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On realization of Siegel domains of the second kind as those of the third kind

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Introduction

Let D be a Siegel domain of the second kind associated with a convex cone V and a V-hermitian form F. Realization of D as a Siegel domain of the third kind was studied by Pyatetski-Shapiro [5], Wolf-Korányi [13] and by Satake [6] when D is symmetric.

Kaneyuki [1] and Takeuchi [8] treated the case where D is homogeneous. Their methods are based on the correspondence between "j-algebras" and "homogeneous Siegel domains of the second kind".

The purpose of the present paper is to prove that D *can be realized as a Siegel domain* $\mathfrak{D}$ *of the third kind in such a way that the group* $Aut(D)$ *acts on* $\mathfrak{D}$ *as quasi-linear transformations.* This is a generalization of a result of Takeuchi [8].[1]

In § 1, we recall some results in [2] and [4] on the structute of the Lie algebra $\mathfrak{g}(D)$ of $\mathrm{Aut}(D)$ and construct a symmetric domain S which corresponds to a semi-simple part of $\mathfrak{g}(D)$. We also recall Tanaka's imbedding of the domain D ([10], [3]).

In § 2, we study a Cartan decomposition of $\mathfrak{g}(D)$ assuming that D is symmetric. Many results in this section can be obtained also from Satake [7]. But our methods and proofs seem to be more direct and simpler.

[1] Takeuchi [7] obtained this result for the identity component of Aut (D) when D is homogeneous.

By using the results in previous sections, we shall study in § 3 the structure of the cone V and introduce a cone V_r and a V_r-hermitian form H.

Finally in § 4, by the same arguments as in Satake [6], we construct a Siegel domain $\mathcal{D}$ of the third kind, with S as a base space, whose fiber is the Siegel domain of the second kind associated with V_r and H. And making use of Tanaka's imbedding, we shall see that D and $\mathcal{D}$ are holomorphically equivalent.

§ 1. Summary of known results.

1. 1. Let R (resp. W) be a real (resp. complex) vector space of a finite dimension. Denote by R_c the complexification of R. For every vector $z\in R_c$, we denote by $\mathrm{Re}\, z$ (resp. by $\mathrm{Im}\, z$) its real (resp. imaginary) part.[2)]

Let D be a Siegel domain of the second kind in R_c+W, due to Pyatetski-Shapiro [5], associated with a convex cone V in R and a V-hermitian form F on W, and let $\mathfrak{g}(D)$ be the Lie algebra of $\mathrm{Aut}(D)$, the group of all holomorphic transformations of the domain D. Denote by E (resp. by I) the element of $\mathfrak{g}(D)$ induced by the following one parameter group in $\mathrm{Aut}(D)$ (with parameter t):

$$z+w\to e^{-2t}z+e^{-t}w \quad (z\in R_c,\ w\in W)$$
$$(\text{resp. } z+w\to z+e^{\sqrt{-1}t}\,w).$$

Then from Kaup-Matsushima-Ochiai [2], the Lie algebra $\mathfrak{g}(D)$ has the following graded structure:

$$\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\mathfrak{g}^{1}+\mathfrak{g}^{2} \quad ([\mathfrak{g}^{\lambda},\mathfrak{g}^{\mu}]\subset\mathfrak{g}^{\lambda+\mu}),$$
$$\mathfrak{g}^{\lambda}=\{X\in\mathfrak{g}(D);\ [E,X]=\lambda X\},$$
$$\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^{0} \quad (\mathfrak{r}^{\lambda}=\mathfrak{r}\cap\mathfrak{g}^{\lambda}),$$

where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}(D)$.

We also know from [2] that both E and I are in the center of $\mathfrak{g}^0$ and that I has the following properties (cf. [3]):

(1. 1) $$ad\, I=0 \text{ on } \mathfrak{g}^{-2}+\mathfrak{g}^{0}+\mathfrak{g}^{2},$$

2) In what follows, for a vector space or a Lie algebra A, we always mean by A_c its complexification.

$$(ad\, I)^2 = -1 \text{ on } \mathfrak{g}^{-1} + \mathfrak{g}^1.$$

The space $\mathfrak{g}^{-2}$ (resp. $\mathfrak{g}^{-1}$) is identified with R (resp. with W) in a natural manner.[3] Then the complex structure of $\mathfrak{g}^{-1}$ is given by $ad\, I$, and the hermitian form F and the domain D are expressed as follows:

$$F(w, w') = \tfrac{1}{4}([[I, w], w'] + \sqrt{-1}[w, w']), \tag{1.2}$$

$$D = \{z + w \in \mathfrak{g}_c^{-2} + \mathfrak{g}^{-1};\ \mathrm{Im}\, z - F(w, w) \in V\}.$$

1.2. We now recall some results in [4]. There exists a semi-simple graded subalgebra $\mathfrak{s} = \sum_{\lambda=-2}^{2} \mathfrak{s}^\lambda$ of $\mathfrak{g}(D)$ with the following properties:

(1.3) i) $\mathfrak{s}^1 = \mathfrak{g}^1$ and $\mathfrak{s}^2 = \mathfrak{g}^2$.

ii) The adjoint representation of $\mathfrak{s}^0$ on $\mathfrak{s}^1 + \mathfrak{s}^2$ is faithful.

Then $\mathfrak{s}^{-1}$ is a complex subspace of $\mathfrak{g}^{-1}$ and

$$\mathfrak{g}^{-2} = \mathfrak{s}^{-2} + \mathfrak{r}^{-2} \text{ (direct sum)} \tag{1.4}$$

$$\mathfrak{g}^{-1} = \mathfrak{s}^{-1} + \mathfrak{r}^{-1} \text{ (direct sum)}.$$

Since $\mathfrak{s}$ is semi-simple, there exists a unique E_s in $\mathfrak{s}^0$ such that

$$\mathfrak{s}^\lambda = \{X \in \mathfrak{s};\ [E_s, X] = \lambda X\}. \tag{1.5}$$

We set

$$\mathfrak{r}_s^{-2} = \{X \in \mathfrak{r}^{-2};\ [E_s, X] = -X\}, \tag{1.6}$$

$$\mathfrak{r}_0^{-2} = \{X \in \mathfrak{r}^{-2};\ [\mathfrak{s}, X] = 0\},$$

$$\mathfrak{r}_s^0 = \{X \in \mathfrak{r}^0;\ [E_s, X] = X\},$$

$$\mathfrak{r}_0^0 = \{X \in \mathfrak{r}^0;\ [\mathfrak{s}, X] = 0\}.$$

We then have

$$\mathfrak{r}^{-2} = \mathfrak{r}_s^{-2} + \mathfrak{r}_0^{-2} \text{ (direct sum)} \tag{1.7}$$

$$\mathfrak{r}^0 = \mathfrak{r}_s^0 + \mathfrak{r}_0^0 \text{ (direct sum)},$$

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[3] $\mathfrak{g}^{-2}$ (resp. $\mathfrak{g}^{-1}$) consists of all elements of $\mathfrak{g}(D)$ induced by the following one parameter group (with parameter t):

$$z + w \longrightarrow z + ta + w \quad (a \in R)$$

$$(\text{resp. } z + w \longrightarrow z + 2\sqrt{-1}F(w, tc) + \sqrt{-1}F(tc, tc) \quad (c \in W)).$$

$$\mathfrak{r}_s^{-2}=[\mathfrak{r}^{-2}, \mathfrak{s}^0]=[\mathfrak{r}^0, \mathfrak{s}^{-2}]\supset[\mathfrak{r}^{-1}, \mathfrak{s}^{-1}]$$

$$\mathfrak{r}_s^0=[\mathfrak{r}^0, \mathfrak{s}^0]=[\mathfrak{r}^{-2}, \mathfrak{s}^2]\supset[\mathfrak{r}^{-1}, \mathfrak{s}^1].$$

Moreover we know

(1.8) $$[E_s, \mathfrak{r}^{-1}]=0.$$

From (1.6), (1.7) and (1.8) we get

(1.9) $$[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]\subset\mathfrak{r}_0^{-2},$$

(1.10) $$[\mathfrak{r}_s^0, \mathfrak{r}_0^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}_s^0]=0$$

$$[\mathfrak{r}_s^0, \mathfrak{r}_s^{-2}]\subset\mathfrak{r}_0^{-2}.$$

For the algebra $\mathfrak{s}$, there exists a semi-simple subalgebra $\mathfrak{c}$ of $\mathfrak{g}^0$ such that

i) $[\mathfrak{s}, \mathfrak{c}]=0$.

ii) $\mathfrak{s}+\mathfrak{c}$ is a direct sum and is a semi-simple part of $\mathfrak{g}(D)$.

Note that the spaces $\mathfrak{r}_s^{-2}, \mathfrak{r}_0^{-2}, \mathfrak{r}_s^0$ and $\mathfrak{r}_0^0$ are stable by $ad\,X$ for $X\in\mathfrak{s}^0+\mathfrak{c}+\mathfrak{r}_0^0$, because $[E_s, \mathfrak{s}^0+\mathfrak{c}+\mathfrak{r}_0^0]=0$.

Let us denote by η_s the projection of $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ onto $\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}$ with respect to the decompositions (1.4). Then from (1.6), we get for any $v\in\mathfrak{g}^{-2}$,

$$\eta_s(v)=\lim_{t\to\infty}\frac{1}{e^{2t}}Ad(\exp(tE_s))v.$$

Therefore if we put

$$V_s=\eta_s(V),\quad S=\eta_s(D),$$

then V_s is contained in $\overline{V}$, because $Ad(\exp X)V=V$ for any $X\in\mathfrak{g}^0$.[4] Hence V_s is an open convex cone is $\mathfrak{s}^{-2}$ containing no entire straight lines. Clearly the restriction F_s of F to $\mathfrak{s}^{-1}\times\mathfrak{s}^{-1}$ is a V_s-hermitian form. One of the main results in [4] is the following.

