alpha_i^{(j)}+\sqrt{-1}\,\beta_i^{(j)})\,v_{1,(j-1)p+i}\|^2dt$$

$$-c'\int_0^h \varphi_n^2(t)\,\|v_{1,(j-1)p+i}\|^2dt\Big\}\equiv\sum_{\substack{1\leq i\leq p\\ 1\leq j\leq m}}e_n^{((j-1)p+i)}.$$

From now on we write v, α, β and e_n instead of $v_{1,(j-1)p+i}$, $\alpha_i^{(j)}$, $\beta_i^{(j)}$, and $e_n^{((j-1)p-i)}$.

Let us introduce the following two standard quantities

$$I_n^2=\int_0^h \varphi_n'^2(t)\,\|v(t)\|^2dt \quad\text{and}\quad \rho_n^2I_n^2=\int_0^h \varphi_n^2(t)\,\|\alpha v(t)\|^2dt.$$

Note that by virtue of (3) in Remark of Lemma 3. 3, the definition of $\alpha_i^{(j)}(x,t;D_x)$ and $\mathrm{supp}[\hat{v}(t)]\subset\{\xi;|\xi|\geq R_0\}$, we have

$$(7)\quad\begin{cases}\dfrac{n^2}{4h^2}\displaystyle\int_0^h \varphi_n^2(t)\,\|v(t)\|^2\leq I_n^2\leq\dfrac{n^2}{h^2}\int_0^h \varphi_n^2(t)\,\|v(t)\|^2dt,\\[2ex] \dfrac{\delta^0}{4}\displaystyle\int_0^h \varphi_n^2(t)\,\|(\Lambda+1)^{1-(1/m)}v(t)\|^2dt\leq\rho_n^2I_n^2\\[2ex] \qquad\leq c\displaystyle\int_0^h \varphi_n^2(t)\,\|(\Lambda+1)^{1-(1/m)}v(t)\|^2dt\quad(1\leq i\leq p_1),\\[2ex] \dfrac{\delta^0}{4}\displaystyle\int_0^h \varphi_n^2(t)\,\|(\Lambda+1)\,v(t)\|^2dt\leq\rho_n^2I_n^2\\[2ex] \qquad\leq c\displaystyle\int_0^h \varphi_n^2(t)\,\|(\Lambda+1)\,v(t)\|^2dt\quad(p_1+1\leq i\leq p),\end{cases}$$

where c depends only on P.

Lemma 4.4. *(A. P. Calderón [1], S. Mizohata [12]) We have the following inequality:*

$$\int_0^h \varphi_n^2(t)\,\|(\partial_t+\alpha+i\beta)v(t)\|^2dt$$

$$\geq(\rho_n-1)^2I_n^2+\frac{1}{n}I_n^2-\frac{2c_1h}{n}\rho_nI_n^2-\frac{c_2h}{n}I_n^2-\varphi_n^2(h)\,|[v(h),\alpha v(h)]|.$$

Proof. The proof is the same as those of A. P. Calderón [1], and of S. Mizohata [12]. We need only take care of the above estimates (7), so we omit it. Q.E.D.

From Lemma 4.4, we have

$$2e_n\equiv\int_0^h\varphi_n^2(t)\,\|(\partial_t+\alpha+i\beta)v(t)\|^2dt-c\int_0^h\varphi_n^2(t)\,\|v(t)\|^2dt$$

$$\geq(\rho_n-1)^2I_n^2+\frac{1}{n}I_n^2-\frac{2c_1h}{n}\rho_nI_n^2-\frac{c_2'h}{n}I_n^2-\varphi_n^2(h)\,|[v(h),\alpha v(h)]|$$

$$\geq\left(\rho_n-1-\frac{c_1h}{n}\right)^2I_n^2+\left(1-2c_1h-c_2'h-\frac{c_1^2h^2}{n}\right)\frac{1}{n}I_n^2$$

$$-\varphi_n^2(h)\,|[v(h),\alpha v(h)]|.$$

From now on, we fix h as small as $(2c_1h+c_2'h+c_1^2h^2)\leq\frac{1}{2}$, then for sufficiently large n, it holds

$$2e_n\geq\left(\rho_n-1-\frac{2c_1h}{n}\right)^2I_n^2+\frac{1}{2n}I_n^2-\varphi_n^2(h)\,\|v(h)\|\cdot\|\alpha v(h)\|.$$

Moreover, if $\rho_n\geq(3/2)$, since $\rho_n-1-(2c_1h/n)\geq(1/3)\rho_n$, we have

$$2e_n\geq\frac{1}{9}\rho_n^2I_n^2+\frac{1}{2n}I_n^2-\varphi_n^2(h)\,\|v(h)\|\cdot\|\alpha v(h)\|,$$

and if $\rho_n\leq(3/2)$, neglecting the term $(\rho_n-1-(2c_1h/n))^2I_n^2$, then we have

$$2e_n\geq\frac{1}{4n}I_n^2+\frac{1}{\rho_n^2}\frac{\rho_n^2}{4n}I_n^2-\varphi_n^2(h)\,\|v(h)\|\cdot\|\alpha v(h)\|$$

$$\geq\frac{1}{4n}I_n^2+\frac{1}{9n}\rho_n^2I_n^2-\varphi_n^2(h)\,\|v(h)\|\cdot\|\alpha v(h)\|.$$

In any case, for sufficiently large n we have

$$2e_n \geq \frac{1}{9n}\ \rho_n^2 I_n^2 + \frac{1}{4n}\ I_n^2 - \varphi_n^2(h)\,||v(h)||\cdot||\alpha v(h)||.$$

If $||v(h)||\cdot||\alpha v(h)||\neq 0$, there exists a positive constant ε such that $v(x,t)\not\equiv 0$ in $\boldsymbol{R}^l\times[0,h-\varepsilon]$. Thus

$$I_n^2 \geq \int_0^{h-\varepsilon} \varphi_n'^2(t)\,||v(t)||^2 dt \geq \varphi_n'^2(h-\varepsilon)\int_0^{h-\varepsilon}||v(t)||^2 dt$$

$$\geq \frac{n^2}{4h^2}\left(1+\frac{\varepsilon}{2h}\right)^n \varphi_n^2(h)\int_0^{h-\varepsilon}||v(t)||^2 dt.$$

Finally for sufficiently large n depending on $v(x,t)$ we have

$$e_n \geq \frac{1}{18n}\ (\rho_n^2 I_n^2 + I_n^2).$$

Then, summing up, we have

$$\int_0^h \varphi_n^2(t)\,||(D_t-\mathscr{D})V_1||^2 dt \geq \sum_{i,j} e_n \geq \frac{1}{18n}\sum_{i,j}(\rho_n^2 I_n^2 + I_n^2).$$

4. 3. Estimate of $\int_0^h \varphi_n^2(t)\,||(D_t-\mathscr{D})V_2||^2 dt$.

