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A survey on the generalized Burgers' equation with a pressure model term

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§ 1. Introduction and Preliminary Lemmas.

We have discussed in [9], [10] on the generalized Bergers' equation (abbreviated below, *G.B.E.*) as a simple model of the fundamental system of equations for compressible viscous fluid. It will be shown in this paper that we also obtain, for *G.B.E.* with a pressure model term, results almost similar to those in [9] and [10]. As for the notations, see [9]. The abbreviation "*m.i.e.a.*" stands for "monotonically increasing in each argument".

Now, the system of differential equations to be discussed on is as follows:

$$(1.1)\quad \begin{cases} \dfrac{\partial v}{\partial t}(x,t)=\dfrac{\mu}{\rho(x,t)}\dfrac{\partial^2}{\partial x^2}v(x,t)-v(x,t)\dfrac{\partial v}{\partial x}(x,t) \\ \quad -\dfrac{K}{\rho(x,t)}\dfrac{\partial}{\partial x}\rho(x,t), \\ \dfrac{\partial}{\partial t}\rho(x,t)+\dfrac{\partial}{\partial x}\{\rho(x,t)v(x,t)\}=0, \quad (v,\ \text{velocity};\ \rho,\ \text{density}; \\ \mu,\ \text{viscosity coefficient (constant)};\ t,\ \text{time};\ K,\ \text{positive constant} \\ \text{such that } P(\text{pressure})=K\rho), \end{cases}$$

where v is a scalar function and $x\in R^1$. For a while, to the end of § 2, we assume for (1.1) the initial condition that

$$(1.1)'\quad v(x,0)=v_0\in H^{2+\alpha},\ \rho(x,0)=\rho_0\in H^{1+\alpha},\ (\alpha\in(0,1),\ 0<\underline{\rho}_0\equiv\inf\rho_0\leq\rho_0\leq\bar{\rho}_0\equiv|\rho_0|^{(0)}<+\infty).$$

In the following, we shall study on the initial value problem for the system of equations (1. 1), especially from a temporally global point of view. First, we prepare some preliminary lemmas.

Lemma 1. 1. *If* $(v, \rho) \in H_T^{2+\alpha} \times B_T{}^1 (0 < T < +\infty)$ *satisfies* (1. 1) - (1. 1)′ *then the function* $y(\tau; x, t)$ *defined by*

$$y(\tau; x, t) \equiv \int_0^\tau \overline{v_{xx}}(\tau'; x, t) \overline{x}_x(\tau'; x, t) d\tau' \tag{1. 2}$$

satisfies an ordinary differential equation

$$\left\{\begin{array}{l} \dfrac{d}{d\tau} y(\tau; x, t) + k\rho(x, t) \overline{x}_x(\tau; x, t)^{-1} y(\tau; x, t) \\ \quad = \rho(x, t) \left\{ \dfrac{d}{d\tau} \overline{v}(\tau; x, t) + k \dfrac{\rho_0{}'(x_0(x, t))}{\rho_0(x_0(x, t))} \right. \\ \qquad \left. \times \dfrac{\partial x_0}{\partial \xi}(\xi = \overline{x}(\tau; x, t), \tau) \right\}, \\ y(\rho; x, t) = 0, \quad \left(k \equiv \dfrac{K}{\mu}\right), \end{array}\right. \tag{1. 2)′$$

where $\overline{x}(\tau; x, t)$ *is the characteristic curve in* ρ *of the latter equation of* (1. 1), *i.e., satisfies*

$$\frac{d}{d\tau} \overline{x}(\tau; x, t) = v(\overline{x}(\tau; x, t), \tau), \quad \overline{x}(t; x, t) = x, \tag{1. 3}$$

and $\overline{v}$, $\overline{v_{xx}}$, *etc., are defined in such a way that, e.g.,*

$$\begin{aligned} &\overline{v}(\tau; x, t) \equiv v(\overline{x}(\tau; x, t), \tau), \\ &\overline{v_{xx}}(\tau; x, t) \equiv v_{xx}(\overline{x}(\tau; x, t), \tau), \quad x_0(x, t) \equiv \overline{x}(0; x, t). \end{aligned} \tag{1. 3)′$$

Proof. The following relations are obvious.

$$\begin{aligned} &\frac{\partial \overline{x}}{\partial x}(\tau; x, t) = \exp\left\{ - \int_\tau^t \overline{v_x}(\tau'; x, t) d\tau' \right\}, \\ &\frac{\partial x_0}{\partial x}(x, t) = \exp\left\{ - \int_0^t \overline{v_x}(\tau'; x, t) d\tau' \right\} = \overline{x}_x(\tau; x, t) \\ &\qquad \times \exp\left\{ - \int_0^\tau \overline{v_x}(\tau'; x, t) d\tau' \right\}, \quad \rho(x, t) = \overline{\rho}(\tau; x, t) \overline{x}_x(\tau; x, t); \\ &x_0(x, t) = x_0(\overline{x}(\tau; x, t), \tau), \quad \overline{v_{xx}}(\tau'; \overline{x}(\tau; x, t), \tau) \end{aligned} \tag{1. 4}$$

$$=\overline{v_{xx}}(\tau'; x, t)\ (0 \leqq \tau' \leqq \tau),\ \text{etc.};$$

$$\frac{\rho_x}{\rho} = \frac{\rho_0'(x_0(x,t))}{\rho_0(x_0(x,t))}\frac{\partial x_0}{\partial x}(x,t) - y(t; x, t),$$

$$y(\tau; \overline{x}(\tau; x, t), \tau) = (\overline{x}_x^{-1}\cdot y)(\tau; x, t).$$

Noting that our discussion is being made along the characteristic curve $\overline{x}(\tau; x, t)$, by (1. 1) and (1.4) we have (1. 2)$'$. Q.E.D.