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4) $\mathfrak{g}^0$ consists of all $A\in\mathfrak{gl}(R_c+W)$ satisfying the followings: $A(R)\subset R$, $A(W)\subset W$, $\exp tA(V)=V$ and $AF(w, w')=F(Aw, w')+F(w, Aw')$. And under the identification of $\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ with $R+W$, the equality; $Ad(\exp A)X=\exp A(X)$ holds for any $X\in\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$.

Theorem 1.1. *S is the symmetric Siegel domain of the second kink in $\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}$ associated with V_s and F_s, and the graded Lie algebra $\mathfrak{s}$ is identified with $\mathfrak{g}(S)$.*

1.3. In this paragraph, we recall Tanaka's imbeddings. (For proofs of the following facts, see [10] and [3].)

Let G be the identity component of $\mathrm{Aut}(D)$ and let $\mathfrak{g}(D)_c$ be the complexification of $\mathfrak{g}(D)$. We denote by G_c the adjoint group of $\mathfrak{g}(D)_c$. Since G is centerless ([2]), we identify the Lie algebra G_c with $\mathfrak{g}(D)_c$ and G with a subgroup of G_c. Define linear transformations Q and $\bar{Q}$ of $\mathfrak{g}_c^{-1}+\mathfrak{g}_c^1$ by

$$(1.11)\qquad Q(X)=\tfrac{1}{2}(X-\sqrt{-1}[I, X])$$
$$\bar{Q}(X)=\tfrac{1}{2}(X+\sqrt{-1}[I, X])\quad \text{for}\quad X\in\mathfrak{g}_c^{-1}+\mathfrak{g}_c^1.$$

We then have for $\lambda=-1, 1$

$$(1.12)\qquad \mathfrak{g}_c^\lambda=Q(\mathfrak{g}^\lambda)+\bar{Q}(\mathfrak{g}^\lambda)\quad \text{(direct sum)},$$
$$Q(\mathfrak{g}^\lambda)=Q(\mathfrak{g}_c^\lambda)=\{X\in\mathfrak{g}_c^\lambda;\ [I, X]=\sqrt{-1}X\}.$$
$$\bar{Q}(\mathfrak{g}^\lambda)=\bar{Q}(\mathfrak{g}_c^\lambda)=\{X\in\mathfrak{g}_c^\lambda;\ [I, X]=-\sqrt{-1}X\}.$$

We set

$$(1.13)\qquad \mathfrak{b}=\bar{Q}(\mathfrak{g}^{-1})+\mathfrak{g}_c^0+\mathfrak{g}_c^1+\mathfrak{g}_c^2.$$

Then $\mathfrak{b}$ is a complex subalgebra of $\mathfrak{g}(D)_c$ and $\dim\mathfrak{b}=\dim\mathfrak{g}(D)_c-\dim D$. Let B be the closed subgroup of G_c defined by

$$(1.14)\qquad B=\{g\in G_c;\ g(\mathfrak{b})=\mathfrak{b}\}.$$

The Lie algebra of B coinsides with $\mathfrak{b}$ as is easily observed. We can now construct a map h of $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ to G_c/B as follows:

$$(1.15)\qquad h(z+w)=\pi\cdot\exp(z+Q(w))\quad (z\in\mathfrak{g}_c^{-2}, w\in\mathfrak{g}^{-1}),$$

where π denotes the projection of G_c onto G_c/B. The map h is holomorphic because $Q([I, w])=\sqrt{-1}Q(w)$. Moreover h is an imbedding of $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ onto an open set of G_c/B and satisfies the following

$$(1.16)\qquad h(g(p))=g\cdot h(p)\quad \text{for}\quad g\in G, p\in D.$$

Remark 1. The mapping h was first constructed by Tanaka [10] when the domain D is homogeneous, and extended to general cases in [3].

§ 2. Symmetric Siegel domains.

2. 1. Throughout this section, we assume that the Siegel domain D is symmetric, which is equivalent to say that $\mathfrak{g}(D)$ is semi-simple.

Lemma 2. 1. *Let $e \in V$. Then there exists a unique e^* in $\mathfrak{g}^2$ such that $[e^*, e] = E$.*

Proof. The uniqueness is obvious, since the mapping: $X \to [e, X]$ of $\mathfrak{g}^2$ to $\mathfrak{g}^0$ is injective (Vey [11]). We shall show the existence. Since D is symmetric, the subalgebra $\mathfrak{g}' = \mathfrak{g}^{-2} + [\mathfrak{g}^{-2}, \mathfrak{g}^2] + \mathfrak{g}^2$ is also semi-simple and E is contained in $[\mathfrak{g}^{-2}, \mathfrak{g}^2]$ ([4]). We denote by ρ the adjoint representaiton of $[\mathfrak{g}^{-2}, \mathfrak{g}^2]$ on $\mathfrak{g}^{-2}$. Let φ_V be the characteristic function of V, which is a positive function defined on V and satisfies the following equality (Vinberg [12]):

$$\varphi_V(ax) = (\det a)^{-1} \varphi_V(x), \tag{2.1}$$

where $a = \exp \rho(A)$ $(A \in [\mathfrak{g}^{-2}, \mathfrak{g}^2])$. Put $M(x) = \log \varphi_V(x)$. Since the killing form α' of $\mathfrak{g}'$ gives a duality between $\mathfrak{g}^{-2}$ and $\mathfrak{g}^2$, we can write in the Taylor series for $M(e+tx)$ as follows:

$$M(e+tx) = M(e) - t\alpha'(\tilde{e}, x) + 0(t^2).$$

where $\tilde{e} \in \mathfrak{g}^2$. Then from (2. 1), we obtain

$$\alpha'(\tilde{e}, [A, e]) = \mathrm{Tr}\, \rho(A) \quad \text{for} \quad A \in [\mathfrak{g}^{-2}, \mathfrak{g}^2].$$

Let $e^* = 4\tilde{e}$. Then $\alpha'([e^*, e], A) = -4 \,\mathrm{Tr}\, \rho(A)$. On the other hand from Tanaka [9], we get $\alpha'(E, A) = 2\, \mathrm{Tr}\, \rho(E) \cdot \rho(A) = -4\, \mathrm{Tr}\, \rho(A)$. Therefore $\alpha'(E - [e^*, e], A) = 0$ for any $A \in [\mathfrak{g}^{-2}, \mathfrak{g}^2]$ and hence $\alpha'(E - [e^*, e], \mathfrak{g}') = 0$. This implies $E = [e^*, e]$. q.e.d.

2. 2. We now investigate Cartan decompositions of the Lie algebra $\mathfrak{g}(D)$. Let $\mathfrak{g}(D) = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition at the point $\sqrt{-1}e \in D (e \in V)$ and let σ be the corresponding Cartan involution.

Then $\mathfrak{k}$ is the isotropy subalgebra of $\mathfrak{g}(D)$ at $\sqrt{-1}e$. Therefore from [2], we have

$$(2.2) \qquad \mathfrak{k}=\mathfrak{k}^0+\mathfrak{m}+\mathfrak{n} \quad \text{(direct sum)},$$
$$\mathfrak{k}^0=\mathfrak{k}\cap\mathfrak{g}^0=\{A\in\mathfrak{g}^0;\ [e, A]=0\},$$
$$\mathfrak{m}=\{X+\tfrac{1}{2}[e, [e, X]];\ X\in\mathfrak{g}^2\},$$
$$\mathfrak{n}=\{Y+[I, [e, Y]];\ Y\in\mathfrak{g}^1\}.$$

We set

$$(2.3) \qquad \tilde{\mathfrak{m}}=\{X-\tfrac{1}{2}[e, [e, X]];\ X\in\mathfrak{g}^2\},$$
$$\tilde{\mathfrak{n}}=\{Y-[I, [e, Y]];\ Y\in\mathfrak{g}^1\}.$$

Lemma 2.2. $\mathfrak{p}\supset\tilde{\mathfrak{n}}+\tilde{\mathfrak{m}}$.

Proof. Let α denote the killing form of $\mathfrak{g}(D)$. Then $\mathfrak{p}=\{X\in\mathfrak{g}(D);\ \alpha(X, \mathfrak{k})=0\}$. Clearly $\alpha(\tilde{\mathfrak{m}}+\tilde{\mathfrak{n}}, \mathfrak{k}^0)=\alpha(\tilde{\mathfrak{m}}, \mathfrak{n})=\alpha(\tilde{\mathfrak{n}}, \mathfrak{m})=0$, because $\alpha(\mathfrak{g}^\lambda, \mathfrak{g}^\mu)=0$ for $\lambda+\mu\neq0$. Let $X, X'\in\mathfrak{g}^2$. Then

$$\alpha(X-\tfrac{1}{2}[e, [e, X]], X'+\tfrac{1}{2}[e, [e, X']])$$
$$=\tfrac{1}{2}\alpha(X, [e, [e, X']])-\tfrac{1}{2}\alpha([e, [e, X], X')=0\,.$$

Therefore we have $\alpha(\tilde{\mathfrak{m}}, \mathfrak{m})=0$ and hence $\tilde{\mathfrak{m}}\subset\mathfrak{p}$. By using (1.1) we can show $\alpha(\tilde{\mathfrak{n}}, \mathfrak{n})=0$ similarly. q.e.d.