Let us remark that $\mathscr{D}(x,t;\xi)$ is not a diagonal matrix in $\{\xi;|\xi|\leq R_o+1\}$. But $\hat{V}_2(\xi,t)$ has compact support for ξ, therefore we have

$$||(\varLambda+1)^j V_2|| \leq (R_o+1)^j ||V_2||.$$

Then we have

$$||(D_t-\mathscr{D})V_2||^2 = ||(\partial_t+(\varLambda+1)I-\tilde{\mathscr{D}})V_2||^2$$

$$\geq \tfrac{1}{2}\,||(\partial_t+\varLambda+1)V_2||^2 - ||\tilde{\mathscr{D}}V_2||^2,$$

where $\tilde{\mathscr{D}} = i\mathscr{D}-(\varLambda+1)I$. Because $\tilde{\mathscr{D}}$ is bounded operator on V_2, by the same argument as in paragraph 4. 2 for $\varphi_n^2(t)\,||(\partial_t+\varLambda+1)v_{2,j}(t)||^2 dt$ $(V_2={}^t(v_{2,1},\cdots,v_{2,mp}))$, we have the following estimate,

$$2e_n \equiv \int_0^h \varphi_n^2(t)\,||(\partial_t+\varLambda+1)v_{2,j}||^2 dt - \text{Const.}(\tilde{\mathscr{D}})\int_0^h \varphi_n^2(t)\,||v_{2,j}||^2 dt$$

$$\geq (\rho_n-1)^2 I_n^2 + \frac{1}{n}\ I_n^2 - \frac{2c_1 h}{n}\rho_n I_n^2 - \frac{c_2 h}{n} I_n^2$$

$$-\varphi_n^2(h)\,||v_{2,j}(h)\,||\alpha v_{2,j}(h)\,||-c(R_o)\int_0^h \varphi_n^2(t)\,||v_{2,j}(t)\,||^2 dt,$$

where

$$\rho_n^2 I_n^2=\int_0^h \varphi_n^2(t)\,||(\Lambda+1)\,v_{2,j}(t)\,||^2 dt \text{ and } I_n^2=\int_0^h \varphi_n'^2(t)\,||v_{2,j}(t)\,||^2 dt.$$

However in this case

$$\rho_n^2 I_n^2 \leq (R_o+1)^2 \int_0^h \varphi_n^2(t)\,||v_{2,j}(t)\,||^2 dt$$

$$\leq \frac{8h^2R_o^2}{n^2}\int_0^h \varphi_n'^2(t)\,||v_{2,j}(t)\,||^2 dt=\frac{8h^2R_o^2}{n^2}I_n^2,$$

So we have

$$2e_n \geq I_n^2-\frac{\text{const.}(R_o)}{n}I_n^2-\varphi_n^2(h)\,||v_{2,j}(h)\,||\cdot||\alpha v_{2,j}(h)\,||.$$

Moreover the same argument as in paragraph 4.2 gives us the following estimate for sufficiently large n depending on R_o and $v_{2,j}$:

$$e_n \geq \tfrac{1}{3}I_n^2 \geq \tfrac{1}{4}(\rho_n^2 I_n^2+I_n^2).$$

Then we have

$$\int_0^h \varphi_n^2(t)\,||(D_t-\mathcal{D})\,V_2||^2 dt \geq \sum_{i,j} e_n \geq \tfrac{1}{4}\sum_{i,j}(\rho_n^2 I_n^2+I_n^2).$$

By paragraphes 4.2, 4.3, Lemma 4.3 and $||V||^2 \leq 2(||V_1||^2+||V_2||^2)$, we have the following result,

(8) $$\int_0^h \varphi_n^2(t)\,||(D_t-\mathcal{D})\,V||^2 dt$$

$$\geq \frac{c_1}{n}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{1-(1/m)}V(t)\,||^2 dt+\frac{c_2 n}{h^2}\int_0^h \varphi_n^2(t)\,||V(t)\,||^2 dt,$$

where c_1 and c_2 depend only on P, and n is sufficiently large.

4.4. Matrices of type $(\mathcal{N})$.

In this paragraph we give the estimates of $\mathcal{N}_t'U$, $(\mathcal{N}H-\mathcal{D}\mathcal{N})\,U$, $\mathcal{N}BU$, and $\mathcal{N}GU$. For detailed proofs, we refer to S. Mizohata-Y. Ohya [13].

Definition 4. 1. Let $A=(a_{ij}(x,t;D_x))$ be a matrix of order mp with elements of ψ.d.o.'s. We say that "A is of type $(\mathscr{N})$" when A satisfies the following conditions.

1) $a_{ij}(x,t;D_x)=a_{ij}^{(1)}(x,t;D_x)$ mod. order -1, where $a_{ij}^{(1)}(x,t;\xi)$ are expanded in the sense of Puiseux in $|\xi|^{-(1/m)}$ for $|\xi|\geq{}^{\exists}R$.

2) $$\mathrm{Order}(a_{ij})\leq\frac{1}{m}\left(\left[\frac{j-1}{p}\right]+1\right)-1.$$

Lemma 4. 5. *When A is of type $(\mathscr{N})$, we have the following estimate,*

$$||AU||\leq c||\mathscr{N}U||+c'||AC^jU|| \quad \text{mod. order } -1,$$

where $\mathrm{order}(C^j)\leq-(j/m)$ *and where c and c', depend only on P and j.*

Proof. Throughout this proof we consider always in the sense of mod. order -1. Let us put $C=\mathscr{M}\mathscr{N}-\mathscr{M}\circ\mathscr{N}$ then $\mathrm{order}(C)\leq-(1/m)$ and $A\mathscr{M}$ is of order 0 because A is of type$(\mathscr{N})$. Our proof is carried out by induction on j. The case $j=1$ follows from the fact that $A=A(\mathscr{M}\circ\mathscr{N})=A\mathscr{M}\mathscr{N}-AC$. Assuming the case $j=k$ is true, let us prove for the case $j=k+1$. First,

$$\begin{aligned}AC^k&=AC^k(\mathscr{M}\circ\mathscr{N})=AC^k\mathscr{M}\mathscr{N}-AC^{k+1}\\&=C^kA\mathscr{M}\mathscr{N}+[A,C^k]\mathscr{M}\mathscr{N}-AC^{k+1}.\end{aligned}$$

In virtue of $\mathrm{order}(C^kA\mathscr{M})\leq-(k/m)$ and $\mathrm{order}([A,C^k]\mathscr{M})\leq-(k+1/m)$, the case $j=k+1$ is proved. Q.E.D.

