J. Math. Kyoto Univ. (JMKYAZ)
16-1 (1976) 51-55

The Veronesean subrings of Gorenstein rings

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(Communicated by Prof. Nagata, Oct. 26, 1974; Revised, Dec. 16, 1974)

Let k be a field and $R=\bigoplus_{n\geq 0}R_n$ be a Noetherian graded ring with $R_0=k$ and suppose that R is generated by R_1 as a k-algebra. For $d>0$ we define $R^{(d)}=\bigoplus_{n\geq 0}R_{nd}$ and call it the Veronesean subring of R of order d. Note that $R^{(d)}$ is Macaulay if R is Macaulay (cf. Proposition 12 of M. Hochster and J. A. Eagon [3]). We put $H(n, R)=[R_n: k]$ $(n\geqq 0)$, and $t(R)=\max\{n\geqq 0$ such that $R_n\neq(0)\}$ if R is Artinian. The purpose of this note is to give

Theorem 1. *Let R be Gorenstein and $\{X_1, X_2, \cdots, X_r\}$ be a homogeneous system of parameters of R $(r=\dim R>0)$. Then for $d>0$, $R^{(d)}$ is Gorenstein $\Leftrightarrow H(1, R)=1$ or $t(R/(X_1, X_2, \cdots, X_r))\equiv\sum_{i=1}^{r}\deg X_i \bmod d$.*

and to show similar results in case R is Artinian.*)

1. The case where R is Artinian.

In this section R is assumed to be Artinian. We denote the k-dimension of the socle of R by $r(R)$ and put $t=t(R)$.

Lemma 1. $H(s, R)=1$ $(s>0)\Rightarrow H(n, R)\leqq 1$ *for every* $n\geqq s$.

Proof. It suffices to prove that $H(s+1, R)\leqq 1$. Since $R_s=R_1R_{s-1}$ and $H(s, R)=1$, we have $R_s=xR_{s-1}$ for some $x\in R_1$. Hence $R_{s+1}=R_1(R_{s-1}x)=R_sx$ and this implies that $H(s+1, R)\leqq 1$. QED.

*) The referee showed to the author that our theorems give a generalization to some results of T. Matsuoka [4]. He proved Th. 1 in case R is a polynomial ring.

Corollary 1. *Suppose that R is Gorenstein and $H(1, R)\geqq 2$. Then $H(s, R)=1 \Leftrightarrow s=t$.*

Proof. $\Rightarrow$) R_t coincides with the socle of R and hence $H(t, R)=1$. $\Rightarrow$) Assume that $s<t$. Then it follows from Lemma 1 that $H(t-1, R)=1$. If we define $R^*=\mathrm{Hom}_k(R, k)$, then R^* becomes a graded R-module with $\{\mathrm{Hom}_k(R_{-n}, k)\}_{n\in Z}$ as its grading (Here we understand $R_n=(0)$ for $n<0$.). Since R is Gorenstein, it is known that $R\simeq R^*(-t)$ as graded R-modules (Here $R^*(-t)$ denotes the graded R-module whose underlying R-module is the same as that of R^* and whose grading is given by $[R^*(-t)]_n=[R^*]_{n-t}$.) by Ch. 4 of S. Goto and K. Watanabe [2]. Therefore $H(n, R)=H(t-n, R)$ for $n=0, 1, \cdots, t$. Thus we have $H(1, R)=H(t-1, R)=1$ and this contradicts the hypothesis of Corollary. QED.

Remark 1. This is incorrect if R is *not* Gorenstein. For example, let k be a field and $R=k[X, Y]/(X^2, XY, Y^{s+1})$ $(s\geqq 2)$. Then $t(R)=s$ and $H(1, R)=2$. But $H(n, R)=1$ for every $n=2, 3, \cdots, s$. Hence the assertion of Corollary does not hold for $s>2$. In this case the socle of R is (x, y^s) and $r(R)=2$ (Here $x=X \bmod (X^2, XY, Y^{s+1})$ and similarly for y.). Moreover the case $s=2$ shows that the converse of Corollary is not true.

Lemma 2. *R_t coincides with the socle of $R \Leftrightarrow 0: R_d=\sum_{n+d\geqq t+1} R_n$ for every $d\geqq 0$.*

Proof. $\Rightarrow$) Put $d=1$. $\Rightarrow$) We have only to prove $0: R_d \supset \sum_{n+d\geqq t+1} R_n$. Assume the contrary and let x be a homogeneous element of $0: R_d$ such that $x\notin \sum_{n+d\geqq t+1} R_n$. Then $0<e<t+1-d$ where $e=\deg x$. If we choose x for which $d+e$ is smallest among the countre examples, then $xR_{d-1}\neq (0)$. Since $xR_{d-1}\subset 0: R_1$, this implies that $(0)\neq xR_{d-1}\subset R_t$. Hence $e+(d-1)=t$. This is the required contradiction. QED.