Since $\sigma=1$ on $\mathfrak{k}$, we have for any $X\in\mathfrak{g}^2$

$$X+\tfrac{1}{2}[e, [e, X]]=\sigma(X)+\tfrac{1}{2}[\sigma(e), [\sigma(e), \sigma(X)]].$$

On the other hand, $\sigma=-1$ on $\mathfrak{p}$. It follows from Lemma 2.2

$$-X+\tfrac{1}{2}[e, [e, X]]=\sigma(X)-\tfrac{1}{2}[\sigma(e), [\sigma(e), \sigma(X)]].$$

Therefore we get

$$(2.4) \qquad \sigma(X)=\tfrac{1}{2}[e, [e, X]] \quad \text{for} \quad X\in\mathfrak{g}^2.$$

Similarly we have

$$(2.5) \qquad \sigma(Y)=[I, [e, X]] \quad \text{for} \quad Y\in\mathfrak{g}^1.$$

Let e^* be as in Lemma 2.1. Then $\tfrac{1}{2}[e^*, [e^*, X]]\in\mathfrak{g}^2$ for any $X\in\mathfrak{g}^{-2}$.

Hence by (2.4) we have

$$\sigma(\tfrac{1}{2}[e^*, [e^*, X]]) = \tfrac{1}{4}[e, [e, [e^*, [e^*, X]]]]$$
$$= \tfrac{1}{4}[e, [e^*, 2X]]$$
$$= X,$$

here we use the fact that E is in the center of $\mathfrak{g}^0$. Since $\sigma^2 = 1$, we have

(2.6) $$\sigma(X) = \tfrac{1}{2}[e^*, [e^*, X]] \quad \text{for} \quad X \in \mathfrak{g}^{-2}.$$

Similarly by using (1.1), we get

(2.7) $$\sigma(Y) = -[I, [e^*, Y]] \quad \text{for} \quad Y \in \mathfrak{g}^{-1}.$$

By (2.4), (2.5), (2.6) and (2.7), we know that $\sigma(\mathfrak{g}^\lambda) = \mathfrak{g}^{-\lambda}$ for $\lambda = -2, -1, 1$ and 2. Hence $\sigma(\mathfrak{g}^0) = \mathfrak{g}^0$, because $\mathfrak{g}^0 = [\mathfrak{g}^{-2}, \mathfrak{g}^2] + [\mathfrak{g}^{-1}, \mathfrak{g}^1]$ ([4]). Clearly $\sigma(e) = -e^*$ and $\sigma(e^*) = -e$. Therefore $\sigma(E) = -E$. We now assert $[e, \mathfrak{g}^2] = [e^*, \mathfrak{g}^{-2}]$. In fact $\mathfrak{g}^2 = [e^*, [e^*, \mathfrak{g}^{-2}]]$. So, $[e, \mathfrak{g}^2] = [e^*, [e, [e^*, \mathfrak{g}^{-2}]]] = [e^*, \mathfrak{g}^{-2}]$, proving our assertion. We set

(2.8) $$\mathfrak{p}^0 = [e, \mathfrak{g}^2] = [e^*, \mathfrak{g}^{-2}].$$

Then $\mathfrak{k}^0 \cap \mathfrak{p}^0 = 0$, because the mapping: $X \to [e, [e, X]]$ of $\mathfrak{g}^2$ to $\mathfrak{g}^{-2}$ is injective (cf. (2.2)). Moreover $\dim \mathfrak{g}^0 = \dim \mathfrak{k}^0 + \dim \mathfrak{g}^{-2} = \dim \mathfrak{k}^0 + \dim \mathfrak{p}^0$. Hence we get $\mathfrak{g}^0 = \mathfrak{k}^0 + \mathfrak{p}^0$ (direct sum). Being invariant by σ, $\mathfrak{p}^0$ is containd in $\mathfrak{p}$. Thus we have proved.

Theorem 2.3. *Let D be a symmetric Siegel domain of the second kind and let $\mathfrak{g}(D) = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition at the point $\sqrt{-1}e (e \in V)$. Then*

$$\mathfrak{k} = \mathfrak{k}^0 + \mathfrak{m} + \mathfrak{n} \quad (\textit{direct sum})$$

$$\mathfrak{p} = \mathfrak{p}^0 + \tilde{\mathfrak{m}} + \tilde{\mathfrak{n}} \quad (\textit{direct sum}),$$

where $\mathfrak{k}^0, \mathfrak{m}, \mathfrak{n}, \tilde{\mathfrak{m}}, \tilde{\mathfrak{n}}$ and $\mathfrak{p}^0$ are given by (2.2), (2.3) and (2.8).

2.3. Let us denote by $\tilde{A}$ the holomorphic vector field on D corresponding to $A \in \mathfrak{g}(D)$. Put $q = \sqrt{-1}e$. It is easy to see that the following equalities hold:

$$(2.9)\qquad \tilde{A}_q=\sqrt{-1}\,\widetilde{[A, e]}_q \quad \text{for} \quad A\in\mathfrak{g}^0,$$
$$\widetilde{[I, Y]}_q=\sqrt{-1}\,\tilde{Y}_q \quad \text{for} \quad Y\in\mathfrak{g}^{-1}.$$

Let J be the complex structure of D. For $X\in\mathfrak{g}^2$, we have $\tilde{X}_q-\widetilde{\sigma(X)}_q=-2\widetilde{\sigma(X)}_q$, because $X+\sigma(X)\in\mathfrak{k}$ and hence $\tilde{X}_q+\widetilde{\sigma(X)}_q=0$. On the other hand by (2.4) and (2.9), $\widetilde{[e, X]}_q=\sqrt{-1}\,\widetilde{[[e, X], e]}_q=-2\sqrt{-1}\,\widetilde{\sigma(X)}_q$. Therefore we have

$$(2.10)\qquad J(\widetilde{X-\sigma(X)})_q=\widetilde{[e, X]}_q \quad \text{for} \quad X\in\mathfrak{g}^2.$$

Similarly for $Y\in\mathfrak{g}^1$, we have $\tilde{Y}_q-\widetilde{\sigma(Y)}_q=-2\widetilde{\sigma(Y)}_q$ and $\widetilde{[I, Y]}_q-\widetilde{\sigma([I, Y])}_q=-2\widetilde{\sigma([I, Y])}_q=-2\widetilde{[I, \sigma(Y)]}_q=-2\sqrt{-1}\,\widetilde{\sigma(Y)}_q$. Therefore

$$(2.11)\qquad J(\widetilde{Y-\sigma(Y)})_q=\widetilde{[I, Y]}_q-\widetilde{\sigma([I, Y])}_q \quad \text{for} \quad Y\in\mathfrak{g}^1.$$

We set

$$(2.12)\qquad Z=\tfrac{1}{2}(I+e-e^*)=\tfrac{1}{2}(I+e+\sigma(e)).$$

Clearly Z is contained in $\mathfrak{k}$.

Proposition 2.4.

(1) *Z belongs to the center of $\mathfrak{k}$.*

(2) *Under the natural identification of $\mathfrak{p}$ with $T_{\sqrt{-1}e}(D)$, the tangent space to D at $\sqrt{-1}e$, the following equality holds*:

$$ad\,ZX=JX \quad \text{for} \quad X\in\mathfrak{p}.$$

Proof. Since $[e, \mathfrak{k}^0]=0$, $\sigma([e, \mathfrak{k}^0])=[-e^*, \mathfrak{k}^0]=0$. Let $X\in\mathfrak{g}^2$. Then by using (1.1), we obtain $[Z, X+\sigma(X)]=\frac{1}{2}([e, X]+[\sigma(e), \sigma(X)])=\frac{1}{2}([e, X]+\sigma([e, X]))=0$, because $[e, X]\in\mathfrak{p}^0$ by Theorem 2.3. Therefore $[Z, \mathfrak{m}]=0$. Next let $Y\in\mathfrak{g}^1$. Then from (1.1), (2.5) and (2.7), $[Z, Y+\sigma(Y)]=\frac{1}{2}([I, Y]+[I, \sigma(Y)]+[e, Y]+\sigma([e, Y]))=\frac{1}{2}([I, Y]-[e, Y]+[e, Y]-[I, Y])=0$. Hence $[Z, \mathfrak{n}]=0$, proving (1).

By direct computations, we have

$$(2.13)\qquad [Z, X-\sigma(X)]=[e, X] \quad \text{for} \quad X\in\mathfrak{g}^2.$$

$$(2.14)\qquad [Z, Y-\sigma(Y)]=[I, Y-\sigma(Y)]$$

$$=[I, Y]-\sigma([I, Y]) \quad \text{for} \quad Y\in\mathfrak{g}^1.$$

Now the statemet (2) follows immediately from (2. 10), (2. 11), (2. 13) and (2. 14). q.e.d.

