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Structure of codimension one foliations which are almost without holonomy

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§ 1. Introduction

There are many results concerning on codimension one foliations without holonomy on closed manifolds. Among them the results of Novikov, Sacksteder and Tischler are remarkable (see Theorems N, N', S, S' and T of § 2). In [3] the author developed a method by which we obtain a comprehensive understanding of the theorem of Sacksteder. In this note we apply the same method to the problems of codimension one foliation without holonomy on manifolds with boundary and we assume the reader the familiarity with [3].

A codimension one foliation on a compact manifold is *almost without holonomy* if the holonomy groups of non-compact leaves are trivial. The Reeb foliation on S^3 and the foliations constructed by Lawson and Tamura on S^{2n+1} are almost without holonomy and these foliations have only proper leaves. Another type of example is obtained as follows. Let $\boldsymbol{T}^n=\boldsymbol{R}^n/\boldsymbol{Z}^n$ be a torus, $\mathcal{F}_\omega$ a codimansion one foliation on $\boldsymbol{T}^n$ defined by $\omega=\sum_{i=1}^n a_i dx^i$, $a_i\in\boldsymbol{R}$, then the foliations obtained from $\mathcal{F}_\omega$ by surgery along closed curves transverse to $\mathcal{F}_\omega$ are almost without holonomy. In this example there exists locally dense leaves if and only if there exists a_i and a_j which are linearly independent over $\boldsymbol{Q}$.

To describe the structure of almost without holonomy foliations, it is convenient to consider the following two types of models. We say that a pair $(M, \mathcal{F})$ is a *model of type i* if $\mathcal{F}$ is a codimension one foliation on a compact manifold M and if the following condition is satisfied.

Type 1. $M=V\times[0,1]$, *where* V *is a closed manifold, and* $\mathcal{F}$ *is the product foliation.*

Type 2. $\mathcal{F}$ *is tangent to* ∂M *and the leaves in the interior of* M *are non-compact with trivial holonomy group.*

We remark that if $(M,\mathcal{F})$ is a model, then $\mathcal{F}$ is transversally orientable.

The following theorem (see [1]) is an easy consequence of the Reeb stability theorem.

Theorem. H. *Let* M *be a compact manifold and* $\mathcal{F}$ *be a codimension one folliation on* M *which is almost without holonomy. Then there exists a foliated manifold* $(M',\mathcal{F}')$ *which is a disjoint countable union of models and a foliation preserving immerision* p *of* M' *onto* M *such that* $p|\mathrm{Int}\,M'$ *is an imbedding of* $\mathrm{int}\,M'$ and $p|\partial M'$ *is a two fold covering onto* $M-p(\mathrm{Int}\,M')$.

By this theorem, to consider the structure of almost without holonomy foliations, it is sufficient to consider the structure of models of type 2. From now on we assume that $(M,\mathcal{F})$ is a model of type 2. Let X be a vector field on M transverse to $\mathcal{F}$ and $\rho(x,t)$, $-a_x\leqq t\leqq b_x$, where a_x or b_x may be infinite, be the (maximal) solution of X passing through $x\in M$. We can assume that X has a periodic solution $\rho(x_0,t)$ for some $x_0\in\mathrm{Int}\,M$. We use the following notations. $\mathring{M}=\mathrm{Int}\,M$, $\mathring{\mathcal{F}}=\mathcal{F}|\mathring{M}$ and $X=\mathring{X}|\mathring{M}$. Contrary to the other authors (for example [6]) we consider the structure of $\mathring{\mathcal{F}}$ at first and then we obtain imformations of $\mathcal{F}$ near ∂M from the structure of $\mathring{\mathcal{F}}$. So we are indifferent to the differentiablity of $\mathcal{F}$ at ∂M and we say that $\mathcal{F}$ is class C^r if $\mathring{\mathcal{F}}$ is class C^r. For simplicity we always assume that $r\geqq 2$.

§ 2. Theorems.

