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On a class of hyperbolic mixed problems

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Introduction Let us consider hyperbolic mixed problem:

$$(P)\begin{cases} Au=f & \text{for } (t, x_1,\dots, x_n)\in(0, \infty)\times\Omega, \\ B_j u=0 \quad (j=1,\dots,\mu) & \text{for } (t, x_1,\dots, x_n)\in(0, \infty)\times\partial\Omega, \\ D_t^j u=0 \quad (j=0, 1,\dots, m-1) & \text{for } t=0, (x_1,\dots, x_n)\in\Omega, \end{cases}$$

where A is a hyperbolic differential operator of order m and $\{B_j\}$ are differential operators of orders $\{r_j\}$. We say that the problem (P) is L^2-well posed if there exists $\gamma_0>0$ as follows: let $\gamma\geqslant\gamma_0$ and $e^{-\gamma t}f\in L^2((0, \infty)\times\Omega)$, then there exists a unique solution u of (P) such that

$$\sum_{j=0}^{m-1}\int_0^\infty e^{-2\gamma t}\|D_t^j u(t, \cdot)\|_{m-1-j}^2\, dt$$

$$\leqslant\frac{C}{\gamma^2}\int_0^\infty e^{-2\gamma t}\|f(t, \cdot)\|^2\, dt,$$

where

$$\|\varphi\|_k^2=\sum_{|\nu|\leqslant k}\|D^\nu\varphi\|_{L^2(\Omega)}^2.$$

Under what conditions on $\{A, B_j\}$, dose our problem (P) become L^2-well posed?

Of course, already we know a class satisfying the uniform Lopatinski condition, which assures the problem (P) to be L^2-well posed.

This class contains typically the Dirichlet problem for the vibration equation, but dose not contain the Neumann problem for the vibration equation, although it is L^2-well posed. There are many results to find out a more general class assuring L^2-well posedness, for example, by Agemi ([1]), Miyatake ([7]) in case of second order equations, and by Shirota ([11]), Agemi ([2]) in case of system or higher order equations. In this paper, we shall also give a class assuring L^2-well posedness in a little more general situation. We shall define a matrix $\tilde{S}$ locally in chapter I, which plays an analogous role to the Lopatinski matrix, and we shall consider a class satisfying condition (S) about $\tilde{S}$ in chapter II.

To fix our considerations, we restrict ourselves to the problem (P) specified as follows:

$$\Omega=R_+^n=\{(x, y);\ x>0,\ y\in R^{n-1}\},$$

$$A=A(t, x, y;\ D_t, D_x, D_y)=\sum_{k+l+|\nu|=m} a_{kl\nu}(t, x, y)D_t^k D_x^l D_y^\nu,$$

$$B_j=B_j(t, y;\ D_t, D_x, D_y)=\sum_{k+l+|\nu|=r_j} b_{kl\nu}^j(t, y)D_t^k D_x^l D_y^\nu,$$

$$D_t=-i\frac{\partial}{\partial t},\quad D_x=-i\frac{\partial}{\partial x},\quad D_y=\left(-i\frac{\partial}{\partial y_1},\dots,\ -i\frac{\partial}{\partial y_{n-1}}\right),$$

where we assume

Assumption (A)

i) $a_{kl\nu}(t, x, y)$, $b_{kl\nu}^j(t, y)$ are constant outside a compact set,

ii) A is strictly hyperbolic with respect to t-direction,

iii) $x=0$ is non-characteristic with respect to $\{A, B_j\}$,

iv) $\{B_j\}$ are normal, i.e. $r_i \neq r_j$ if $i\neq j$ and $0\leqslant r_j\leqslant m-1$,

v) A satisfies Agmon's #-condition ([4]), i.e. real roots of $A(t, 0, y;\ \tau, \xi, \eta)=0$ with respect to ξ are simple or double for $(\tau, \eta)\in R^n-\{0\}$.

Chapter I Necessary conditions for L^2-well posedness.

§1. Preliminaries

If a variable coefficient problem is L^2-well posed, then the corresponding constant coefficient problems, whose coefficients are fixed at each point on $x=0$, are also L^2-well posed ([3], [6]). Therefore we consider only of constant coefficient problems in this chapter.

Now let us introduce a supplementary problem for ordinary differential equations with parameters (τ, η):

$$(\hat{P}) \begin{cases} A(\tau, D_x, \eta)v = g & \text{for} \quad x>0, \\ B_j(\tau, D_x, \eta)v = 0 \quad (j=1,\dots,\mu) & \text{for} \quad x=0, \end{cases}$$

then we have

Theorem (**Agemi-Shirota** ([3])) *(P) is L^2-well posed if and only if the uniform estimates for $(\hat{P})$ hold in* $\mathrm{Im}\,\tau<0$, $\eta\in R^{n-1}$, $|\tau|^2+|\eta|^2=1$, *that is*

$$|\mathrm{Im}\,\tau|\,\|v\|_{m-1} \leqslant C\|g\| \qquad \textit{for} \quad \mathrm{Im}\,\tau<0,\ \eta\in R^{n-1},\ |\tau|^2+|\eta|^2=1,$$

where $v\in H^m(R^1_+)$ satisfies $(\hat{P})$ and C is independent not only of v but also independent of (τ, η).

Next we shall introduce the Lopatinski determinant. Since A is hyperbolic, roots ξ of $A(\tau, \xi, \eta)=0$ are non-real for $\mathrm{Im}\,\tau<0$, $\eta\in R^{n-1}$, which we denote by $\xi_1^+(\tau, \eta),\dots, \xi_\mu^+(\tau, \eta)$ with positive imaginary parts and $\xi_1^-(\tau, \eta),\dots, \xi_{m-\mu}^-(\tau, \eta)$ with negative imaginary parts. Of course $\{\xi_j^\pm(\tau, \eta)\}$ have their continuous extensions on $\mathrm{Im}\,\tau=0$, $\eta\in R^{n-1}$. Here we define the Lopatinski determinant by

$$R(\tau, \eta) = \det\left(\frac{1}{2\pi i}\oint \frac{B_j(\tau, \xi, \eta)\xi^{k-1}}{A_+(\tau, \xi, \eta)}d\xi\right)_{j,k=1,\cdots,\mu}$$

for $\mathrm{Im}\,\tau\leqslant 0$ and $\eta\in R^{n-1}$, where

$$A_+(\tau, \xi, \eta) = \prod_{j=1}^{\mu}(\xi-\xi_j^+(\tau, \eta)).$$

Here we have

Theorem (**Hersh**([**5**])) Let (P) be L^2-well posed, then

$$R(\tau, \eta) \neq 0 \quad \textit{for} \quad \operatorname{Im} \tau < 0, \quad \eta \in R^{n-1},$$

which is called Lopatinski's condition or Hersh's condition.

By the well known analysis ([8]), $R(\tau_0, \eta_0) \neq 0$, $((\tau_0, \eta_0) \in R^n)$ assures the uniform estimates for $(\hat{P})$ in a neighbourhood of (τ_0, η_0) in $\operatorname{Im} \tau \leqslant 0$, $\eta \in R^{n-1}$. What means the uniform estimate for $(\hat{P})$ in a neighbourhood of $(\tau_0, \eta_0) \in R^n$, where $R(\tau_0, \eta_0) = 0$? Hereafter, our consideration will be restricted to such a point $X_0 = (\tau_0, \eta_0) \in R^n$ and its neighbourhood U in $C^1 \times R^{n-1}$. We denote

$$U_\pm = U \cap \{\operatorname{Im} \tau \gtrless 0, \ \eta \in R^{n-1}\}.$$

We may think that $\{\xi_1^+(X_0) = \xi_1^-(X_0), \dots, \xi_1^+(X_0) = \xi_d^-(X_0)\}$ are real double roots, $\{\xi_{d+1}^\pm(X_0), \dots, \xi_{d+s_\pm}^\pm(X_0)\}$ are real simple roots, and the others are non-real roots $(m = 2d + s_+ + s_- + 2M$, $\mu = d + s_+ + M)$. We denote for $X = (\tau, \eta) \in U_-$

$$E_\pm(X; \xi) = \prod_{j=1}^{M} (\xi - \xi_{d+s_\pm+j}^\pm(X)),$$

and

$$\begin{cases} P_j^\pm(X, \xi) = \dfrac{A(X, \xi)}{\xi - \xi_j^\pm(X)} & j = 1, \dots, d, \\[2ex] Q_j^\pm(X, \xi) = \dfrac{A(X, \xi)}{\xi - \xi_{d+j}^\pm} & j = 1, \dots, s_\pm, \\[2ex] R_j^\pm(X, \xi) = \xi^{j-1} \dfrac{A(X, \xi)}{E_\pm(X, \xi)} & j = 1, \dots, M. \end{cases}$$

Moreover we denote

$$P^\pm(X, \xi) = \begin{pmatrix} P_1^\pm(X, \xi) \\ \vdots \\ P_d^\pm(X, \xi) \end{pmatrix}, \qquad Q^\pm(X, \xi) = \begin{pmatrix} Q_1^\pm(X, \xi) \\ \vdots \\ Q_{s_\pm}^\pm(X, \xi) \end{pmatrix},$$

$$R^{\pm}(X,\xi)=\begin{pmatrix} R_1^{\pm}(X,\xi) \\ \vdots \\ R_M^{\pm}(X,\xi) \end{pmatrix},$$

and

$$J_{\pm}(X,\xi)=\begin{pmatrix} P^{\pm}(X,\xi) \\ Q^{\pm}(X,\xi) \\ R^{\pm}(X,\xi) \end{pmatrix}, \qquad J(X,\xi)=\begin{pmatrix} J_+(X,\xi) \\ J_-(X,\xi) \end{pmatrix}.$$

Now we consider a linear space L of 1-variable polynomials of degree less than m, in general. Let $A(\xi)$ be a fixed polynomial of degree m. Then we can define bilinear form $<\ ,\ >_A$ in L by

$$<P, Q>_A=\frac{1}{2\pi i}\oint\frac{P(\xi)Q(\xi)}{A(\xi)}d\xi,$$

where the integral is taken along a curve enclosing all the zeros of $A(\xi)$. We say that P and Q are orthogonal with respect to A, if $<P, Q>_A=0$. Let us denote

$$P(\xi)=\sum_{j=1}^{m} p_j\xi^{\ -1}, \qquad Q(\xi)=\sum_{j=1}^{m} q_j\xi^{j-1},$$

then we have

$$<P, Q>_A=(p_1,\dots, p_m)\tilde{A}\begin{pmatrix} q_1 \\ q_m \end{pmatrix},$$

where

$$\tilde{A}=\left(\frac{1}{2\pi i}\oint\frac{\xi^{j+k-2}}{A(\xi)}d\xi\right)_{j,k=1\dots,m}, \quad |\tilde{A}|\neq 0.$$

Moreover let us define

$$<\begin{pmatrix} P_1 \\ \vdots \\ P_s \end{pmatrix}, \begin{pmatrix} Q_1 \\ \vdots \\ Q_t \end{pmatrix}>_A=\begin{pmatrix} <P_1, Q_1>_A\cdots <P_1, Q_t>_A \\ \vdots \qquad\qquad \vdots \\ <P_s, Q_1>_A\cdots <P_s, Q_t>_A \end{pmatrix}$$

$$=\begin{pmatrix} p_{11}\cdots\cdots p_{1m} \\ \vdots \qquad\quad \vdots \\ p_{s1}\cdots\cdots p_{sm} \end{pmatrix}\tilde{A}\begin{pmatrix} q_{11}\cdots\cdots q_{t1} \\ \vdots \qquad\quad \vdots \\ q_{1m}\cdots\cdots q_{tm} \end{pmatrix},$$

where

$$P_i(\xi)=\sum_{j=1}^{m} p_{ij}\xi^{j-1}, \qquad Q_i(\xi)=\sum_{j=1}^{m} q_{ij}\xi^{j-1}.$$

We denote $\mathscr{L}\{P_1,\dots, P_s\}$ the linear sub-space spaned by $\{P_1, \dots, P_s\}$. Then we have

Lemma 1.1.