We set

$$\mathfrak{p}_-=\{X\in\mathfrak{p}_c;\ [Z, X]=-\sqrt{-1}X\}, \tag{2.15}$$

$$\mathfrak{p}_+=\{X\in\mathfrak{p}_c;\ [Z, X]=\sqrt{-1}X\}.$$

Then the following equalities hold:

$$\mathfrak{p}_c=\mathfrak{p}_+ + \mathfrak{p}_- \quad \text{(direct sum)}, \tag{2.16}$$

$$\mathfrak{p}_+=\{X-\sqrt{-1}[Z, X];\ X\in\mathfrak{p}_c\},$$

$$\mathfrak{p}_-=\{X+\sqrt{-1}[Z, X];\ X\in\mathfrak{p}_c\}.$$

Proposition 2. 5. *The following equlity holds:*

$$Ad(\exp\sqrt{-1}e)\mathfrak{b}=\mathfrak{k}_c+\mathfrak{p}_-,$$

where $\mathfrak{b}$ *is the subalgebra of* $\mathfrak{g}(D)_c$ *given by (1. 13).*

Proof. Let $X\in\mathfrak{g}_c^2$. By (2. 13) we have $Ad(\exp\sqrt{-1}e)X=X+\sqrt{-1}[e, X]-\frac{1}{2}[e, [e, X]]=X-\sigma(X)+\sqrt{-1}\, ad\, Z(X-\sigma(X))$. Therefore by (2. 16), $Ad(\exp\sqrt{-1}e)\mathfrak{g}_c^2\subset\mathfrak{p}_-$. Next let $Y\in\mathfrak{g}_c^1$. Then $Ad(\exp\sqrt{-1}e)Y-\sigma(Ad(\exp\sqrt{-1}e)Y)=Y+\sqrt{-1}[e, Y]-\sigma(Y)-\sqrt{-1}\times[\sigma(e), \sigma(Y)]=Y-\sigma(Y)+\sqrt{-1}([I, Y]+[e, Y])=Y-\sigma(Y)+\sqrt{-1}\times ad\, Z(Y-\sigma(Y))$. This implies $Ad(\exp\sqrt{-1}e)\mathfrak{g}_c^1\subset\mathfrak{k}_c+\mathfrak{p}_-$. Let $A=A_1+A_2\in\mathfrak{g}_c^0$, where $A_1\in\mathfrak{k}_c^0$ and $A_2\in\mathfrak{p}_c^0$. It follows $Ad(\exp\sqrt{-1}e)A\equiv A_2+\sqrt{-1}[e, A_2] (\mathrm{mod}\ \mathfrak{k}_c)$. And $A_2+\sqrt{-1}[e, A_2]-\sigma(A_2+\sqrt{-1}[e, A_2])=2A_2+\sqrt{-1}[e, A_2]-\sqrt{-1}[e^*, A_2]=2(A_2+\sqrt{-1}[Z, A_2])\in\mathfrak{p}_-$. Hence $Ad(\exp\sqrt{-1}e)\mathfrak{g}_c^0\subset\mathfrak{k}_c+\mathfrak{p}_-$. Finally for $Y\in\mathfrak{g}_c^{-1}$, $Ad(\exp\sqrt{-1}e)\bar{Q}(Y)=\bar{Q}(Y)$. Since $\sigma(I)=I$, it follows $\bar{Q}(Y)-\sigma\bar{Q}(Y)=\bar{Q}(Y-\sigma(Y))$. Hence by (2. 14), we get $ad\, Z(\bar{Q}(Y)-\sigma(\bar{Q}(Y)))=ad\, I(\bar{Q}(Y-\sigma(Y))=-\sqrt{-1}(\bar{Q}(Y)-\sigma\bar{Q}(Y))$. Therefore $\bar{Q}(Y)-\sigma(\bar{Q}(Y))\in\mathfrak{p}_-$ and hence $\bar{Q}(\mathfrak{g}_c^{-1})\subset\mathfrak{k}_c+\mathfrak{p}_-$. Thus we have proved $Ad(\exp\sqrt{-1}e)\mathfrak{b}\subset\mathfrak{k}_c+\mathfrak{p}_-$. Since $\dim(\mathfrak{k}_c+\mathfrak{p}_-)=\dim\mathfrak{g}(D)_c-\dim D=\dim\mathfrak{b}$, we can conclude $Ad(\exp\sqrt{-1}e)\mathfrak{b}=\mathfrak{k}_c+\mathfrak{p}_-$. q.e.d.

§ 3. The structure of the cone V.

3.1. We return to general cases. Let η_r denote the projection of $\mathfrak{g}^{-2}$ onto $\mathfrak{r}_0^{-2}$ with respect to the decomposition $\mathfrak{g}^{-2}=\mathfrak{s}^{-2}+\mathfrak{r}_0^{-2}+\mathfrak{r}_s^{-2}$. Put

$$V_r=\eta_r(V).$$

By (1.6) we get for any $v\in\mathfrak{g}^{-2}$

$$\eta_r(v)=\lim_{t\to\infty} Ad(\exp tE_s)v\,.$$

Therefore $V_r\subset\overline{V}$. From this fact, it follows that V_r is an open convex cone in $\mathfrak{r}_0^{-2}$ containing no entire straight lines. It is clear from (1.2) and (1.9) that

(3.1) $\qquad F(w, w)=\frac{1}{4}[[I, w], w]\in\overline{V}_r \quad \text{for} \quad w\in\mathfrak{r}^{-1}.$

Let $v=e+a+b\in\mathfrak{g}^{-2}$, where $e\in V_s$, $a\in\mathfrak{r}_0^{-2}$ and $b\in\mathfrak{r}_s^{-2}$. Since the domain S, constructed in § 1, is symmetric, there exists e^* in $\mathfrak{s}^2(=\mathfrak{g}^2)$ such that $E_s=[e^*, e]$ by Lemma 2.1. We then have by (1.6), (1.7) and (1.10)

(3.2) $\qquad Ad(\exp[e^*, b])e=e-b+\frac{1}{2}[[b, e^*], b],$

(3.3) $\qquad Ad(\exp[e^*, b])v=e+a-\frac{1}{2}[[b, e^*], b].$

Since V_s is contained in $\overline{V}$, we get from (3.2)

(3.4) $\qquad \frac{1}{2}[[b, e^*], b]\in\overline{V}_r \quad \text{for any} \quad b\in\mathfrak{r}_s^{-2}.$

And by (3.3)

(3.5) $\qquad a-\frac{1}{2}[[b, e^*], b]\in V_r \quad \text{if} \quad v\in V.$

Let $\eta=\eta_s+\eta_r$, i.e., the projection of $\mathfrak{g}^{-2}$ to $\mathfrak{s}^{-2}+\mathfrak{r}_0^{-2}$. Since $x+y\in V$ if $x\in V$ and $y\in\overline{V}$, we know from (3.2) and (3.4) that $\eta(V)$ is contained in V.

Lemma 3.1. $V_s+V_r=\eta(V)$.

Proof. Clearly $V_s+V_r\supset\eta(V)$. Conversely let $e\in V_s$ and $a\in V_r$. Then $e+a\in\overline{V}$. Therefore $e+a\in\eta(\overline{V})\subset\overline{\eta(V)}$. Hence $V_s+V_r\subset\overline{\eta(V)}$. Therefore V_s+V_r is the interior of $\overline{\eta(V)}$. This implies V_s+V_r

$=\eta(V)$. q.e.d.

Proposition 3.2. *Let* $v=e+a+b\in\mathfrak{g}^{-2}$, *where* $e\in\mathfrak{s}^{-2}$, $a\in\mathfrak{r}_0^{-2}$ *and* $b\in\mathfrak{r}_s^{-2}$. *Then* $v\in V$ *is and only if* $e\in V_s$ *and* $a-\frac{1}{2}[[b, e^*], b]\in V_r$.

Proof. The "if" part is already proved (cf. (3.5)). Suppose that $e\in V_s$ and $a-\frac{1}{2}[[b, e^*], b]\in V_r$. Then from Lemma 3.1, $u=e+a-\frac{1}{2}[[b, e^*], b]\in\eta(V)\subset V$. Since $ad(\exp(-[e^*, b]))u=v$, we get $v\in V$. q.e.d.

If we set

$$D_o=\eta_s^{-1}(\sqrt{-1}e)\quad(e\in V_s), \tag{3.6}$$

Then we get immediately from proposition 3.2

Corollary 3.3.

$$D_o=\{u+v+w+\sqrt{-1}e;\ u\in(\mathfrak{r}_0^{-2})_c,\ v\in(\mathfrak{r}_s^{-2})_c,\ w\in\mathfrak{r}^{-1}$$
$$\operatorname{Im} u-\tfrac{1}{4}[[I, w], w]-\tfrac{1}{2}[[\operatorname{Im} v, e^*], \operatorname{Im} v]\in V_r\}.$$

3.2. By (1.7), $ad\,e$ (resp. $ad\,e^*$) gives a linear mapping of $\mathfrak{r}_s^0$ (resp. $\mathfrak{r}_s^{-2}$) to $\mathfrak{r}_s^{-2}$ (resp. to $\mathfrak{r}_s^0$). Let $X\in\mathfrak{r}_s^{-2}$ and $Y\in\mathfrak{r}_s^0$. By (1.6), $[e, [e^*, X]]=[[e, e^*], X]=-[E_s, X]=X$ and $[e^*, [e, Y]]=[[e^*, e], Y]=[E_s, Y]=Y$. Thereby

$$ad\,e\cdot ad\,e^*=1\quad\text{on}\quad\mathfrak{r}_s^{-2}, \tag{3.7}$$
$$ad\,e^*\cdot ad\,e=1\quad\text{on}\quad\mathfrak{r}_s^0.$$

In particular, $ad\,e^*$ (resp. $ad\,e$) gives an isomorphism of $\mathfrak{r}_s^{-2}$ (resp. of $\mathfrak{r}_s^0$) onto $\mathfrak{r}_s^0$ (resp. onto $\mathfrak{r}_s^{-2}$).

Lemma 3.4.

(1) *Let* $b\in\mathfrak{r}_s^{-2}$. *Then* $[[b, e^*], b]\in\overline{V}_r$ *and* $[[b, e^*], b]=0$ *implies* $b=0$.

(2) *Let* $c\in\mathfrak{r}_s^0$. *Then* $[[e, c], c]\in\overline{V}_r$ *and* $[[e, c], c]=0$ *implies* $c=0$.