Remark. The lemma above shows that $y(t; x, t)$ is expressed as follows:

(1. 5)
$$y(t; v, t) = \rho(x,t)\cdot[v(x,t) - \exp\left\{-k\rho(x,t)\int_0^t \overline{x}_x(\tau; x, t)^{-1}d\tau\right\}$$
$$\times v_0(x_0(x,t)) - k\int_0^t \exp\left\{-k\rho(x,t)\int_\tau^t \overline{x}_x(\tau'; x, t)^{-1}d\tau'\right\}$$
$$\times\left[\rho(x,t)\overline{v}(\tau; x, t)\overline{x}_x(\tau; x, t)^{-1}\right.$$
$$\left.-\frac{\rho_0'(x_0(x,t))}{\rho_0(x_0(x,t))}\exp\left\{-\int_0^\tau \overline{v_x}(\tau'; x, t)d\tau'\right\}\right]d\tau].$$

Lemma 1. 2. *If a function* $u(x,t)$ *defined on* $R^1\times[0,T]$ *has* $\partial/\partial x\ u(x,t)$ *and, moreover, satisfies*

(1. 6)
$$|u(x,t) - u(x,t)| \leqq C_1|t-t'|^\alpha,$$
$$|u_x(x,t) - u_x(x',t)| \leqq C_2|x-x'|^\beta,$$
$$(C_1 \text{ and } C_2,\ \text{constants};\ \alpha,\ \beta\in(0,1]),$$

then it holds that

(1. 7)
$$|u_x(x,t) - u_x(x,t')| \leqq C_3|t-t'|^{\alpha\beta/1+\beta},$$

where C_3 *is a constant depending on* C_1, C_2, *and* β, *especially, monotonically increasing in* C_1 *and* C_2, *respectively.*

Proof. It is obvious that

$$\left|\left[\int_{x'}^{x''}\{u_x(x,t) - u_x(x',t')\}dx\right]_{t=t'}^{t=t''}\right| = |u(x'',t'')$$
$$-u(x',t'') + u(x',t') - u(x'',t')| \leqq 2C_1|t''-t'|^\alpha,$$

$$\left|\int_{x'}^{x''}\{u_x(x,t')-u_x(x',t')\}dx\right|\leqq C_2|x'-x''|^{1+\beta},$$

and that

$$\left|\int_{x'}^{x''}\{u_x(x,t'')-u_x(x',t')\}dx\right|\geqq|x''-x'|$$
$$\times[|u_x(x',t'')-u_x(x',t')|-C_2|x''-x'|^{\beta}].$$

Therefore, we have

$$|u_x(x',t'')-u_x(x',t')|\leqq 2C_1\frac{|t''-t'|^{\alpha}}{|x''-x'|}+2C_2|x''-x'|^{\beta},$$

where we note that x'' is arbitrary. If we define $f(s)$ by

$$f(s)=\frac{A}{s}+Bs^{\beta}\quad(A\equiv 2C_1|t''-t'|^{\alpha},\ B\equiv 2C_2),$$

then $f(s)$ takes at $s=(A/\beta B)^{1/1+\beta}\equiv s_0$ its minimum value (if $C_2>0$)

(1.7)′ $f(s_0)=[2C_1^{\beta/1+\beta}C_2^{1/1+\beta}(\beta^{1/1+\beta}+\beta^{-\beta/1+\beta})]\cdot|t''-t'|^{\alpha\beta/1+\beta}.$

We define C_3 by $[\cdots]$ in (1.7)′, including the case $C_2=0$. Then, by the above discussion follows our assertion. Q.E.D.

Lemma 1. 3. *If* $(v,\rho)\in H_T^{2+\alpha}\times B_T^{1}$ *satisfies* (1.1)-(1.1)′, *then it holds that*

(1.8) $$|\rho|_{x,T}^{(L)},\ |\rho|_{t,T}^{(1/2)}\leqq C_4(T,[\rho]_T,|v|_T^{(0)})\quad(<+\infty),$$

where $[\rho]_T\equiv|\rho|_T^{(0)}+|\rho^{-1}|_T^{(0)}$ *and* C_4 *is a non-negative value depending on* T, *etc., and m.i.e.a.*

Proof. Let $w(x,t)$ be defined by

$$w(x,t)=\int_{x_1}^{x}\rho(x',t)dx'.$$

Then, it follows that

$$w_t(x,t)=\int_{x_1}^{x}\rho_t(x',t)dx'=-\int_{x_1}^{x}(\rho v)_{x'}dx'=(\rho v)\Big|_{x'=x}^{x'=x_1}.$$

Therefore, we have

$$|w(x,t')-w(x,t'')|\leqq|w_t|_T^{(0)}|t''-t'|\leqq 2|\rho v|_T^{(0)}|t''-t'|.$$

On the other hand, there are relations

$$\rho = w_x, \quad \rho_x = w_{xx}.$$

Here, by (1.4) and (1.5) it holds that

$$|\rho_x|_T^{(0)} \leqq C_5(T, [\rho]_T, |v|_T^{(0)}) < +\infty, \tag{1.9}$$

where C_5 has the same property as C_4. Applying Lemma 1.2 to $w(x, t)$ $(\alpha = \beta = 1)$, we have the inequality (1.8). Q.E.D.

Lemma 1.4. *Let $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ satisfy (1.1)-(1.1)′. Then, for the fundamental solution $\Gamma(x, t; \xi, \tau)$ of a linear parabolic equation*

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2}{\partial x^2} u - v \frac{\partial}{\partial x} u \tag{1.10}$$

it holds that

$$\left| \frac{\partial}{\partial x} \Gamma(x, t; \xi, \tau) \right| \leqq C_6(T, [\rho]_T^{(\alpha)}, |v|_T^{(0)}) \tag{1.11}$$

$$\times (t - \tau)^{-1} \exp\left\{ -(a_0 \cdot |\rho^{-1}|_T^{(0)})^{-1} \cdot \frac{(x - \xi)^2}{t - \tau} \right\} \ (a_0 > 0),$$

where $[\rho]_T^{(\alpha)} = [\rho]_T + |\rho^{-1}|_T^{(\alpha)}$ and C_6 is m.i.e.a.

