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The derivation algebras of the classical infinite Lie algebras

By

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Introduction

We mean by a classical infinite Lie algebra one of the following Lie algebras which arise from primitive infinite Lie tansformation groups:

(I) the Lie algebra of all vector fields,

(II) the Lie algebra of vector fields of divergence zero,

(III) the Lie algebra of vector fields of constant divergence,

(IV) the Lie algebra of vector fields preserving a hamiltonian structure (the hamiltonian Lie algebra),

(V) the Lie algebra of vector fields preserving a hamiltonian structure up to constant factors,

(VI) the Lie algebra of vector fields preserving a contact structure (the contact Lie algebra).

Though we have not clalified the category to which vector fields belong, specifying it, we may speak of *formal* or *global* classical infinite Lie algebras. More precisely, formal algebras are those algebras consisting of formal (i.e. formal power series) vector fields, and global algebras are those algebras consisting of vector fields which are defined globally on certain differentiable manifolds.

In this paper we shall determine completely the derivation algebras of the classical infinite Lie algebras, both formally and globally.

The first cohomology group $H^1(L, L)$ of a Lie algebra L with adjoint representation is ipso facto the derivation algebra $D(L)$ of L factored by the ideal of $D(L)$ consisting of inner derivations.

Therefore the first cohomology groups of the classical infinite Lie algebras will be determined at the same time.

The present paper is composed of two chapters.

Chapter I is devoted to the study of formal version. Let $L_{gl}(n)$, $L_{sl}(n)$, $L_{csl}(n)$, $L_{sp}(2n)$, $L_{csp}(2n)$ and $L_{ct}(2n+1)$ be the formal classical infinite Lie algebras corresponding to (I), (II), $\cdots$ (VI). We shall prove the following

Theorem I. *The derivation algebras $D(L)$ of the formal classical infinite Lie algebras L are as follows*:

i) $D(L_{gl}(n))=L_{gl}(n)$.

ii) $D(L_{sl}(n))=D(L_{csl}(n))=L_{csl}(n)$.

iii). $D(L_{sp}(2n))=D(L_{csp}(2n))=L_{csp}(2n)$.

iv) $D(L_{ct}(2n+1))=L_{ct}(2n+1)$.

The formal classical infinite Lie algebras are defined purely algebraically, and our proof is algebraic and elementary only except that we use some knowledge of the structures of those Lie algebras.

After preparing the manuscript, the author was informed that the first cohomology groups of the formal classical infinite Lie algebras had been determined by C. Freifeld [2].

Chapter II is devoted to the study of the global version. We shall obtain the results parallel to those of Chapter I. Denote by $\mathcal{L}_{gl}(M)$, $\mathcal{L}_{sl}(M, \Omega)$, $\mathcal{L}_{csl}(M, \Omega)$, $\mathcal{L}_{sp}(M, \omega)$, $\mathcal{L}_{csp}(M, \omega)$, $\mathcal{L}_{ct}(M, \theta)$, the global classical infinite Lie algebras corresponding to (I), (II), $\cdots$ (VI). Then we shall prove

Theorem II. *The derivation algebras $D(\mathcal{L})$ of the global classical infinite Lie algebras $\mathcal{L}$ are as follows*:

i) $D(\mathcal{L}_{gl}(M))=\mathcal{L}_{gl}(M)$,

ii) $D(\mathcal{L}_{sl}(M, \Omega))=D(\mathcal{L}_{csl}(M, \Omega))=\mathcal{L}_{csl}(M, \Omega)$,

iii) $D(\mathcal{L}_{sp}(M, \omega))=D(\mathcal{L}_{csp}(M, \omega))=\mathcal{L}_{csp}(M, \omega)$,

iv) $D(\mathcal{L}_{ct}(M, \theta))=\mathcal{L}_{ct}(M, \theta)$.

Since the global algebras are deeply related to the formal algebras, the formal results in Chapter I give us a clear perspective and much information to study the global version.

The principle of the proof of Theorem II is very simple: Let $\mathcal{L}$ be a global classical infinite Lie algebra and α be a derivation of $\mathcal{L}$. We show that α is a local operator and further that at any point p it induces the derivation α_p of the formal algebra of $\mathcal{L}$ at p. By the formal results we see that there exists a formal vector field ζ_p such that $\alpha_p(\xi_p)=[\zeta_p, \xi_p]$ for all formal vector fields ξ_p of $\mathcal{L}$ at p. To show that ζ_p really defines a global smooth vector field on the manifold, we are led to solve certain partial differential equation, the solvability of which is assured by the formal integrability and the uniqueness of the formal solution.

Recently F. Takens [7] has proved the reseult for $\mathcal{L}_{gl}(M)$ and A. Abez, A. Lichnerowicz and A. Diaz-Miranda [1] and Y. Kanie [3] for $\mathcal{L}_{sp}(M, \omega)$ and $\mathcal{L}_{csp}(M, \omega)$, but their proofs seem to be rather complicated and use the case by case analysis depending on the peculiarity of each structure concerned.

Chapter I. Formal Version

1. Let k be the complex number field $\boldsymbol{C}$ or the real number field $\boldsymbol{R}$. The following Lie algebras are called *(formal) classical infinite Lie algebras* over k:

(I) $L_{gl}(n)$: the Lie algebra of all formal (i.e. formal power series with coefficients in k) vector fields in n-verialbes $x^1, x^2, \cdots x^n$.

(II) $L_{sl}(n)$: the Lie algebra of formal vector fields in n-veriables $x^1, x^2, \cdots x^n$, preserving the volume form $dx^1\wedge dx^2\wedge\cdots\wedge dx^n$.

(III) $L_{csl}(n)$: the Lie algebra of formal vector fields in n-variables $x^1, x^2, \cdots x^n$, preserving the volume form $dx^1\wedge dx^2\wedge\cdots\wedge dx^n$ up to constant factors.

(IV) $L_{sp}(2n)$: the Lie algebra of formal vector fields in $2n$-variables $x^1, x^2, \cdots x^{2n}$, preserving the symplectic form $\sum_{i=1}^n dx^i\wedge dx^{i+n}$.

(V) $L_{csp}(2n)$: the Lie algebra of formal vector fields in $2n$-variables $x^1, x^2, \cdots x^{2n}$, preserving the symplectic form $\sum_{i=1}^n dx^i\wedge dx^{i+n}$ up to constant factors.

(VI) $L_{ct}(2n+1)$: the Lie algebra of formal vector fields in

$(2n+1)$-variables $x^0, x^1, \cdots x^{2n}$, preserving the contact form $dx^0+\frac{1}{2}\sum_{i=1}^n x^i dx^{n+i}-x^{n+i}dx^i$ up to functional factors.

In this chapter we shall determine the derivation algebras of these Lie algebras.