Lemma 4. 6. *$\mathscr{N}_t'$, $(\mathscr{N}H-\mathscr{D}\mathscr{N})$, $\mathscr{N}B$, and $\mathscr{N}G$ are all of type$(\mathscr{N})$.*

Proof. Let us note that $\mathscr{N}$ is of type$(\mathscr{N})$. Since G is of order at most -1, $\mathscr{N}G$ is of type$(\mathscr{N})$. Concerning $\mathscr{N}_t'$, obviously it is of type$(\mathscr{N})$. Concerning $\mathscr{N}B$, since B is a triangular matrix of order 0, $\mathscr{N}B$ is of type$(\mathscr{N})$, too. As regards $\mathscr{N}H-\mathscr{D}\mathscr{N}$, we devide this as $(\mathscr{N}H-\mathscr{N}\circ H)+(\mathscr{D}\circ\mathscr{N}-\mathscr{D}\mathscr{N})$. Then the former is of type$(\mathscr{N})$ because H is triangular mod. order 0, and the latter is also of type$(\mathscr{N})$

since $\mathscr{D}$ is a diagonal matrix of order 1 in a n.b.d. of the point at infinity. Q.E.D.

We are now in a position to prove Proposition 1. In view of (6), the estimate (8) and Lemmas 4.5, 4.6 give

$$(9)\quad E_n \geq \frac{c_1}{n}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{1-(1/m)}\mathscr{N}U||^2dt + \frac{c_2 n}{h^2}\int_0^h \varphi_n^2(t)\,||\mathscr{N}U||^2dt$$

$$-c_3\int_0^h \varphi_n^2(t)\,||\mathscr{N}U||^2dt - c_4\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1}U||^2dt$$

$$-c_5\int_0^h \varphi_n^2(t)\,||U_0||^2dt$$

$$\geq \frac{c_1}{n}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{1-(1/m)}\mathscr{N}U||^2dt + \frac{c_2 n}{2h^2}\int_0^h \varphi_n^2(t)\,||\mathscr{N}U||^2dt$$

$$-c_4\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1}U||^2dt - c_5\int_0^h \varphi_n^2(t)\,||U_0||^2dt.$$

Let us recall that we have started from $PU=D_tU-HU-BU-GU-U_0$ (see (2) in paragraph 3.2). Because T and T' are of order at most $mp-2$, by Lemma 4.2 and Corollary 3.2 we have the following estimate,

$$E_n \geq \frac{c_1'}{n}\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||\partial_t^{mp-j}u||_{j-1}^2dt$$

$$+\frac{c_2' n}{h^2}\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1+(1/m)}\partial_t^{mp-j}u||_{j-1}^2dt$$

$$-c'\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1}\partial_t^{mp-j}u||_{j-1}^2dt$$

$$\geq \frac{c_1'}{n}\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||\partial_t^{mp-j}u||_{j-1}^2dt$$

$$+\frac{c_2' n}{2h^2}\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1+(1/m)}\partial_t^{mp-j}u||_{j-1}^2dt,$$

(n is sufficiently large).

Thus we have proved Proposition 1.

4.5. Proof of Theorem 1 under the condition (A).

By $\tilde{P}U=0$, we have $E_n=0$, so by Proposition 1 we have

$$\sum_{j=1}^{mp}\int_0^h \varphi_n^2(t)\,||(\Lambda+1)^{-1+(1/m)}\partial_t^{mp-j}u(t)||_{j-1}^2dt=0,$$

that is, $u(x,t)\equiv 0$ in $\boldsymbol{R}^l\times[0,h]$. This is equivalent to saying that in the original coordinates $u(x,t)\equiv 0$ in

$$\Omega=\{(x,t);\ |t|\leq -|x|^2+h\}.$$

5. Proof of Theorem 1 under the condition ($\boldsymbol{B}_1$) or ($\boldsymbol{B}_2$).

The condition (B_1) implies that $P_{mp-1}\equiv QP_p$ (mod. order $mp-2$) where Q is a homogeneous partial differential operator of order $(m-1)p-1$, and the condition (B_2) asserts further that $Q(0,0;\xi,\tau)|_{\tau=\lambda_j(0,0;\xi)}\neq 0$ for all $\xi\in\boldsymbol{R}^l-\{0\}$.

When we perform a Holmgren's transformation and modify the solution and coefficients in the same way as paragraph 3.1, the conditions in the theorem are invariant in a neighbourhood of the origin.

So if we regard $P_pu=v$ as the unknown function, we have

$$Pu\equiv(P_p^{m-1}+Q)v+R'u=0.$$

Hence if $m\geq 3$ we apply the estimate under the condition (A) and if $m=2$ we do the modified Calderón's estimate with our weight function, then we have

$$E_n\geq\frac{c_0}{n}\sum_{j=1}^{(m-1)p}\int_0^h\varphi_n^2(t)\,||\partial_t^{(m-1)p-j}v(t)||_{j-1}^2dt$$
$$+\frac{c_0n}{h^2}\sum_{j=1}^{(m-1)p}\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{-1+(1/m)}\partial_t^{(m-1)p-j}v(t)||_{j-1}^2dt$$
$$-c_1\sum_{j=1}^{mp-1}\int_0^h\varphi_n^2(t)\,||\partial_t^{mp-j-1}u(t)||_{j-1}^2dt-c_2\varphi_n^2(h)\,||V(h)||\cdot||V(h)||_1.$$