i) *Let* $\{Q_1,\dots, Q_m\}$ *be a base of* L, *then*

$$\dim \mathscr{L}\{P_1,\dots, P_s\}=\operatorname{rank} <\begin{pmatrix} P_1\\ \vdots\\ P_s\end{pmatrix}, \begin{pmatrix} Q_1\\ \vdots\\ Q_m\end{pmatrix}>_A.$$

ii) *Let* $\{Q_1,\dots, Q_m\}$, $\{Q'_1,\dots, Q'_m\}$ *be two bases, then*

$$\begin{pmatrix} P_1\\ \vdots\\ P_s\end{pmatrix}=<\begin{pmatrix} P_1\\ \vdots\\ P_s\end{pmatrix}, \begin{pmatrix} Q'_1\\ \vdots\\ Q'_m\end{pmatrix}>_A<\begin{pmatrix} Q_1\\ \vdots\\ Q_m\end{pmatrix}, \begin{pmatrix} Q'_1\\ \vdots\\ Q'_m\end{pmatrix}>_A^{-1}\begin{pmatrix} Q_1\\ \vdots\\ Q_m\end{pmatrix}.$$

iii) *Let* $\{Q_1,\dots, Q_m\}$ *be a base of* L, *and let* $\{Q'_{t+1},\dots, Q'_m\}$ *be linearly independent and orthogonal to* $\{Q_1,\dots, Q_t\}$ *with respect to* A, *then*

$$\operatorname{rank} <\begin{pmatrix} P_1\\ \vdots\\ P_s\end{pmatrix}, \begin{pmatrix} Q_1\\ \vdots\\ Q_t\end{pmatrix}>_A=\dim \mathscr{L}\{P_1,\dots, P_s\}-$$

$$-\dim(\mathscr{L}\{P_1,\dots, P_s\}\cap \mathscr{L}\{Q'_{t+1},\dots, Q'_m\}).$$

Proof. ii) Let

$$\begin{pmatrix} P_1\\ \vdots\\ P_s\end{pmatrix}=\mathscr{P}\begin{pmatrix} Q_1\\ \vdots\\ Q_m\end{pmatrix},$$

then we have

$$<\begin{pmatrix} P_1\\ \vdots\\ P_s\end{pmatrix}, \begin{pmatrix} Q'_1\\ \vdots\\ Q'_m\end{pmatrix}>_A=\mathscr{P}<\begin{pmatrix} Q_1\\ \vdots\\ Q_m\end{pmatrix}, \begin{pmatrix} Q'_1\\ \vdots\\ Q'_m\end{pmatrix}>_A,$$

hence

$$\mathscr{P}=<\begin{pmatrix}P_1\\ \vdots\\ P_s\end{pmatrix},\begin{pmatrix}Q'_1\\ \vdots\\ Q'_m\end{pmatrix}>_A<\begin{pmatrix}Q_1\\ \vdots\\ Q_m\end{pmatrix},\begin{pmatrix}Q'_1\\ \vdots\\ Q'_m\end{pmatrix}>_A^{-1}.$$

iii) We have

$$\dim\mathscr{L}\{P_1,\dots,P_s,Q'_{t+1},\dots,Q'_n\}=\operatorname{rank}<\begin{pmatrix}P_1\\ \vdots\\ P_s\\ Q'_{t+1}\\ \vdots\\ Q'_m\end{pmatrix},\begin{pmatrix}Q_1\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ Q_m\end{pmatrix}>_A$$

$$=\operatorname{rank}\begin{pmatrix}<\begin{pmatrix}P_1\\ \vdots\\ P_s\end{pmatrix},\begin{pmatrix}Q_1\\ \vdots\\ Q_t\end{pmatrix}>_A & <\begin{pmatrix}P_1\\ \vdots\\ P_s\end{pmatrix},\begin{pmatrix}Q_{t+1}\\ \vdots\\ Q_m\end{pmatrix}>_A\\ 0 & <\begin{pmatrix}Q'_{t+1}\\ \vdots\\ Q'_m\end{pmatrix},\begin{pmatrix}Q_{t+1}\\ \vdots\\ Q_m\end{pmatrix}>_A\end{pmatrix}$$

$$=\operatorname{rank}<\begin{pmatrix}P_1\\ \vdots\\ P_s\end{pmatrix},\begin{pmatrix}Q_1\\ \vdots\\ P_s\end{pmatrix}>_A+(m-t).$$

On the other hands, we have

$$\dim\mathscr{L}\{P_1,\dots,P_s,Q'_{t+1},\dots,Q'_m\}=\dim\mathscr{L}\{P_1,\dots,P_s\}$$
$$+\dim\mathscr{L}\{Q'_{t+1},\dots,Q'_m\}-\dim(\mathscr{L}\{P_1,\dots,P_s\}\cap\mathscr{L}\{Q'_{t+1},\dots,Q'_m\})$$
$$=\dim\mathscr{L}\{P_1,\dots,P_s\}+(m-t)-\dim(\mathscr{L}\{P_1,\dots,P_s\}\cap\mathscr{L}\{Q'_{t+1},\dots,Q'_m\}).$$

(Q. E. D.)

Now we return to our special case, then $J(X,\xi)$ is a base of L in U_-, and

$$L=\mathscr{L}(P_1^+)\oplus\cdots\oplus\mathscr{L}(P_d^+)\oplus\mathscr{L}(Q_1^+)\oplus\cdots\oplus\mathscr{L}(Q_{s_+}^+)\oplus\mathscr{L}(R^+)$$
$$\oplus\mathscr{L}(P_1^-)\oplus\cdots\oplus\mathscr{L}(P_d^-)\oplus\mathscr{L}(Q_1^-)\oplus\cdots\oplus\mathscr{L}(Q_{s_-}^-)\oplus\mathscr{L}(R^-),$$

where the direct sum is orthogonal with respect to $A(X,\xi)$ in U_-. Now we denote

$$J_{\pm}(X)=<J_{\pm}(X,\xi),\,J_{\pm}(X,\xi)>_{A(X,\xi)},$$

$$J(X)=<J(X,\xi),\,J(X,\xi)>_{A(X,\xi)}=\begin{pmatrix} J_+(X) & \\ & J_-(X)\end{pmatrix},$$

then we have

$$B(X,\xi)=\begin{pmatrix} B_1(X,\xi) \\ \vdots \\ B_\mu(X,\xi)\end{pmatrix}=<B(X,\xi),\,J(X,\xi)>_{A(X,\xi)}$$

$$\times<J(X,\xi),\,J(X,\xi)>^{-1}_{A(X,\xi)}J(X,\xi)$$

$$=<B(X,\xi),\,J_+(X,\xi)>_{A(X,\xi)}J_+(X)^{-1}J_+(X,\xi)$$

$$+<B(X,\xi),\,J_-(X,\xi)>_{A(X,\xi)}J_-(X)^{-1}J_-(X,\xi)$$

$$=B_+(X)J_+(X)^{-1}J_+(X,\xi)+B_-(X)J_-(X)^{-1}J_-(X,\xi).$$

Since

$$B^+(X)=<B(X,\xi),\,J_+(X,\xi)>_{A(X,\xi)}=<B(X,\xi),\,\frac{J_+(X,\xi)}{A_-(X,\xi)}>_{A_+(X,\xi)},$$

and

$$R(X)=|<B(X,\xi),\begin{pmatrix}1\\ \vdots \\ \xi^{\mu-1}\end{pmatrix}>_{A_+(X,\xi)}|,$$

$B_+(X)$ is a non-singular matrix in U_- from Hersh's theorem, hence $\{B(X,\xi),\,J_-(X,\xi)\}$ becomes a base of L in U_-. Therefore we have

$$J_+(X,\xi)=S(X)J_-(X,\xi)+T(X)B(X,\xi)\qquad\text{in}\quad U_-,$$

where

$$\begin{cases} S(X)=-J_+(X)B_+(X)^{-1}B_-(X)J_-(X)^{-1}, \\ T(X)=J_+(X)B_+(X)^{-1}. \end{cases}$$

We denote more precisely

$$\begin{pmatrix} P^+(X, \xi) \\ Q^+(X, \xi) \\ R^+(X, \xi) \end{pmatrix} = \begin{pmatrix} S_{11}(X) & S_{12}(X) & S_{13}(X) \\ S_{21}(X) & S_{22}(X) & S_{23}(X) \\ S_{31}(X) & S_{32}(X) & S_{33}(X) \end{pmatrix} \begin{pmatrix} P^-(X, \xi) \\ Q^-(X, \xi) \\ R^-(X, \xi) \end{pmatrix} + \begin{pmatrix} T_1(X) \\ T_2(X) \\ T_3(X) \end{pmatrix} B(X, \xi),$$

then we have

Theorem (Sakamoto ([9])) *In order that the uniform estimate for* $(\hat{P})$ *holds in* U_-, *it is necessary that*

$$\begin{pmatrix} S_{11}(X) & \gamma^{-\frac{1}{2}}S_{12}(X) & S_{13}(X) \\ S_{21}(X) & S_{22}(X) & \gamma^{\frac{1}{2}}S_{23}(X) \\ \gamma^{\frac{1}{2}}S_{31}(X) & \gamma^{\frac{1}{2}}S_{32}(X) & \gamma S_{33}(X) \end{pmatrix}, \quad \begin{pmatrix} T_1(X) \\ T_2(X) \\ T_3(X) \end{pmatrix}$$

are bounded in I_-, *where*

$$I_- = \{X = (\tau_0 - i\gamma, \eta_0) \in U_-\}.$$

§2. Definitions of $\tilde{S}$ and $\tilde{T}$

In the previous section we used $J(X, \xi)$ as a base of L in U_-, which is no more a base at $X = X_0$ if $d > 0$. We shall choose $\tilde{J}(X, \xi)$ as an another base of L in U in this section.

Now we denote for $X = (\tau, \eta) \in U$

$$H_j(X, \xi) = (\xi - \xi_j^+(X))(\xi - \xi_j^-(X)) = (\xi - \alpha_j(X))^2 - \beta_j(X), \qquad j = 1, \dots, d,$$

then $\alpha_j(X)$, $\beta_j(X)$ are holomorphic in U, real valued if X is real, and $\frac{\partial \beta_j}{\partial \tau}(X) \neq 0$, because of strictly hyperbolicity of A. We denote $\varepsilon_j = \operatorname{sgn} \frac{\partial \beta_j}{\partial \tau}(X_0)$. Let us choose the branch of $\sqrt{\beta_j(X)}$ such that

$$\begin{cases} \varepsilon_j \operatorname{Re}\sqrt{\beta_j(X)} < 0 & \text{if} \quad \beta_j(X) > 0 \quad \text{or} \quad \operatorname{Im}\beta_j(X) \neq 0, \\ \operatorname{Im}\sqrt{\beta_j(X)} \geqslant 0 & \text{if} \quad \beta_j(X) \leqslant 0, \end{cases}$$

then we have

$$\operatorname{Im}\sqrt{\beta_j(X)}\geqslant 0 \quad \text{in} \quad \overline{U_-},$$

and

$$\xi_j^{\pm}(X)=\alpha_j(X)\pm\sqrt{\beta_j(X)} \quad \text{in} \quad \overline{U_-}.$$

Here we denote

$$\begin{cases} P_j(X, \xi)=(\xi-\alpha_j(X))\dfrac{A(X, \xi)}{H_j(X, \xi)}, & j=1,\cdots, d, \\ P'_j(X, \xi)=\dfrac{A(X, \xi)}{H_j(X, \xi)}, & j=1,\cdots, d, \end{cases}$$

and

$$P(X, \xi)=\begin{pmatrix} P_1(X, \xi) \\ \vdots \\ P_d(X, \xi) \end{pmatrix}, \qquad P'(X, \xi)=\begin{pmatrix} P'_1(X, \xi) \\ \vdots \\ P'_d(X, \xi) \end{pmatrix},$$

then we have

$$L=\mathscr{L}\{P_1, P'_d\}\oplus\cdots\oplus\mathscr{L}\{P_d, P'_d\}\oplus\mathscr{L}(Q_1^+)\oplus\cdots\oplus\mathscr{L}(Q_{s_+}^+)\oplus\mathscr{L}(R^+)$$
$$\oplus\mathscr{L}(Q_1^-)\oplus\cdots\oplus\mathscr{L}(Q_{s_-}^-)\oplus\mathscr{L}(R^-)$$

in U, where the direct sum is orthogonal with respect to A in U. Let us denote

$$\sqrt{\beta(X)}=\begin{pmatrix} \sqrt{\beta_1(X)} & & \\ & \ddots & \\ & & \sqrt{\beta_d(X)} \end{pmatrix},$$

then we have

$$\begin{pmatrix} P^+(X, \xi) \\ P^-(X, \xi) \end{pmatrix}=\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} P(X, \xi) \\ \sqrt{\beta(X)}P'(X, \xi) \end{pmatrix} \quad \text{in } U,$$

i.e.