Proposition. *Let R be Gorenstein and $0<d\leqq t$. Put $s=\min\{n\in dZ$ such that $n+d\geqq t+1\}$. Then the socle of $R^{(d)}$*

coincides with R_s. *Therefore* $r(R^{(d)}) = H(s, R)$. *In particular* $R^{(d)}$ *is Gorenstein* $\Leftrightarrow$ $H(s, R) = 1$.

Proof. The socle of $R^{(d)} = (0: R_d) \cap R^{(d)}$

$$= \sum_{n+d \geqq t+1} R_n \cap R^{(d)} \qquad \text{(by Lemma 2)}$$

$$= R_s . \qquad \text{QED.}$$

Theorem 2. *Assume that* R *is Gorenstein.*

(1) *Let* $H(1, R) \leqq 1$. *Then* $R^{(d)}$ *is Gorenstein for every* $d > 0$.

(2) *Suppose that* $H(1, R) \geqq 2$. *Then*

(a) $R^{(d)}$ *is Gorenstein for every* $d > t$.

(b) *Let* $t \geqq d > 0$. *Then* $R^{(d)}$ *is Gorenstein* $\Leftrightarrow$ $t \equiv 0 \bmod d$.

Proof. (1) In this case $R^{(d)}$ is a PIR and hence it is Gorenstein. (2) (a) For $d > t$ we have $R^{(d)} = k$. (b) Let $t \geqq d > 0$. Then $R^{(d)}$ is Gorenstein $\Leftrightarrow$ $H(s, R) = 1$ where $s = \min\{n \in d\boldsymbol{Z}$ such that $n + d \geqq t + 1\}$. The latter is equivalent to say that $s = t$ by Corollary of Lemma 1. And this means that $t \equiv 0 \bmod d$. QED.

2. Proof of Theorem 1.

In this section R is assumed to be Macaulay and $\{X_1, X_2, \cdots, X_r\}$ denotes a homogeneous system of parameters of $R (r = \dim R > 0)$.

Lemma 3. $t(R/(X_1^d, X_2^d, \cdots, X_r^d)) = t(R/(X_1, X_2, \cdots, X_r)) + \sum_{i=1}^r (\deg X_i)(d-1)$ *for every* $d > 0$.

Proof. Put $P = k[X_1, X_2, \cdots, X_r]$. Then P is a graded subring of R. Since R is Macaulay, $\{X_1, X_2, \cdots, X_r\}$ are algebraically independent over k and R is a finitely generated free graded P-module (By a free P-module we understand a graded P-module which is isomorphic to a direct sum of graded P-modules of the form $P(n)$, $n \in \boldsymbol{Z}$. Recall that $P(n)$ denotes the graded P-module whose underlying P-module is the same as that of P and whose grading is given by $[P(n)]_m = P_{n+m}$.). In fact, if $\{e_i\}_{i \in I}$ is a family of homogeneous elements of R which constitutes a k-basis of $R/(X_1, X_2, \cdots, X_r)$ mod $(X_1, X_2, \cdots, X_r)$, then $\{e_i\}_{i\in I}$ forms a free basis of R as a P-module. Thus we have $R\cong\bigoplus_{n=0}^{t}P(-n)^{a_n}$ as graded P-modules for some $\{a_n\}_{n=0,1,\cdots,t}$ (Here $t=t(R/(X_1, X_2, \cdots, X_r))$.). Therefore we obtain $R/(X_1^d, X_2^d, \cdots, X_r^d)\cong\bigoplus_{n=0}^{t}[P/(X_1^d, X_2^d, \cdots, X_r^d)](-n)^{a_n}$ and hence $t(R/(X_1^d, X_2^d, \cdots, X_r^d))=t(P/(X_1^d, X_2^d, \cdots, X_r^d))+t=\sum_{i=1}^{r}(\deg X_i)(d-1)+t$.

QED.