2. Here we recall briefly a few fundamental facts about the structures of the classical infinite Lie algebras. Details are refered to [4], [5], and [6].

a) Let L be a classical infinite Lie algebra. L has the filtration $\{L_p\}_{p\in \boldsymbol{Z}}$, defined as follows.

$$\begin{cases} L_p=L \quad \text{for} \quad p\leq -1 \\ L_0=\{X\in L \mid \text{ the value } X_0 \text{ of } X \text{ at the origin}=0\} \\ L_p=\{X\in L_{p-1} \mid [X, L]\subset L_{p-1}\} \quad (p\geq 1) \end{cases}$$

Specially for the contact Lie algebra $L_{ct}(2n+1)$ we define another filtration $\{\bar{L}_p\}$ which is more convienient than usual one. It is defined inductively as follows:

$$\begin{cases} \bar{L}_p=L \quad \text{for} \quad p\leq -2 \\ \bar{L}_{-1}=\{X\in L \mid \langle X, \theta\rangle_0=0, \text{ where } \theta=dx^0+\frac{1}{2}\sum x^i dx^{i+n}-x^{i+n}dx^i\}. \\ \bar{L}_0=\{X\in L \mid X_0=0\} \\ L_p=\{X\in \bar{L}_{p-1} \mid [X, \bar{L}_{-1}]\subset \bar{L}_{p-1}\} \quad (p\geq 1) \end{cases}$$

This filtration is compatible with the usual one: We have

$$\bar{L}_p\supset L_{p+1} \quad \text{and} \quad L_p\supset \bar{L}_{2p+1}$$

Since we exclusively use the filtration $\{\bar{L}_p\}$ for the contact Lie algebra, we denote it by the same letter $\{L_p\}$ by abuse of language. The filtrarion L_p satisfies

$$[L_p, L_q]\subset L_{p+q} \quad \text{for all } p, q\in \boldsymbol{Z}.$$

We topologize L by assigning $\{L_p\}$ as a system of fundamental neighpourhoods of L. Then L is a topological Lie algebra and it is separated and complete.

b) The graded Lie algebra $gr(L)=\sum_{p\in \boldsymbol{Z}}\mathfrak{g}_p(L)$, where $\mathfrak{g}_p(L)$

$=L_p/L_{p+1}$, satisfies the following conditions:

i) $\dim \mathfrak{g}_p<\infty$ for all $p\in \boldsymbol{Z}$ and $[\mathfrak{g}_p, \mathfrak{g}_q]\subset \mathfrak{g}_{p+q}$ for all $p, q\in \boldsymbol{Z}$.

ii) There is a positive nteger μ such that $g_{-\mu}\neq 0$ and $\mathfrak{g}_q=0$ f,r all $q<-\mu$. ($\mu=2$ for $L_{ct}(2n+1)$ and $\mu=1$ for the other classical infinite Lie algebras.)

iii) The subalgebra $\mathfrak{m}=\sum_{p<0}\mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$.

iv) For any $p\geq 0$, the condition that $x_p\in \mathfrak{g}_p$ and $[x_p, \mathfrak{m}]=0$ implies $x_p=0$.

c) Any classical infinite Lie algebra L is isomorphic to the comletion $\prod_{p\in \boldsymbol{Z}}\mathfrak{g}_p(L)$ of the graded Lie algebra $\sum_{p\in \boldsymbol{Z}}\mathfrak{g}_p(L)$ of L.

Hereafter we identify L with the direct product $\prod_{p\in \boldsymbol{Z}}\mathfrak{g}_p(L)$ and each $\mathfrak{g}_p(L)$ is considered to be imbedded in L.

d) The subalgebra $\mathfrak{g}_0(L)$ is reductive, and either $\mathfrak{g}_0(L)$ is simple or $\mathfrak{g}_0(L)$ is a direct sum of a simple ideal $\mathfrak{h}_0(L)$ and 1-dimensional center $\mathfrak{z}_0(L)$.

If $L=L_{gl}(n)$, $L_{csl}(n)$, $L_{csp}(2n)$ or $L_{ct}(2n+1)$, $\mathfrak{z}_0(L)$ contains a unique element I such that

$$[I, x_p]=px_p \text{ for all } x_p\in \mathfrak{g}_p(L).$$

e) $L_{sl}(n)$ (resp. $L_{sp}(2n)$) is an ideal of $L_{csl}(n)$ (resp. $L_{csp}(2n)$) and

$$L_{csl}(n)=L_{sl}(n)+kI$$

$$L_{csp}(2n)=L_{sp}(2n)+kI$$

f) The graded Lie algebra $\sum_{p\in \boldsymbol{Z}}\mathfrak{g}_p(L)$ is determined by its lower subspaces $\{\mathfrak{g}_p\}_{p\leq p_0}$ for some p_0. If $L=L_{sl}(n)$ or $L_{sp}(2n)$, then for $p\geq 1$ $\mathfrak{g}_p(L)$ is identified with the prolongation $\mathfrak{g}_{p-1}(L)^{(1)}$, where $\mathfrak{g}_{p-1}(L)^{(1)}$ is the subspace of $\mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{p-1})$ consisteing of those T such that

$$[T(x), y]=[T(y), x] \text{ for all } x, y\in \mathfrak{g}_{-1}.$$

g) $[L_p, L_q]=L_{p+q}$ for all $p, q>0$.
Moreover $\dim L/[L, L]\leq 1$, equality holds if and only if $L=L_{csl}(n)$ or $L_{csp}(2n)$.

3. From now on L always represents a classical infinite Lie algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$.

Proposition 1. *Every derivation of L is continous.*

Proof. Since L_p constitutes a fundamental system of neighbourhoods of the origin of L, it suffices to prove that for a derivation α of L and for any integer r there exists an integer s such that $\alpha(L_s)\subset L_r$.

Let r' be an integer such that $r'>0$ and $r'\geq r+2$, and put $s=2r'$. Then by 2-g) we have $L_s=[L_{r'}, L_{r'}]$. Since $[\alpha(L_{r'}), L_{r'}]\subset[L_{-2}, L_{r'}]\subset L_r$, we see that $\alpha(L_s)\subset L_r$. q.e.d.

A derivation α of $L=\prod\mathfrak{g}_p(L)$ is said to be *of degree k* if $\alpha(\mathfrak{g}_p(L))\subset\mathfrak{g}_{p+k}(L)$ for all $P\in\boldsymbol{Z}$.

We denote by $\hat{L}$ the Lie algebra containing L defined as follows: $\hat{L}_{sl}(n)=L_{csl}(n)$, $\hat{L}_{sp}(2n)=L_{csp}(2n)$, and $\hat{L}=L$ for the others.

Proposition 2. *Let L be a cleassical infinite Lie algebra and α be a derivation of degree 0. Then there exists an element u_0 of $\mathfrak{g}_0(\hat{L})$ such that $\alpha(x)=[u_0, x]$ for all $x\in L$.*

Proof. First assume that the ground field of L is the complex number field $\boldsymbol{C}$.

Let $\mathfrak{h}_0$, be the simple part of $\mathfrak{g}_0(=\mathfrak{g}_0(L))$. Since $[\mathfrak{g}_0, \mathfrak{g}_0]=\mathfrak{h}_0$, α induces a derivation of $\mathfrak{h}_0$. $\mathfrak{h}_0$ being simple, there is $v_0\in\mathfrak{h}_0$ such that $\alpha-\mathrm{ad}\, v_0=0$ on $\mathfrak{h}_0$.

Let β be the restriction of $\alpha-\mathrm{ad}\, v_0$ to $\mathfrak{g}_{-1}$, then

$$\beta([x_0, x_{-1}])=[x_0, \beta(x_{-1})] \text{ for all } x_0\in\mathfrak{h}_0 \text{ and } x_{-1}\in\mathfrak{g}_{-1}.$$

This implies that the map $\beta:\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ commutes with the representation of $\mathfrak{h}_0$ to $\mathfrak{g}_{-1}$. Since $\mathfrak{g}_{-1}$ is $\mathfrak{h}_0$-irreducible, we see from Schur's lemma that there exists a complex number λ such that

$$\beta+\lambda id_{\mathfrak{g}_{-1}}=0.$$

Put $u_0=v_0+\lambda I$, where I is the element of $\mathfrak{g}_0(L)$ determined by $[I, x_p]=px_p$ for all $x_p\in\mathfrak{g}_p(L)$. Then we have $\alpha-\mathrm{ad}\, u_0=0$ on $\mathfrak{g}_{-1}$ and $\mathfrak{h}_0$. Since $\mathfrak{m}=\sum_{p<0}\mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$, $\alpha-\mathrm{ad}\, u_0$ vanishes on $\mathfrak{m}$, which, combined with the fact that $\alpha-\text{ad}\ u_0$ is a derivation of degree 0 and 2-b) implies that $\alpha-\text{ad}\ u_0=0$ on $\mathfrak{g}_p$ forall p. Since the derivation $\alpha-\text{ad}\ u_0$ is continuous, we see that $\alpha=\text{ad}\ u_0$. Thus we have completed the proof in the complex case.