Proof. The fact $[[b, e^*], b] \in \overline{V}_r$ is already proved (cf. (3.4)). Suppose $[[b, e^*], b]=0$. Then by (3.2), $e-b$ is contained in $\overline{V}$. Since $\lim_{t\to\infty} e^t Ad(\exp tE_s)(e-b) = -b$, we get $-b \in \overline{V}$. Similarly we have $b \in \overline{V}$, because $[[-b, e^*], -b]=0$. Now $b=0$ follows immediately from the fact that $\overline{V}$ contains no entire straight lines. Hence we have proved (1). We can write $c=[e^*, b'], b' \in \mathfrak{r}_s^{-2}$. Then $[[e, c], c]=[[e, [e^*, b']], [e^*, b']]=[b', [e^*, b']]=[[b', e^*], b']$. Therefore the assertion (2) follows from (3.7) and (1). q.e.d.

Now we set

(3.8) $$\mathcal{U}=\mathfrak{r}_0^{-2},$$

$$\mathcal{W}=\mathfrak{r}_s^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}_s^{0},$$

(3.9) $$j_0=ad(I+e-e^*).$$

It follows from (1.1) and (3.7) that $j_0^2=-1$ on $\mathcal{W}$. Hence we can write

(3.10) $$\mathcal{W}_c=\mathcal{W}_+ + \mathcal{W}_- \quad \text{(direct sum)},$$

$$\mathcal{W}_+=\{w \in \mathcal{W}_c; j_0 w=\sqrt{-1}w\},$$

$$\mathcal{W}_-=\{w \in \mathcal{W}_c; j_0 w=-\sqrt{-1}w\},$$

$$\overline{\mathcal{W}}_+=\mathcal{W}_-$$

Clearly the following equalities hold:

(3.11) $$[\mathcal{W}_+, \mathcal{W}_+]=[\mathcal{W}_-, \mathcal{W}_-]=0.$$

Define a $\mathcal{U}_c$-valued skew-symmetric biliear form $\mathcal{A}$ on $\mathcal{W}_c$ by

(3.12) $$\mathcal{A}(w, w')=\tfrac{1}{4}[w, w'] \quad (w, w' \in \mathcal{W}_c).$$

Proposition 3.5. *Let* $H(w, w')=2\sqrt{-1}\mathcal{A}(w, \bar{w}')$ *for* $w, w' \in \mathcal{W}_+$. *Then H is a V_r-hermitian form on $\mathcal{W}_+$.*

Proof. Each element w of $\mathcal{W}_+$ can be written as $w=w_1+\sqrt{-1} \times [e^*, w_1]+w_2-\sqrt{-1}[e, w_2]+Q(w_3)$, where $w_1 \in \mathfrak{r}_s^{-2}$, $w_2 \in \mathfrak{r}_s^{0}$ and $w_3 \in \mathfrak{r}^{-1}$. Then by using (1.10) and (3.11), we get

$$H(w, w)=[[w_1, e^*], w_1]+[[e, w_2], w_2]+\tfrac{1}{4}[[I, w_3], w_3].$$

Hence by (3.1) and Lemma 3.4, we know that $H(w, w)\in\overline{V}_r$ and that $H(w, w)=0$ means $w_1=w_2=w_3=0$. q.e.d.

§ 4. Realization of D as a Siegel domain of the third kind.

4.1. Let S be the symmetric domain constructed in § 1 and let $\mathfrak{g}(S)=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition at $\sqrt{-1}e(e\in V_s)$. There exists a unique I_s in $\mathfrak{s}^0$ such that $ad\, I_s=ad\, I$ on $\mathfrak{s}(=\mathfrak{g}(S))$. Then by Proposition 2.4, $Z=\frac{1}{2}(I_s+e-e^*)$ is in the center of $\mathfrak{k}$ and $ad\, Z$ gives a complex structure on $\mathfrak{p}$ which coincides with one of the domain S under the natural identification of $\mathfrak{p}$ with the tangent space to S at $\sqrt{-1}e$. Let $\mathfrak{p}_+$ and $\mathfrak{p}_-$ be the subspace of $\mathfrak{p}_c$ given by (2.15) for the domain S. Note that

$$\begin{aligned}(4.1)\qquad &[\mathfrak{p}_+, \mathfrak{p}_+]=[\mathfrak{p}_-, \mathfrak{p}_-]=0\,,\\ &[\mathfrak{p}_+, \mathfrak{p}_-]\subset\mathfrak{k}_c\,,\\ &\overline{\mathfrak{p}_+}=\mathfrak{p}_-\,,\\ &[\mathfrak{k}_c, \mathfrak{p}_\pm]\subset\mathfrak{p}_\pm\end{aligned}$$

Let G^s, G_c^s, K, K_c, P_+ and P_- be the connected subgroups of G_c corresponding to the subalgebras $\mathfrak{s}, \mathfrak{s}_c, \mathfrak{k}, \mathfrak{k}_c, \mathfrak{p}_+$ and $\mathfrak{p}_-$. Then $P_\pm$, K_c and K_cP_- are closed complex subgroups of G_c^s. Moreover $\exp: \mathfrak{p}_\pm\to P_\pm$ are holomorphic diffeomorphisms. It is also well known that the mapping defined by

$$P_+\times K_c\times P_-\ni(a, b, c)\to abc\in G_c^s$$

is a holomorphic diffeomorphism onto an open set of G_c^s and that $G^s\subset P_+K_cP_-$. Therefore for each point gK in $S=G^s/K$, there corresponds a unique z in $\mathfrak{p}_+$ such that $\exp z=$ the P_+-part of g, and the assignment: $gK\to z$ gives a holomorphic imbedding of S onto a bounded domain $\mathcal{S}$ in $\mathfrak{p}_+$. This is called the Harish-Chandra imbedding.

Let $z, z'\in\mathcal{S}$. Then we can write $\exp z=g\cdot k\cdot p$, $\exp z'=g'\cdot k'\cdot p'$, where $g, g'\in G^s$, $k, k'\in K_c$ and $p, p'\in P_-$. It follows that $(\exp\bar{z}')^{-1}\cdot\exp z=\bar{p}'^{-1}\cdot\bar{k}'^{-1}\cdot g'^{-1}\cdot g\cdot k\cdot p$. Since $g'^{-1}\cdot g\in P_+K_cP_-$, $(\exp\bar{z}')^{-1}\cdot\exp z\in P_+K_cP_-$. Thus we can define a mapping $\mathcal{K}(z, z')$: $\mathcal{S}\times\mathcal{S}\to K_c$ by

$$\mathcal{K}(z, z')^{-1} = \text{the } K_c\text{-part of } (\exp \bar{z}')^{-1} \cdot \exp z .$$

It is easy to check the following equality (Satake [6]):

$$(4.2) \qquad \mathcal{K}(z', z) = \overline{\mathcal{K}(z, z')^{-1}}.$$

We also define a mapping $\mathcal{J}(g, z): G^s \times \mathcal{S} \to K_c$ by

$$\mathcal{J}(g, z) = \text{the } K_c\text{-part of } g \cdot \exp z .$$

The group G^s acts on $\mathcal{S}$ in obvious manner. Then $g \cdot \exp z \equiv \exp g(z) \cdot \mathcal{J}(g, z) \pmod{P_-}$. By a simple calculation we have (Satake [6])

$$(4.3) \qquad \mathcal{K}(g(z), g(z')) = \mathcal{J}(g, z) \cdot \mathcal{K}(z, z') \cdot \overline{\mathcal{J}(g, z)}^{-1}.$$

If we put $\mathcal{K}(z) = \mathcal{K}(z, z)$, then by (4.2) and (4.3) we know

$$(4.4) \qquad \mathcal{K}(z) = \overline{\mathcal{K}(z)}^{-1} \quad (z \in \mathcal{S}),$$

$$\mathcal{K}(g(0)) = \mathcal{J}(g, 0) \cdot \overline{\mathcal{J}(g, 0)}^{-1} \quad (g \in G^s).$$

4.2. Let $\mathcal{U}_c$, $\mathcal{W}_+$ and $\mathcal{W}_-$ be as in § 3. Clearly $[\mathfrak{s}_c, \mathcal{U}_c] = 0$ and $[\mathfrak{s}_c, \mathcal{W}_c] \subset \mathcal{W}_c$ by (1.6), (1.7) and (3.8). Since $j_0 X = ad(I + e - e^*) X = ad(I_s + e - e^*) X$ for $X \in \mathfrak{s}$, we easily have the followings:

$$(4.5) \qquad [\mathfrak{p}_+, \mathcal{W}_+] = [\mathfrak{p}_-, \mathcal{W}_-] = 0 ,$$

$$[\mathfrak{p}_+, \mathcal{W}_-] \subset \mathcal{W}_+ ,$$

$$[\mathfrak{p}_-, \mathcal{W}_+] \subset \mathcal{W}_- ,$$

$$[\mathfrak{k}_c, \mathcal{W}_\pm] \subset \mathcal{W}_\pm .$$

In what follows, we simply write the actions of $g \in G_c^s$ and $z \in \mathfrak{s}_c$ on $\mathcal{W}_c$ as gw and zw ($w \in \mathcal{W}_c$). Since $[\mathfrak{s}_c, \mathcal{U}_c] = 0$, following equalities hold:

$$(4.6) \qquad \mathcal{A}(gw, gw') = \mathcal{A}(w, w').$$

$$\mathcal{A}(zw, w') + \mathcal{A}(w, zw') = 0 \quad (g \in G_c^s, z \in \mathfrak{s}_c, w, w' \in \mathcal{W}_c),$$

where $\mathcal{A}$ is the skew-symmetric bilinear form on $\mathcal{W}_c$ defined by (3.12). We now define for each $z \in \mathcal{S}$ a $\mathcal{U}_c$-valued form $L_z(w, w')$ on $\mathcal{W}_+$ by

$$L_z(w, w') = 2\sqrt{-1}\mathcal{A}(\overline{\mathcal{K}(z)} w, \bar{w}') \quad (w, w' \in \mathcal{W}_+).$$

Then by (4.4) and (4.6), $L_z(w, w')$ is hermitian.