Hence, $\partial_t^kv=\partial_t^kP_pu=P_p\partial_t^ku+[\partial_t^k,P_p]u$, where $[\partial_t^k,P_p]$ is of order $p+k-1$, so we have

$$E_n\geq\frac{c_0}{n}\sum_{j=1}^{(m-1)p}\int_0^h\varphi_n^2(t)\,||P_p\partial_t^{(m-1)p-j}u(t)||_{j-1}^2dt$$
$$-c_1'\sum_{j=1}^{mp-1}\int_0^h\varphi_n^2(t)\,||\partial_t^{mp-j-1}u(t)||_{j-1}^2dt$$
$$-c_2'(2h)^{-2n}||V(h)||\cdot||V(h)||_1.$$

Once again we apply modified Calderón's estimate with our weight function regarding $\partial_t^{(m-1)p-j}u=v_j'$ as a function, then

$$\int_0^h \varphi_n^2(t)\,||P_p v_j'(t)||_{j-1}^2 dt \geq c\int_0^h \varphi_n^2(t)\,||(D_t-H'(x,t;D_x))\,V_j'(t)||_{j-1}^2 dt$$

$$\geq \frac{c'n}{h^2}\int_0^h \varphi_n^2(t)\,||V_j'(t)||_{j-1}^2 dt - c_2'\varphi_n^2(h)\,||V_j'(h)||_{j-1}||V_j'(h)||_j$$

$$\geq \frac{c''n}{h^2}\sum_{k=1}^{p}\int_0^h \varphi_n^2(t)\,||\partial_t^{mp-j-k}u(t)||_{j+k-2}^2 dt$$

$$-c_2'(2h)^{-2n}||V_j'(h)||_{j-1}||V_j'(h)||_j.$$

Therefore

$$E_n \geq \frac{c}{h^2}\sum_{j=1}^{mp-1}\int_0^h \varphi_n^2(t)\,||\partial_t^{mp-j-1}u(t)||_{j-1}^2 dt$$

$$-c_1'\sum_{j=1}^{mp-1}\int_0^h \varphi_n^2(t)\,||\partial_t^{mp-j-1}u(t)||_{j-1}^2 dt$$

$$-c''(2h)^{-2n}(||V(h)||\cdot||V(h)||_1+\sum_{j=1}^{(m-1)p}||V_j'(h)||_{j-1}||V_j'(h)||_j).$$

Here we replace h with smaller h' depending only on P, then we have

$$E_n \geq \frac{c}{2h'^2}\sum_{j=1}^{mp-1}\int_0^{h'} \varphi_n^2(t)\,||\partial_t^{mp-j-1}u(t)||_{j-1}^2 dt$$

$$-c_2''(h')^{-2n}(||V(h')||\,||V(h')||_1+\sum_{j=1}^{(m-1)p}||V_j'(h')||_{j-1}||V_j'(h')||_j).$$

If

$$||V(h')||\cdot||V(h')||_1+\sum_{j=1}^{(m-1)p}||V_j'(h')||_{j-1}||V_j'(h')||_j\neq 0,$$

We have $u\not\equiv 0$ in $\boldsymbol{R}_x^l\times[0,h']$. So there exists a positive constant ε depending on u such that $u(x,t)\not\equiv 0$ in $\boldsymbol{R}_x^l\times[0,h'-\varepsilon]$. By

$$\int_0^{h'} \varphi_n^2(t)\,||u(t)||_{mp-2}^2 dt \geq \int_0^{h'-\varepsilon} \varphi_n^2(t)\,||u(t)||^2 dt$$

$$\geq (2h'-\varepsilon)^{-2n}\int_0^{h'-\varepsilon}||u(t)||^2 dt>0,$$

we have

$$(2h')^{-2n}(||V(h')||\cdot||V(h')||_1+\sum_{j=1}^{(m-1)p}V_j'(h')||_{j-1}||V_j'(h')||_j)$$

$$=\circ\left(\frac{c}{2h'^2}\int_0^{h'}\varphi_n^2(t)\,||u(t)||^2_{mp-2}dt\right),$$

with respect to n. This asserts that

$$E_n\geq\frac{c}{4h'^2}\sum_{j=1}^{mp-1}\int_0^{h'}\varphi_n^2(t)\,||\partial_t^{mp-j-1}u(t)||^2_{j-1}dt$$

for sufficiently large n.

On the other hand, from the equation we have $E_n=0$. Therefore we have $u(x, t)\equiv 0$ in $\boldsymbol{R}_x^l\times[0, h']$. This completes the proof of Theorem 1.

Appendices

A. 1. An extension of Theorem 1.

We can immediately extend Theorem 1 to the following form. For simplicity we announce only the case corresponding to the condition (A).

Let us consider the following operator,

$$P=P_p^{m_1}P_q^{m_2}P_r+P_{s-1}+R,$$

where $m_1>m_2\geq 2$, p, q, $r\geq 0$ and $s=pm_1+qm_2+r>0$. And here $P_p(x, t;\partial_x, \partial_t)$, $P_q(x, t;\partial_x, \partial_t)$ and $P_r(x, t;\partial_x, \partial_t)$ are homogeneous partial differential operators of order p, q, and r respectively with $C^{l+\max\{s,6\}}$ coefficients, $P_{s-1}(x, t;\partial_x, \partial_t)$ is a homogeneous partial differential operator of order $s-1$ with $C^{l+\max\{s-1,5\}}$ coefficients, and $R(x, t;\partial_x, \partial_t)$ is a partial differential operator of order at most $s-2$ with bounded measurable coefficients.

Moreover we assume the following four conditions.

Assumptions.

1) The characteristic roots $\{\lambda_j\}_{1\leq j\leq p+q+r}$ are always distinct and real or non-real. Let λ_j stand for the real characteristic roots of P_p when $1\leq j\leq p_1$, the non-real roots of P_p when $p_1+1\leq j\leq p$, the real roots of P_q when $p+1\leq j\leq p+q_1$, the non-real roots of P_q when $p+q_1+1\leq j\leq p+q$, the real roots of P_r when $p+q+1\leq j\leq p+q+r_1$ and the non-real roots of P_r when $p+q+r_1+1\leq j\leq s$.

2) $\sum_{j=1}^{[k/p]} \frac{(m_1-j)p}{m_1} + \sum_{j=1}^{[k/q]} \frac{(m_2-j)q}{m_2} < k \quad (1 \leq k \leq \min\{m_1 p, m_2 q\})$ [7].