In the real case we just note that α induces the derivation α^c of the complexification L^c of L, and that L^c is also classical infinite Lie algebra over $\boldsymbol{C}$ of the same type as L. Thus we can find an element u_0 of $\hat{L}^c$ such that $\alpha^c=\text{ad}\ u_0$. As easily seen, u_0 proves to be an element of L, which proves our proposition.

Proposition 3. *Let L be a classical infinite Lie algebra and α be a derivation of degree k. If $k\neq 0$, then there exists an element u_k of $\mathfrak{g}_k(L)$ such that $\alpha=\text{ad}\ u_k$.*

Proof. If $L=L_{gl}(n)$, $L_{csp}(n)$, or $L_{ct}(2n+1)$, the fact that I is contained in L facilitates the proof.

Put $u_k=-(1/k)\alpha(I)$, and $\beta=\alpha-ad\ u_k$. Then $u_k\in\mathfrak{g}_k(L)$ and β is a derivation of degree k, moreover $\beta(I)=0$. Hence we have

$$\beta([I, x_p])=[I, \beta(x_p)] \text{ for all } x_p\in\mathfrak{g}_p.$$

On the other hand $\beta([I, x_p])=p\beta(x_p)$ and $[I, \beta(x_p)]=(p+k)\beta(x_p)$, from which it follows that $\beta(x_p)=0$ for all $x_p\in\mathfrak{g}_p$. By the continuity of β we have $\beta=0$, that is $\alpha=\text{ad}\ u_k$, which proves Proposition 3 in our case.

The rest of this section is devoted to the proof of Proposition 3 for the Lie algebras $L_{sl}(n)$ and $L_{sp}(2n)$.

Now we assume that $L=L_{sl}(n)$ or $L_{sp}(2n)$.

Lemma 1. *Let α be a derivation of L.* If α vanishes on $\mathfrak{g}_p$ *for $p\leq 0$, then $\alpha=0$.*

Proof. By our assumption we have

(1) $$[\alpha(x_1), x_{-1}]=0 \text{ for } x_1\in\mathfrak{g}_1,\ x_{-1}\in\mathfrak{g}_{-1}.$$

(2) $$\alpha([x_0, x_1])=[x_0, \alpha(x_1)] \text{ for } x_1\in\mathfrak{g}_1,\ x_0\in\mathfrak{g}_0.$$

From (1) we see that $\alpha(x_1)\in\mathfrak{g}_{-1}$ for all $x_1\in\mathfrak{g}_1$, and therefore the restriction $\alpha|\mathfrak{g}_1$ maps $\mathfrak{g}_1$ into $\mathfrak{g}_{-1}$. (2) implies that $\mathfrak{g}_1$ commutes with the representations of $\mathfrak{g}_0$ to $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$. It is known that every $\mathfrak{g}_p, (p \geq -1)$ is $\mathfrak{g}_0$-irreducible and that $\mathfrak{g}_p$ and $\mathfrak{g}_q$ are never isomorphic to each other if $p \neq q$. Hence we have $\alpha|\mathfrak{g}_1 = 0$. By the same way, or by using the fact that the Lie algebra generated by $\{\mathfrak{g}_p\}_{p \leq 1}$ is dense in L, we have our conclusion.

Corollary. *For $k < -1$, any derivations of L of degree k are trivial.*

Proof. Let α be a derivation of degree k. Since $\mathfrak{g}_q = 0$ for $q < -1$, we have $\alpha(\mathfrak{g}_p) = 0$ for $p \leq 0$. Hence we have $\alpha = 0$ by Lemma 1.

Lemma 2. *Let α be a derivation of degree -1. Then there exists a $u_{-1} \in \mathfrak{g}_{-1}$ such that $\alpha = \mathrm{ad}\, u_{-1}$.*

Proof. Denote by α' the restriction of α to $\mathfrak{g}_0$. The formula:

$$\alpha'([x_0, y_0]) = [\alpha'(x_0), y_0] + [x_0, \alpha'(y_1)] \quad \text{for } x_0, y_0 \in \mathfrak{g}_0\,,$$

implies that α' is closed, regarded as an element of $C^1(\mathfrak{g}_0, \mathfrak{g}_1)$, where $\sum_{q \geq 0} C^q(\mathfrak{g}_0, \mathfrak{g}_{-1})$ is the complex associated to the representation of $\mathfrak{g}_0$ to $\mathfrak{g}_{-1}$. It is well known that the 1-st cohomology group $H^1(\mathfrak{g}_0, \mathfrak{g}_{-1})$ vanishies for any semi-simple Lie algebra $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is simple, this applies to this case. Thus we can find a $u_{-1} \in \mathfrak{g}_{-1}$ such that

$$\alpha'(x_0) = [u_{-1}, x_0] \quad \text{for all } x_0 \in \mathfrak{g}_0.$$

Put $\beta = \alpha - ad\, u_{-1}$ then β vanishes on $\mathfrak{g}_0$ and also on $\mathfrak{g}_{-1}$, for β is of degree -1. From this and Lemma 1 it follows that $\alpha = \mathrm{ad}\, u_{-1}$.

q.e.d.

Lemma 3. *Suppose that $k > 0$ and that α be a derivation of degree k. Then there exists a $u_k \in \mathfrak{g}_k$ such that $\alpha = \mathrm{ad}\, u_k$.*

Proof. Let α' be the restriction of α to $\mathfrak{g}_{-1}$, then α' is a map from $\mathfrak{g}_{-1}$ to $\mathfrak{g}_{k-1}$ and satisfies

$$[\alpha'(x_{-1}), y_{-1}] + [x_{-1}, \alpha'(y_{-1})] = 0 \quad \text{for } x_{-1}, y_{-1} \in \mathfrak{g}_{-1},$$

which implies that α' is an element of the prolongation $\mathfrak{g}_{k-1}^{(1)}$ of $\mathfrak{g}_{k-1}$. For the algebras $L_{sl}(n)$ and $L_{sp}(2n)$, it holds that $\mathfrak{g}_{p-1}^{(1)}=\mathfrak{g}_p$ for all $p\geq 1$. Hence we can find $u_k\in\mathfrak{g}_k$ such that $\alpha - ad\, u_k$ vanishes on $\mathfrak{g}_{-1}$. It easily follows that $\alpha = ad\, u_k$. q.e.d.

Combining Corollary, Lemma 2 and Lemma 3, we have proved Proposition 3 for $L_{sl}(n)$ and $L_{sp}(2n)$, and the proof of Proposition 3 is complete.