Lemma 4.1 (cf. [6]).

(1) *$L_z(w, w')$ is a V_r-hermitian form on $\mathcal{W}_+$.*

(2) *$L_z(w, z\bar{w}')$ is a symmetric bilinear form on $\mathcal{W}_+$.*

Proof. We can take an element $g \in G'$ such that $g(0) = z$. Now (1) follows immediately from (4.4), (4.6) and from Proposition 3.5.

To prove (2), we first show that the following equalities hold:

$$(1 - z\bar{z})w_+ = \mathcal{K}(z)w_+ \quad \text{for} \quad w_+ \in \mathcal{W}_+, \tag{4.7}$$

$$(1 - \bar{z}z)w_- = \mathcal{K}(z)^{-1}w_- \quad \text{for} \quad w_- \in \mathcal{W}_-. \tag{4.8}$$

In fact, there exist $z_1, z_2 \in \mathfrak{p}_+$ such that $(\exp \bar{z})^{-1} \cdot \exp z = \exp z_1 \cdot \mathcal{K}(z)^{-1} \cdot \exp \bar{z}_2$. Hence by (4.5),

$$\begin{aligned} w_- + zw_- - \bar{z}zw_- &= (\exp \bar{z})^{-1} \cdot (\exp z)w_- \\ &= \exp z_1 \cdot \mathcal{K}(z)^{-1}w_- \\ &= \mathcal{K}(z)^{-1}w_- + z_1\mathcal{K}(z)^{-1}w_- . \end{aligned}$$

Comparing the $\mathcal{W}_-$-parts we get (4.8). The equality (4.7) follows from (4.4) and (4.8). From (4.7), $\bar{z}(1 - z\bar{z}) = \bar{z}\mathcal{K}(z)$ on $\mathcal{W}_+$. And from (4.8), $(1 - \bar{z}z)\bar{z} = \mathcal{K}(z)^{-1} \cdot \bar{z}$ on $\mathcal{W}_+$. Therefore $\bar{z} \cdot \mathcal{K}(z) = \mathcal{K}(z)^{-1} \cdot \bar{z}$ and hence $\mathcal{K}(z) \cdot \bar{z} = \bar{z} \cdot \overline{\mathcal{K}(z)}$ on $\mathcal{W}_+$. It follows

$$\begin{aligned} L_z(w, z\bar{w}') &= 2\sqrt{-1}\mathcal{A}(\overline{\mathcal{K}(z)}w, \bar{z}w') \\ &= 2\sqrt{-1}\mathcal{A}(w, \mathcal{K}(z)\bar{z}w') \\ &= 2\sqrt{-1}\mathcal{A}(w, \bar{z} \cdot \overline{\mathcal{K}(z)}w') \\ &= -2\sqrt{-1}\mathcal{A}(\bar{z}w, \overline{\mathcal{K}(z)}w') \\ &= 2\sqrt{-1}\mathcal{A}(\overline{\mathcal{K}(z)}w', \bar{z}w) \\ &= L_z(w', z\bar{w}). \end{aligned}$$

q.e.d.

We now set for $z \in \mathcal{S}$,

$$\mathcal{L}_z(w, w') = L_z(w, w') + L_z(w, z\bar{w}') \quad (w, w' \in \mathcal{W}_+).$$

Then $\mathcal{L}_z$ is a non-degenerate semi-hermitian form on $\mathcal{W}_+$ in the sence of Pyatetski-Shapiro [5]. Indeed, suppose that there exists $w_0 \in \mathcal{W}_+$

such that $\mathcal{L}_z(w, w_o) = 0$ for any $w \in \mathcal{W}_+$. Then $w_o + z\bar{w}_o = 0$ and hence $\bar{z}w_o + \bar{z}z\bar{w}_o = 0$. It follows that $(1 - \bar{z}z)\bar{w}_o = \bar{w}_o - \bar{z}z\bar{w}_o = \overline{(w_o + z\bar{w}_o)} - (\bar{z}w_o + \bar{z}z\bar{w}_o) = 0$. Since $1 - \bar{z}z$ is non-singular (cf. Proof of Lemma 4.1), we get $w_o = 0$. Therefore $\mathcal{L}_z(w, w')$ is non-singular. Thereby we can define a Siegel domain $\mathcal{D}$ of the third kind by

$$(4.9) \qquad \mathcal{D} = \{(u, w, z) \in \mathcal{U}_c \times \mathcal{W}_+ \times \mathcal{S};\ \mathrm{Im}\, u - \mathrm{Re}\, \mathcal{L}_z(w, w) \in V_r\}.$$

Let ξ denote the natural projection of $\mathcal{D}$ onto $\mathcal{S}$ and let $\mathcal{D}_o = \xi^{-1}(0)$. Since $\mathcal{L}_0(w, w') = H(w, w')$, we get

Proposition 4.2. *The fiber $\mathcal{D}_o$ is the Siegel domain of the second kind associated with the cone V_r and the V_r-hermitian form H on $\mathcal{W}_+$ given in Proposition 3.5.*

4.3. Let B be the subgroup of G_c given by (1.14) and let $B_o = \delta B \delta^{-1}$, where $\delta = \exp \sqrt{-1}e$. We set

$$\mathfrak{t} = \mathfrak{c} + \mathfrak{r}_0^0,$$

where $\mathfrak{c}$ and $\mathfrak{r}_0^0$ are subalgebras of $\mathfrak{g}^0$ as in § 1. Then $\mathfrak{t}$ satisfies $[\mathfrak{s}, \mathfrak{t}] = 0$.

Lemma 4.3. *Under the notations above, the Lie algebra $\mathfrak{b}_o$ of B_o coincides with $\mathcal{W}_- + \mathfrak{t}_c + \mathfrak{k}_c + \mathfrak{p}_-$.*

Proof. By (1.4), (1.7) and (1.13), the Lie algebra $\mathfrak{b}$ of B is decomposed in the following form:

$$\mathfrak{b} = \bar{Q}(\mathfrak{r}^{-1}) + (\mathfrak{r}_s^0)_c + \mathfrak{t}_c + \bar{Q}(\mathfrak{s}^{-1}) + \mathfrak{s}_c^0 + \mathfrak{s}_c^1 + \mathfrak{s}_c^2.$$

By Proposition 2.5, $Ad\,\delta(\bar{Q}(\mathfrak{s}^{-1}) + \mathfrak{s}_c^0 + \mathfrak{s}_c^1 + \mathfrak{s}_c^2) = \mathfrak{k}_c + \mathfrak{p}_-$. Clearly $Ad\,\delta(\bar{Q}(\mathfrak{r}^{-1}) + \mathfrak{t}_c) = \bar{Q}(\mathfrak{r}^{-1}) + \mathfrak{t}_c \subset \mathcal{W}_- + \mathfrak{t}_c$. Let $x \in (\mathfrak{r}_s^0)_c$. Then $Ad\,\delta x = x + \sqrt{-1}j_o x \in \mathcal{W}_-$. Hence we have proved $Ad\,\delta\mathfrak{b} \subset \mathfrak{b}_o$. By considering the equality $\dim \bar{Q}(\mathfrak{r}^{-1}) + \dim(\mathfrak{r}_s^0)_c = \dim \mathcal{W}_-$, we get $Ad\,\delta\mathfrak{b} = \mathfrak{b}_o$.

q.e.d.

Let h_o be a holomorphic mapping: $\mathcal{U}_c \times \mathcal{W}_+ \times \mathfrak{p}_+ \to G_c/B_o$ given by

$$(4.10) \qquad h_o(u, w, z) = \pi_o \cdot \exp u \cdot \exp w \cdot \exp z\,,$$

where π_0 denotes the projection of G_c onto G_c/B_0. Note that $\mathcal{U}_c + \mathcal{W}_+ + \mathfrak{p}_+$ is an abelian subalgebra of $\mathfrak{g}(D)_c$.

Lemma 4.4. *h_0 is a holomorphic diffeomorphism of $\mathcal{U}_c \times \mathcal{W}_+ \times \mathfrak{p}_+$ onto an open set of G_c/B_0.*

Proof. It is sufficient to prove that h_0 is injective. Now suppose that $a = \exp u \cdot \exp w \cdot \exp z \in B_0$. Let $E' = E - E_s$. Since $[E', \mathfrak{s}] = 0$, E' is contained in $\mathfrak{t}$. Therefore $Ad\, a\, E' = E' + 2u + w \in \mathfrak{b}_0$, because $\mathfrak{t} \subset \mathfrak{b}_0$. Hence by Lemma 4.3, $u = w = 0$. Recall that $Z = \frac{1}{2}(I_s + e - e^*)$ is in $\mathfrak{k}$ and hence in $\mathfrak{b}_0$. Therefore $Ad\, a\, Z = Z + [z, Z] = Z - \sqrt{-1} z \in \mathfrak{b}_0$. This implies $z = 0$. q.e.d.

Since $[[\mathcal{W}, \mathcal{W}], \mathcal{W}] = 0$, we can see the following (cf. [1] or [10]):

$$(4.11) \quad \exp(w + w') = \exp w \cdot \exp w' \cdot \exp \tfrac{1}{2}[w', w] \quad (w, w' \in \mathcal{W}_c).$$

For an element w of $\mathcal{W}_c$, denote by w_+ (resp. by w_-) its $\mathcal{W}_+$-(resp. $\mathcal{W}_-$-) component.