3) $P_{s-1}(0, 0; \xi, \tau)|_{\tau=\lambda_j(0,0;\xi)} \neq 0 \quad \xi \in \boldsymbol{R}^l - \{0\}, \quad 1 \leq j \leq p+q.$

4)
$$\frac{P_{s-1}(0, 0; \xi, \tau)|_{\tau=\lambda_j(0,0;\xi)}}{\prod_{\substack{k\neq i \\ 1\leq k\leq p}} (\lambda_i - \lambda_k)^{m_1} \prod_{\substack{k\neq i \\ p+1\leq k\leq p+q}} (\lambda_i - \lambda_k)^{m_2} \prod_{p+q+1\leq k\leq s} (\lambda_i - \lambda_k)}$$

is not pure imaginary at the origin for $1 \leq i \leq p_1$, $p+1 \leq i \leq p+q_1$.

Theorem 2. *If all the conditions are satisfied, there exists a neighbourhood Ω of the origin such that the solution $u(x, t) \in C^s$ of $Pu=0$, $\partial_t^j u|_{t=0}=0$ $(0 \leq j \leq s-1)$, vanishes identicrlly in Ω.*

Remark. A final extension of Theorem 1 and 2 will be announced in Zentralblatt für Mathematik.

A. 2. Uniqueness theorem under another condition.

We proved a uniqueness theorem under the condition (A), (B_1) or (B_2). We are tempted to consider the following condition:

(C) $$\partial_\tau^j P_{mp-1}(x, t; \xi, \tau)|_{\tau=\lambda_k(x,t;\xi)} \equiv 0 \qquad (j=0, 1).$$

In general, (C) is not sufficient to get uniqueness theorem. In fact, in Example 3 the operator of following type is contained,

$$\Delta^m + P_{[m/2]+1}.$$

But if we assume similar conditions to (C) not only on the homogeneous term of order $2m-1$ but also on the terms of order at least $[m/2]+1$, we have the following theorem.

Let us consider the following operator:

(A·1) $$P = \prod_{k=1}^m P_k + Q_1 \prod_{k=2}^m P_k + \cdots + Q_j \prod_{k=j+1}^m P_k + \cdots + Q_{m-1} P_m + R,$$

where we assume 1) P_j $(1 \leq j \leq m)$ is a partial differential operator of order p_j whose real characteristic roots are simple and non-real ones are either simple or always double, and whose coefficients belong to $C^{\max\{q_j, 1+\sigma\}}$ respectively, where $q_j = \sum_{k=1}^{j-1} p_k$. 2) Q_j $(1 \leq j \leq m-1)$

[7] If especially $q/p < 2/m_1(m_2-1)$ or $p/q < 2/m_2(m_1-1)$, the condition 2) is satisfied.

is a partial differential operator of order $\sum_{k=1}^{j}(p_k-1)$ with bounded measurable coefficients. 3) R is a partial differential operator of order $p-m$ with bounded measurable coefficients, where $p=\sum_{k=1}^{m}p_k$.

Then we have

Theorem 3. *There exists a n.b.d. Ω of the origin such that all solutions of*

$$\begin{cases} Pu=0 \\ \partial_t^j u|_{t=0}=0 \qquad (0\leq j\leq p-1) \end{cases}$$

vanishes identically there.

Remark. In (A. 1) if we assume 1′) instead of 1), we can get the same result as Theorem 3 by using Mizohata's estimate [11].

1′) P_j is elliptic of order 4 with C^{l+4}-characteristic roots $(1\leq j\leq m)$.[8] Here we do not assume that the multiplicities of characteristic roots are constant, that is, more precisely we assume only that they are at most double.

Before proving Theorem 3, we state a lemma.

Lemma A. 1. *When P satisfies the above condition 1) and $u(x,t)\in C^p$ satisfies $\partial_t^j u|_{t=0}=0$ $(0\leq j\leq p-1)$, we have the following estimate for sufficiently small h.*

$$E_n\equiv\int_0^h \varphi_n^2(t)\,||Pu(t)||^2dt\geq\frac{c}{h^2}\sum_{k=1}^{p}\int_0^h \varphi_n^2(t)\,||\partial_t^{p-k}u(t)||_{k-1}^2dt.$$

Proof. Let us put $u_j=(\Lambda+1)^{p-j}D_t^{j-1}u$ $(1\leq j\leq p)$, then we can reduce Pu to a first order system:

$$\text{(A. 2)} \qquad \tilde{P}U=D_tU-H\Lambda U+BU,$$

where $U={}^t(u_1,\cdots,u_p)$ and $||Pu||=||\tilde{P}U||$.

Let $\{\lambda_j(x,t;\xi)\}_{1\leq j\leq p}$ be eigenvalues of $H(x,t;\xi)$. Because of $\operatorname{rank}(\lambda_jI-H)=p-1$, we have a smooth $\mathscr{N}(x,t;\xi)$ such that $\mathscr{N}H\mathscr{N}^{-1}$ is a Jordan's normal form. Operating $\mathscr{N}$ to (A. 2), we have following

[8] We assume P_j has real coefficients when $l=1$ $(1\leq j\leq m)$. Then for every $l(\geq1)$, two λ_j have positive imaginary parts and the other two have negative imaginary parts. Further we can have the same result if P_j is of order 2 instead of order 4.

(A. 3) $$\mathcal{N}\tilde{P}U=D_tV-\mathcal{D}V+B'U,$$

where $V=\mathcal{N}U$ and $\mathcal{D}=\begin{pmatrix} \lambda_1 & \Lambda & & & & & & & \\ 0 & \lambda_1 & & & & & & & \\ & & \lambda_2 & \Lambda & & & & 0 & \\ & & 0 & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_r & \Lambda & & \\ & & & & & 0 & \lambda_r & & \\ & & & & & & & \lambda_{2r+1} & \\ & & 0 & & & & & & \ddots \\ & & & & & & & & \lambda_p \end{pmatrix}.$

Now we can use modified Caldrón's estimate by multiplying $1/\sqrt{n}$ to the $(2j-1)$-th components of $\mathcal{N}\tilde{P}U$ $(1\leq j\leq r)$. Q.E.D.

Remark. Under the condition 1′) by modified Mizohata's estimate we can get the same estimate as Lemma A. 1 without reducing to a system.