4. Now we are in a position to prove our theorem.

Theorem. *The derivation algebras $D(L)$ of the classical infinite Lie algebras L over $\boldsymbol{C}$ or $\boldsymbol{R}$ are as follows*:

i) $D(L_{gl}(n)) = L_{gl}(n)$.

ii) $D(L_{sl}(n)) = D(L_{csl}(n)) = L_{csl}(n)$.

iii) $D(L_{sp}(2n)) = D(L_{csp}(2n)) = L_{csp}(2n)$.

iv) $D(L_{ct}(2n+1)) = L_{ct}(2n+1)$.

Proof. We show that for any derivation α of L there exists one and only one $u\in\hat{L}$ such that

$$\alpha(x) = [u, x] \quad \text{for } x\in L\,.$$

Denote by $\alpha_p{}^{(k)}$ the $\mathrm{Hom}(\mathfrak{g}_p(L), \mathfrak{g}_{p+k}(L))$-component of α. The continuous derivation $\alpha^{(k)}$ of L determined by $\alpha^{(k)}|\mathfrak{g}_p = \alpha_p{}^{(k)}$ is a derivation of L of degree k. By Proposition 2 and Proposition 3 we can find a $u_k\in\mathfrak{g}_k(\hat{L})$ for each k such that

$$\alpha^{(k)}(x) = [u_k, x] \quad \text{for all } x\in L\,.$$

The direct product $u=\prod_k u_k$ is an element of $\hat{L}$. The continuity of α assures that u satisfies the required property.

The uniqueness of u follows from the obvious fact that the condition "$u\in\hat{L}$ and $[u, L]=0$" implies $u=0$, q.e.d.

The first cohomology group $H^1(L, L)$ of the Lie algebra L with adjoint representation is immeadiately from the definition seen to be the derivation algebra $D(L)$ factored by the ideal of $D(L)$ consisting of inner derivations.

Thus we have the following

Corollary. *Let L be the classical infinite Lie algebra over k, where $k=\boldsymbol{C}$ or $\boldsymbol{R}$, and $H^1(L, L)$ be the 1-st cohomology group of L with adjoint representation. Then we have*

i) $H(L, L)=0$ *if* $L=L_{gl}(n), L_{csl}(n), L_{csp}(2n)$ *or* $L_{ct}(2n+1)$.

ii) $H(L, L)=k$ *if* $L=L_{sl}(n)$ *or* $L_{sp}(2n)$.

Chapter II. Global Version

1. In this chapter we consider a global version of the results in Chapter I. Throughout this chapter manifolds are assumed to be connected, paracompact, and of class C^∞, and vector fields, forms and functions on them are all assumed to be of class C^∞ and defined globally on them even if it is not stated explicitely.

Our objects are the following Lie algebras which we call *(global) classical infinite Lie algebras*:

(I) The Lie algebra $\mathcal{L}_{gl}(M)$ of all smooth vector fields on a smooth manifold M.

Suppose that it is given a volume form Ω on M.

(II) $\mathcal{L}_{sl}(M, \Omega)$ is the Lie algebra consisting of smooth vector fields X on M satisfying $L_X\Omega=0$, where L_X denotes the Lie derivative along X.

(III) $\mathcal{L}_{csl}(M, \omega)$ is the Lie algebra consisting of smooth vector fields X on M satisfying $L_X\Omega=c\Omega$, where c is some constant depending on X.

Suppose that (M, ω) is a symplectic manifold of dimension $2n$, that is, there is given a closed 2-form ω on M with $\overbrace{\omega\wedge\omega\wedge\cdots\wedge\omega}^{\text{n-times}}\neq 0$ everywhere.

(IV) $\mathcal{L}_{sp}(M, \omega)$ is the Lie algebra consisting of smooth vector fields X on M satisfying $\mathcal{L}_X\omega=0$.

(V) $\mathcal{L}_{csp}(M, \omega)$ is the Lie algebra consisting of smooth vector fields X on M satisfying $L_X\omega=c\omega$, where c is some constant depending on X.

If (M, θ) is a $(2n+1)$-dimensional contact manifold, that is, there is given a 1-form with $\theta\wedge \overbrace{d\theta\wedge\cdots\wedge d\theta}^{\text{n-times}}\neq 0$ everywhere.

(VI) $\mathcal{L}_{ct}(M, \theta)$ is the Lie algebra consisting of smooth vector fields X on M satisfying $L_X\theta=\rho\theta$, where ρ is some function depending on X.

Hereafter $\mathcal{L}_{gl}(\mathrm{M})$, $\mathcal{L}_{sl}(M, \Omega)$, $\cdots$, $\mathcal{L}_{ct}(M, \theta)$ are often abbriviated as $\mathcal{L}_{gl}(M)$, $\mathcal{L}_{sl}(M)$, $\cdots$, $\mathcal{L}_{ct}(M)$, or more simply as $\mathcal{L}_{gl}$, $\mathcal{L}_{sl}$, $\cdots$, $\mathcal{L}_{ct}$.

For an open set U of M, $\mathcal{L}_{gl}(U)$, $\mathcal{L}_{sl}(U)$, $\cdots$, $\mathcal{L}_{ct}(U)$ always mean $\mathcal{L}_{gl}(U)$, $\mathcal{L}_{sl}(U, \Omega|U)$, $\cdots$, $\mathcal{L}_{ct}(U, \theta|U)$.

2. We begin with introducing some general properties of the global classical infinite Lie algebras.

Proposition 1. *There are canonical isomorphisms between*

i) $\mathcal{L}_{sl}(M, \Omega)$ *and the space of closed* $(n-1)$*-forms on* M, *where* $n=\dim M$,

ii) $\mathcal{L}_{sp}(M, \omega)$ *and the space of closed* 1*-forms on* M,

iii) $\mathcal{L}_{ct}(M, \theta)$ *and the space of smooth functions on* M.

Proof. The isomorphisms are given by

i) $X\to X\lrcorner\Omega$ for $X\in\mathcal{L}_{sl}(M, \Omega)$,

ii) $X\to X\lrcorner\omega$ for $X\in\mathcal{L}_{sp}(M, \omega)$,

iii) $X\to X\lrcorner\theta$ for $X\in\mathcal{L}_{ct}(M, \theta)$,

where $X\lrcorner$ denotes the interior product by X. Non-degeneracy of the forms assures that the maps are isomorphisms.

Proposition 2. $\mathcal{L}_{sl}(M, \Omega)$ *(reep.* $\mathcal{L}_{sp}(M, \omega)$*) is an ideal of* $\mathcal{L}_{csl}(M, \Omega)$ *(resp.* $\mathcal{L}_{csp}(M, \omega)$*) of codimension 1 if* Ω *(resp.* ω*) is exact. If it is not exact,* $\mathcal{L}_{csl}(M, \Omega)$ *(resp.* $\mathcal{L}_{csp}(M, \omega)$*) coincides with* $\mathcal{L}_{sl}(M, \Omega)$ *(resp.* $\mathcal{L}_{sp}(M, \omega)$*).*

Remark. By this reason, hereafter we always assume that Ω (resp. ω) is exact whenever we speak of $\mathcal{L}_{csl}(M, \Omega)$ (resp. $\mathcal{L}_{csp}(M, \omega)$).

Proof. Since $\mathcal{L}_{sl}$ is the kernel of the Lie homomorphism which maps $X \in \mathcal{L}_{csl}$ to $c_X \in \boldsymbol{R}$, where $L_X \Omega = c_X \Omega$, $\mathcal{L}_{sl}$ is an ideal of $\mathcal{L}_{csl}$ of condimension at most 1.

If there exists an $X \in \mathcal{L}_{csl}$ with $c_X \neq 0$, then Ω is exact, because $\Omega = c_X^{-1} L_X \Omega = d(c_X^{-1} X \lrcorner \Omega)$.