Proof of Theorem 1. Suppose that $H(1, R)=1$. Then $R=k[X]$ (a polynomial ring over k with a variable X) and hence $R^{(d)}=k[X^d]$ for every $d>0$. Therefore we assume that $H(1, R)\geqq 2$. The assertion is obvious in case $d=1$ and hence we assume that $d\geqq 2$. Now suppose that $r\geqq 2$. In this case $t(R/(X_1^d, X_2^d, \cdots, X_r^d))\geqq r(d-1)\geqq d$ by Lemma 3. Note that if we put $A=R/(X_1^d, X_2^d, \cdots, X_r^d)$ then $R^{(d)}/(X_1^d, X_2^d, \cdots, X_r^d)R^{(d)}\cong A^{(d)}$. Since $R^{(d)}$ is Macaulay and $\{X_1^d, X_2^d, \cdots, X_r^d\}$ is a system of parameters of $R^{(d)}$, $R^{(d)}$ is Gorenstein *globally* if and only if $A^{(d)}$ is Gorenstein (cf. Corollary (4.2) of Ch. 1, S. Goto and K. Watanabe [2]). Thus Theorem 2 is applicable since $H(1, R)=H(1, A)\geqq 2$ and we have the assertion by virture of Lemma 3 if $r\geqq 2$. For $r=1$, put $X=X_1$. In the following we will prove that $t(A)\geqq d$. Assume the contrary. Then $d>t(A)\geqq(\deg X)(d-1)$. Hence $\deg X=1$ and $t(A)=d-1$. Therefore $R_d=kX^d$ and so $H(d, R)=1$. On the other hand, since X is R-regular and $XR_i\subset R_{i+1}$, we have that $H(i, R)\leqq H(i+1, R)$ $(i=0, 1, 2, \cdots)$. Hence $H(1, R)\leqq H(d, R)$ and therefore $H(1, R)=1$. This is a contradiction. Thus we have $t(A)\geqq d$ and the assertion follows form Theorem 2.

QED.

Remark 2. With the same hypothesis as Theorem 1 we define $g(R)=\{d>0$ such that $R^{(d)}$ is Gorenstein$\}$. Then

(a) $g(R)<\infty\Leftrightarrow H(1, R)\geqq 2$ and $t(R/(X_1, X_2, \cdots, X_r))\neq\sum_{i=1}^{r}\deg X_i$.

(b) $g(R)=\infty\Leftrightarrow R^{(d)}$ is Gorenstein for every $d>0$.

Examples. (1) Let k be a field and $R=k[X_1, X_2, \cdots, X_r]$ $(r>0)$ be a polynomial ring over k with r variables $\{X_i\}_{i=1,2,\cdots,r}$. Then for $d>0$, $R^{(d)}$ is Gorenstein $\Leftrightarrow$ $r=1$ or $r\equiv 0 \bmod d$. (This example is given by T. Matsuoka [4].)

(2) Let k be a field and $R=k[\{X_{ij}\}_{1\leqq i,j\leqq r}]/\mathfrak{A}_2$ $(r\geqq 2)$ where $k[\{X_{ij}\}_{1\leqq i,j\leqq r}]$ is a polynomial ring over k with r^2 variables $\{X_{ij}\}_{1\leqq i,j\leqq r}$ and $\mathfrak{A}_2$ denotes the ideal generated by all the 2×2 minors of the matrix $[X_{ij}]$. Then R is Gorenstein and $H(1,R)=r^2$ (cf. S. Goto and K. Watanabe [2]). If we put $f_t=\sum_{i+j=t} X_{ij} \bmod \mathfrak{A}_2$ $(t=2,3,\cdots,2r)$, $\{f_t\}_{t=2,3,\dots,2r}$ constitutes a homogeneous system of parameters of R and $t(R/(f_2,f_3,\cdots,f_{2r}))=r-1$ (cf. S. Goto [1]). Hence for $d>0$, $R^{(d)}$ is Gorenstein $\Leftrightarrow$ $r\equiv 0 \bmod d$.

(3) Let k be a field and $R=k[X_1,X_2,\cdots,X_r,X_{r+1}]/(X_{r+1}^{n+1}-f(X_1,X_2,\cdots,X_r,X_{r+1}))$ $(r>0, n>0)$ where $f(X_1,X_2,\cdots,X_r,X_{r+1})$ denotes a homogeneous polynomial of degree $n+1$ which does not contain the term X_{r+1}^{n+1}. Then $H(1,R)=r+1$ and $\dim R=r$. Moreover if we put $x_i=X_i \bmod (X_{r+1}^{n+1}-f(X_1,X_2,\cdots,X_r,X_{r+1}))$, $t(R/(x_1,x_2,\cdots,x_r))=n$. Hence for $d>0$, $R^{(d)}$ is Gorenstein $\Leftrightarrow$ $n\equiv r \bmod d$. Thus, if $n=r$, $R^{(d)}$ is Gorenstein for every $d>0$.

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