Proof. $\Gamma(x, t; \xi, \tau)$ is expressed by using as a parametrix the fundamental solution $\Gamma_0(x, t; \xi, \tau)$ of the linear equation

$$w_t = \frac{\mu}{\rho} w_{xx} \tag{1.12}$$

as follows:

$$\Gamma(x, t; \xi, \tau) = \Gamma_0(x, t; \xi, \tau) \tag{1.13}$$

$$+ \int_\tau^t d\tau' \int_{R^1} \Gamma_0(x, t\, \xi', \tau') \times \Phi(\xi', \tau'; \xi, \tau) d\xi'.$$

Φ satisfies the integral equation

$$\Phi(x, t; \xi, \tau) = \tilde{k}(x, t; \xi, \tau) \tag{1.14}$$

$$+ \int_\tau^t d\tau' \int_{R^1} \tilde{k}(x, t; \xi', \tau') \times \Phi(\xi', \tau'; \xi, \tau) d\xi',$$

where $\tilde{k}(\cdots)$ is defined by

$$\tilde{k}(x, t; \xi, \tau) = v(x, t)\frac{\partial}{\partial x}\Gamma_0(x, t; \xi, \tau). \tag{1.15}$$

Φ is solved as a Neumann series. If we estimate Γ_x on the basis of (1.13) $\sim$ (1.15) (also, cf. [9]), we obtain easily the estimate (1.11).

Q.E.D.

§ 2. Discussions.

Theorem 2. 1. *Let* (v, ρ) *and* $(v^*, \rho^*) \in H_T^{2+\alpha} \times B_T^1$ *satisfy* (1.1)-(1.1)′. *Then,* $(v, \rho) = (v^*, \rho^*)$.

Proof. The theorem is proved in a way similar to that in [9] by using (1.2), (1.5), and Lemma 1.3. Q.E.D.

Theorem 2. 2. *For some* $T \in (0, +\infty)$, *there exists a unique solution* $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ *satisfying* (1.1)-(1.1)′.

Proof. The theorem is proved almost in the same way as in [7], [9]. Q.E.D.

Now, we make a step toward demonstrating several lemmas that show as a result what is essential in the temporally global problem of the system of equations (1.1).

Lemma 2. 1. *Let* $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ *satisfiy* (1.1)-(1.1)′. *Then,* $|v|_T^{(0)}$ *can be estimated from above in terms of* T *and* $[\rho]_T$.

Proof. We can express v by use of $\Gamma(x, t; \xi, \tau)$ in Lemma 1.4 as follows:

$$v(x, t) = \int_{R^1}\Gamma(x, t; \xi, 0)v_0(\xi)d\xi + \int_0^t d\tau\int_{R^1}\Gamma(x, t; \xi, \tau)\left\{-K\frac{\rho_\xi(\xi, \tau)}{\rho(\xi, \tau)}\right\}d\xi. \tag{2.1}$$

In virtue of (1.2), (1.4), and (1.5), ρ_ξ/ρ is to be expressed by using $y(\tau; \xi, \tau)$. On this occasion, $\bar{x}_\xi(\tau'; \xi, \tau)^{-1} (0 \leqq \tau' \leqq \tau \leqq T)$ is estimated in such a way that

$$(2.2)\qquad |\bar{x}_\xi(\tau';\xi,\tau)^{-1}|\leqq|\rho|_T^{(0)}\cdot|\rho^{-1}|_T^{(0)}\leqq[\rho]_T^2.$$

Thus, we have the assertion of the lemma, noting that

$$(2.3)\qquad \int_{R^1}\Gamma(x,t;\tau,\xi)d\xi=1,$$

and that the inequality

$$(2.4)\qquad 0\leqq y(t)\leqq a+b\int_0^t y(\tau)d\tau+c\int_0^t d\tau\int_0^\tau y(\tau')d\tau'\quad(a,b,c\geqq 0)$$

implies the following relation

$$(2.5)\qquad 0\leqq y(t)-B\int_0^t y(\tau)d\tau\leqq a+A\left[\int_0^t d\tau\,(y(\tau)-B\times\int_0^\tau y(\tau')d\tau')\right],$$

where A and B are the roots of $\xi^2-b\xi-c=0$ such that $A\geqq 0\geqq B$. From (2.5) follows

$$(2.6)\qquad 0\leqq y(t)\leqq y(t)+(-B)\int_0^t y(\tau)d\tau\leqq a\cdot e^{At}.$$

Finally, we have an estimate

$$(2.7)\qquad |v|_T^{(0)}\leqq C_7(T,[\rho]_T)<+\infty,$$

where C_7 is a non-negative finite value *m.i.e.a.* Q.E.D.

Lemma 2.2. *Let* $(v,\rho)\in H_T^{2+\alpha}\times B_T^1$ *satisfy* (1.1)-(1.1)'. *Then,* $|v_x|_T^{(0)}$ *is estimated in terms of* $|v|_T^{(0)}$, $|\rho|_T^{(0)}$, $\|\rho^{-1}\|_T^{(\alpha)}$, *and* T.