Proof of Theorem 3. We prove Theorem 3 by induction. Let us assume

$$E_n\geq\frac{c}{h^{2(r-1)}}\sum_{k=r-1}^{q_r}\int_0^h\varphi_n^2||\partial_t^{q_r-k}\prod_{j=r}^m P_ju||_{k-r+1}^2dt$$
$$-\sum_{j=r}^{m-1}\int_0^h\varphi_n^2||Q_j\prod_{k=j+1}^m P_ku||^2dt-\int_0^h\varphi_n^2||Ru||^2dt.$$

Then by the inequality

$$\sum_{k=r-1}^{q_r}||\partial_t^{q_r-k}\prod_{j=r}^m P_ju||_{k-r+1}^2\geq\sum_{k=r-1}^{q_r}||P_r(\partial_t^{q_r-k}\prod_{j=r+1}^m P_ju)||_{k-r+1}^2$$
$$-c\sum_{k=r}^{q_{r+1}}||\partial_t^{q_{r+1}-k}\prod_{j=r+1}^m P_ju||_{k-r}^2dt,$$

and Lemma A. 1, we have

$$E_n\geq\frac{c}{h^{2r}}\sum_{k=r}^{q_{r+1}}\int_0^h\varphi_n^2||\partial_t^{q_{r+1}-k}\prod_{j=r+1}^m P_ju||_{k-r}^2dt$$
$$-\sum_{j=r+1}^{m-1}\int_0^h\varphi_n^2||Q_j\prod_{k=j+1}^m P_ku||^2dt-\int_0^h\varphi_n^2||Ru||^2dt.$$

At last, we reach to $r=m$, that is,

$$E_n \geq \frac{c}{h^{2(m-1)}} \sum_{j=m-1}^{p-p_m} \int_0^h \varphi_n{}^2 \|\partial_t{}^{p-p_m-k} P_m u\|_{k-m+1}^2 dt - \int_0^h \varphi_n{}^2 \|Ru\|^2 dt.$$

Once again we apply Lemma A. 1, we have

$$E_n \geq \frac{c}{h^{2m}} \sum_{j=m}^{p} \int_0^h \varphi_n{}^2 \|\partial_t{}^{p-k} u\|_{k-m}^2 dt.$$

On the other hand, from the equation, we see $E_n=0$. This implies that $u\equiv 0$ in a n.b.d. of the origin. Q.E.D.

A. 3. On the unique continuation in two-dimensional space.

We are interested in the unique continuation property for operators without any essential assumption for l.o.t., especially those operators having Δ^m as their leading term. If $m\leq 3$ we have affirmative results in more general cases by A. P. Calderón [1], S. Mizohata [11], and K. Watanabe [15].

However Exampl 3 contains the following case:

$$\Delta^4 + (\text{l.o.t. with } C^{1+\sigma} \text{ coefficients}) \qquad (\sigma>0).$$

This suggests that we need assume sufficient differentiability for coefficients. But by Example 2 we have the following case where unique continuation is incorrect:

$$\Delta^6 + (\text{l.o.t. with } C^\infty\text{-coefficients}).$$

If we suppose that only $\Delta^m +$ (homogeneous terms of order $2m-1$) has sufficiently smooth coefficients, Example 3 contains the following case:

$$\Delta^5 + (\text{l.o.t. of order 8 with Hölder-continuous coefficients}).$$

Therefore we restrict ourselves to operators whose principal parts are Δ^4. Moreover, we assume severely that homogeneous terms of order 7 are of constant coefficients. Then we have an affirmative result when $l=1$, that is, $x\in \boldsymbol{R}^1$. Let us consider the following operator:

$$L=\Delta^4 + P_7(\partial_x, \partial_t) + R(x,t;\partial_x,\partial_t),$$

where P_7 is a homogeneous partial differential operator of order 7 with real constant coefficients, and R is a partial differential operator of order at most 6 with bounded measurable coefficients.

We have the following theorem as a corollary to Theorems 1 and 3.

Theorem 4. *If* $Lu=0$ $(u\in C^8(\boldsymbol{R}^2))$ *and* $u(x,y)=0$ *on an open set, then we have* $u(x,y)\equiv 0$ *in* $\boldsymbol{R}^2$.

Proof. If $u\not\equiv 0$ in $\boldsymbol{R}^2$, there is an open ball B such that $u\equiv 0$ in B and $\text{supp}[u]\cap\bar{B}\neq\phi$. Note that we can take a point $A\in \text{supp}[u]\cap\bar{B}$.

In the above operator, we have only the following three cases:

Case 1. $P_7(1,\sqrt{-1})\neq 0$,

Case 2. $P_7(1,\sqrt{-1})=0$ and $\partial_r P_7(1,\sqrt{-1})\neq 0$,

Case 3. $\partial_r{}^j P_7(1,\sqrt{-1})=0 \quad (j=0,1)$,

where r is the radius of B. Let us transform the coordinates (x,y) to (z,t) in a n.b.d. of A mapping A and the boundary of B to the origin and $t=0$, respectively. Then we can apply Theorem 1 in Cases 1 and 2, and Theorem 3 in Case 3. That is contrary to $u\not\equiv 0$ in a n.b.d. of A. Q.E.D.

A.4. Case of first order systems.

We give remarks in case of first order systems:

$$P=\partial_t-H(x,t;\partial_x)+B(x,t),$$

where H and B are $p\times p$ homogeneous matrices of order 1 and 0, respectively. When we establish Theorems 1, 2, and 3, we reduce equations to first order systems and make those principal parts diagonal or triangular by smooth regular matrix $\mathcal{N}$. But in case of matrices, if we do not assume nothing about the structure of the matrices we can not find in general a continuous regular matrices $\mathcal{N}(x,t;\xi)$ with respect to x and t such that $\mathcal{N}H\mathcal{N}^{-1}$ is triangular.

Example A.1. Let λ be an arbitrary number and $(x,t)\in\boldsymbol{R}^1\times\boldsymbol{R}^1$.

$$H_1=\begin{pmatrix} \lambda-xt & -t^2 \\ x^2 & \lambda+xt \end{pmatrix}.$$

Although H_1 is analytic, it has not a continuous matrix $\mathscr{N}$ at the origin, (which will be called "triangulator") that $\mathscr{N}H\mathscr{N}^{-1}=\begin{pmatrix} \lambda & * \\ 0 & \lambda \end{pmatrix}$.

Even in the case of one variable, we have the following

Example A. 2.

$$H_2=\begin{pmatrix} \lambda-\exp(-1/t^2)\sin(1/t) & -\exp(-1/t^2) \\ \exp(-1/t^2)\sin^2(1/t) & \lambda+\exp(-1/t^2)\sin(1/t) \end{pmatrix}.$$

This matrix has not continuous "triangulator" at the origin, too. Let us remark that all elements of H_2 belong to C^∞ class.