Lemma 4.5 (cf. [6]). *Every $g \in G^s$ leaves $h_0(\mathcal{U}_c \times \mathcal{W}_+ \times \mathcal{S})$ invariant and hence induces a holomorphic transformation $\tilde{g}$ of $\mathcal{U}_c \times \mathcal{W}_+ \times \mathcal{S}$. Let $\tilde{g}(u, w, z) = (u', w', z')$. Then*

$$\begin{cases} z' = g(z), \\ w' = (gw)_+ - z'(gw)_- = \mathcal{J}(g, z) w, \\ u' = u - \frac{1}{2}[w', gw]. \end{cases}$$

Proof. By using (4.11), we obtain

$$\begin{aligned} & g \cdot \exp u \cdot \exp w \cdot \exp z \\ & = \exp u \cdot \exp gw \cdot g \cdot \exp z \\ & \equiv \exp u \cdot \exp gw \cdot \exp g(z) \quad (\mathrm{mod}\ K_c P_-) \\ & = \exp(u - \tfrac{1}{2}[(gw)_+, (gw)_-]) \cdot \exp(gw)_+ \cdot \exp(gw)_- \cdot \exp g(z). \end{aligned}$$

And

$\exp(gw)_{-}\cdot\exp g(z)$

$$=\exp g(z)\cdot\exp((gw)_{-}-g(z)(gw)_{-})$$

$$=\exp g(z)\cdot\exp(-g(z)(gw)_{-})\cdot\exp(gw)_{-}\cdot\exp\tfrac{1}{2}[g(z)(gw)_{-},(gw)_{-}].$$

Therefore

$$g\cdot\exp u\cdot\exp w\cdot\exp z$$

$$\equiv\exp\left(u-\tfrac{1}{2}[(gw)_{+},(gw)_{-}]+\tfrac{1}{2}[g(z)(gw)_{-},(gw)_{-}]\right)$$

$$\cdot\exp((gw)_{+}-g(z)(gw)_{-})\cdot\exp g(z)\qquad(\mathrm{mod}\ B_{0}).$$

Hence $z'=g(z)$, $w'=(gw)_{+}-g(z)(gw)_{-}$ and $u'=u-\frac{1}{2}[w',(gw)_{-}]=u-\frac{1}{2}[w',gw]$. It remains to show $(gw)_{+}-g(z)(gw)_{-}=\mathcal{J}(g,z)w$. We can write $g\cdot\exp z=\exp g(z)\cdot\mathcal{J}(g,z)\cdot\exp\bar{z}_{1}(z_{1}\in\mathfrak{p}_{+})$. Then $gw=g\cdot\exp z\ w=\mathcal{J}(g,z)w+\mathcal{J}(g,z)\bar{z}_{1}w+g(z)\mathcal{J}(g,z)\bar{z}_{1}w$. Therefore $(gw)_{+}=\mathcal{J}(g,z)w+g(z)\mathcal{J}(g,z)\bar{z}_{1}w$ and $(gw)_{-}=\mathcal{J}(g,z)\bar{z}_{1}w$. Hence we have $w'=(gw)_{+}-g(z)(gw)_{-}=\mathcal{J}(g,z)w$. q.e.d.

Next we verify

Lemma 4.6 (cf. [6]). *Let $g\in G^{s}$ and let $\tilde{g}(u,w,z)=(u',w',z')$. Then* $\mathrm{Im}\,u-\mathrm{Re}\,\mathcal{L}_{z}(w,w)=\mathrm{Im}\,u'-\mathrm{Re}\,\mathcal{L}_{z'}(w',w')$.

Proof. We first assume that $z=0$. By Lemma 4.5, $u'=u-\frac{1}{2}\times[\mathcal{J}w,gw]$, $w'=\mathcal{J}w$ and $z'=g(0)$, here we put $\mathcal{J}=\mathcal{J}(g,0)$. Therefore $\mathrm{Im}\,u'-\mathrm{Re}\,\mathcal{L}_{z'}(w',w')=\mathrm{Im}\,u-\frac{1}{2}\,\mathrm{Im}[\mathcal{J}w,gw]-\mathrm{Re}\,\mathcal{L}_{z'}(\mathcal{J}w,\mathcal{J}w)$. By direct calculations,

$$\mathrm{Re}\,\mathcal{L}_{z'}(\mathcal{J}w,\mathcal{J}w)$$

$$=2\sqrt{-1}\mathcal{A}(\mathcal{J}w,\mathcal{K}(z')\bar{\mathcal{J}}\bar{w})-2\,\mathrm{Im}\,\mathcal{A}(\mathcal{J}w,\mathcal{K}(z')\bar{z}'\mathcal{J}w)$$

$$=2\sqrt{-1}\mathcal{A}(w,\bar{w})-2\,\mathrm{Im}\,\mathcal{A}(\mathcal{J}w,\bar{z}'\overline{\mathcal{K}(z')}\mathcal{J}w)$$

$$=\mathrm{Re}\,\mathcal{L}_{0}(w,w)-\tfrac{1}{2}\,\mathrm{Im}[\mathcal{J}w,\bar{z}'\bar{\mathcal{J}}w],$$

here we used the facts that $\mathcal{K}(z')=\mathcal{J}\bar{\mathcal{J}}^{-1}$ and $\bar{z}'\overline{\mathcal{K}(z')}=\mathcal{K}(z')\bar{z}'$ on $\mathcal{W}_{+}$ (cf. Proof of Lemma 4.5). Since $g=\exp z'\cdot\mathcal{J}\cdot\exp\bar{z}''\ (z''\in\mathfrak{p}_{+})$, $g=\bar{g}=\exp\bar{z}'\cdot\bar{\mathcal{J}}\cdot\exp z''$. Hence $gw=\bar{\mathcal{J}}w+\bar{z}'\bar{\mathcal{J}}w$ and $[\mathcal{J}w,\bar{z}'\bar{\mathcal{J}}w]=[\mathcal{J}w,gw]$. Combining these equalities, we get $\mathrm{Im}\,u'-\mathrm{Re}\,\mathcal{L}_{z'}(w',$

$w') = \operatorname{Im} u - \operatorname{Re} \mathcal{L}_0(w, w)$. Since $\mathcal{S}$ is homogeneous, for any $z \in \mathcal{S}$ there exist $f \in G^s$, $u_0 \in \mathcal{U}_c$ and $w_0 \in \mathcal{W}_+$ such that $\tilde{f}(u_0, w_0, 0) = (u, w, z)$. Hence $\widetilde{gf}(u_0, w_0, 0) = (u', w', z')$. It follows $\operatorname{Im} u' - \operatorname{Re} \mathcal{L}_{z'}(w', w') = \operatorname{Im} u_0 - \operatorname{Re} \mathcal{L}_0(w_0, w_0) = \operatorname{Im} u - \operatorname{Re} \mathcal{L}_z(w, w)$. q.e.d.

By Lemma 4.6, we know that each $\tilde{g}(g \in G^s)$ leaves D invariant. Moreover by Lemma 4.5 we know that $\tilde{g}$ acts as a quasi-linear transformation in the sence of Pyatetski-Shapiro [5].

4.4. Let $\tilde{\delta}$ be a holomorphic diffeomorphism of G_c/B onto G_c/B_0 given by

$$G_c/B \ni gB \overset{\tilde{\delta}}{\rightarrow} g\delta^{-1}B_0 \in G_c/B_0 ,$$

where $\delta = \exp \sqrt{-1}e$. Clearly $\tilde{\delta}$ is compatible with the action of $f \in G$, i.e., $\tilde{\delta}(fp) = f\tilde{\delta}(p)$ $(p \in G_c/B)$. We are now in a position to prove

Theorem 4.7. *Let $\mathcal{D}$ be the Siegel domain of the third kind defined by (4.9) and let h (resp. h_0) be the imbedding of D (resp. of $\mathcal{D}$) into G_c/B (resp. into G_c/B_0) given by (1.15) (resp. by (4.10)). Then*

$$h_0(\mathcal{D}) = \tilde{\delta}h(D).$$

Proof. First we show that $h_0(\mathcal{D}_0) = \tilde{\delta}h(D_0)$. Let $u \in (\mathfrak{r}_0^{-2})_c$, $v \in (\mathfrak{r}_s^{-2})_c$ and $w \in \mathfrak{r}^{-1}$. Then

$$\begin{aligned}
&\tilde{\delta}h(u+v+w+\sqrt{-1}e) \\
&\qquad \equiv \exp u \cdot \exp v \cdot \exp Q(w) \\
&\qquad \equiv \exp u \cdot \exp v_+ \cdot \exp v_- \cdot \exp \tfrac{1}{2}[v_-, v_+] \cdot \exp Q(w) \\
&\qquad \equiv \exp\left(u - \tfrac{1}{2}[v_+, v_-]\right) \cdot \exp(v_+ + Q(w)) \quad (\operatorname{mod} B_0).
\end{aligned}$$

Therefore $\tilde{\delta}h(u+v+w+\sqrt{-1}e) = h_0(u', w', 0)$, where

$$\left\{\begin{aligned} u' &= u - \tfrac{1}{2}[v_+, v_-], \\ w' &= v_+ + Q(w). \end{aligned}\right. \tag{4.12}$$