Theorem I. *Let* $y_0=\rho(x_0,t)$, x_0, $y_0\in M$, *then there exists a continuous function* $f_{x_0,t}$ *on* L_{x_0} *such that* $f_{x_0,t}(x_0)=t$ *and the map* $\phi_{x_0,t}$ *defined by* $\phi_{x_0,t}(x)=\rho(x,f_{x_0,t}(x))$, $x\in L_{x_0}$, *defines a diffeomorphism from* L_{x_0} *to* L_{y_0}.

Theorem N. *Let* $(\widetilde{M}, \widetilde{\mathcal{F}}, \widetilde{X})$ *be the universal cover of* $(\mathring{M}, \mathring{\mathcal{F}}, \mathring{X})$. *Then* $\widetilde{M}=\widetilde{L}\times \boldsymbol{R}$, *where* $\widetilde{L}$ *is the universal covering of a leaf of* $\mathring{\mathcal{F}}$, $\widetilde{\mathcal{F}}$ *is the product foliation and* $\tilde{x}\times \boldsymbol{R}(\tilde{x}\in\widetilde{L})$ *is the trajectory of* $\tilde{x}$ *passing through* $\tilde{x}$.

Theorem S. *There exists a foliation preserving topological flow* ρ' *on* M *such that* $\rho'(x, t)=x$ *for any* $x\in\partial M$ *and* $t\in\boldsymbol{R}$, $\rho'(\ ,t)$ *sends each leaf of* $\mathring{\mathcal{F}}$ *diffeomorphically onto a leaf of* $\mathring{\mathcal{F}}$ *and the orbits of* ρ' *are same as the orbits of* $\mathring{X}$ *in* $\mathring{M}$.

Theorem S′. *There exists a non-singular closed one form* ω *on* $\mathring{M}$ *and a homeomorphism* h *of* M *such that* $\omega=0$ *defines a model* $(M, \mathcal{F}_\omega)$ *of type 2*, h *sends each leaf of* $\mathcal{F}$ *diffeomorphically onto a leaf of* $\mathcal{F}_\omega$, h *is isotopic to the identity of* M *and* $h|\partial M$ *is the identity. Moreover* h *can be choosen arbitrarily near to the identity of* M.

Theorem N′. *Let* L *be a leaf of* $\mathring{\mathcal{F}}$ *then the following sequence is exact.*

$$1\longrightarrow\pi_1(L)\xrightarrow{i_*}\pi_1(M)\xrightarrow{j}\boldsymbol{R}$$

where $j(\alpha)=\int_\alpha\omega$ *for* $\alpha\in\pi_1(M)$ *and* ω *is the form of Theorem S′. All leaves of* $\mathring{\mathcal{F}}$ *are proper iff the rank of* $\operatorname{Im} j=1$ *and if the rank of* $\operatorname{Im} j>1$, *all leaves of* $\mathring{\mathcal{F}}$ *are everywhere dense in* $\mathring{M}$.

Theorem T. $\mathring{M}$ *is a locally trivial fibration over* S^1 *and leaves of* $\mathring{\mathcal{F}}$ *are covering spaces of the fibre of the fibration.*

Corollary 1. *Let* L *be a leaf of* $\mathring{\mathcal{F}}$ *then* $\pi_i(L)\cong\pi_i(M)$ *for* $i>1$.

Theorem I′. *Let* V *be a component of* ∂M *and* y *be a point of* V. *Suppose that* X *is an inward normal at* y. *Then the holonomy* $\Phi_y(l)$, *where* l *is a representative of an element of* $\pi_1(V, y)$, *is defined on* $(0, b_y)$ *and there exists an injective homomorphism* $\psi: \boldsymbol{R}\to\operatorname{Homeo}((0, b_y))$ *such that the following diagram commutes.*

$$\begin{array}{ccc}
\pi_1(V, y) & \xrightarrow{\Phi_y} & \mathrm{Diff}([0, b_y)) \subset \mathrm{Homeo}((0, b_y)) \\
\downarrow i_* & & \nearrow \psi \\
\pi_1(M, y) & \xrightarrow{\ j\ } & \boldsymbol{R}
\end{array}$$

where j is the homomorphism defined in Theorem N'.