Conversely suppose that Ω is exact, say $\Omega = d\Omega'$. Let X_0 be the vector field determined by $X_0 \lrcorner \Omega = \Omega'$. Then we have $L_{X_0} \Omega = \Omega$, which shows that $\mathcal{L}_{csl} / \mathcal{L}_{sl} \cong \boldsymbol{R}$.

The assertion for $\mathcal{L}_{csp}$ can be proved quite analogously.

q.e.d.

Let us introduce filtrations of the global calssical infinite Lie algebras which connect the global algebras with the corresponding formal algebras.

Let $\mathcal{L}$ represents one of the classical infinite Lie algebras. For any point p of M, a filtration $\{\mathcal{L}_p^k\}_{k \in \boldsymbol{Z}}$ of $\mathcal{L}$ is defined as follows:

$$\begin{cases} \mathcal{L}_p^k = \{X \in \mathcal{L} \mid j_p^k(X) = 0\} & \text{for} \quad k \geq 0 \\ \mathcal{L}_p^k = \mathcal{L} & \text{for} \quad k \leq -1, \end{cases}$$

where $j_p^k(X)$ denotes the k-th jet of X at p.

Specially for the contact Lie algebra $\mathcal{L}_{ct}(M, \theta)$ we use another filtration, which we denote by the same letter $\{\mathcal{L}_p^k\}$. It is defined inductively as follows:

$$\begin{cases} \mathcal{L}_p^k = \mathcal{L} & \text{for} \quad k \leq -2 \\ \mathcal{L}_p^{-1} = \{X \in \mathcal{L} \mid \langle X, \theta \rangle_p = 0\} \\ \mathcal{L}_p^0 = \{X \in \mathcal{L} \mid X_p = 0\} \\ \mathcal{L}_p^k = \{X \in \mathcal{L}_p^{k-1} \mid [X, \mathcal{L}_p^{-1}] \subset \mathcal{L}_p^{k-1}\} & \text{for} \quad k \geq 1. \end{cases}$$

The filtration $\{\mathcal{L}_p^k\}$ of $\mathcal{L}$ satisfies

$$\begin{cases} \mathcal{L}_p^k \supset \mathcal{L}_p^{k+1} \text{ for all } k \in \boldsymbol{Z}. \\ [\mathcal{L}_p^k, \mathcal{L}_p^l] \subset \mathcal{L}_p^{k+l} \text{ for all } k, l \in \boldsymbol{Z} \end{cases}$$

Let $j_p(\mathcal{L})$ be the projective limit $\lim_k \mathcal{L}/\mathcal{L}_p^k$ and denote by j_p the canonical projection from $\mathcal{L}$ to $j_p(\mathcal{L})$. $j_p(\mathcal{L})$ inherits a Lie algebra structure and a filtration $\{j_p(\mathcal{L}_p{}^k)\}_{k\in Z}$, which make it a topological Lie algebra called the *formal algebra* of of $\mathcal{L}$ at p.

Proposition 3. *The formal algebra of a global classical infinite Lie algebra at any point is isomorphic to the corresponding formal classical infinite Lie algebra.*

Here we agree that the corresponding formal classical infinite Lie algebras of $\mathcal{L}_{gl}, \mathcal{L}_{sl}, \cdots, \mathcal{L}_{ct}$ are respectively $L_{gl}, L_{sl}, \cdots, L_{ct}$.

It is a well-known fact that for each of a volume form Ω, a sympletic form ω, and a contact form θ, there exists a coordiante neighbourhood and a coordinate system such that it is expressed in the following form:

$$\Omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n ,$$

$$\omega = \sum_{i=1}^{n} dx^i \wedge dx^{i+n}$$

$$\theta = dx^0 + \tfrac{1}{2} \sum_{i=1}^{n} x^i dx^{i+n} - x^{i+n} dx^i$$

We mean by $\mathcal{L}$-coordinate system a coordinate system by which the defining form of $\mathcal{L}$ has the above standard representation. $\mathcal{L}$-coordinate neighbourhood is a coordinate neighbourhood on which an $\mathcal{L}$-coordinate system is defined.

Proposition 3 follows from the existence of $\mathcal{L}$-coordinate systems and from Proposition 4, which will be proved in the next section.

3. From now on we use the following convention.

$\mathcal{L}$ always represents a global classical infinite Lie algebra. We denote by $\check{\mathcal{L}}$ and $\hat{\mathcal{L}}$ respectively the ideal of $\mathcal{L}$ and the Lie algebra containing $\mathcal{L}$ defined as follows: $\check{\mathcal{L}}_{csl} = \mathcal{L}_{sl}$, $\check{\mathcal{L}}_{csp} = \mathcal{L}_{sp}$, and $\check{\mathcal{L}} = \mathcal{L}$ otherwise, and $\hat{\mathcal{L}}_{sl} = \mathcal{L}_{csl}$, $\hat{\mathcal{L}}_{sp} = \mathcal{L}_{csp}$ and $\hat{\mathcal{L}} = \mathcal{L}$ otherwise. To exhibit the base space we often write as $\mathcal{L}(M)$, $\check{\mathcal{L}}(U)$, etc.

First of all observe that any local vector field of $\mathcal{L}$ can be extended globally. More precisely,

Proposion 4. *Let $\mathcal{L}(M)$ be as above, and U be an open subset* *of M. For any* $X\in\mathcal{L}(U)$ *and for any* $p\in U$, *there exists an* $\tilde{X}\in\mathcal{L}(M)$ *such that* $\tilde{X}=X$ *in a neighpourhood of p. Moreover if* $X\in\check{\mathcal{L}}(U)$, *then* $\tilde{X}$ *can be taken so as to satisfy* $\operatorname{supp}\tilde{X}\subset U$.

Proof. Say $X\in\mathcal{L}_{sl}(U)$. Let φ be the isomorphism from $\mathcal{L}_{sl}$ to the space of closed $(n-1)$-form. Since $\varphi(X)$ is closed, there exists an $(n-2)$-form η on M such that $d\eta=\varphi(X)$ on some neighbourhood of p and $\operatorname{supp}\eta\subset U$. $\tilde{X}=\varphi^{-1}(d\eta)$ satisfies the desired properties.

Let $X\in\mathcal{L}_{csl}(U,\Omega|_U)$. Recall that in this case we always assume Ω to be exact. (See Remark after Prop. 2). Thus there exists $X_0\in\mathcal{L}_{csl}(M)$ such that $L_{X_0}\Omega=\Omega$. Since $X-c_XX_0\in\mathcal{L}_{sl}(U)$ where $L_X\Omega=c_X\Omega$, there exists $\tilde{X}$ such that $\tilde{X}=X$ in a neighbourhood of p

Proof for the other cases is quite similar. q.e.d.

Proposition 5. *For any* $X\in\mathcal{L}$, *the condition that* $[X,\check{\mathcal{L}}]=0$ *implies* $X=0$.

Proof. This follows from Proposition 4 and the corresponding fact on the formal algebra of $\mathcal{L}$.

Proposition 6. *Any derivation* α *of* $\mathcal{L}$ *is a local operator, that is, if a vector field X of* $\mathcal{L}$ *vanishes on some open set U of M, then* $\alpha(X)$ *also vanishes on U.*

Proof. Suppose that $Y\in\check{\mathcal{L}}(M)$ and $\operatorname{supp}Y\subset U$. Then we have $[X,Y]=0$ and then $\alpha[X,Y]=0$. On the othe hand

$$\alpha([X,Y])=[\alpha(X),Y]+[X,\alpha(Y)].$$

Since $[X,\alpha(Y)]|_U=0$, we have $[\alpha(X),Y]|_U=0$. From this fact, taking accout of Proposition 4 and Proposition 5 we see that $\alpha(X)=0$ on U. q.e.d.