$$2\begin{pmatrix} P(X, \xi) \\ \sqrt{\beta(X)}P'(X, \xi) \end{pmatrix}=\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} P^+(X, \xi) \\ P^-(X, \xi) \end{pmatrix} \quad \text{in } U_-.$$

Since

$$P^+(X,\xi)=S_{11}(X)P^-(X,\xi)+S_{12}(X)Q^-(X,\xi)+S_{13}(X)R^-(X,\xi)$$
$$+T_1(X)B(X,\xi) \quad \text{in} \quad U_-,$$

we have

$$2\begin{pmatrix} P(X,\xi) \\ \sqrt{\beta(X)}P'(X) \end{pmatrix}=\begin{pmatrix} S_{11}(X)+I & S_{12}(X) & S_{13}(X) & T_1(X) \\ S_{11}(X)-I & S_{12}(X) & S_{13}(X) & T_1(X) \end{pmatrix}\begin{pmatrix} P^-(X,\xi) \\ Q^-(X,\xi) \\ R^-(X,\xi) \\ B(X,\xi) \end{pmatrix}.$$

Since we may assume that

$$|S_{11}(X_0)+\Delta|\neq 0, \quad \Delta=\begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}\begin{matrix} \} d_0 \\ \\ \} d-d_0 \end{matrix},$$

we denote

$$\hat{P}(X,\xi)=\begin{pmatrix} P_1(X,\xi) \\ \vdots \\ P_{d_0}(X,\xi) \end{pmatrix}, \qquad \hat{P}'(X,\xi)=\begin{pmatrix} P'_1(X,\xi) \\ \vdots \\ P'_{d_0}(X,\xi) \end{pmatrix},$$

$$\check{P}(X,\xi)=\begin{pmatrix} P_{d_0+1}(X,\xi) \\ \vdots \\ P_d(X,\xi) \end{pmatrix}, \quad \check{P}'(X,\xi)=\begin{pmatrix} P'_{d_0+1}(X,\xi) \\ \vdots \\ P'_d(X,\xi) \end{pmatrix},$$

and

$$\sqrt{\hat{\beta}(X)}=\begin{pmatrix} \sqrt{\beta_1(X)} & & \\ & \ddots & \\ & & \sqrt{\beta_{d_0}(X)} \end{pmatrix}, \quad \sqrt{\check{\beta}(X)}=\begin{pmatrix} \sqrt{\beta_{d_0+1}(X)} & & \\ & \ddots & \\ & & \sqrt{\beta_d(X)} \end{pmatrix},$$

then we have

$$\begin{cases} 2\begin{pmatrix} \sqrt{\hat{\beta}(X)} & \\ & I \end{pmatrix}\begin{pmatrix} \hat{P}'(X,\xi) \\ \check{P}(X,\xi) \end{pmatrix} = (S_{11}(X)+\Delta)P^-(X,\xi)+S_{12}(X)Q^-(X,\xi) \\ \qquad\qquad +S_{13}(X)R^-(X,\xi)+T_1(X)B(X,\xi), \\ 2\begin{pmatrix} I & \\ & -\sqrt{\check{\beta}(X)} \end{pmatrix}\begin{pmatrix} \hat{P}(X,\xi) \\ \check{P}'(X,\xi) \end{pmatrix} = (S_{11}(X)-\Delta)P^-(X,\xi)+S_{12}(X)Q^-(X,\xi) \\ \qquad\qquad +S_{13}(X)R^-(X,\xi)+T_1(X)B(X,\xi), \end{cases}$$

hence we have

$$\left(\begin{array}{cc:c} 2I & & \\ & 2\sqrt{\check{\beta}} & \\ \hdashline & & I \end{array}\right)\begin{pmatrix} \hat{P} \\ \check{P}' \\ Q^+ \\ R^+ \end{pmatrix}$$

$$= \left(\begin{array}{c:ccc} S_{11}-\Delta & S_{12} & S_{13} & T_1 \\ \hdashline S_{21} & S_{22} & S_{23} & T_2 \\ S_{31} & S_{32} & S_{33} & T_3 \end{array}\right)$$

$$\times \left(\begin{array}{c:ccc} (S_{11}+\Delta)^{-1} & -(S_{11}+\Delta)^{-1}S_{12} & -(S_{11}+\Delta)^{-1}S_{13} & -(S_{11}+\Delta)^{-1}T_1 \\ \hdashline & & I & \end{array}\right)$$

$$\times \left(\begin{array}{cc:c} 2\sqrt{\hat{\beta}} & & \\ & 2I & \\ \hdashline & & I \end{array}\right)\begin{pmatrix} \hat{P}' \\ \check{P} \\ Q^- \\ R^- \\ B \end{pmatrix}.$$

Denoting

$$\tilde{J}_+(X,\xi) = \begin{pmatrix} \hat{P}(X,\xi) \\ \check{P}'(X,\xi) \\ Q^+(X,\xi) \\ R^+(X,\xi) \end{pmatrix}, \quad \tilde{J}_-(X,\xi) = \begin{pmatrix} \hat{P}'(X,\xi) \\ \check{P}(X,\xi) \\ Q^-(X,\xi) \\ R^-(X,\xi) \end{pmatrix}, \quad \tilde{J}(X.\xi) = \begin{pmatrix} \tilde{J}_+(X,\xi) \\ \tilde{J}_-(X,\xi) \end{pmatrix},$$

we have

$$\tilde{J}_+(X, \xi)=\tilde{S}(X)\tilde{J}_-(X, \xi)+\tilde{T}(X)B(X, \xi) \qquad \text{in} \quad U_-,$$

or more precisely

$$\begin{pmatrix} \hat{P}(X, \xi) \\ \check{P}'(X, \xi) \\ Q^+(X, \xi) \\ R^+(X, \xi) \end{pmatrix} = \begin{pmatrix} \tilde{S}_{00}(X)\tilde{S}_{01}(X)\tilde{S}_{02}(X)\tilde{S}_{03}(X) \\ \tilde{S}_{10}(X)\tilde{S}_{11}(X)\tilde{S}_{12}(X)\tilde{S}_{13}(X) \\ \tilde{S}_{20}(X)\tilde{S}_{21}(X)\tilde{S}_{22}(X)\tilde{S}_{23}(X) \\ \tilde{S}_{30}(X)\tilde{S}_{31}(X)\tilde{S}_{32}(X)\tilde{S}_{33}(X) \end{pmatrix} \begin{pmatrix} \hat{P}'(X, \xi) \\ \check{P}(X, \xi) \\ Q^-(X, \xi) \\ R^-(X, \xi) \end{pmatrix}$$

$$+\begin{pmatrix} \vec{T}_0(X) \\ \tilde{T}_1(X) \\ \tilde{T}_2(X) \\ \tilde{T}_3(X) \end{pmatrix} B(X, \xi) \qquad \text{in} \quad U_-.$$

On the other hands, we denote

$$c_j(X)=\frac{<P'_j(X, \xi),\ P'_j\ (X, \xi)>_{A(X,\xi)}}{<P_j(X, \xi),\ P'_j\ (X, \xi)>_{A(X,\xi)}},$$

$$P_{j0}(X, \xi)=P_j(X, \xi)-\beta_j(X)c_j(X)P'_j(X, \xi),$$

$$P'_{j0}(X, \xi)=P'_j(X, \xi)-c_j(X)P_j(X, \xi),$$

and

$$\tilde{J}_{+0}(X, \xi)=\begin{pmatrix} \hat{P}'_0(X, \xi) \\ \check{P}_0(X, \xi) \\ Q^+(X, \xi) \\ R^+(X, \xi) \end{pmatrix}, \quad \tilde{J}_{-0}(X, \xi)=\begin{pmatrix} \hat{P}_0(X, \xi) \\ \check{P}'_0(X, \xi) \\ Q^-(X, \xi) \\ R^-(X, \xi) \end{pmatrix},$$

$$\tilde{J}_0(X, \xi)=\begin{pmatrix} \tilde{J}_{+0}(X, \xi) \\ \tilde{J}_{-0}(X, \xi) \end{pmatrix},$$

then we have

$$<\tilde{J}_{\pm}(X, \xi), \tilde{J}_{\mp 0}(X, \xi)>_{A(X,\xi)}=0 \quad \text{in} \quad U.$$

Since $\tilde{J}(X, \xi)$ and $\tilde{J}_0(X, \xi)$ are two bases of L in U, we have from lemma 1.1

$$\begin{aligned} B(X, \xi) &= <B(X, \xi), \tilde{J}_{+0}(X, \xi)>_{A(X,\xi)}<\tilde{J}_+(X, \xi), \tilde{J}_{+0}(X, \xi)>^{-1}_{A(X,\xi)} \\ &\qquad \times \tilde{J}_+(X, \xi) \\ &\quad + <B(X, \xi), \tilde{J}_{-0}(X, \xi)>_{A(X,\xi)}<\tilde{J}_-(X, \xi), \tilde{J}_{-0}(X, \xi)>^{-1}_{A(X,\xi)}\cdot \tilde{J}_-(X, \xi) \\ &= \tilde{B}_+(X)\tilde{J}_+(X)^{-1}\tilde{J}_+(X, \xi)+\tilde{B}_-(X)\tilde{J}_-(X)^{-1}\tilde{J}_-(X, \xi). \end{aligned}$$

Here we have

$$\begin{cases} \tilde{S}(X)=-\tilde{J}_+(X)\tilde{B}_+(X)^{-1}\tilde{B}_-(X)\tilde{J}_-(X)^{-1} \\ \tilde{T}(X)=\tilde{J}_+(X)\tilde{B}_+(X)^{-1} \end{cases} \quad \text{in} \quad U_-,$$

where the second terms are rational functions in U. Hence we have from the last theorem in § 1

Lemma 1.2. *In order that the uniform estimate for* $(\hat{P})$ *holds in* U_-, *it is necessary that*

$$\begin{pmatrix} \gamma^{-1}\tilde{S}_{00}(X) & \tilde{S}_{01}(X) & \gamma^{-1}\tilde{S}_{02}(X) & \tilde{S}_{03}(X) \\ \tilde{S}_{10}(X) & \tilde{S}_{11}(X) & \tilde{S}_{12}(X) & \tilde{S}_{13}(X) \\ \gamma^{-1}\tilde{S}_{20}(X) & \tilde{S}_{21}(X) & \tilde{S}_{22}(X) & \tilde{S}_{23}(X) \\ \tilde{S}_{30}(X) & \tilde{S}_{31}(X) & \tilde{S}_{32}(X) & \gamma\tilde{S}_{33}(X) \end{pmatrix}, \quad \begin{pmatrix} \tilde{T}_0(X) \\ \tilde{T}_1(X) \\ \tilde{T}_2(X) \\ \gamma\tilde{T}_3(X) \end{pmatrix}$$

are bounded on l_-.

Now we define

Condition (N):

i) there exist $\lim_{l_-\ni X\to X_0} \tilde{S}_{ij}(X)=\tilde{S}_{ij}(X_0)$ for $(i, j)\neq(3, 3)$, and

$$\tilde{S}_{00}(X_0)=\tilde{S}_{02}(X_0)=\tilde{S}_{20}(X_0)=0,$$

ii) there exists $\lim_{I_- \ni X \to X_0} \tilde{S}_{33}(X).(\tau - \sigma_0)$.