Proof. We express $v(x,t)$ in the form

$$(2.8)\qquad v(x,t)=v_0+\int_0^t d\tau\int_{R^1}\Gamma(x,t;\xi,\tau)\times\left\{\frac{v_0''(\xi)}{\rho(\xi,\tau)}-v(\xi,\tau)v_0'(\xi)-K\frac{\rho_\xi(\xi,\tau)}{\rho(\xi,\tau)}\right\}_I d\xi.$$

Hence, we have

$$(2.9)\qquad v_x(x,t)=v_0'+\int_0^t d\tau\int_{R^1}\Gamma_x(x,t;\xi,\tau)\cdot\{\cdots\}_I d\xi.$$

Thus, by Lemma 1.4 we have $(\lambda(\cdots)\equiv a_0\cdot|\rho^{-1}|_T^{(0)})$

$$(2.10)\qquad |v_x|_T^{(0)}\leqq|v_0'|^{(0)}+2C_6(\cdots)\cdot(T\cdot\lambda(\cdots))^{1/2}\times|\{\cdots\}_I|_T^{(0)},$$

and, in the same way as in the preceding lemma, it follows that

(2.11) $\quad |\{\cdots\}_I|_T^{(0)} \leqq C_8(T, |v|_T^{(0)}, |\rho|_T^{(0)}, |\rho^{-1}|_T^{(0)}) < +\infty,$

where C_8 is a value *m.i.e.a.* By (1.11), (2.10), and (2.11), we obtain our assertion. Q.E.D.

Lemma 2.3. *Let* $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ *satisfy* (1.1)-(1.1)′. *Then,* $\|\rho^{-1}\|_T^{(\alpha)}$ *is estimated in terms of* $[\rho]_T$ *and* T.

Proof.

(2.12) $\quad \|\rho^{-1}\|_T^{(\alpha)} = |\rho^{-1}|_T^{(0)} + |\rho^{-1}|_{x,T}^{(\alpha)} + |\rho^{-1}|_{t,T}^{(\alpha/2)}, \quad (\alpha \in (0, 1)).$

First, from the relation

$$(\rho^{-1})_x = -\rho^{-2} \cdot \rho_x,$$

by (1.4), (1.5), (2.2), etc., it is known that $|\rho^{-1}|_{x,T}^{(\alpha)}$ is estimated from above in terms of $|v|_T^{(0)}$, $[\rho]_T$, and T, i.e.,

(2.13) $\quad |\rho^{-1}|_{x,T}^{(\alpha)} \leqq C_9(T, |v|_T^{(0)} [\rho]_T) < +\infty,$

where C_9 is *m.i.e.a.* Next, by Lemma 1.3 it holds that

$$|\rho(x, t)^{-1} - \rho(x, t')^{-1}| = \rho(x, t)^{-1} \rho(x, t')^{-1}$$
$$\times |\rho(x, t') - \rho(x, t)| \leqq (|\rho^{-1}|_T^{(0)})^2$$
$$\times C_4(T, |v|_T^{(0)}, [\rho]_T) \cdot |t' - t|^{1/2}.$$

Therefore,

$$|\rho(x, t)^{-1} - \rho(x, t')^{-1}| \leqq 2(|\rho^{-1}|_T^{(0)})^{1+\alpha} \cdot |t - t'|^{\alpha/2}$$
$$\times C_4(T, |v|_T^{(0)}, [\rho]_T)^{\alpha}.$$

Hence, it follows that

(2.14) $\quad |\rho^{-1}|_{t,T}^{(\alpha/2)} \leqq C_9'(T, |v|_T^{(0)}, [\rho]_T) < +\infty,$

where C_9' is *m.i.e.a.* Since, by Lemma 2.2, $|v|_T^{(0)}$ is estimated is terms of $[\rho]_T$ and T, we have the assertion of the lemma. Q.E.D.

Lemma 2.4. *For* $(v, \rho) \in H_T^{2+\alpha} \times B_T^1$ *satisfying* (1.1)-(1.1)′, $\|v\|_T^{(2+\alpha)}$ *can be estimated in terms of* $|\rho|_T^{(0)}$, $|\rho^{-1}|_T^{(0)}$, *and* T.

Proof. By using the fundamental solution Γ_0 of the linear equation (1.12), we can express $v(x, t)$ in the following way:

$$(2.15)\qquad v(x,t)=v_0(x)+\int_0^t d\tau\int_{R^1}\Gamma_0(x,t;\xi,\tau)\left\{\frac{v_0''(\xi)}{\rho(\xi,\tau)}\right.$$

$$\left.-v(\xi,\tau)v_\xi(\xi,\tau)-K\frac{\rho_\xi}{\rho(\xi,\tau)}\right\}_{II}d\xi.$$

In a way analogous to that in [9], we have

$$(2.16)\qquad \|v\|_T^{(1+\alpha)}\leqq 3\|v_0\|^{(2)}+C_{10}(T;\ [\rho^{-1}]_T^{(\alpha)})\,|\{\cdots\}_{II}|_T^{(0)}.$$

Hence, it follows that

$$(2.17)\qquad \|\{\cdots\}_{II}\|_T^{(\alpha)}\leqq C_{10}'(T;\ [\rho^{-1}]_T^{(\alpha)},\ \|\rho\|_T^{(\alpha)},\ \|v\|_T^{(1+\alpha)})<+\infty,$$

where C_{10} and C_{10}' are *m.i.e.a.*, respectively. For $\|\rho\|_T^{(\alpha)}$, we have also an estimate such that

$$(2.18)\qquad \|\rho\|_T^{(\alpha)}\leqq C_{10}''(T,\ |v|_T^{(0)},\ [\rho]_T)<+\infty,$$

where C_{10}'' is *m.i.e.a.* Therefore, finally, by the lemmas 2. 1, 2. 2, and 2. 3, and by (2. 16), (2. 17), and (2. 18) it holds that

$$(2.19)\qquad \|v\|_T^{(2+\alpha)}\leqq C_{11}(T,\ |\rho|_T^{(0)},\ |\rho^{-1}|_T^{(0)})<+\infty,$$

where C_{11} is *m.i.e.a.* Q.E.D.

The preceding lemma denotes that, in order to have an a priori estimate for $\|v\|_T^{(2+\alpha)}$, it sufices to have such ones for $|\rho|_T^{(0)}$ and $|1/\rho|_T^{(0)}$. It is known that ρ is expressed in the form

$$(2.20)\qquad \rho(x,t)=\rho_0(x_0(x,t))\exp\left\{-\int_0^t\overline{v_x}(\tau;x,,t)d\tau\right\}$$

$$=\rho_0(x_0(x,t))\frac{\partial x_0}{\partial x}(x,t),$$

which implies that, for the above-mentioned purpose, it suffices to have a priori estimates for $\exp\{\pm\int_0^t\overline{v_x}(\tau;x,t)d\tau\}|_T^{(0)}$.