By Proposition 4 and Proposition 6 we have

Proposition 7. *Let* α *be a derivation of* $\mathcal{L}(M)$. Then for any open set U of M, α induces the derivation α_U of $\mathcal{L}(U)$ such that

$$\alpha_U(X|_U)=\alpha(X)|_U \text{ for all } X\in\mathcal{L}(M).$$

4. In this section we show that derivations of $\mathcal{L}$ induce the derivations of the formal algebra of $\mathcal{L}$. In order to prove it we establish the following local properties of $\mathcal{L}$.

Proposition 8. *Let $\mathcal{L}$ be a global classical infinite Lie algebra and U be a contractible $\mathcal{L}$-coordinate neighbourhood. Let $\{\mathcal{L}^k(U)_p\}$ be the filtration of $\mathcal{L}(U)$ at $p\in U$. Then for any integer r there exists an integer s such that*

$$\mathcal{L}^s(U)_p\subset[\mathcal{L}^r(U)_p, \mathcal{L}^r(U)_p].$$

We see easily that for $k\geq 1$ $\mathcal{L}^k_{csl}(U)_p=\mathcal{L}^k_{sl}(U)_p$ and $\mathcal{L}^k_{csp}(U)_p=\mathcal{L}^k_{sp}(U)_p$. Hence it is sufficient to prove this proposition only for $\mathcal{L}_{gl}$, $\mathcal{L}_{sl}$, $\mathcal{L}_{sp}$ and $\mathcal{L}_{ct}$. The proof for $\mathcal{L}_{gl}$ is not difficult and the proof for $\mathcal{L}_{sp}(U)$ is almost covered by that for $\mathcal{L}_{ct}$. Therefore we omit the proof for $\mathcal{L}_{gl}$ and $\mathcal{L}_{sp}$.

Proof of Proposition 8. (*for the contact algebra* $\mathcal{L}_{ct}$)

Let $(x_1, x_2, \cdots, x_n, y_1, \cdots, y_n, z)$ be coordinate system on U with $\theta=dz+\frac{1}{2}\sum_{i=1}^n x_i dy_i-y_i dx_i$. We may assume that p is the origin 0 of the coordinates.

Denote by $\mathcal{F}$ the ring of all smooth functions on U, and let φ be the canonical isomorphism of $\mathcal{L}$ onto $\mathcal{F}$ (See Proposition 1.). φ induces a Lie algebra structure on $\mathcal{F}$, of which bracket operation is called generalized Poisson bracket and will be denoted by $\{\ ,\ \}$. Then it has the following coordinate representation:

$$\{f, g\}=\sum_{i=1}^n \frac{\delta f}{\delta x_i}\frac{\delta g}{\delta y_i}-\frac{\delta f}{\delta y_i}\frac{\delta g}{\delta x_i}-\frac{\partial f}{\partial z}g+f\frac{\partial g}{\partial z},$$

where

$$\frac{\delta}{\delta x_i}=\frac{\partial}{\partial x_i}+\frac{1}{2}y_i\frac{\partial}{\partial z}, \quad \frac{\delta}{\delta y_i}=\frac{\partial}{\partial y_i}-\frac{1}{2}x_i\frac{\partial}{\partial z}.$$

To introduce a filtration of $\mathcal{F}$, we define $\operatorname{ord}(f)$ for any $f\in\mathcal{F}$. For a monomial $x^\alpha y^\beta z^\gamma$, where $\alpha=(\alpha_1\cdots\alpha_n)$, $\beta=(\beta_1\cdots\beta_n)$ and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ etc, $\operatorname{ord}(x^\alpha y^\beta z^\gamma)$ is defined as

$$\operatorname{ord}(x^{\alpha}y^{\beta}z^{\gamma}) = |\alpha| + |\beta| + 2\gamma - 2$$

where $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $|\beta| = \sum_{i=1}^{n} \beta_i$. For any function $f \in \mathcal{F}$, $\operatorname{ord}(f)$ is defined to be the minimum of orders of non-zero monomials which apper in the Taylor expansion of f around 0.

Then we define a filtration $\{\mathcal{F}^k\}$ of $\mathcal{F}$ by,

$$\mathcal{F}^k = \{f \in \mathcal{F} \mid \operatorname{ord}(f) \geq k\} \quad \text{for } k \in \boldsymbol{Z}.$$

This is the filtration which corresponds to that of $\mathcal{L}$, namely we see that $\varphi(\mathcal{L}_0^k) = \mathcal{F}^k$.

Hence we can reformulate the proposition in the following dual from.

Lemma 1. *For any integer r there exists an integer s such that $\mathcal{F}^s \subset \{\mathcal{F}^r, \mathcal{F}^r\}$.*

Proof. We may assume that $r > 0$. Let s be an integer satisfying

$$s \geq 2(n+1)r + 2(2n+5). \tag{1}$$

Any $f \in \mathcal{F}^s$ can be written as

$$f = \sum_{|\alpha| + |\beta| + 2\gamma - 2 \geq s} x^{\alpha}y^{\beta}z^{\gamma} f_{\alpha\beta\gamma}, \quad \text{where } f_{\alpha\beta\gamma} \in \mathcal{F}. \tag{2}$$

Hence it suffices to show that each $x^{\alpha}y^{\beta}z^{\gamma}f$, where $|\alpha| + |\beta| + 2\gamma - 2 \geq s$, is included in $\{\mathcal{F}^r, \mathcal{F}^r\}$.

From (1) we see that one of $\{\cdots\alpha_i\cdots\beta_i\cdots\}$, is not less than $r+2$, or $\gamma \geq r+6$.

In the first case we may without loss of generality that $\alpha_1 \geq r+2$. We consider a following equation with u unknown.

$$\{x_1^{r+2}, u\} = x^{\alpha}y^{\beta}z^{\gamma}f, \tag{3}$$

which reduces by a simple calculation to

$$\left((r+2)\frac{\partial}{\partial y_1} + \frac{1}{2}\, x_1 \frac{\partial}{\partial z}\right)u = x_1^{\alpha_1 - r - 1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} y^{\beta} z^{\gamma} f. \tag{4}$$

A solution g of (4) with $g(0) = 0$ can be obtained by integration. Thus $x^{\alpha}y^{\beta}z^{\gamma}f = \{x_1^{r+2}, g\}$ and $\operatorname{ord}(x_1^{r+2}) = r$ and

$$\begin{aligned}\operatorname{ord}(g) &= |\alpha| - (r+1) + |\beta| + 2\gamma + 1 - 2 \\ &\geq s - r \\ &\geq r\end{aligned}$$

In the latter case where $\gamma \geq r+6$. Let λ be an integer such that $\frac{1}{4}(r+2)+1 \geq \lambda \geq \frac{1}{4}(r+2)$. Now we consider a following differential equa tion for u:

$$\{z^{2\lambda}, z^{2\lambda}u\} = x^{\alpha}y^{\beta}z^{\gamma}f \tag{5}$$

By a simple calculation the equation (5) reduces to

$$\lambda \sum_{i=1}^{n} \left(x_i \frac{\partial u}{\partial x_i} + y_i \frac{\partial u}{\partial y_i} \right) + z \frac{\partial u}{\partial z} = zh\,, \tag{6}$$

where we put $h = x^{\alpha}y^{\beta}z^{\gamma-4\lambda}f$. Differential equation (6) is easily integrated, in fact, we see that

$$g = \int_0^1 h(t^{\lambda}x_1, \cdots t^{\lambda}x_n, \cdots t^{\lambda}y_n, \cdots tz)\, z dt \tag{7}$$

is a solution of (6). Hence we have

$$x^{\alpha}y^{\beta}z^{\gamma}f = \{z^{2\lambda}, z^{2\lambda}g\}$$

and we see easily that $\operatorname{ord}(z^{2\lambda}) = 4\lambda - 2 \geq r$. and that

$$\begin{aligned}\operatorname{ord}(z^{2\lambda}g) &\geq |\alpha| + |\beta| + 2\gamma - 4\lambda \\ &\geq s - (4\lambda - 2) \\ &\geq s - r - 4\,. \\ &\geq r\,.\end{aligned}$$

Thus proof of the lemma is complete.