Then we have from lemma 1.2

Theorem I *Condition* (*N*) *is a necessary condition for* (*P*) *to be L^2-well posed.*

Let

$$\dim\{\mathscr{L}(B(X_0, \xi)) \cap \mathscr{L}(R^-(X_0, \xi))\} = M_0,$$

then we may assume that

$$\hat{R}^-(X, \xi) = \begin{pmatrix} R_1^-(X, \xi) \\ \vdots \\ R_{M_0}^-(X, \xi) \end{pmatrix}, \qquad \check{R}^-(X, \xi) = \begin{pmatrix} R_{M_0+1}^-(X, \xi) \\ \vdots \\ R_M^-(X, \xi) \end{pmatrix},$$

and

$$\hat{R}^-(X_0, \xi) \in \mathscr{L}\{B(X_0, \xi), \check{R}^-(X_0, \xi)\}.$$

Moreover, we may assume that

$$\hat{R}^+(X, \xi) = \begin{pmatrix} R_1^+(X, \xi) \\ \vdots \\ R_{M_0}^+(X\xi) \end{pmatrix}, \qquad \check{R}^+(X, \xi) = \begin{pmatrix} R_{M_0+1}^+(X, \xi) \\ \vdots \\ R_M^+(X, \xi) \end{pmatrix},$$

and

$$\dim \mathscr{L}\{B(X_0, \xi), \check{R}^-(X_0, \xi), \check{P}(X_0, \xi), Q^-(X_0, \xi), \hat{P}'(X_0, \xi), \hat{R}^+(X_0, \xi)\} = m.$$

Now we denote

$$\tilde{\tilde{J}}_+(X, \xi) = \begin{pmatrix} \hat{P}(X, \xi) \\ \check{P}'(X, \xi) \\ Q^+(X, \xi) \\ \check{R}^+(X, \xi) \\ \hat{R}^-(X, \xi) \end{pmatrix}, \qquad \tilde{\tilde{J}}_-(X, \xi) = \begin{pmatrix} \hat{P}'(X, \xi) \\ \check{P}(X, \xi) \\ Q^-(X, \xi) \\ \check{R}^-(X, \xi) \\ \hat{R}^+(X, \xi) \end{pmatrix},$$

then we have

$$\tilde{\tilde{J}}_+(X,\xi)=\tilde{\tilde{S}}(X)\tilde{\tilde{J}}_-(X,\xi)+\tilde{\tilde{T}}(X)B(X,\xi)\qquad \text{in}\quad U,$$

or more precisely

$$\begin{pmatrix}\hat{P}\\ \check{P}'\\ Q^+\\ \check{R}^+\\ \hat{R}^-\end{pmatrix}=\begin{pmatrix}\tilde{\tilde{S}}_{00}&\tilde{\tilde{S}}_{01}&\tilde{\tilde{S}}_{02}&\tilde{\tilde{S}}_{03}&\tilde{\tilde{S}}_{04}\\ \tilde{\tilde{S}}_{10}&\tilde{\tilde{S}}_{11}&\tilde{\tilde{S}}_{12}&\tilde{\tilde{S}}_{13}&\tilde{\tilde{S}}_{14}\\ \tilde{\tilde{S}}_{20}&\tilde{\tilde{S}}_{21}&\tilde{\tilde{S}}_{22}&\tilde{\tilde{S}}_{23}&\tilde{\tilde{S}}_{24}\\ \tilde{\tilde{S}}_{30}&\tilde{\tilde{S}}_{31}&\tilde{\tilde{S}}_{32}&\tilde{\tilde{S}}_{33}&\tilde{\tilde{S}}_{34}\\ \tilde{\tilde{S}}_{40}&\tilde{\tilde{S}}_{41}&\tilde{\tilde{S}}_{42}&\tilde{\tilde{S}}_{43}&\tilde{\tilde{S}}_{44}\end{pmatrix}\begin{pmatrix}\hat{P}'\\ \check{P}\\ Q^-\\ \check{R}^-\\ \hat{R}^+\end{pmatrix}+\begin{pmatrix}\tilde{\tilde{T}}_0\\ \tilde{\tilde{T}}_1\\ \tilde{\tilde{T}}_2\\ \tilde{\tilde{T}}_3\\ \tilde{\tilde{T}}_4\end{pmatrix}B\qquad \text{in } U,$$

where $\tilde{\tilde{S}}_{ij}(X)$, $\tilde{\tilde{T}}_i(X)$ are smooth in U, and

$$\tilde{\tilde{S}}_{40}(X_0)=\tilde{\tilde{S}}_{41}(X_0)=\tilde{\tilde{S}}_{42}(X_0)=\tilde{\tilde{S}}_{44}(X_0)=0.$$

Here we remark that

$$\tilde{S}_{33}=\begin{pmatrix}\tilde{\tilde{S}}_{33}-\tilde{\tilde{S}}_{34}\tilde{\tilde{S}}_{44}^{-1}\tilde{\tilde{S}}_{43} & \tilde{\tilde{S}}_{34}\tilde{\tilde{S}}_{44}^{-1}\\ -\tilde{\tilde{S}}_{44}^{-1}\tilde{\tilde{S}}_{43} & \tilde{\tilde{S}}_{44}^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}I&\tilde{\tilde{S}}_{34}\\ 0&I\end{pmatrix}\begin{pmatrix}I&0\\ 0&\tilde{\tilde{S}}_{44}^{-1}\end{pmatrix}\begin{pmatrix}I&0\\ -\tilde{\tilde{S}}_{34}&I\end{pmatrix}+\begin{pmatrix}-I+\tilde{\tilde{S}}_{33}&0\\ 0&0\end{pmatrix},$$

$$\tilde{T}_3=\begin{pmatrix}\tilde{\tilde{T}}_3-\tilde{\tilde{S}}_{34}\tilde{\tilde{S}}_{44}^{-1}\tilde{\tilde{T}}_4\\ -\tilde{\tilde{S}}_{44}^{-1}\tilde{\tilde{T}}_4\end{pmatrix}=\begin{pmatrix}I&\tilde{\tilde{S}}_{34}\\ 0&I\end{pmatrix}\begin{pmatrix}I&0\\ 0&\tilde{\tilde{S}}_{44}^{-1}\end{pmatrix}\begin{pmatrix}\tilde{\tilde{T}}_3\\ -\tilde{\tilde{T}}_4\end{pmatrix}.$$

Remark. Condition (N)–i) is equivalent to each one of the following i)′∼i)‴:

$$\text{i)}'\begin{cases}\dim\mathscr{L}\{B,R^-,P\}=\dim\mathscr{L}\{B,\check{R}^-,\check{P}\}=\mu+M-M_0+d-d_0 \\ \hfill \text{at } X=X_0,\\ \dim\mathscr{L}\{B,R^-,P,Q^-,Q^+\}=\dim\mathscr{L}\{B,\check{R}^-,\check{P},Q^-\}\\ \qquad =\mu+M-M_0+d-d_0+s_- \quad \text{at } X=X_0,\\ \dim\mathscr{L}\{B,R^-,P,Q^-,Q^+,P'\}=\dim\mathscr{L}\{B,\check{R}^-,\check{P},Q^-,\hat{P}'\}\\ \qquad =m-M_0 \quad \text{at } X=X_0,\end{cases}$$

$$
\text{i)}'' \begin{cases}
\dim\{\mathscr{L}(B)\cap\mathscr{L}(R^-,P)\}=\dim\{\mathscr{L}(B)\cap\mathscr{L}(R^-,P,Q^-)\} \\
\qquad\qquad =M_0+d_0 \quad \text{at } X=X_0, \\
\dim\{\mathscr{L}(B)\cap\mathscr{L}(R^-,P,Q^-,Q^+)\}=M_0+d_0+s_+ \quad \text{at } X=X_0, \\
\dim\{\mathscr{L}(B)\cap\mathscr{L}(R^-,P,Q^-,Q^+,P')\}=M_0+d+s_+ \\
\qquad\qquad \text{at } X=X_0,
\end{cases}
$$

$$
\text{i)}''' \begin{cases}
\operatorname{rank}<B,\begin{pmatrix}P\\Q^+\\Q^-\\R^+\end{pmatrix}>_A=\operatorname{rank}<B,\begin{pmatrix}P\\Q^+\\R^+\end{pmatrix}>_A=\mu-M_0-d_0 \quad \text{at } X=X_0, \\
\operatorname{rank}<B,\begin{pmatrix}P\\R^+\end{pmatrix}>_A=\mu-M_0-d_0-s_+ \quad \text{at } X=X_0, \\
\operatorname{rank}<B,R^+>_A=\mu-M_0-d-s_+=M-M_0 \quad \text{at } X=X_0.
\end{cases}
$$

Now we denote

$$\tilde{R}(X)=|\tilde{B}_+(X)|,$$

then we have

$$\tilde{R}=|\tilde{B}_+|=|<B,\tilde{J}_{+0}>_A|=|<\tilde{J}_-,\tilde{J}_{-0}>_A|^{-1}|<\begin{pmatrix}B\\ \tilde{J}_-\end{pmatrix},\begin{pmatrix}\tilde{J}_{+0}\\ \tilde{J}_{-0}\end{pmatrix}>_A|$$

$$=|<\tilde{J}_-,\tilde{J}_{-0}>_A|^{-1}|\tilde{\tilde{\tilde{S}}}_{44}||<\begin{pmatrix}B\\ \tilde{\tilde{\tilde{J}}}_-\end{pmatrix},\begin{pmatrix}\tilde{J}_{+0}\\ \tilde{J}_{-0}\end{pmatrix}>_A|,$$

hence we have

$$\tilde{R}(X)=c(X)|\tilde{\tilde{\tilde{S}}}_{44}(X)| \qquad (c(X)\neq 0).$$

Lemma 1.3. *Condition* (N) *is equivalent to condition* $(N)'$: *it holds at* $X=X_0$

$$0)\quad \operatorname{rank}<B,\begin{pmatrix}P\\P'\\Q^+\\Q^-\\R^+\end{pmatrix}>_A=\mu-M_0,$$

i) $\operatorname{rank} < B, \begin{pmatrix} P \\ Q^+ \\ Q^- \\ R^+ \end{pmatrix} >_A = = \mu - M_0 - d_0,$

ii) $\operatorname{rank} < B, \begin{pmatrix} P \\ R^+ \end{pmatrix} >_A = \mu - M_0 - d_0 - s_+,$

iii) $\operatorname{rank} < B, R^+ >_A = M - M_0,$

iv) $\tilde{R} = \frac{\partial}{\partial \tau} \tilde{R} = \cdots = \left(\frac{\partial}{\partial \tau}\right)^{M_0 - 1} \tilde{R} = 0$ and $\left(\frac{\partial}{\partial \tau}\right)^{M_0} \tilde{R} \neq 0.$

Finally, we consider of $\tilde{S}_{00}$. Since

$$\begin{pmatrix} \hat{P}_- \\ \check{P}_- \end{pmatrix} = \begin{pmatrix} \hat{P} \\ \check{P} \end{pmatrix} - \sqrt{\beta} \begin{pmatrix} \hat{P}' \\ \check{P}' \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{S}_{00} - \sqrt{\hat{\beta}} & \tilde{S}_{01} & \tilde{S}_{02} & \tilde{S}_{03} & \tilde{T}_0 \\ -\sqrt{\check{\beta}\tilde{S}_{10}} & I - \sqrt{\check{\beta}\tilde{S}_{11}} & -\sqrt{\check{\beta}\tilde{S}_{12}} & -\sqrt{\check{\beta}\tilde{S}_{13}} & -\sqrt{\check{\beta}\tilde{T}_1} \end{pmatrix} \begin{pmatrix} \hat{P}' \\ \check{P} \\ Q^- \\ R^- \\ B \end{pmatrix},$$

we have

$$|B_+| = | < B, J_+ >_A | = | < J_-, J'_- >_A |^{-1} \left| < \begin{pmatrix} B \\ J_- \end{pmatrix}, \begin{pmatrix} J_+ \\ J'_- \end{pmatrix} >_A \right|$$

$$= | < J_-, J'_- >_A |^{-1} \begin{vmatrix} \tilde{S}_{00} - \sqrt{\hat{\beta}} & \tilde{S}_{01} \\ -\sqrt{\check{B}\tilde{S}_{10}} & I - \sqrt{\check{\beta}\tilde{S}_{11}} \end{vmatrix} \left| < \begin{pmatrix} B \\ \check{J}_- \end{pmatrix}, \begin{pmatrix} J_+ \\ J'_1 \end{pmatrix} >_A \right|,$$

where

$$J'_- = \begin{pmatrix} P' \\ Q^- \\ R^- \end{pmatrix},$$

hence we have

$$|B_+| = c|H|\,|\tilde{B}_+|,$$

where

$$H = \begin{pmatrix} \tilde{S}_{00} - \sqrt{\beta} & \tilde{S}_{01} \\ -\sqrt{\check{\beta}}\tilde{S}_{10} & I - \sqrt{\check{\beta}}\tilde{S}_{11} \end{pmatrix}, \qquad c \neq 0.$$