But, corresponding to Mizohata's results [11] (see also B. Malgrange [9]), we have the following theorem.

We assume that $H(x, t; \partial_x)$ has simple real characteristic roots and at most double $C^{1+\sigma}$ non-real characteristic roots and has $C^{1+\sigma}$ coefficients, and that $B(x, t)$ has Lipschitz continuous elements.[9] Then we have

Theorem 5. *There exists a n.b.d. Ω of the origin such that all solutions $u(x, t)\in C^1$ of*

$$\begin{cases} Pu=0 \\ u(x, 0)=0, \end{cases}$$

vanish in Ω.

Proof. By localization in ξ and modification of coefficients in [12], we can assume that λ_{2j-1} coincides with only λ_{2j} $(1\leq j\leq r)$ and λ_j $(2r+1\leq j\leq p)$ is simple. We can have smooth regular matrix $\mathscr{N}$ such that $\mathscr{N}H_0\mathscr{N}^{-1}=\mathscr{C}$, where $H=H_0\Lambda$ and

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9) All assumptions are held after Holmgren's transformation. On the other hand, dividing U to U_1 and U_2 as in paragraph 4, we need the smoothness of characteristic roots only up to order $1+\sigma$ $(\sigma>0)$.

$$\mathscr{C}=\left(\begin{array}{cc:cc:ccc} d_{11} & d_{12} & & & & & \\ d_{21} & d_{22} & & & & 0 & \\ \hdashline & & d_{33} & d_{34} & & & \\ & & d_{43} & d_{44} & & & \\ \hdashline & & & & \ddots & & \\ & 0 & & & & \lambda_{2r+1} & \\ & & & & & & \ddots \\ & & & & & & \lambda_p \end{array}\right).$$

We operate $\mathscr{N}(x, t; D_x)$ to the equation, then we have

$$D_t\mathscr{N}U-\mathscr{C}\Lambda\mathscr{N}U+B'U=0. \tag{A. 4}$$

Moreover we operate L to (A. 4) in distribution sense, where

$$L=\left(\begin{array}{c:c} (\Lambda+1)^{-1}I_{2r} & 0 \\ \hdashline 0 & I_{p-2r} \end{array}\right)\left(\begin{array}{cc:cc:c:c} D_t-d_{22} & d_{12} & & & & \\ d_{21} & D_t-d_{11} & & 0 & & 0 \\ \hdashline & & D_t-d_{44} & d_{34} & & \\ & & d_{43} & D_t-d_{33} & & \\ \hdashline & 0 & & & \ddots & \\ \hdashline & & 0 & & & I_{p-2r} \end{array}\right),$$

I_{2r} and I_{p-2r} are the unit matrices of order $2r$ and $p-2r$ respectively. Finally we get the following

$$\mathscr{D}V-B''U=0, \tag{A. 5}$$

where $V={}^t(v_1, \cdots, v_p)=\mathscr{N}U$ and

$$\mathscr{D}=\begin{pmatrix} (\Lambda+1)^{-1}(D_t-\lambda_1)(D_t-\lambda_2) & & & & & & 0 \\ & (\Lambda+1)^{-1}(D_t-\lambda_1)(D_t-\lambda_2) & & & & & \\ & & (\Lambda+1)^{-1}(D_t-\lambda_3)(D_t-\lambda_4) & & & & \\ & & & (\Lambda+1)^{-1}(D_t-\lambda_3)(D_t-\lambda_4) & & & \\ & & & & \ddots & & \\ & & & & & D_t-\lambda_{2r+1} & \\ 0 & & & & & & \ddots \\ & & & & & & D_t-\lambda_p \end{pmatrix}.$$

If we expand all components appearing in (A. 5), we see that all terms except $D_t^2(\Lambda+1)^{-1}v_j$ belong to L^2. Then, from the equation, $D_t^2(\Lambda+1)^{-1}v_j$ is also an element of L^2 $(1\leq j\leq 2r)$. On the other hand, for $2r+1\leq j\leq p$, operating $(\Lambda+1)^{-1}$ to the equation, we see

$$||D_t(\Lambda+1)^{-1}v_j||\leq||(\Lambda+1)^{-1}\lambda_j v_j||+c\sum_{k=1}^{p}||(\Lambda+1)^{-1}v_k||\leq c'\sum_{k=1}^{p}||v_k||.$$

By applying twice modified Calderón's estimate, we have

$$E_n\equiv\int_0^h\varphi_n^2(t)||\mathcal{D}V(t)-B''U(t)||^2dt$$

$$\geq\frac{c}{h^2}\sum_{j=1}^{p}\int_0^h\varphi_n^2(t)||v_j(t)||^2dt \text{ (see S. Mizohata[11]).}$$

Then, from (A. 5), we can see $u_j(t)\equiv 0$ in a n.b.d. of the origin.

Q.E.D.

A. 5. Remark on the localization in ξ.

In proving Theorem 1 and 2, in case of $l=2$, we must localize $P(x,t;\xi,\tau)$ in R_ξ^2. Let $\alpha_i(\xi)$ be a partition of the unity of S_ξ^1. The modification of the symbols out of supp$[\alpha_i]$ is as same as S. Mizohat [12]. Operating $\alpha_i(D_x)$ to the equation, we have

$$\alpha_i Pu\equiv P_p^m(\alpha_i u)+P_{mp-1}(\alpha_i u)$$

$$-\sqrt{-1}\,mP_p^{m-1}(\mathrm{grad}_x P_p\cdot\mathrm{grad}\,\alpha_i)u+\tilde{R}u=0,$$

where $\tilde{R}$ is of order $mp-2$.