Proof of Proposition 8 (*for* $\mathcal{L}_{sl}$).

Let $(x_1, x_2, \cdots, x_n)$ be a coordinate system on U such that $\Omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ and p is the origin 0 of the coordinates. We denote by $\mathcal{L}$ the Lie algebra $\mathcal{L}_{sl}(U)$. Let C^{n-1} and Λ^{n-2} be respectively the space of closed $(n-1)$-forms and the space of $(n-2)$-forms on U, and let φ be the isomorphism of C^{n-1} onto $\mathcal{L}$ as in Proposition 1.

Since U is a contractible domain, we can define a homotopy integral operator K from C^{n-1} to Λ^{n-2} such that $d\cdot K=id_{C^{n-1}}$ and further that

$$j_0^k(K\alpha)=0 \qquad \text{if } j_0^{k-1}(\alpha)=0\,,$$

for $\alpha\in C^{n-1}$ and $k\geq 0$.

Set $\#=\varphi\circ d$ and $b=K\circ\varphi^{-1}$, then we have

$$\#\circ b=id_{\mathcal{L}}$$

$$\xi^{\#}(=\#(\xi))\in L_0^{k-1} \qquad \text{if } \xi\in\Lambda^{n-2} \text{ and } j_0^k(\xi)=0$$

$$j_0^{k+1}(X^b)=0 \qquad \text{if } X\in L_0^k,\ \text{where } X^b=b(X).$$

Hence to obtain the proposition it is sufficient to prove the following lemma.

Lemma 2. *For any integer r there exists an integer s such that the following holds: Any* $\theta^{\#}$, *with* $\theta\in\Lambda^{n-2}$ *and* $j_0^s(\theta)=0$, *can be written as*

$$\theta^{\#}=\sum_{l;\,\text{finite}}[\xi_l^{\#},\eta_l^{\#}]$$

for some $\xi_l,\ \eta_l\in\Lambda^{n-2}$ *with* $j_0^r(\xi_l)=j_0^r(\eta_l)=0$.

Proof. Any θ^{n-2} is written as

$$\theta=\sum_{i<j}f_{ij}\theta_{ij}\,,$$

where $\theta_{ij}=(-1)^{i+j+1}dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n$. If $j_0^s(\theta)=0$, then $j_0^s(f_{ij})=0$ for all i,j, and f_{ij} is written as

$$\mathrm{f}_{ij}=\sum_{|\alpha|=s+1}x^\alpha f_{ij:\alpha}\,,$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$.

Thus it suffices to prove that $(x^\alpha f\theta_{ij})^{\#}$, where $|\alpha|=s+1$, is written in a form $[\xi^{\#},\eta^{\#}]$ with $j_0^r(\xi)=j_0^r(\eta)=0$, provied s is large enough.

Take s so as to satisfy

$$s\geq\max(n(r+1)-1,\ 2r-1),$$

then we see that for some k, $\alpha_k\geq r+1$.

Case 1. $k\neq i$ and $k\neq j$.

Let l be an integer such that $1 \leq l \leq n$ and $l \neq k$. For simplicity we assume that $k < l$. Put $\xi = x_k^{r+1} \theta_{kl}$, and $\eta = g\theta_{ij}$, where g is given by

$$g(x_1, x_2, \cdots x_n) = -\int_0^{x_l} \frac{x^\alpha f}{(r+1)x_k^r} dx_l .$$

By a simple calculation we see that

$$[\xi^\#, \eta^\#] = \left(-(r+1)x_k^r \frac{\partial g}{\partial x_l} \theta_{ij}\right)^\# .$$

Hence we have $[\xi^\#, \eta^\#] = (x^\alpha f \theta_{ij})^\#$. It is easy to see that $j_0^r(\xi) = j_0^r(\eta) = 0$.

Case 2. $k = i$ or j.

Observe that the following formula holds,

$$[(g\theta_{ij})^\#, (h\theta_{ij})^\#] = -(\{g, h\}_{ij}\theta_{ij})^\# ,$$

where $\{g, h\}_{ij}$ is the Poisson bracket of g and h with respect to x_i, x_j, that is,

$$\{g, h\}_{ij} = \frac{\partial g}{\partial x_i}\frac{\partial h}{\partial x_j} - \frac{\partial h}{\partial x_i}\frac{\partial g}{\partial x_j}$$

In the same way as the proof for the contact algebra (or better for the hamiltonian algebra) we can find g, h such that

$$-\{g, h\}_{ij} = x^\alpha f$$

and $j_0^r(g) = j_0^r(h) = 0$.

We have completed the proof of Proposition 8.

From Proposition 8, taking account of the structures of the formal algebras, we have the following more detailed result.

Proposition 9. *Let $\mathcal{L}$ be a global classical Lie algebra, U be a simply connected $\mathcal{L}$-coordinate neighbourhood of a point p. Then it hold that*

$$[\mathcal{L}^s(U)_p, \mathcal{L}^r(U)_p] = \mathcal{L}^{r+s}(U)_p \text{ for all } r, s > 0,$$

and further that

$$[\mathcal{L}(U), \mathcal{L}'(U)] = \check{\mathcal{L}}(U).$$

Now we can prove the following

Proposition 10. *Let* α *be a derivation of* $\mathcal{L}$. *Then for any point* p *there is induced a unique continuous derivation* α_p of the formal algebra $j_p(\mathcal{L})$ *of* $\mathcal{L}$ *at* p *such that the following diagram commutes*:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\alpha} & \mathcal{L} \\ j_p \downarrow & & \downarrow j_p \\ j_p(\mathcal{L}) & \xrightarrow{\alpha_p} & j_p(\mathcal{L}) \end{array}$$

Proof. It is sufficient to show that for any $\mathcal{L}_p^r$, there exists s such that $\alpha(\mathcal{L}_p^s) \subset \mathcal{L}_p^r$. This follows from Proposition 8 and Proposition 4.

5. Let $\mathcal{L}(M)$ be a global classical infinite Lie algebra on M and α be a derivation of $\mathcal{L}(M)$.

The following equation (E) for unknown vector field Z,

$$(E) \qquad [Z, X] = \alpha(X) \quad \text{for all } X \in \mathcal{L}(M),$$

has a unique formal solution at any point, that is, for any point $p \in M$ there exists a unique formal vector field ζ_p at p such that

$$[\zeta_p, j_p(X)] = j_p(\alpha(X)) \quad \text{for all } X \in \mathcal{L}(M).$$

In fact, Let α_p be the continuous derivation of $j_p(\mathcal{L})$ induced from α, we see, by Proposition 3 and the theorem in Chapter I, that there exists a unique ζ_p satisfying

$$[\zeta_p, j_p(X)] = \alpha_p(j_p(X)) \quad \text{for all } X \in \mathcal{L}(M).$$

Since $\alpha_p(j_p(X)) = j_p(\alpha(X))$, we have our assertion.