Now we denote diagonal elements of $\tilde{S}_{00}$ by

$$\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_{d_0} \end{pmatrix},$$

and denote $\tilde{S}_{00}^{(0)} = \tilde{S}_{00} - \delta$, then we have

Lemma 1.4. *Let (P) be L^2-well posed. If we assume that $|B_+| \neq 0$ in U_+ and j-th row of $(\tilde{S}_{00}^{(0)}\ \tilde{S}_{01})$ is zero on $\delta_j - \sqrt{\beta_j} = 0$, then*

$$\delta_j - \sqrt{\beta_j} \neq 0 \quad in \quad U \cap \{\operatorname{Im}\tau \neq 0 \quad or \quad \beta_j > 0\} \quad (j = 1, \dots, d_0),$$

that is,

$$(*) \qquad \varepsilon_j \operatorname{Re}\delta_j \geqslant 0 \quad on \quad z_j \equiv \beta_j - \delta_j^2 = 0 \quad (j = 1, \dots, d_0).$$

Proof. If $\delta_j - \sqrt{\beta_j} = 0$, then $|H| = 0$, therefore $|B_+| = 0$. Hence we have

$$\delta_j - \sqrt{\beta_j} \neq 0 \quad \text{in} \quad U_+ \cup U_-.$$

Moreover, we have from condition (N)

$$P_j^- \notin \mathscr{L}\{Q^-, R^-, B\} \quad \text{for} \quad \beta_j > 0,$$

that is, j-th row of H is non-zero, hence we have

$$\delta_j - \sqrt{\beta_j} \neq 0 \quad \text{for} \quad \beta_j > 0. \qquad (Q.E.D)$$

Lemma 1.5. *(*) is equivalent to*

$$(*)' \qquad \varepsilon_j \operatorname{Re}\delta_j \geqslant 0 \quad on \quad v_j \equiv \beta_j + (\operatorname{Im}\delta_j)^2 = 0.$$

Proof. Since

$$z_j=\beta_j-\delta_j^2=\nu_j-\{(\mathrm{Re}\,\delta_j)^2+2i\,\mathrm{Re}\,\delta_j\,\mathrm{Im}\,\delta_j\},$$

we have

$$\delta_j=\delta_j|_{z_j=0}+\tilde{\delta}_j z_j$$

and

$$\mathrm{Re}\,\delta_j=\mathrm{Re}\,\delta_j|_{z_j=0}+\mathrm{Re}(\tilde{\delta}_j\nu_j)-\mathrm{Re}\,\tilde{\delta}_j(\mathrm{Re}\,\delta_j)^2+2\,\mathrm{Im}\,\tilde{\delta}_j\,\mathrm{Re}\,\delta_j\,\mathrm{Im}\,\delta_j,$$

therefore

$$\mathrm{Re}\,\delta_j\{1+\mathrm{Re}\,\tilde{\delta}_j\,\mathrm{Re}\,\delta_j-2\,\mathrm{Im}\,\tilde{\delta}_j\,\mathrm{Im}\,\delta_j\}=\mathrm{Re}\,\delta_j|_{z_j=0}+\mathrm{Re}(\tilde{\delta}_j\nu_j),$$

which implies the equivalence of $(*)$ and $(*)'$. (Q.E.D.)

Remark. If $M=0$ at $X=X_0$, then $|B_+(X)|=(-1)^{r_1+\cdots+r_\mu}|B_+(-X)|$, therefore we have $|B_+(X)|\neq 0$ in U_+ from Hersh's condition.

Chapter II Sufficient conditions for L^2-well posedness

§1. Condition (S)

Hereafter, we deal with variable coefficient problems. Hence let us denote $X=(t, y; \tau, \eta)$ and let U be a neighbourhood of $X_0=(t_0, y_0; \tau_0, \eta_0)$ in $R^n\times(C^1\times R^{n-1})$. Moreover, let $\tilde{X}=(x, t, y; \tau, \eta)$ and let $\tilde{U}$ be a neighbourhood of $\tilde{X}_0=(0, X_0)$ in $\overline{R_+^{n+1}}\times(C^1\times R^{n-1})$.

Now, let us consider the matrix $\tilde{S}(X)$ in U, which is defined in chapter I such that

$$\tilde{J}_+(X, \xi)=\tilde{S}(X)\tilde{J}_-(X, \xi)+\tilde{T}(X)B(X, \xi),$$

more precisely

$$\begin{pmatrix} \hat{P}(X, \xi) \\ \check{P}'(X, \xi) \\ Q^+(X, \xi) \\ R^+(X, \xi) \end{pmatrix}=\begin{pmatrix} \tilde{S}_{00}(X) & \tilde{S}_{01}(X) & \tilde{S}_{02}(X) & \tilde{S}_{03}(X) \\ \tilde{S}_{10}(X) & \tilde{S}_{11}(X) & \tilde{S}_{12}(X) & \tilde{S}_{13}(X) \\ \tilde{S}_{20}(X) & \tilde{S}_{21}(X) & \tilde{S}_{22}(X) & \tilde{S}_{23}(X) \\ \tilde{S}_{30}(X) & \tilde{S}_{31}(X) & \tilde{S}_{32}(X) & \tilde{S}_{33}(X) \end{pmatrix}\begin{pmatrix} \hat{P}'(X, \xi) \\ \check{P}(X, \xi) \\ Q^-(X, \xi) \\ R^-(X, \xi) \end{pmatrix}$$

$$+\begin{pmatrix} \tilde{T}_0(X) \\ \tilde{T}_1(X) \\ \tilde{T}_2(X) \\ \tilde{T}_3(X) \end{pmatrix} B(X, \xi).$$

Let

$$\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_{d_0} \end{pmatrix}$$

be diagonal elements of $\tilde{S}_{00}$, and

$$\nu = \begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_{d_0} \end{pmatrix}, \qquad \nu_j = \beta_j + (\operatorname{Im} \delta_j)^2 ,$$

$$\rho = \begin{pmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_{d_0} \end{pmatrix}, \qquad \rho_j = \varepsilon_j \{\operatorname{Re} \delta_j\}_{\nu_j = 0} .$$

Let us assume $\rho \geqslant 0$. In general, let $A(X)$ be a $k \times l$-matrix, then we say that A is bounded with respect to (ν, ρ), if there exists a smooth decomposition of A in $U_0 = U \cap \{\operatorname{Im} \tau = 0\}$ such that

$$A = \begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix},$$

$$\begin{cases} A_0 = A_{00} + A_{01}\nu + \nu A_{02} + \nu A_{03}\nu, \\ A_1 = A_{11} + \nu A_{13}, \\ A_2 = A_{22} + A_{23}\nu, \end{cases}$$

where A_0 is a $d_0 \times d_0$-matrix and

$$|||A|||_{U_0} = \sup_{U_0} \|\rho^{-\frac{1}{2}} A_{00} \rho^{-\frac{1}{2}}\| + \sup_{U_0} \|\rho^{-\frac{1}{2}} (A_{01} A_{11})\|$$

$$+ \sup_{U_0} \left\| \begin{pmatrix} A_{02} \\ A_{22} \end{pmatrix} \rho^{-\frac{1}{2}} \right\| + \sup_{U_0} \left\| \begin{pmatrix} A_{03} & A_{13} \\ A_{23} & A_3 \end{pmatrix} \right\| < +\infty (\| \ \|: \text{matrix norm}).$$

Now we denote $|||A|||_U=|||A|||_{U_0}+\sup_U\{|\mathrm{Im}\,\tau|^{-1}\|A-A_{\mathrm{Im}\,\tau=0}\|\}$, then we say that A is small with respect to (ν, ρ), if A is bounded with respect to (ν, ρ) and $|||A|||_U\to 0$ as $\mathrm{diam}(U)\to 0$. Moreover we say that A is positive with respect to (ν, ρ), if A is hermitian and bounded with respect to (ν, ρ) and $A_{00}\geqslant c\rho\,(c>0)$.

Remark.

i) Let D be diagonal, then AD and DA are bounded (resp. small) with respect to (ν, ρ), if A is bounded (resp. small) with respect to (ν, ρ). Especially when $D(X_0)=0$, AD and DA are small with respect to (ν, ρ), if A is bounded with respect to (ν, ρ).

ii) Let A_1 and A_2 are bounded with respect to (ν, ρ), then A_1A_2 is also bounded with respect to (ν, ρ). Especially when A_1 is $k\times d_0$-matrix and A_2 is $d_0\times l$-matrix, then A_1A_2 becomes small with respect to (ν, ρ).

Now we define condition (S):

Condition (S.1) $R(X)\neq 0$ for $X=(t, y; \tau, \eta)\in R^n\times(C^1\times R^{n-1})\cap\{\mathrm{Im}\,\tau<0\}$.

Condition (S.2) Let X_0 be real, then there exists a neighbourhood U of X_0, where

i) $$\tilde S_{33}(X)=C_1(X)\varphi(X)^{-1}C_2(X)+C_0(X),$$

where $C_0(X)$, $C_1(X)$, $C_2(X)$, $\varphi(X)$ are smooth, $|C_1(X)|\neq 0$, $|C_2(X)|\neq 0$, and

$$\varphi(X)=\begin{pmatrix} 1 \ddots 1 \;\; M-M_0 & \\ & \varphi_1(X) \\ & & \ddots \\ & & & \varphi_{M_0}(X)\end{pmatrix},\quad \varphi_j(X)=\tau-\tau_j(t, y:\eta),\ \ \mathrm{Im}\,\tau_j\geqslant 0,$$

ii) $$\begin{pmatrix} \tilde S_{00} & \tilde S_{01} & \tilde S_{02} & \tilde S_{03} \\ \tilde S_{10} & S_{11} & \tilde S_{12} & \tilde S_{13} \\ \tilde S_{20} & \tilde S_{21} & \tilde S_{22} & \tilde S_{23} \\ \tilde S_{30} & \tilde S_{31} & \tilde S_{32} & 0 \end{pmatrix}$$ are smooth,

iii) $\rho \geqslant 0$,

iv) there exists a smooth positive diagonal matrix

$$\psi = \begin{pmatrix} \psi_1 & & \\ & \ddots & \\ & & \psi_{d_0} \end{pmatrix}$$

such that the hermitian part of $\psi \in \tilde{S}_{00}$ is positive with respect to (ν, ρ),

v) $$\begin{bmatrix} \tilde{S}_{00}^{(0)} & \tilde{S}_{01} & \tilde{S}_{02} \\ \tilde{S}_{10} & \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{20} & \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix}$$

is bounded with respect to (ν, ρ), where $\tilde{S}_{00}^{(0)} = \tilde{S}_{00} - \delta$.

Our aim in this chapter is to obtain

Theorem II *Condition(S) is a sufficient condition for the problem(P) to be L^2-well posed.*

Now we state some notations used in the following. If $e^{-\gamma t}u \in H^m(R_+^{n+1})$, where $(t, x, y) \in R^1 \times R_+^1 \times R^{n-1} = R_+^{n+1}$, then we say that $u \in \mathscr{H}_\gamma^m(R_+^{n+1})$, where inner products and norms are given by

$$\begin{aligned} &(u, v)_{m,\gamma} \\ &= \sum_{j+k+|\nu|<m} (e^{-\gamma t} D_t^j D_x^k D_y^\nu u(t, x, y),\ e^{-\gamma t} D_t^j\ D_x^k D_y^\nu v(t, x, y))_{L^2(R_+^{n+1})} \\ &= \sum_{j+k+|\nu|<m} ((D_t - i\gamma)^j D_x^k D_y^\nu (e^{-\gamma t} u),\ (D_t - i\gamma)^j D_x^k D_y^\nu (e^{-\gamma t} v))_{L^2(R_+^{n+1})}. \end{aligned}$$

$$|u|_{m,\gamma}^2 = (u, u)_{m,\gamma},\ (u, v)_\gamma = (u, v)_{0,\gamma},\ |u|_\gamma = |u|_{0,\gamma}.$$