Since $v_+=\frac{1}{2}(v+\sqrt{-1}[e^*, v])$ and $v_-=\frac{1}{2}(v-\sqrt{-1}[e^*, v])$, we get $\frac{1}{2}[v_+, v_-]=\sqrt{-1}/4[[e^*, v], v]$ and $\frac{1}{2}[v_+, \bar{v}_+]=\sqrt{-1}/4[[e^*, v], \bar{v}]$. It follows

$$\begin{aligned}
&\mathrm{Im}\, u' - \mathrm{Re}\, \mathcal{L}_0(w', w') \\
&= \mathrm{Im}\, u - \frac{1}{4}\mathrm{Re}[[e^*, v], v] - \frac{\sqrt{-1}}{2}[v_+, \bar{v}_+] - \frac{\sqrt{-1}}{2}[Q(w), \bar{Q}(w)] \\
&= \mathrm{Im}\, u - \frac{1}{4}\mathrm{Re}[[e^*, v], v] + \frac{1}{4}[[e^*, v], \bar{v}] - \frac{1}{4}[[I, w], w] \\
&= \mathrm{Im}\, u - \frac{1}{4}[[I, w], w] - \frac{1}{2}[[\mathrm{Im}\, v, e^*], \mathrm{Im}\, v].
\end{aligned}$$

Hence by Corollary 3. 3, $(u', w', 0)\in\mathcal{D}_0$ if and only if $u+v+w+\sqrt{-1}e \in D_0$. Since for any $(u', w')\in\mathcal{U}_c\times\mathcal{W}_+$ there exist unique $u\in(\mathfrak{r}_0^{-2})_c$ $(=\mathcal{U}_c)$, $v\in(\mathfrak{r}_s^{-2})_c$ and $w\in\mathfrak{r}^{-1}$ satisfying (4. 12), we get $h_0(\mathcal{D}_0) =\tilde{\delta}h(D_0)$. Clearly $D=G^sD_0$ and $\mathcal{D}=\tilde{G}^s\mathcal{D}_0$. Hence $h_0(\mathcal{D})=G^sh_0(\mathcal{D}_0) =G^s\tilde{\delta}h(D_0)=\tilde{\delta}h(G^sD_0)=\tilde{\delta}h(D)$. q.e.d.

4. 5. Since $D\cong\mathcal{D}$ by Theorem 4. 7, every $g\in\mathrm{Aut}(D)$ corresponds to a holomorphic transformation $\tilde{g}$ of $\mathcal{D}$. Then for $g\in G$ and $p\in\mathcal{D}$, the equality; $h_0(\tilde{g}(p))=gh_0(p)$ holds, because the mappings h and $\tilde{\delta}$ are compatible with the action of G.

Lemma 4. 7. *Let T be the connected subgroup of G corresponding to the subalgebra $\mathfrak{t}=\mathfrak{c}+\mathfrak{r}_0^0$. Then for each $t\in T$, $\tilde{t}$ is a quasi-linear transformation.*

Proof. Let $(u, w, z)\in\mathcal{D}$. Then

$$t\cdot\exp u\cdot\exp w\cdot\exp z\equiv\exp(Ad\,t\,u)\cdot\exp(Ad\,t\,w)\cdot\exp z \quad (\mathrm{mod}\ B_0).$$

Since $Ad\,t\circ j_0=j_0\circ Ad\,t$, we know $Ad\,t\,w\in\mathcal{W}_+$. Clearly $Ad\,t\,u\in\mathcal{U}_c$. Therefore $\tilde{t}$ is a quasi-linear transformation of $\mathcal{D}$. q.e.d.

Next we consider the action of the connected subgroup of G corresponding to the subalgebra $\mathcal{U}+\mathcal{W}$. It is easy to see that this group coincides with $\exp\mathcal{U}\cdot\exp\mathcal{W}$.

Lemma 4. 9. *Let* $f = \exp a \cdot \exp b$ $(a \in \mathcal{U}, b \in \mathcal{W})$, *and let* $\tilde{f}(u, w, z) = (u', w', z')$. *Then*

$$\begin{cases} z' = z \\ w' = w + b_+ - zb_- \\ u' = u + a + \frac{1}{2}[b, w] + \frac{1}{2}[b_-, b_+ + w] - \frac{1}{2}[b_-, zb_-]. \end{cases}$$

In particular, $\tilde{f}$ *is a parallel transformation in Pyatetski-Shapiro's sence* ([5]).

Proof. By using (4. 11), one has

$$\begin{aligned} & f \cdot \exp u \cdot \exp w \cdot \exp z \\ & \quad = \exp(a+u) \cdot \exp(b+w) \cdot \exp \tfrac{1}{2}[b, w] \cdot \exp z \\ & \quad = \exp(a + u + \tfrac{1}{2}[b, w]) \cdot \exp(b_+ + w) \cdot \exp b_- \cdot \exp \tfrac{1}{2}[b_-, b_+ + w] \cdot \exp z \\ & \quad = \exp(a + u + \tfrac{1}{2}[b, w] + \tfrac{1}{2}[b_-, b_+ + w]) \cdot \exp(b_+ + w) \cdot \exp b_- \cdot \exp z\,. \end{aligned}$$

Since $\exp b_- \cdot \exp z = \exp z \cdot \exp(b_- - zb_-) \equiv \exp z \cdot \exp(-zb_-) \cdot \exp \frac{1}{2}[zb_-, b_-]$ $(\mathrm{mod}\ B_o)$, we get

$$\begin{aligned} & f \cdot \exp u \cdot \exp w \cdot \exp z \\ & \quad = \exp(a + u + \tfrac{1}{2}[b, w] + \tfrac{1}{2}[b_-, b_+ + w] - \tfrac{1}{2}[b_-, zb_-]) \\ & \qquad \cdot \exp(b + w - zb_-) \cdot \exp z \quad (\mathrm{mod}\ B_o). \end{aligned}$$

q.e.d.

4. 6. Define a Subgroup $GL(D)$ of $\mathrm{Aut}(D)$ by

$$GL(D) = \{f \in GL(R_c + W);\ f(D) = D\}.$$

Then $\mathrm{Aut}(D) = G \cdot GL(D)$ ([2] or [3]) and the Lie algebra of $GL(D)$ is $\mathfrak{g}^0$([2]). By virtue of Lemma 4. 5, Lemma 4. 8 and Lemma 4. 9, each element of G corresponds to a quasi-linear transformation of $\mathcal{D}$. Therefore it remains to investigate the action of $GL(D)$ on $\mathcal{D}$.

For any $g \in GL(D)$, let us denote by $\tau(g)$ the isomorphism of G_c given by

$$\tau(g) a = Ad(g)\, a\, Ad(g)^{-1} \quad (a \in G_c).$$

Then $\tau(g) B = B$ and hence $\tau(g)$ induces an automorphism (denoted by the same letter $\tau(g)$) of G_c/B. From the definition of Tanaka's imbedding, we have

$$h(g(p)) = \tau(g)h(p).$$

We now put $\mathfrak{s}' = Ad\, g\mathfrak{s}$. Then $\mathfrak{s}'$ is also a semi-simple graded subalgebra of $\mathfrak{g}(D)$ satisfying (1.3). Hence there exists X in $\mathfrak{g}^0$ such that $Ad(\exp X)\mathfrak{s}' = \mathfrak{s}$ ([4]). Put $g' = \exp X \cdot g$. Clearly $g' \in GL(D)$ and $Ad\, g'\mathfrak{s} = \mathfrak{s}$. It follows $Ad\, g'\, V_s = V_s$ and hence there exists $Y_1, \cdots, X_m \in \mathfrak{s}^0$ such that $Ad(\exp X_1 \cdots \exp Y_m) Ad\, g'\, e = e$. Let $g'' = \exp Y_1 \cdots \exp Y_m \cdot g'$. Then g'' is an element of $GL(D)$ having the following properties:

$$\text{a)} \quad Ad\, g''\mathfrak{s}^\lambda = \mathfrak{s}^\lambda \quad \text{and} \quad Ad\, g'' e = e\,.$$

Moreover it is not difficult to show the equality;

$$\text{b)} \quad Ad\, g'' E_s = E_s\,.$$

By using a) and b), we can see

$$\text{c)} \quad Ad\, g'' e^* = e^*.$$

From a), b) and c), we know that the spaces $\mathfrak{p}_+$, $\mathfrak{p}_-$, $\mathcal{U}$, $\mathcal{W}_+$ and $\mathcal{W}_-$ are stable under $Ad\, g''$. Furthermore $\tau(g'')B_0 = B_0$ and hence $\tau(g'')$ induces an autsmorphism of G_c/B_0, which is denoted by the same letter $\tau(g'')$. Obviously $\tau(g'')\circ\tilde{\delta} = \tilde{\delta}\circ\tau(g'')$. As a consequence we get for any $p \in D$,

$$\tilde{\delta} h(g''(p)) = \tilde{\delta}\circ\tau(g'')\cdot h(p) = \tau(g'')\tilde{\delta} h(p).$$

Hence for any $(u, z, w) \in \mathcal{D}$,

$$\begin{aligned} h_0(\tilde{g}''(u, z, w)) &= \pi_0 \tau(g'')\,(\exp u \cdot \exp w \cdot \exp z) \\ &= \pi_0 \exp(Ad\, g'' u) \cdot \exp(Ad\, g'' w) \cdot \exp(Ad\, g'' z). \end{aligned}$$

This equality leads us to say that $\tilde{g}''$ (and therefore $\tilde{g}$) is a quasi-linear transformation of $\mathcal{D}$. Thus we have proved the following.

Theorem 4.10. *In the realization of D as $\mathcal{D}$, each element of Aut(D) corresponds to a quasi-linear transformation and each element of exp $\mathcal{U}$ exp $\mathcal{W}$ induces a parallel transformation of $\mathcal{D}$.*

Remark 2. It is not difficult to see that when the domain D is homogeneous, our realization coincides with one given in Corollary 2, II-37 in Takeuchi [8].

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