§ 3. Main Theorem.

Let $(v,t)\in H_T^{2+\alpha}\times B_T^1$ satisfy (1. 1)-(1. 1)′. The expression in the characteristic co-ordinates (x_0,t_0) (cf. [9]) of the system of equations (1. 1) and the initial condition (1. 1)′ is as follows:

(3.1)
$$\begin{cases} \hat{v}_{t_0}(x_0, t_0) = \dfrac{\mu}{\rho_0(x_0)}\left(\dfrac{\hat{v}_{x_0}(x_0, t_0)}{1+\omega(x_0, t_0)}\right)_{x_0} \\ \quad -\dfrac{K}{\rho_0(x_0)}\left(\dfrac{\rho_0(x_0)}{1+\omega(x_0, t_0)}\right)_{x_0}, \\ \left(\hat{\rho}(x_0, t_0) = \dfrac{\rho_0(x_0)}{1+\omega(x_0, t_0)}\right), \end{cases}$$

(3.1)′ $\hat{v}(x_0, 0) = v_0(x_0) \in H^{2+\alpha} (\rho_0(x_0) \in H^{1+\alpha}, 0 < \bar{\rho}_0 \leqq \rho_0 \leqq \bar{\bar{\rho}}_0 < +\infty)$,

where

(3.2)
$$\begin{cases} x_0 \equiv x_0(x, t),\ t_0 \equiv t \ \text{(therefore, } x = x(x_0, t_0),\ t = t_0); \\ \hat{v}(x_0, t_0) \equiv v(x(x_0, t_0),\ t = t_0),\ \hat{\rho}(x_0, t_0) \equiv \rho(x(x_0, t_0), \\ t = t_0);\ \omega(x_0, t_0) = \displaystyle\int_0^{t_0} \hat{v}_{x_0}(x_0, t_0')\, dt_0'. \end{cases}$$

Note that

(3.2)′
$$\frac{1}{1+\omega} = \frac{\partial x_0}{\partial x}(x, t) = \exp\left\{-\int_0^t \overline{v_x}(\tau; x, t)\, d\tau\right\},$$

$$\frac{\rho_x}{\rho} = \frac{1}{\rho_0}\left(\frac{\rho_0}{1+\omega}\right)_{x_0}.$$

Here, we remark that $v \in H_T^{2+\alpha}$ implies $\hat{v} \in H_T^{2+\alpha}$. Directly from (3.1)-(3.1)′, we obtain an equality

(3.3)
$$\frac{\partial}{\partial t_0}\int_a^{x_0} \frac{\rho_0}{\mu}(\hat{v} - v_0)(x_0', t_0)\, dx_0' = \left.\frac{\hat{v}_{k_0} - k\rho_0}{1+\omega}\right|_{x_0 = a}^{x_0 = x_0}.$$

where $k = K/\mu$. Define $Y^a(x_0, t_0)$ by

(3.4)
$$Y^a(x_0, t_0) \equiv \int_a^{x_0} \frac{\rho_0}{\mu}(\hat{v} - v_0)(x_0', t_0)\, dx_0' - \int_0^{t_0}\left[\frac{\hat{v}_{x_0} - k\rho_0}{1+\omega}\right]_{x_0 = a} dt_0'.$$

Then, it holds that

(3.5)
$$Y^a_{x_0}(x_0, t_0) = \frac{\rho_0}{\mu}(\hat{v} - v_0), \quad \hat{v}_{x_0} = \left(\frac{\mu}{\rho_0} Y^a_{x_0}\right)_{x_0} + v_0'.$$

Y^a satisfies the following equation

$$(3.6)\quad \begin{cases} \dfrac{\partial}{\partial t_0} Y^a = \dfrac{\hat{v}_{x_0} - k\rho_0}{1+\omega} = \dfrac{\mu}{1+\omega}\left(\dfrac{Y^a_{x_0}}{\rho_0}\right)_{x_0} + \dfrac{v_0' - k\rho_0}{1+\omega} \equiv \mathcal{L}(Y^a) + \dfrac{v_0' - k\rho_0}{1+\omega} \\ Y^a(x_0, 0) = 0, \end{cases}$$

where we consider $\mathcal{L}$ as a linear operator. Since it holds by (3. 2)′ that

$$\left|\frac{v_0' - k\rho_0}{1+\omega}\right| \leqq |v_0' - k\rho_0|^{(0)} \exp\{T|v_x|_T^{(0)}\}$$

and since, by (3. 4), Y^a satisfies Täcklind's condition, we know, from the expression of Y^a by use of the fundamental solution G for $\mathcal{L}_1 = \mathcal{L} - \partial/\partial t_0$ as a linear operator, that

$$(3.7)\qquad Y^a = Y^{a'} \quad (a \text{ and } a', \text{ arbitrary real numbers}),$$

$$|Y^a(x_0, t_0)|^{(0)} < +\infty \ (0 \leqq t_0 \leqq T).$$

By virtue of (3. 7), we put

$$(3.7)'\qquad Y \equiv -Y^a = -Y^{a'}.$$

Then, Y satisfies

$$(3.8)\quad \begin{cases} Y_{t_0} = \dfrac{\mu}{1+\omega}\left(\dfrac{Y_{x_0}}{\rho_0}\right)_{x_0} + \dfrac{k\rho_0 - v_0'}{1+\omega} = \mathcal{L}(Y) + \dfrac{k\rho_0 - v_0'}{1+\omega}, \\ Y(x_0, 0) = 0. \end{cases}$$

Lemma 3. 1. *Y is related to $(1+\omega)^{-1}$ in such a way that*

$$(3.9)\qquad (1+\omega)^{-1} = e^Y\left\{1 + k\rho_0 \int_0^{t_0} e^{Y(x_0, t_0')} dt_0'\right\}^{-1}.$$

Proof. It is obvious that

$$Y_{t_0} = \frac{k\rho_0 - \hat{v}_{x_0}}{1+\omega} = \frac{k\rho_0}{1+\omega} - \frac{(1+\omega)_{t_0}}{1+\omega}.$$

Hence,

$$(1+\omega)_{t_0} + (1+\omega) Y_{t_0} = k\rho_0,$$

from which we obtain (3·9). Q.E.D.