Next, we define by $a(t, y; D_t', D_y, \gamma)$ a pseudo-differential operator in R^n with symbol $a(t, y; \sigma, \eta, \gamma)$:

$$a(t, y; D_t', D_y, \gamma)u(t, y) = e^{\gamma t} F_{t,y}^{-1}[a(t, y; \sigma, \eta, \gamma) F_{\sigma,\eta}[e^{-\gamma t} u(t, y)]],$$

where F is the Fourier transform and γ is a parameter.
We say that $a(t, y; D_t', D_y, \gamma)$ is of order s if $a(t, y; \sigma, \eta, \gamma)$ is homogeneous of degree s with respect to (σ, η, γ). We denote

$$\wedge(\sigma, \eta, \gamma)=\sigma^2+|\eta|^2+\gamma^2,$$

and we say that $u\in\mathscr{H}_\gamma^s(R^n)$ if $e^{-\gamma t}\wedge^s u\in L^2(R^n)$, where

$$<u, v>_{s,\gamma}=(e^{-\gamma t}\wedge^s u, e^{-\gamma t}\wedge^s u)_{L^2(R^n)}, \quad <u>_{s,\gamma}^2=<u, u>_{s,\gamma},$$

$$<u, v>_\gamma=<u, v>_{0,\gamma}, \qquad <u>_\gamma=<u>_{0,\gamma}.$$

Moreover, we define pseudo-differential operators in $\overline{R_+^{n+1}}$ as differential operators of 1-variable x with coefficients of pseudo-differential operators in R^n:

$$P(t, x, y; D_t', D_x, D_y, \gamma)=a_0(t, x, y; D_t', D_y, \gamma)D_x^m$$
$$+a_1(t, x, y; D_t', D_y, \gamma)D_x^{m-1}+\cdots+a_m(t, x, y; D_t', D_y, \gamma),$$

where $a_j(t, x, y; D_t', D_y, \gamma)$ are pseudo-differential operators in R^n with parameter x. We say that P is of order s, if a_j are of orders $s-m+j$. If $a(X)$ is defined in U, which is homogeneous with respect to (σ, η, γ) of degree s there, then it can be extended to be a symbol of a pseudo-differential operator of order s. Moreover in our case, multiplying by $\wedge^s$ if necessary, we regard that pseudo-differential operators in $\overline{R_+^{n+1}}$, such as $\tilde{J}_+$, B, are of order $m-1$, and pseudo-differential operators in R^n, such as $\alpha, \beta, \nu, \rho, \varphi,\dots$, are of order 0. Hereafter, we assume that the functions, on which pseudo-differential operators act, are restricted to the following ones:

$$u(t, x, y)=a(t, x, y; D_t', D_y, \gamma)v(t, x, y),$$

where the support of $a(\tilde{X})$ belongs to a small neighbourhood of $\tilde{X}_0$. We denote by μ positive numbers such that

$$\mu\longrightarrow 0 \quad \text{as} \quad \operatorname{diam}(\tilde{U})\longrightarrow 0.$$

Now we denote for $U=(u_1,\dots, u_k)$

$$\ll U \gg^2 = \left\{ \mathrm{Re} < \rho \begin{pmatrix} u_1 \\ \vdots \\ u_{d_0} \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_{d_0} \end{pmatrix} >_\gamma + < \nu \begin{pmatrix} u_1 \\ \vdots \\ u_{d_0} \end{pmatrix} >_\gamma^2 + \gamma < \begin{pmatrix} u_1 \\ \vdots \\ u_{d_0} \end{pmatrix} >_{-\frac{1}{2},\gamma}^2 \right\}$$

$$+ < \begin{pmatrix} u_{d_0+1} \\ \vdots \\ u_k \end{pmatrix} >_\gamma^2 \qquad (\gamma \geqslant \gamma_0),$$

then $\ll U \gg \leqslant \mu < U >_\gamma$ and

Lemma 2.1.

i) *let A be bounded with respect to* (ν, ρ), *then*

$$|< AU, V>_\gamma| \leqslant C \ll U \gg \cdot \ll V \gg,$$

ii) *let A be small with respect to* (ν, ρ), *then*

$$|< AU, V>_\gamma| \leqslant \mu \ll U \gg \cdot \ll V \gg.$$

Next, we consider of the singularities of $\tilde{S}(X)$ and $\tilde{T}(X)$. Since there exists a smooth $m \times \mu$-matrix $C(X)$ such that

$$I = (\tilde{B}_+(X), \tilde{B}_-(X)) C(X),$$

we have

$$\tilde{T}(X) = \tilde{J}_+(X) \tilde{B}_+^{-1}(X) = \tilde{J}_+(X) \tilde{B}_+^{-1}(X) (\tilde{B}_+(X), \tilde{B}_-(X)) C(X)$$

$$= (\tilde{J}_+(X), -\tilde{S}(X) \tilde{J}_-(X)) C(X),$$

hence $\tilde{T}_0(X)$, $\tilde{T}_1(X)$ and $\tilde{T}_2(X)$ are smooth and

$$\tilde{T}_3(X) = C_1(X) \varphi(X)^{-1} C_3(X),$$

where $C_3(X)$ is smooth. Therefore we have

$$\varphi(X) C_1^{-1}(X) R^+(X, \xi)$$

$$= \varphi(X) C_1^{-1}(X) (\tilde{S}_{30}(X) \ \ \tilde{S}_{31}(X) \ \ \tilde{S}_{32}(X) \ \ C_0(X)) \begin{pmatrix} \hat{P}'(X, \xi) \\ \check{P}(X, \xi) \\ Q^-(X, \xi) \\ R^-(X, \xi) \end{pmatrix}$$

$$+(C_2(X)\ \ C_3(X))\begin{pmatrix} R^-(X,\xi) \\ B(X,\xi) \end{pmatrix}.$$

Since

$$\varphi_j(X)=\wedge^{-1}(\sigma,\eta,\gamma)(\sigma-i\gamma-\tau_j(t,y;\eta)),\quad \operatorname{Im}\tau_j(t,y;\eta)\geqslant 0,$$

we have

$$<\varphi_j(t,y;D_t',D_y,\gamma)u>^2_{\frac{1}{2},\gamma}=<(D_t'-\tau_j(t,y;D_y))u>^2_{-\frac{1}{2},\gamma}$$

$$+i\gamma\{<(D_t'-\tau_j(t,y;D_y))u,\ u>_{-\frac{1}{2},\gamma}-<u,(D_t'-\tau_j(t,y;D_y))u>_{-\frac{1}{2},\gamma}\}$$

$$+\gamma^2<u>^2_{-\frac{1}{2},\gamma}\geqslant(\gamma^2-C)<u>^2_{-\frac{1}{2},\gamma},$$

therefore

$$<\varphi(t,y;D_t',D_y,\gamma)^{-1}U>_{-\frac{1}{2},\gamma}\leqslant\frac{C}{\gamma}<U>_{\frac{1}{2},\gamma}\qquad(\gamma\geqslant\gamma_0).$$

Here we can define

$$\begin{cases} \tilde{S}_{33}(t,y;D_t',D_y,\gamma) \\ \quad =C_1(t,y;D_t',D_y,\gamma)\varphi(t,y;D_t',D_y,\gamma)^{-1}C_2(t,y;D_t',D_y,\gamma) \\ \qquad\qquad\qquad\qquad +C_0(t,y;D_t',D_y,\gamma), \\ \tilde{T}_3(t,y;D_t',D_y,\gamma) \\ \quad =C_1(t,y;D_t',D_y,\gamma)\varphi(t,y;D_t',D_y,\gamma)^{-1}C_3(t,y;D_t',D_y,\gamma), \end{cases}$$

then we have

Lemma 2.2. *Let us assume condition* (*S*), *then we have*

$$R^+u=(\tilde{S}_{30}\tilde{S}_{31}\tilde{S}_{32}\tilde{S}_{33})\tilde{J}_-u+\tilde{T}_3Bu+\cdots\qquad on\quad x=0,$$

where

$$<\tilde{S}_{33}U>_{-\frac{1}{2},\gamma}+<\tilde{T}_3U>_{-\frac{1}{2},\gamma}\leqslant\frac{C}{\gamma}<U>_{\frac{1}{2},\gamma},$$

and

$$<\cdots>_{-\frac{1}{2},\gamma}\leqslant\frac{C}{\gamma}<u>_{m-1-\frac{1}{2},\gamma}.$$

Let us denote

$$F^2=\frac{1}{\gamma}|Au|_\gamma^2+\frac{1}{\gamma}<Bu>_{\frac{1}{2},\gamma}^2+|u|_{m-1,\gamma}^2,$$

$$\ll \tilde{J}_-u\gg_{(\frac{1}{2})}^2=\ll \hat{P}'u\gg^2+<\check{P}u>_\gamma^2+<Q^-u>_\gamma^2+\frac{1}{\gamma}<R^-u>_{\frac{1}{2},\gamma}^2,$$

$$\ll \tilde{J}_+u\gg_{(-\frac{1}{2})}^2=\ll \hat{P}u\gg^2+<\check{P}'u>_\gamma^2+<Q^+u>_\gamma^2+\gamma<R^+u>_{-\frac{1}{2},\gamma}^2,$$

then $\ll \tilde{J}_-u\gg_{(\frac{1}{2})}\leqslant CF$ will be shown in the following sections. For simplicity, we denote lower order terms by I_{m-1}, $I'_{m-1-\frac{1}{2}}$ such that

$$|I_{m-1}|\leqslant C|u|_{m-1,\gamma}^2,\quad |I'_{m-1-\frac{1}{2}}|\leqslant C\sum_{j=0}^{m-1}<D_x^ju>_{m-1-\frac{1}{2}-,\gamma}^2,$$

where

$$|I_{m-1}|+|I'_{m-1-\frac{1}{2}}|\leqslant CF^2.$$

Lemm 2.3. *Let us assume condition* (S), *then we have*

i) $\ll \tilde{J}_+u\gg_{(-\frac{1}{2})}^2\leqslant C\{\ll \tilde{J}_-u\gg_{(\frac{1}{2})}^2+F^2\}$,

ii) $\ll \hat{P}u\gg^2\leqslant\mu\ll \tilde{J}_-u\gg_{(\frac{1}{2})}^2+CF^2$.

Proof. Since

$$\begin{pmatrix}\hat{P}-\delta\\ \check{P}'\\ Q^+\end{pmatrix}u=\begin{pmatrix}\tilde{S}_{00}^{(0)} & \tilde{S}_{01} & \tilde{S}_{02}\\ \tilde{S}_{10} & \tilde{S}_{11} & \tilde{S}_{12}\\ \tilde{S}_{20} & \tilde{S}_{21} & \tilde{S}_{22}\end{pmatrix}\begin{pmatrix}\hat{P}'\\ \check{P}\\ Q^-\end{pmatrix}u+\begin{pmatrix}\tilde{S}_{03} & \tilde{T}_0\\ \tilde{S}_{13} & \tilde{T}_1\\ \tilde{S}_{23} & \tilde{T}_2\end{pmatrix}\begin{pmatrix}R^-\\ B\end{pmatrix}u$$

$$+\{\text{lower order terms}\},$$

we have from lemma 2.1

$$<\begin{pmatrix}\hat{P}-\delta\\ \check{P}'\\ Q^+\end{pmatrix}u>_\gamma\leqslant C(\ll \tilde{J}_-u\gg_{(\frac{1}{2})}+F)$$

and

$$\ll \hat{P}u \gg \leqslant \ll \delta\hat{P}'u \gg + \mu < \hat{P}u - \delta\hat{P}'u >_{\gamma} \leqslant \mu \ll \tilde{J}_{-}u \gg_{(\frac{1}{2})} + CF.$$

Moreover, we have from lemma 2.2

$$< R^{+}u >^{2}_{-\frac{1}{2},\gamma} \leqslant C(\ll \tilde{J}_{-}u \gg^{2}_{(\frac{1}{2})} + F^{2}). \qquad \text{(Q. E. D.)}$$

Finally, we remark the ellipticity of E, then we have

Lemma 2.4.

i) $\frac{1}{\gamma} < R^{-}u >^{2}_{\frac{1}{2},\gamma} + | R^{-}u |^{2}_{\gamma} \leqslant \frac{C}{\gamma}(| Au |^{2}_{\gamma} + | u |^{2}_{m-1,\gamma})$,

ii) $\gamma | R^{+}u |^{2}_{\gamma} \leqslant C\{\gamma < R^{+}u >^{2}_{-\frac{1}{2},\gamma} + \frac{1}{\gamma}(| Au |^{2}_{\gamma} + | u |^{2}_{m-1,\gamma})\}$.