Corollary 2. *The holonomy group of V is abelian and Archemedean.*

Remarks (i) The invariants defined in [6] coincide with the cohomology classes $\{\omega\}$ and $i^*\{\omega\}$, where i is the inclusion of a component of ∂M into M and ω is the form defined in Theorem S'.

(ii) Theorems I, N and Corollary 2 are valid for C^1 case.

(iii) The bahaviour of a foliation near a compact leaf with abelian holonomy group is studied in [7] and [2].

§ 3. Indication of Proof.

The proofs of Theorems in § 2 are almost the same as the proofs of corresponding theorems of [3]. Only the Theorems S', T and I' will require some explanations.

To prove Theorem S' it is sufficient to show that by changing the differentiable structure of M, the foliation $\mathring{\mathcal{F}}$ becomes a foliation defined by a closed one form. As in the proof of Proposition 5.1. of [3] let U_λ be a distinguished neighborhood of $(M, \mathcal{F})$. If $U_\lambda \cap \partial M = \phi$ we define n-th coordinate function $\bar{x}_\lambda^n$ as in [3]. If $U_\lambda \cap \partial M \neq \phi$ and X is inward on $U_\lambda \cap \partial M$ then we replace $\bar{x}_\lambda^n$ by $\exp \bar{x}_\lambda^n$ for points of $U_\lambda \cap \mathring{M}$, if X is outward on $U_\lambda \cap \partial M$ then we replace $\bar{x}_\lambda^n$ by $\exp(-\bar{x}_\lambda^n)$ for points of $U_\lambda \cap \mathring{M}$ and on $U_\lambda \cap \partial M$ we define n-th coordiante function to be zero. Then, since $\bar{x}_\lambda^n = \bar{x}_\mu^n + c_{\mu\lambda}$ for points of $U_\lambda \cap U_\mu \cap \mathring{M}$, it is easy to see that this coordinate system defines a differentiable structure on M and $\mathring{\mathcal{F}}$ is defined by $\omega = d\bar{x}_\lambda^n$ on $U_\lambda \cap \mathring{M}$. The rest of the proof is the same as the proof of Theorem 1.2. of [3].

To prove Theorem T, we can assume that $\mathring{\mathcal{F}}$ is defined by a nonsingular one form ω on $\mathring{M}$. If the rank of $H^1(M, \boldsymbol{R})$ is one then by Theorem N' all leaves of $\mathring{\mathcal{F}}$ are propor and $\mathring{\mathcal{F}}$ is induced from a fibration over S^1. So we assume that the rank of $H^1(M, \boldsymbol{R})$ is greater than one. Let $\omega_1, \omega_2, \cdots, \omega_k$ be closed one forms which form a basis of $H^1(M, \boldsymbol{R})$ then the norm of ω_i defined by a riemannian metric on M is bounded. On the other hand, the norm of ω at a point $x \in \mathring{M}$ tends to infinity when x tends to a point of ∂M. It is easy to see that, for sufficiently small a_i, the form $\omega' = \omega + \sum_{i=1}^{k} a_i \omega_i$ is non-singular one form on $\mathring{M}$ and the foliation defined by $\omega' = 0$ defines a model $(M, \mathcal{F}_{\omega'})$ of type 2. Then, since $k \geqq 2$, we can choose small a_i $i = 1, 2, \cdots, k$, so that the image of j', where $j'(\alpha) = \int_\alpha \omega'$ for $\alpha \in \pi_1(\mathring{M})$, has the rank one. By Theorem N', M is a fibtation over S^1, moreover we can choose a_i $i = 1, 2, \cdots, k$ so that the final assertion of Theorem T holds by the method of [5].