Lemma 3. 1 shows together with Lemma 2. 4 that, in order to have a priori estimates from above and below for $(1+\omega)^{-1}$, it suffices to have such ones for Y.

Lemma 3. 2. *It holds that*

$$\int_{R^1} G(x_0, t_0; \xi, \tau) \cdot \left(\int_{\xi}^{x_0} \rho_0(\xi') d\xi' \right) d\xi = 0. \tag{3.10}$$

Proof. $S(x_0, t_0) \equiv \int_a^{x_0} \rho_0(\xi') d\xi'$ Täcklind's condition and the equation

$$S_{t_0} = \mathcal{L}(S)\ (=0),\ [S(x_0, t_0) = S(x_0, 0)]. \tag{3.11}$$

Therefore, it follows that

$$S(x_0, t_0) = \int_a^{x_0} \rho_0(\xi) d\xi = \int_{R^1} G(x_0, t_0 : \xi, \tau) \times \left(\int_a^{\xi} \rho_0(\xi') d\xi' \right) d\xi, \quad (0 \leqq \tau < t_0 \leqq T). \tag{3.12}$$

On the other hand,

$$S(x_0, t_0) = \int_{R^1} G(x_0, t_0; \xi, \tau) S(x_0, t_0) d\xi. \tag{3.12$'$}$$

From (3. 12) and (3. 12)$'$ follows (3. 10). Q.E.D.

Lemma 3. 3. *If w_0 is such that $w_0' \in H^0$ and $w_0' \geqq 0$, then it holds that*

$$\int_{R^1} G(x_0, t_0 : \xi, \tau) \left(\int_{\xi}^{x_0} \rho_0(\xi') w_0(\xi') d\xi' \right) d\xi \leqq 0. \tag{3.13}$$

Proof. By the preceding lemma, we have

$$\begin{aligned} &\int_{-\infty}^{x_0} G d\xi \int_{\xi}^{x_0} \rho_0(\xi') w_0(\xi') d\xi' \\ &\leqq \int_{-\infty}^{x_0} G d\xi \int_{\xi}^{x_0} \rho_0(\xi') w_0(x_0) d\xi' \\ &= -\int_{x_0}^{+\infty} G d\xi \int_{\xi}^{x_0} \rho_0(\xi') w_0(x_0) d\xi' \\ &\leqq -\int_{x_0}^{+\infty} G d\xi \int_{\xi}^{x_0} \rho_0(\xi') w_0(\xi') d\xi'. \end{aligned} \tag{3.14}$$

From the relation between the most right-hand side and the most left-hand one of (3.14) follows (3.13). Q.E.D.

Now, in addition to the assumption (1.1)′ (or (3.1)′) on the initial condition (i.e., $(v_0, \rho_0) \in H^{2+\alpha} \times H^{1+\alpha}$), we assume, moreover, that

(3.15)
$$\begin{cases} v_0 \text{ has an expression such that } v_0 = v_{01} + v_{02} \ (v_{01} \text{ and} \\ v_{02} \in H^{2+\alpha},\ v'_{01} \geqq 0,\ v_{02} \in L^1(R^1)), \\ \rho_0' \in L^1(R^1) \text{ (which guarantees the existence of } \rho_0(\pm\infty) \\ \qquad \equiv \lim_{x_0 \to \pm\infty} \rho_0(x_0)), \\ \int_{-\infty}^{x_0} |\rho_0(\xi) - \rho_0(-\infty)| d\xi < +\infty \text{ (for an arbitrary } x_0 \in R^1), \\ \qquad k\rho_0 - v'_{01} \geqq 0, \\ u_0(x_0) \equiv v_{01}(x_0) - \int_{-\infty}^{x_0} k(\rho_0(\xi) - \rho_0(-\infty)) d\xi \text{ has such an ex-} \\ \text{pression that } u_0 = u_{01} + u_{02} (u'_{01} \in H^0,\ u'_{01} \geqq 0,\ u_{02} \in L^1(R^1), \\ [\text{Remark: } u_0' \in H^{1+\alpha} \subset H^0]. \end{cases}$$

It is easy to see that the above-mentioned assumption is consistent. $Y(x_0, t_0)$ is expressed by using the fundamental solution G for $\mathcal{L}_1$ as follows:

$$(3.16) \quad Y(x_0, t_0) = \int_0^{t_0} d\tau \int_{R^1} G(x_0, t_0; \xi, \tau) \left(\frac{k\rho_0 - v_0'}{1+\omega}\right)(\xi, \tau) d\xi$$
$$= \int_0^{t_0} d\tau \int_{R^1} G \frac{k\rho_0 - v'_{01} - v'_{02}}{1+\omega} d\xi.$$

Therefore, we have, by the non-negativity of G and $k\rho_0 - v'_{01}/1+\omega$,

$$(3.16)' \quad Y^* \equiv Y + \int_0^{t_0} d\tau \int_{R^1} G \frac{v'_{02}}{1+\omega} d\xi = \int_0^{t_0} d\tau \int_{R^1} G \frac{k\rho_0 - v'_{01}}{1+\omega} d\xi \geqq 0.$$