Applying (i) *of lemma* 2.3 *on* $\gamma < R^{+}u >^{2}_{-\frac{1}{2},\gamma}$, *we have*

Corollary

$$\gamma(| R^{+}u |^{2}_{\gamma} + | R^{-}u |^{2}_{\gamma}) \leqslant C\{\ll \tilde{J}_{-}u \gg^{2}_{(\frac{1}{2})} + F^{2}\}.$$

§2. Green's formulas

At first, we consider of $Q^{\pm}u$. Integrating by parts, we have

$$((D_x - \xi^{\pm}_{d+j})Q^{\pm}_{j}u,\ Q^{\pm}_{j}u)_{\gamma} - (Q^{\pm}_{j}u,\ (D_x - \xi^{\pm}_{d+j})Q^{\pm}_{j}u)_{\gamma}$$

$$= i < Q^{\pm}_{j}u,\ Q^{\pm}_{j}u >_{\gamma} - 2i(\mathrm{Im}\, \xi^{\pm}_{d+j}Q^{\pm}_{j}u,\ Q^{\pm}_{j}u)_{\gamma} + I_{m-1},$$

where $\xi^{\pm}_{d+j}$ are pseudo-differential operators of order 1. Hence we have

Green's formula (Q):

$$(Au,\ Q^{\pm}_{j}u)_{\gamma} - (Q^{\pm}_{j}u,\ Au)_{\gamma}$$

$$= i < Q^{\pm}_{j}u,\ Q^{\pm}_{j}u >_{\gamma} - 2i(\mathrm{Im}\, \xi^{\pm}_{d+j}Q^{\pm}_{j}u,\ Q^{\pm}_{j}u)_{\gamma} + I_{m-1} \quad (j = 1,\dots, s_{\pm}).$$

Here we have

Lmame 2.5.

i) $\gamma|Q^-u|_\gamma^2+<Q^-u>_\gamma^2\leqslant C\Big(\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\Big)$,

ii) $\gamma|Q^+u|_\gamma^2\leqslant C\Big\{<Q^+u>_\gamma^2+\Big(\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\Big)\Big\}$.

Applying (i) of lemma 2.3 on $<Q^+u>_\gamma^2$, we have

Corollary

$$\gamma(Q^+u|_\gamma^2+|Q^-u|_\gamma^2)\leqslant C\{\ll\tilde{J}_-u\gg_{(\frac{1}{2})}^2+F^2\}.$$

In the analogous way, we have

Green's formula (P);

i) $(Au,\ P_ju)_\gamma-(P_ju,\ Au)_\gamma=i\{<P_ju,\ P_ju>_\gamma+<\mathrm{Re}\,\beta_jP'_ju,\ P'_ju>_\gamma\}$

$-i\{2(\mathrm{Im}\,\alpha_j\wedge P_ju,\ P_ju)_\gamma+2(\mathrm{Re}\,\beta_j\,\mathrm{Im}\,\alpha_j\wedge P'_ju,\ P'_j)_\gamma$

$+(\mathrm{Im}\,\beta_j\wedge P_ju,\ P'_ju)_\gamma+(\mathrm{Im}\,\beta_j\wedge P'_ju,\ P_ju)_\gamma\}+I_{m-1}$,

ii) $(Au,\ P'_ju)_\gamma-(P'_ju,\ Au)_\gamma=i\{<P_ju,\ P'_ju>_\gamma+<P'_ju,\ P_ju>_\gamma\}$

$-2i\{(\mathrm{Im}\,\alpha_j\wedge P_ju,\ P'_ju)_\gamma+(\mathrm{Im}\,\alpha_j\wedge P'_ju,\ P_ju)_\gamma+(\mathrm{Im}\,\beta_j\wedge P'_ju,\ P'_ju)_\gamma$

$+I_{m-1}$,

iii) $|P_ju|_\gamma^2=i<P'_ju,\ P_ju>_{-\frac{1}{2},\gamma}-2i(\mathrm{Im}\,\alpha_jP'_ju,\ P_ju)_\gamma$

$+(P'_ju,\ \beta_jP_ju)_\gamma+(\wedge^{-1}P'_ju,\ Au)_\gamma+\frac{1}{\gamma}I_{m-1}$,

iv) $i<P'_ju>_{-\frac{1}{2},\gamma}^2=(P_ju,\ P'_ju)_\gamma-(P'_ju,\ P_ju)_\gamma+2i(\mathrm{Im}\,\alpha_jP'_ju,\ P'_ju)_\gamma+I_{m-1}$.

Remark. Let χ be a pseudo-differential operator of order 0 with real symbol, then above formulas (i)~(iv) are also valid even if inner products $<u,\ v>_\gamma$, $(u,\ v)_\gamma$ are changed into weighted ones $<\chi u,\ v>_\gamma$, $(\chi u,\ v)_\gamma$.

From Green's formula (P) for $d_0+1 \leqslant j \leqslant d$, we have

i)' $<\check{P}u>_\gamma^2 \leqslant \mu <\check{P}'u>_\gamma^2$

$$+C\{\gamma|\check{P}u|_\gamma^2+\gamma|\check{P}u|_\gamma|\check{P}'u|_\gamma+\mu\gamma|\check{P}'u|_\gamma^2+\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\},$$

ii)' $\gamma|\check{P}'u|_\gamma^2 \leqslant C\{<\check{P}u>_\gamma<\check{P}'u>_\gamma+\gamma|\check{P}u|_\gamma^2+\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\},$

iii)' $\gamma|\check{P}u|_\gamma^2 \leqslant \mu\{<\check{P}u>_\gamma^2+<\check{P}'u>_\gamma^2+\gamma|\check{P}'u|_\gamma^2\}$

$$+C\left(\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\right).$$

Hence we have

Lemma 2.6.

$$<\check{P}u>_\gamma^2+\gamma(|\check{P}u|_\gamma^2+|\check{P}'u|_\gamma^2) \leqslant \mu <\check{P}'u>_\gamma^2+C\left(\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\right).$$

Applying (i) of lemma 2.3 on $<\check{P}'u>_\gamma^2$, we have

Corollary

$$<\check{P}u>_\gamma^2+\gamma(|\check{P}u|_\gamma^2+|\check{P}'u|_\gamma^2) \leqslant \mu \ll \check{J}_-u \gg_{(\frac{1}{2})}^2+CF^2.$$

Finally, we denote

$$\beta=\begin{pmatrix}\beta_1 & & \\ & \ddots & \\ & & \beta_{d_0}\end{pmatrix}, \quad \chi=\begin{pmatrix}\chi_1 & & \\ & \ddots & \\ & & \chi_{d_0}\end{pmatrix}, \quad \varepsilon=\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_{d_0}\end{pmatrix},$$

then we have from Green's formula (P) (weighted with χ) for $1 \leqslant j \leqslant d_0$,

i)'' $|<\chi\hat{P}u, \hat{P}u>_\gamma+<\chi \operatorname{Re}\beta\hat{P}'u, \hat{P}'u>_\gamma|$

$$\leqslant C\{\gamma|\hat{P}u|_\gamma^2+\gamma|\hat{P}u|_\gamma|\hat{P}'u|_\gamma+\mu\gamma|\hat{P}'u|_\gamma^2+\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\},$$

ii)'' $|<\psi\varepsilon\hat{P}u, \hat{P}'u>_\gamma+<\psi\varepsilon\hat{P}'u, \hat{P}u>_\gamma-2(\psi\varepsilon \operatorname{Im}\beta\wedge\hat{P}'u, \hat{P}'u)_\gamma|$

$$\leqslant C\{\gamma|\hat{P}u|_\gamma|\hat{P}'u|_\gamma+\mu\gamma|\hat{P}'u|_\gamma^2+|Au|_\gamma|\hat{P}'u|_\gamma+|u|_{m-1,\gamma}^2\},$$

iii)″ $\gamma|\hat{P}u|_\gamma^2 \leqslant C\{\gamma<\hat{P}'u>_{-\frac{1}{2},\gamma}<\hat{P}u>_{-\frac{1}{2},\gamma}+\mu\gamma|\hat{P}'u|_\gamma^2$

$$+\frac{1}{\gamma}|Au|_\gamma^2+|u|_{m-1,\gamma}^2\},$$

iv)″ $\gamma<\hat{P}'u>_{-\frac{1}{2},\gamma}^2 \leqslant C\{\gamma|\hat{P}u|_\gamma|\hat{P}'u|_\gamma+|u|_{m-1,\gamma}^2\}.$

Applying (ii) of lemma 2.3 on $\gamma<\hat{P}u>_{-\frac{1}{2},\gamma}^2$, we have

Lemma 2.7.

i) $$|<\chi\hat{P}u,\ \hat{P}u>_\gamma+<\chi\,\mathrm{Re}\,\beta\,\hat{P}'u,\ \hat{P}'u>_\gamma|$$

$$\leqslant\mu(\ll\tilde{J}_-u\gg_{(\frac{1}{2})}^2+\gamma|\hat{P}'u|_\gamma^2)+CF^2,$$

ii) $$|\mathrm{Re}<\varepsilon\hat{P}u,\ \hat{P}'u>_\gamma-(\varepsilon\,\mathrm{Im}\,\beta\wedge\hat{P}'u,\ \hat{P}'u)_\gamma|$$

$$\leqslant\mu(\ll\tilde{J}_-u\gg_{(\frac{1}{2})}^2+\gamma|\hat{P}'u|_\gamma^2)+C_\mu\cdot F^2,$$

iii) $$\gamma|\hat{P}'u|_\gamma^2\leqslant\mu(\ll\tilde{J}_-u\gg_{(\frac{1}{2})}^2+\gamma|\hat{P}'u|_\gamma^2)+CF^2,$$

iv) $$\gamma<\hat{P}'u>_{-\frac{1}{2},\gamma}^2\leqslant\mu(\ll\tilde{J}_-u\gg_{(\frac{1}{2})}^2+\gamma|\hat{P}'u|_\gamma^2)+CF^2.$$

§3. Energy estimates

We shall obtain energy estimates for $\hat{P}u$ and $\hat{P}'u$ in this section.

Lemma 2.8. *Let us assume condition* (*S*), *then we have*

$$|<\chi\hat{P}u,\ \hat{P}u>_\gamma-<\chi\delta\hat{P}'u,\ \delta\hat{P}'u>_\gamma|$$

$$\leqslant\mu\ll\tilde{J}_-u\gg_{(\frac{1}{2})}^2+CF^2.$$

Proof. Since

$$\hat{P}u=\delta\hat{P}'u+(\tilde{S}_{00}^{(0)}\ \tilde{S}_{01}\ \tilde{S}_{02}\ \tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}\tilde{J}_-u\\ Bu\end{pmatrix}+\{\text{lower order terms}\},$$

we have

$$<\chi \hat{P}u, \hat{P}u>_\gamma - <\chi\delta\hat{P}'u, \delta\hat{P}'u>_\gamma$$

$$=2\,\mathrm{Re}<\chi\delta\hat{P}'u, (\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02}\ \tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}\tilde{J}_-u\\ Bu\end{pmatrix}>_\gamma$$

$$+<\chi(\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02}\ \tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}\tilde{J}_-u\\ Bu\end{pmatrix}, (\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02}\ \tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}\tilde{J}_-u\\ Bu\end{pmatrix}>_\gamma$$

$$+I'_{m-1-\frac{1}{2}}$$

$$=2\,\mathrm{Re}<\hat{P}'u, \chi\bar{\delta}(\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02})\begin{pmatrix}\hat{P}'u\\ \check{P}u\\ Q^-u\end{pmatrix}>_\gamma$$

$$+2\,\mathrm{Re}<\chi\delta\hat{P}'u, (\tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}R^-u\\ Bu\end{pmatrix}>_\gamma$$

$$+<\begin{pmatrix}\hat{P}'u\\ \check{P}u\\ Q^-u\end{pmatrix}, (\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02})^*\chi(\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{03})\begin{pmatrix}\hat{P}'u\\ \check{P}u\\ Q^-u\end{pmatrix}>_\gamma$$

$$+2\,\mathrm{Re}<\chi(\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02})\begin{pmatrix}\hat{P}'u\\ \check{P}u\\ Q^-u\end{pmatrix}, (\tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}R^-u\\ Bu\end{pmatrix}>_\gamma$$

$$+<\chi(\tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}R^-u\\ Bu\end{pmatrix}, (\tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix}R^-u\\ Bu\end{pmatrix}>_\gamma$$

$$+I'_{m-1-\frac{1}{2}}$$

$$=K_1+K_2+K_3+K_4+K_5+I'_{m-1-\frac{1}{2}}.$$

Since

$$\chi\bar{\delta}(\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02}), \qquad (\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02})^*\chi(\tilde{S}^{(0)}_{00}\ \tilde{S}_{01}\ \tilde{S}_{02})$$

are small with respect to (ν, ρ), we have from (ii) of lemma 2.1

$$|K_1|+|K_3|\leqslant\mu\ll\hat{P}'u\gg\ll\begin{pmatrix}\hat{P}'u\\ \check{P}u\\ Q^-u\end{pmatrix}\gg\leqslant\mu\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}\ .$$