To prove Theorem I$'$ we use the notion of characteristic map defined in [3]. Let x_0 be a point of $\mathring{M}$ such that the solution $\rho(x_0, \boldsymbol{R})$ of X is periodic of period one. There exists t_0 such that $\rho(y, t_0)$ belongs to L_{x_0} and we fix a leaf curve l_0 from $\rho(y, t_0)$ to x_0. We define a diffeomorphism $\phi : (0, b_y) \to \boldsymbol{R}$ by $\phi(t) = \bar{\Theta}(l_0)(t - t_0)$. Let l be a closed curve in V which represent an element α of $\pi_1(V, y)$ then it is easy to see that $\Phi(\bar{\Theta}(l)(t_0))$ belongs to $\bar{G}_{x_0}$ and is independent of the choice of l. We have the following commutative diagram.

$$\begin{array}{ccc} \pi_1(V, y) & \xrightarrow{\ \Phi_y\ } & \mathrm{Diff}((0, b_y)) \\ \Big\downarrow p & & \Big\downarrow \phi_* \\ \bar{G}_{x_0} & \xrightarrow{\ \bar{\chi}\ } & \mathrm{Diff}(R) \end{array}$$

where $p(\alpha) = \phi(\bar{\Theta}(l)(t_0))$ for $l \in \alpha \in \pi_1(V, y)$ and $\phi_* g = \phi g \phi^{-1}$ for $g \in \mathrm{Diff}((0, b_y))$. On the other hand it is easy to construct an injective homomorphism $\psi' : R \to \mathcal{H}^p(R)$ by using a linearlization map of $\bar{\chi}(\bar{G}_{x_0})$ such that the following diagram commutes.

$$\begin{array}{ccccc} \bar{G}_{x_0} & \xrightarrow{\ \bar{\chi}\ } & \mathrm{Diff}^{\,p}(R) & \subset & \mathcal{H}^{p}(R) \\ \Big\uparrow \delta & & \Big\downarrow \gamma & \nearrow \psi' & \\ \pi_1(\mathring{M}, x_0) & \xrightarrow{\ i\ } & \boldsymbol{R} & & \end{array}$$

Finally from the difinitions of δ and p the following diagram commutes.

$$\begin{array}{ccc} \pi_1(V, y) & \xrightarrow{\quad p \quad} & \\ \Big\downarrow i_* & & \bar{G}_{x_0} \\ \pi_1(M, x_0) & \xrightarrow{\quad \partial \quad} & \end{array}$$

where i_* is defined by the curve l' from x_0 to y which is the composition of l_0^{-1} and $\rho(y, [0, t_0])^{-1}$. Combining these diagrams we obtain the theorem.

§ 4. Reeb foliations.

We say that a model $(M, \mathcal{F})$ of type 2 is a *Reeb foliation* if the leaves of $\mathring{\mathcal{F}}$ are homeomorphic to $\boldsymbol{R}^n$. Then from Theorem N′ and Corollary 1 we have the following result.

Theorem I″. *Let* $(M, \mathcal{F})$ *be a Reeb folian then*

(i) *M is homotopy equivalent to* $\boldsymbol{T}^k$, $1 \leqq k \leqq n+1$.

(ii) $H^*(\partial M : \boldsymbol{Z}) \cong H^*(\boldsymbol{T}^k \times S^{n-k} : \boldsymbol{Z})$

(iii) *The leaves of $\mathring{\mathcal{F}}$ are proper iff $k=1$ and otherwise all leaves are dense in $\mathring{M}$.*

We remark that on $\boldsymbol{T}^k \times V^{n+1-k}$, where V^{n+1-k} is a contractible manifold, there exists a Reeb foliation if $k \geqq 2$ and $n+1-k \geqq 5$. This follows from the following lemmas.

Lemma 1. *Let M be a closed manifold which is a principal S^1-bundle and F a codimension one foliation on M which is transverse to fibres S^1 and is invariant under the action of S^1. Let N be a compact manifold with boundary. Then there exists a model $(M \times N, \mathcal{F}')$ of type 2 such that the leaves of $\mathring{\mathcal{F}}'$ are $L \times \mathring{N}$ where L is a leaf of $\mathcal{F}$.*

Lemma 2. *Let V^n be a contractible manifold then $V^n \times R$ is diffeomorphic to $\boldsymbol{R}^{n+1}$ if $n \geqq 5$.*

For a proof of Lemma 2 see [8]. When $k=n+1$, then M is diffeomorphic to $\boldsymbol{T}^{n+1}$ for $n \neq 4$ (see [5]). The case $k=1$ will be treated in [4].

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