We put

$$(3.17) \qquad Q \equiv Y^* - Y.$$

By integrating by parts the integrand $G(v'_{02}/1+\omega)$ twice in ξ, we have

$$(3.18) \quad Q(x_0, t_0) = \int_0^{t_0} d\tau \int_{R^1} \left(\frac{\mu}{\rho_0}\left(\frac{G}{1+\omega}\right)_\xi\right)_\xi \left(\int_a^\xi \frac{\rho_0 v_0}{\mu} d\xi'\right) d\xi$$

$$= \int_0^{t_0} d\tau \int_{R_1} \left\{ -\frac{\partial}{\partial\tau} G(x_0, t_0; \xi, \tau) \right\} \left(\int_a^{\xi} \frac{\rho_0 v_0}{\mu} d\xi' \right) d\xi$$

$$= -\int_a^{x_0} \frac{\rho_0 v_0}{\mu} d\xi' + \int_{R^1} G(x_0, t_0; \xi, 0) \left(\int_a^{\xi} \frac{\rho_0 v_0}{\mu} d\xi' \right) d\xi$$

$$= -\int_{R^1} G(x_0, t_0; \xi, 0) \left(\int_{\xi}^{x_0} \frac{\rho_0 v_0}{\mu} d\xi' \right) d\xi,$$

where we have made use of the well-known fact that the formally adjoint operator $\mathcal{L}_1{}^*$ for $\mathcal{L}_1$ has the form

$$\mathcal{L}_1{}^*(u) = \left(\frac{\mu}{\rho_0} \left(\frac{u(x_0, t_0)}{1+\omega} \right)_{x_0} \right)_{x_0} + \frac{\partial}{\partial t_0} u(x_0, t_0) \tag{3.19}$$

and that, for the fundamental solution $G^*(x_0, t_0; \xi, \tau)$ $(t_0 < \tau)$ for $\mathcal{L}_1{}^*$, we have an equality (cf. [4], [10])

$$G^*(x_0, t_0; \xi, \tau) = G(\xi, \tau; x_0, t_0) \quad (t_0 < \tau). \tag{3.20}$$

By (3.18) we have an a priori estimate for $Q(x_0, t_0)$

$$|Q|_T{}^{(0)} \leqq \mu^{-1} \cdot \|\rho_0 v_{02}\|_{L^1(R^1)} \quad (< +\infty). \tag{3.21}$$

Similary we have for $Y^* (\bar{P} \equiv k\rho_0(-\infty))$

$$Y^*(x_0, t_0) = \int_0^{t_0} d\tau \int_{R^1} G \left\{ \frac{k(\rho_0 - \rho_0(-\infty)) - v_{01}'}{1+\omega} - \frac{\bar{P}\omega}{1+\omega} + \bar{P} \right\} \tag{3.22}$$

$$\times d\xi = \bar{P} t_0 + \int_0^{t_0} d\tau \int_{R^1} \left(\frac{\mu}{\rho_0} \left(\frac{G}{1+\omega} \right)_{\xi} \right)_{\xi} d\xi$$

$$\times \int_a^{\xi} \frac{\rho_0}{\mu} \left\{ -u_0(\xi') - \bar{P} \int_0^{\tau} \hat{v}(\xi', \tau') d\tau' \right\} d\xi'$$

$$= \bar{P} t_0 + \int_0^{t_0} d\tau \int_{R^1} \left(-\frac{\partial G}{\partial \tau} \right) d\xi \int_a^{\xi} \frac{\rho_0}{\mu} \{\cdots\} d\xi'$$

$$= \bar{P} t_0 + \frac{1}{\mu} \int_{R^1} G(x_0, t_0; \xi, 0) d\xi \int_{\xi}^{x_0} \rho_0 u_0 d\xi'$$

$$- \int_0^{t_0} d\tau \int_{R^1} G(x_0, t_0; \xi, \tau) d\xi \bar{P} \int_{\xi}^{x_0} \frac{\rho_0}{\mu} \hat{v}(\xi', \tau) d\xi'$$

$$\left(\text{Remark: } \hat{v} = -\frac{\mu}{\rho_0} Y_{\xi} + v_0 \right)$$

$$=\bar{P}t_0+\mu^{-1}\int_{R^1}G(x_0,t_0;\xi,0)d\xi\Big(\int_{\xi}^{x_0}\rho_0u_0d\xi'\Big)$$

$$+\frac{\bar{P}}{\mu}\int_0^{t_0}d\tau\int_{R^1}G(x_0,t_0;\xi,\tau)d\xi\Big(\int_{\xi}^{x_0}\rho_0v_0d\xi'\Big)$$

$$+\bar{P}\int_0^{t_0}d\tau\int_{R^1}G(x_0,t_0;\xi,\tau)Y(\xi,\tau)d\xi$$

$$-\bar{P}\int_0^{t_0}d\tau\, Y(x_0,\tau)$$

$$=\bar{P}t_0+\mu^{-1}\int_{R^1}G(x_0,t_0;\xi,0)\,d\xi\int_{\xi}^{x_0}\rho_0(u_{01}+u_{02})d\xi'$$

$$+\frac{\bar{P}}{\mu}\int_0^{t_0}d\tau\int_{R^1}G(x_0,t_0;\xi,\tau)\,d\xi\int_{\xi}^{x_0}\rho_0(v_{01}+v_{02})d\xi'$$

$$+\bar{P}\int_0^{t_0}d\tau\int_{R^1}G(x_0,t_0;\xi,\tau)d\xi(Y^*-Q)(\xi,\tau)$$

$$-\bar{P}\int_0^{t_0}d\tau(Y^*-Q)(x_0,\tau).$$

Thus, by Lemma 3. 3 and by the properties of Y^*, Q, v_0, and u_0, it follows that

$$0\leqq|Y^*|_{t_0}^{(0)}\leqq A_1t_0+B_1+\bar{P}\int_0^{t_0}|Y^*|_{\tau}^{(0)}d\tau, \tag{3.23}$$

where

$$A_1=\bar{P}\Big(1+3\Big\|\frac{\rho_0v_{02}}{\mu}\Big\|_{L^1(R^1)}\Big),\quad B_1=\Big\|\frac{\rho_0u_{02}}{\mu}\Big\|_{L^1(R^1)}. \tag{3.24}$$