Now we prove that (E) has a smooth global solution. By virtue of the uniqueness of the formal solution of (E), it suffices to prove it locally.

Let U be an $\mathcal{L}$-coordinate neighbourhood of M, and consider the following equation (E_U) for unknown vector field Z on U,

$$(E_U) \qquad [Z, X] = \alpha_U(X) \quad \text{for all } X \in \mathcal{L}(U),$$

where α_U is the derivation of $\mathcal{L}(U)$ induced from α

Since X runs in the infinite dimensional space $\mathcal{L}(U)$, we introduce an appropriate finite dimensional subalgebra $\mathcal{H}$ of $\mathcal{L}(U)$ and reduce (E_U) to a differential equation (E'_U).

Let $(x_1, x_2, \cdots, x_n)$ be an $\mathcal{L}$-coordinate system of U. $\mathcal{H}$ is defined to be the subspace of $\mathcal{L}(U)$ consisting of those vector fields $X=\sum_{i=1}^n P_i(\partial/\partial x_i)$ such that each P_i is a polynomials in $x_1, x_2, \cdots, x_n$ of at most degree 1. Then $\mathcal{H}$ is a finite dimensional subalgebra of $\mathcal{L}(U)$, and we see easily that

$$(1) \qquad j_p(\mathcal{H}) + j_p(\mathcal{L}^1(U)_p) = j_p(\mathcal{L}(U)) \quad \text{for any } p \in U.$$

Lemma 3. *Let Z be a local vector field around p and assume that for some integer $k \geq 1$ $j_p{}^{k-1}[Z, X]=0$ for all $X \in \mathcal{H}$,* then $j_p{}^k(Z) = 0$.

This lemma follows from (1) and the corresponding facts on the formal algebra of $\mathcal{L}$, and we omit the proof.

Now we consider the following equation for Z,

$$(E'_U) \qquad [Z, X] = \alpha_U(X) \quad \text{for all } X \in \mathcal{H},$$

which is an inhomogeneous partial differential equation of 1-st order.

Proposition 11. *The differential equation (E_U') has a unique smooth solution, and the solution Z satisfies (E_U) and $Z \in \hat{\mathcal{L}}(U)$.*

Proof. First of all we note that (E_U') has a formal solution at any point of U, and further we see by Lemma 3 that the formal solution is unique. From these facts we can conclude that (E_U') has a smooth solution.

We now see it more precisely. Let T be tangent bundle of U and $j^1(T)$ be the 1-st jet bundle of T. Hom $(\mathcal{H}, T)$ is a vector bundle over U, $\mathcal{H}$ being regarded as the trivial bundle over U.

The differential equation (E_U') comes from the following differential operator:

$$\phi : J^1(T) \to \mathrm{Hom}(\mathcal{H}, T),$$

where ϕ is the bundle map defined by

$$\phi_p(j_p{}^1 Z)(X) = [Z, X]_p, \quad X \in \mathcal{H}.$$

From Lemma 3 we have immediately

Lemma 4. *$\phi_p : J^1(T)_p \to \mathrm{Hom}(\mathcal{H}, T)_p$ is injective for all p.*

For the given derivation α, we define a cross-section $\tilde{\alpha}$ of $\mathrm{Hom}(\mathcal{H}, T)$ by $\tilde{\alpha}_p(X) = \alpha(X)_p$ for $p \in U$ and $X \in \mathcal{H}$.

Lemma 5. *Ther exists a smooth cross-section σ of $J^1(T)$ such that $\phi \circ \sigma = \tilde{\alpha}$.*

Since (E_U') has a unique formal solution, such σ exists. The smoothness of σ is ensured by Lemma 4.

Let π_1 be the projection of $J^1(T)$ to T, and put $Z = \pi_1 \circ \sigma$. We claim that Z is a solution of (E_U'). This will follow immediately if we see that $j^1(Z) = \sigma$, and it is equivalent to say that $j^1\sigma \in J^2(T)$, where $J^2(T)$ is regarded as the subbundle of $J^1(J^1(T))$. Let ζ_p be the formal solution of (E_U') at p, and $\zeta_p{}^2$ be the projection of ζ_p to $J_p{}^2(T)$. It will not be difficult to see that $\zeta_p{}^2 = j_p{}^1\sigma$. Hence $j^1\sigma \in J^2(T)$ and we see that Z is a solution of (E_U').

It is clear from the above argument that Z is uniquely determined and that Z satisfies (E_U). The fact that $Z \in \hat{\mathcal{L}}(U)$ follows from the theorem in Chapter I. This completes the proof of Proposition 11.

By Proposition 11 we see that there is one and only one smooth vector field Z on M which satisfies (E). Moreover we see that $Z \in \hat{\mathcal{L}}(M)$.

Thus we have the following theorem.

Theorem. *The derivation algebras $D(\mathcal{L})$ of the global classical infinite Lie algebras $\mathcal{L}$ are as follows:*

i) $D(\mathcal{L})=\mathcal{L}$ *if* $\mathcal{L}=\mathcal{L}_{gl}(M)$, $\mathcal{L}_{ct}(M,\theta)$, $\mathcal{L}_{csl}(M,\Omega)$, *or* $\mathcal{L}_{csp}(M,\omega)$.

ii) $D(\mathcal{L}_{sl}(M,\Omega)=\mathcal{L}_{csl}(M,\Omega)$.

iii) $D(\mathcal{L}_{sp}(M,\omega))=\mathcal{L}_{csp}(M,\omega)$.

As a corollary of the theorem we have

Corollary. *Let* $\mathcal{L}$ *be a global classical infinite Lie algebra, then the* 1-*st cohomology group* $H^1(\mathcal{L},\mathcal{L})$ *of* $\mathcal{L}$ *with adjoint representation is as follows*:

i) $H^1(\mathcal{L},\mathcal{L})=0$ *if* $\mathcal{L}=\mathcal{L}_{gl}(M)$, $\mathcal{L}_{ct}(M,\theta)$, $\mathcal{L}_{csl}(M,\Omega)$, *or* $\mathcal{L}_{csp}(M,\omega)$.

ii) $$H^1(\mathcal{L}_{sl}(M,\Omega),\mathcal{L}_{sl}(M,\Omega))=\begin{cases}\mathbf{R} & \textit{if } \Omega \textit{ is exact}\\ 0 & \textit{if } \Omega \textit{ is not exact}.\end{cases}$$

iii) $$H^1(\mathcal{L}_{sp}(M,\omega),\mathcal{L}_{sp}(M,\omega))=\begin{cases}\mathbf{R} & \textit{if } \omega \textit{ is exact}\\ 0 & \textit{if } \omega \textit{ is not exact}.\end{cases}$$

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Added in Proof.

The same result of the theorem in Chapter II has been obtained by other authors: Prof. A. Lichnerowicz let me know that he had determined the 1-st cohomology groups also for contact and unimodular cases in J. Math. pure et appl., 53 (1974), 459-484, and Ann. Inst. Fourier, Grenoble 24, 3 (1974), 219-266. Recently Y. Kanie has extended his result [3] of hamiltonian case to contact and unimodular cases in Publ. RIMS, Kyoto Univ. 11 (1975), 213-245. Compared with their proofs depending on case by case analysis, our proof seems to be more systematic and simple.