Let us reduce this equation to a first order system as the same way as in paragraph 3. 2. Put

$$u_k=(\Lambda+1)^{mp-k}\partial_{k-1}\cdots\partial_0\alpha_i u \quad (1\leq k\leq mp),$$

$$u_{jk}=(\Lambda+1)^{mp-k}\partial_{k-1}\cdots\partial_0(\partial_{\xi_j}\alpha_i)(\Lambda+1)u \quad (j=1,2,\ \ 1\leq k\leq mp).$$

Then the above equation is reduced to a system of the following form:

$$P_i U\equiv D_t U-HU-BU-GU-\sum_{j=1}^{2}K_jU_j-U_0=0,$$

where $U=(u_k)$ and $U_j=(u_{jk})$, (see (2) in paragraph 3. 2). Note that K_jU_j corresponds to the term $-\sqrt{-1}\,mP_p^{m-1}(\partial_{x_j}P_p)(\partial_{\xi_j}\alpha_i)u$ in the above equation. Let $k_{ik}^{(j)}$ be the (i, k) component of K_j, then $k_{ik}^{(j)}$ is zero when $1 \leq i \leq mp-1$ or when $i = mp$ and $1 \leq k \leq (m-1)p$. For the rest, the order is at most 0. Let us operate the diagonalizator $\mathcal{N}$ of H to $P_i U = 0$. We have

$$\mathcal{N} P_i U \equiv D_t \mathcal{N} U - \mathcal{D} \mathcal{N} U - \mathcal{N}_t{}' U - (\mathcal{N} H - \mathcal{D} \mathcal{N}) U - \mathcal{N} B U - \mathcal{N} G U$$
$$- \mathcal{N} U_0 - \sum_{j=1}^{2} (\mathcal{N} K_j \mathcal{M}) \mathcal{N} U_j - \sum_{j=1}^{2} \mathcal{N} K_j (\mathcal{M} \mathcal{N} - \mathcal{M} \circ \mathcal{N}) U_j = 0,$$

(see (5) in paragraph 4.1). Let us remark that by virtue of the structure of K_j, $\mathcal{N} K_j \mathcal{M}$ and $\mathcal{N} K_j (\mathcal{M}\mathcal{N} - \mathcal{M} \circ \mathcal{N})$ are of order 0 and -1, respectively.

After we estimate $(D_t - \mathcal{D}) \mathcal{N} U$, $\mathcal{N}_t{}' U$, $(\mathcal{N} H - \mathcal{D} \mathcal{N}) U$, $\mathcal{N} B U$, and $\mathcal{N} G U$ as the same way as from paragraph 4.1 to 4.4, we see

$$E_{i0} \equiv \int_0^h \varphi_n{}^2(t) \|\mathcal{N} P_i U\|^2 dt \geq \frac{c_1}{n} \int_0^h \varphi_n{}^2(t) \|(\Lambda+1)^{1-(1/m)} \mathcal{N} U\|^2 dt$$
$$+ \frac{c_2 n}{2h^2} \int_0^h \varphi_n{}^2(t) \|\mathcal{N} U\|^2 dt - c_4 \int_0^h \varphi_n{}^2(t) \|(\Lambda+1)^{-1} U\|^2 dt$$
$$- c_5 \int_0^h \varphi_n{}^2(t) \|U_0\|^2 dt - c_6 \sum_{j=1}^{2} \int_0^h \varphi_n{}^2(t) \|\mathcal{N} U_j\|^2 dt$$
$$- c_7 \sum_{j=1}^{2} \int_0^h \varphi_n{}^2(t) \|(\Lambda+1)^{-1} U_j\|^2 dt,$$

(see (9) in paragraph 4.4). Here,

$$\frac{c_1}{n} \int_0^h \varphi_n{}^2(t) \|(\Lambda+1)^{1-(1/m)} \mathcal{N} U\|^2 dt + \frac{c_2 n}{2h^2} \int_0^h \varphi_n{}^2(t) \|\mathcal{N} U\|^2 dt$$
$$\geq \frac{c}{h} \int_0^h \varphi_n{}^2(t) \|(\Lambda+1)^{1/2(1-(1/m))} \mathcal{N} U\|^2 dt.$$

Moreover we consider $(\partial_{\xi_j} \alpha_i)(\Lambda+1)^{1/2(1+(1/m))} P u$. We have

$$(\partial_{\xi_j} \alpha_i)(\Lambda+1)^{1/2(1+(1/m))} P u \equiv (P_p{}^m + P_{mp-1})(\partial_{\xi_j} \alpha_i)(\Lambda+1)^{1/2(1+(1/m))} u + R_j u = 0,$$

where R_j is of order $mp - (3/2) + 1/(2m)$, $(j = 1, 2)$. This is reduced to the following

$$P_{ij} \tilde{U}_j \equiv D_t \mathcal{N} \tilde{U}_j - \mathcal{D} \mathcal{N} \tilde{U}_j - \mathcal{N}_t{}' \tilde{U}_j - (\mathcal{N} H - \mathcal{D} \mathcal{N}) \tilde{U}_j - \mathcal{N} B \tilde{U}_j$$
$$- \mathcal{N} G \tilde{U}_j - \mathcal{N} U_0{}' = 0, \quad \text{where} \quad \tilde{U}_j = (\Lambda+1)^{-1/2(1-(1/m))} U_j.$$

Then we have the similar estimate to that of $\mathcal{N}P_iU$,

$$E_{ij}\equiv\int_0^h\varphi_n^2(t)\,||\mathcal{N}P_{ij}\tilde{U}_j||^2dt\geq\frac{c'}{h}\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{1/2(1-(1/m))}\mathcal{N}\tilde{U}_j||^2dt$$

$$-c_4'\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{-1}\tilde{U}_j||^2dt$$

$$-c_5'\int_0^h\varphi_n^2(t)\,||U_0'||^2dt\qquad(j=1,2),$$

Summing up for j, we see that

$$\sum_{j=0}^{2}E_{ij}\geq\frac{c}{h}\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{1/2(1-(1/m))}\mathcal{N}U||^2dt$$

$$-c_4\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{-1}U||^2dt$$

$$-c_5\int_0^h\varphi_n^2(t)\,||U_0||^2dt-c_5'\int_0^h\varphi_n^2(t)\,||U_0'||^2dt$$

$$\geq\frac{c}{h}\sum_{k=1}^{mp}\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{-1/2(1-(1/m))}\partial_t^{mp-k}\alpha_iu||_{k-1}^2dt$$

$$-c_0\sum_{k=1}^{mp}\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{-1/2(1-(1/m))}\partial_t^{mp-k}u||_{k-1}^2dt,$$

(see Corolary 3.2 and Lemma 4.2). Then, summing up for i,

$$\sum_{i,j}E_{ij}\geq\frac{\tilde{c}}{h}\sum_{k=1}^{mp}\int_0^h\varphi_n^2(t)\,||(\Lambda+1)^{-1/2(1-(1/m))}\partial_t^{mp-k}u||_{k-1}^2dt,$$

(h is sufficiently small and n is sufficiently large).

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