Hence, we have a priori estimates such that

$$\begin{aligned}&|Y^*|_T^{(0)}\leqq(A_1+B_1)e^{\bar{P}T}-A_1,\\ &|Y|_T^{(0)}\leqq|Y^*|_T^{(0)}+|Q|_T^{(0)}\leqq(A_1+B_1)e^{\bar{P}T}-A_1\\ &\qquad+\Big\|\frac{\rho_0v_{02}}{\mu}\Big\|_{L^1(R^1)},\quad (N.B.: Y=Y^*-Q,\ Y^*\geqq0).\end{aligned} \tag{3.25}$$

Therefore, by Lemma 3. 1 we have:

Lemma 3. 4. *Under the assumptions* (1. 1)'-(3. 15) *on the initsal condition, we have an a priori estimate*

(3.26) $[\rho]_T=|\rho|_T^{(0)}+\left|\frac{1}{\rho}\right|_T^{(0)}\leqq C(T)<+\infty\,(C(T)\nearrow \text{ as } T\nearrow).$

Finally, we obtain:

Theorem 3. 1. *Under the assumptions* (1. 1)′-(3. 15) *on the initial condition for* (1. 1), *G.B.E. with a pressure model term, there exists a unique temporally global solution* (v, ρ) *of* (1. 1) *which belongs to* $H_T^{2+\alpha}\times B_T^1$ *for an arbitrary* $T\in(0, +\infty)$,

Corollary of Theorem 3. 1. *Under the same assuptions as above, there exists a unique regvlar solution* (v, ρ) *of* (1. 1) *in* $R^1\times[0, +\infty)$ *such that, for an arbitary* $T\in(0, +\infty)$, v *and* v_x *are bounded in* $R^1\times[0, T]$.

Proof. For example, see [9]. Q.E.D.

Epilogue. As regards the case that $v_0'\leqq 0$ and $v_0'\not\equiv 0$, we know only that there are global solutions of the form

$$v(x, t)=v_0(x-ct),\ \rho(x, t)=\rho_0(x-ct),\quad (\text{c. constant}).$$

The functions v_0 and ρ_0 can be known by substituting $v_0(x-ct)$ and $\rho_0(x-ct)$ into (1. 1) and solving a system of ordinary differential equations in v_0 and ρ_0. Except for such solutions, we do not know any result as yet. It is certain that (3. 8) and (3. 9) are very essential in our problem.

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Corrigenda and Addenda to the Author's Paper [7].

Corrigenda:

1) $|h(x, t)|_T^{(0)}=\sup_{(x,t)\in\overline{R^2_T}}|h(x, t)|.$

2) *Proof of Lemma* 1. 1. From the equation

$$det(\sigma(x,t)\cdot P_0(i\xi)-\lambda I)=-(\lambda+\sigma|\xi|^2)^2$$

$$\times\left(\lambda+\frac{4}{3}\sigma|\xi|^2\right)=0 \text{ (we denote the roots by } \lambda_i(i=1,2,3))$$

we have $\lambda_1=\lambda_2=-\sigma(x,t)|\xi|^2$ and $\lambda_3=-(4/3)\sigma(x,t)|\xi|^2$. Therefore, it holds that

$$\max_i \sup_{|\xi|=1} \mathrm{Re}\,\lambda_i(\xi;x,t)=-\sigma(x,t)\leqq-\sigma_0<0. \qquad \text{Q.E.D.}$$

3) *Lemma* 2. 2. *For the matrix* $e^{t\sigma(y,\tau)P_0(i\zeta)}$, *it holds that*

$$|e^{t\sigma(y,\tau)P_0(i\zeta)}|\leqq 3\sqrt{2}\,\{1+2t\sigma(y,\tau)|P_0(i\zeta)|$$

$$+4t^2\sigma(y,\tau)^2\cdot|P_0(i\zeta)|^2\}\exp\{t\sigma(y,\tau)\max_i \mathrm{Re}\,\lambda_i^{(0)}(\zeta)\}.$$

4) *Lemma* 2. 3. (*which follows directly from* 2) (*above*)).

$$\max_j \mathrm{Re}\,\lambda_j(\zeta=\xi+i\eta;y,\tau)\leqq\sigma(y,\tau)\left\{-|\xi|^2+\frac{4}{3}|\eta|^2\right\}.$$

[Thus, from Lemma 2. 4 on, we should take $\delta=2$ and $a=4/3$.]

Addenda:

1) $H_T{}^n\equiv\{h(x,t): D_x{}^r D_t{}^s h\,(|r|+2s\leqq n)$ *are continuous*, $\|h\|_T{}^{(n)\dagger)}<+\infty\}$,

$$B_T{}^n\equiv\{w(x,t): D_x{}^r D_t{}^s w\,(|r|+s\leqq n) \text{ are continuous},$$

$$\sum_{|r|+s=0}^{n}|D_x{}^r D_t{}^s w|_T{}^{(0)}<+\infty\},$$

$$B_T{}^{n+\alpha}\equiv\{w(x,t): w\in B_T{}^n,\ \sum_{|r|+s=n}|D_x{}^r D_t{}^s w|_T{}^{(\alpha)}<+\infty\}.$$

2) *Lemma* 4. 7. [$C_{16}^{(|m|)}$ *etc. are defined in* $\sigma_0{}^{-1}>0$, $\sigma_1>0$, $|\sigma|_T{}^{(\alpha)}\geqq 0$, $T\geqq 0$; *therefore, their values at* $T=0$ *are positsve.*]

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†) $\|h\|_T^{(n)}\equiv\sum_{|r|+2s=0}^{n}|D_x^r D_t^s h|_T^{(0)}$

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