On the other hands, we have

$$|K_2|+|K_4|\leqslant\mu<\begin{pmatrix}\hat{P}'u\\ \check{P}u\\ Q^-u\end{pmatrix}>_{-\frac{1}{2},\gamma}<\begin{pmatrix}R^-u\\ Bu\end{pmatrix}>_{\frac{1}{2},\gamma}$$

$$\leqslant\mu\Big(\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}+\frac{1}{\gamma}<Bu>^2_{\frac{1}{2},\gamma}\Big)$$

and

$$|K_5|\leqslant C<\begin{pmatrix}R^-u\\ Bu\end{pmatrix}>^2_\gamma\leqslant\mu\frac{1}{\gamma}<\begin{pmatrix}R^-u\\ Bu\end{pmatrix}>^2_{\frac{1}{2},\gamma}$$

$$\leqslant\mu\Big(\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}+\frac{1}{\gamma}<Bu>^2_{\frac{1}{2},\gamma}\Big).\qquad\text{(Q. E. D.)}$$

Applying (i) of lemma 2.7, we have

Corollary

$$|<\chi\nu P'u,\ P'u>_\gamma|\leqslant\mu\{\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}+\gamma|\hat{P}'u|^2_\gamma\}+CF^2.$$

Lemma 2.9. *Let us assume condition* (S), *then we have*

$$\mathrm{Re}<\psi\varepsilon\hat{P}u,\ \hat{P}'u>_\gamma\geqslant c\,\mathrm{Re}<\rho\hat{P}'u,\ \hat{P}'u>_\gamma$$

$$-C\Big\{<\nu\hat{P}'u>^2_\gamma+\gamma<\hat{P}'u>^2_{-\frac{1}{2},\gamma}+<\check{P}u>^2_\gamma+<Q^-u>^2_\gamma$$

$$+\frac{1}{\gamma}<R^-u>^2_{\frac{1}{2},\gamma}+F^2\Big\}.$$

Proof. We have

$$<\psi\varepsilon\hat{P}u,\ \hat{P}'u>_\gamma-<\psi\varepsilon\tilde{S}_{00}\hat{P}'u,\ \hat{P}'u>_\gamma$$

$$= <\psi\varepsilon(\tilde{S}_{01}\ \ \tilde{S}_{02}\ \tilde{S}_{03}\ \tilde{T}_0)\begin{pmatrix} \check{P}u \\ Q^-u \\ R^-u \\ Bu \end{pmatrix}, \hat{P}'u>_\gamma + I'_{m-1-\frac{1}{2}}$$

$$= <\psi\varepsilon(\tilde{S}_{01}\ \ \tilde{S}_{02})\begin{pmatrix} \check{P}u \\ Q^-u \end{pmatrix}, \hat{P}'u>_\gamma$$

$$+ <\psi\varepsilon(\tilde{S}_{03}\ \ \tilde{T}_0)\begin{pmatrix} R^-u \\ Bu \end{pmatrix}, P'u>_\gamma + I'_{m-1-\frac{1}{2}}$$

$$= L_1 + L_2 + I'_{m-1-\frac{1}{2}}\,.$$

Since $(0\ \tilde{S}_{01}\ \tilde{S}_{02})$ is bounded with respect to (v, ρ), we have from (i) of lemma 2.1

$$|L_1| \leqslant C < \begin{pmatrix} \check{P}u \\ Q^-u \end{pmatrix} >_\gamma \ll \hat{P}'u \gg .$$

Moreover, since

$$|L_2| \leqslant C < \begin{pmatrix} R^-u \\ Bu \end{pmatrix} >_{\frac{1}{2},\gamma} < \hat{P}'u >_{-\frac{1}{2},\gamma}\,,$$

we have

$$|<\psi\varepsilon \hat{P}u,\ \hat{P}'u>_\gamma - <\psi\varepsilon\, \tilde{S}_{00}\hat{P}'u,\ \hat{P}'u>_\gamma|$$

$$\leqslant C \ll \hat{P}'u \gg \left\{ < \begin{pmatrix} \check{P}u \\ Q^-u \end{pmatrix} >_\gamma^2 + \frac{1}{\gamma} < R^-u >_{\frac{1}{2},\gamma}^2 \right\}^{\frac{1}{2}} + CF^2.$$

On the other hands, we have

$$\psi\varepsilon\, \tilde{S}_{00} + (\psi\varepsilon\, \tilde{S}_{00})^* = A_0 + v A_1 + A_2 v + v A_3 v + \gamma A_4,$$

where

$$\rho^{-\frac{1}{2}}A_0\rho^{-\frac{1}{2}},\quad A_1\rho^{-\frac{1}{2}},\quad \rho^{-\frac{1}{2}}A_2,\quad A_3,\quad A_4$$

are bounded and

$$A_0 \geqslant c\rho \qquad (c>0).$$

Hence we have

$$<\{\psi\varepsilon\tilde{S}_{00}+(\psi\varepsilon\tilde{S}_{00})^*\}\hat{P}'u,\ \hat{P}'u>_\gamma$$

$$\geqslant c\,\mathrm{Re}<\rho\hat{P}'u,\ \hat{P}'u>_\gamma - C\{<v\hat{P}'u>_\gamma^2+\gamma<\hat{P}'u>^2_{-\frac{1}{2},\gamma}+F^2\}.$$

(Q. E. D.)

Applying corollary of lemma 2.8, (iv) of lemma 2.7, lemma 2.5 and lemma 2.4 respectively on $<v\hat{P}'u>_\gamma^2$, $\gamma<\hat{P}'u>^2_{-\frac{1}{2},\gamma}$, $<Q^-u>_\gamma^2$ and $\frac{1}{\gamma}<R^-u>^2_{\frac{1}{2},\gamma}$, then we have

$$\mathrm{Re}<\psi\varepsilon\hat{P}u,\ \hat{P}'u>_\gamma \geqslant c\,\mathrm{Re}<\rho\hat{P}'u,\ \hat{P}'u>_\gamma-\mu\{\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}+\gamma|\hat{P}'u|_\gamma^2\}-CF^2.$$

Therefore we have, applying (ii) of lemma 2.7,

Corollary.

$$\mathrm{Re}<\rho\hat{P}'u,\ \hat{P}'u>_\gamma+\gamma|P'u|_\gamma^2 \leqslant \mu\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}+C_\mu F^2.$$

Now we sum up the above results of lemma 2.4, lemma 2.5, corollary of lemma 2.6, (ii) and (iv) of lemma 2.7, corollary of lemma 2.8 and corollary of lemma 2.9, then we have

$$\ll\tilde{J}_-u\gg^2_{(\frac{1}{2})}+\gamma|P'u|_\gamma^2+\gamma|Pu|_\gamma^2 \leqslant CF^2,$$

if we take the diameter of U sufficiently small. Summing up volume estimates of corollaries of lemmas 2.4 and 2.5, we have

Proposition 2.1. *Let us assume condition* (S), *then we have*

$$\gamma|u|_{m-1,\gamma} \leqslant C(|Au|_\gamma+<Bu>_{\frac{1}{2},\gamma}) \qquad \textit{for}\quad \gamma\geqslant\gamma_0.$$

§4. Adjoint problems

In this section, we shall obtain energy estimates also for adjoint problems of (P), which assure the existence of solutions for (P). At first, we shall define an adjoint problem of (P). Let $\{r_j\}_{j=1,\dots,\mu}$ be orders of $\{B_j\}_{j=1,\dots,\mu}$, then we can find $\{r_j\}_{j=\mu+1,\dots,m}$ such that

$$\{r_1,\dots, r_m\}=\{0, 1,\dots, m-1\}.$$

Let

$$C=\begin{pmatrix} D_x^{r_{\mu+1}} \\ \vdots \\ D_x^{r_m} \end{pmatrix},$$

then there exist differential operators $\{C', B'\}$ such that

$$(Au, v)-(u, A^*v)=i\{<Bu, C'v>+<Cu, B'v>\},$$

where A^* is the formal adjoint of A. We say that $\{A^*, B'\}$ is an adjoint system of $\{A, B\}$, and

$$(P')\quad \begin{cases} A^*v=g & \text{for } (x, y)\in R_+^n,\ t<t_0, \\ B'_jv=0\ (j=1,\dots, m-\mu) & \text{for } x=0,\ y\in R^{n-1},\ t<t_0, \\ D_t^jv=0\ (j=0,\dots, m-1) & \text{for } (x, y)\in R_+^n,\ t=t_0, \end{cases}$$

is an adjoint problem of (P).

Let $B'_0(X, \xi)$, $C'_0(X, \xi)$ be principal parts of $B'(X, \xi)$, $C'(X, \xi)$, and let $\bar{B}'_0(X, \xi)$, $\bar{C}'_0(X, \xi)$ be polynomials of (τ, ξ, η) with complex conjugate coefficients to $B'_0(X, \xi)$, $C'_0(X, \xi)$, then we have

$$\frac{A(X, \xi)-A(X, \xi')}{\xi-\xi'}={}^t\begin{pmatrix} \bar{C}'_0(X, \xi') \\ \bar{B}'_0(X, \xi') \end{pmatrix}\begin{pmatrix} B(X, \xi) \\ C(X, \xi) \end{pmatrix}.$$

On the other hands, we have

$$\frac{A(X, \xi)-A(X, \xi')}{\xi-\xi'}={}^t\begin{pmatrix} \tilde{J}_{+0}(X, \xi') \\ \tilde{J}_{-0}(X, \xi') \end{pmatrix}\begin{pmatrix} \tilde{J}_+^{-1}(X) & \\ & \tilde{J}_-^{-1}(X) \end{pmatrix}\begin{pmatrix} \tilde{J}_+(X, \xi) \\ \tilde{J}_-(X, \xi) \end{pmatrix}$$

$$= {}^t\begin{pmatrix} \tilde{J}_{+0}(X, \xi') \\ \tilde{J}_{-0}(X, \xi') \end{pmatrix} \begin{pmatrix} \tilde{J}_+^{-1}(X) & \\ & \tilde{J}_-^{-1}(X) \end{pmatrix} \begin{pmatrix} \tilde{T}(X) & \tilde{S}(X) \\ 0 & I \end{pmatrix} \begin{pmatrix} B(X, \xi) \\ \tilde{J}_-(X, \xi) \end{pmatrix},$$

hence we define

$$\begin{pmatrix} \tilde{C}'(\bar{X}, \bar{\xi}) \\ \tilde{B}'(\bar{X}, \bar{\xi}) \end{pmatrix} = \begin{pmatrix} \tilde{T}^*(X) & 0 \\ \tilde{S}^*(X) & I \end{pmatrix} \begin{pmatrix} \tilde{J}_+^{-1*}(X) & \\ & \tilde{J}_-^{-1*}(X) \end{pmatrix} \overline{\begin{pmatrix} \tilde{J}_{+0}(X, \xi) \\ \tilde{J}_{-0}(X, \xi) \end{pmatrix}}.$$

Now let us denote

$$\tilde{J}_-(X, \xi) = (\mathscr{J}_+(X), \mathscr{J}_-(X)) \begin{pmatrix} B(X, \xi) \\ C(X, \xi) \end{pmatrix},$$

then we have

$$\begin{pmatrix} B(x, \xi) \\ \tilde{J}_-(X, \xi) \end{pmatrix} = \begin{pmatrix} I & 0 \\ \mathscr{J}_+(X) & \mathscr{J}_-(X) \end{pmatrix} \begin{pmatrix} B(X, \xi) \\ C(X, \xi) \end{pmatrix}$$

and

$$\begin{pmatrix} C_0'(\bar{X}, \bar{\xi}) \\ B_0'(\bar{X}, \bar{\xi}) \end{pmatrix} = \begin{pmatrix} I & \mathscr{J}_+^*(X) \\ 0 & \mathscr{J}_-^*(X) \end{pmatrix} \begin{pmatrix} \tilde{C}'(\bar{X}, \bar{\xi}) \\ \tilde{B}'(\bar{X}, \bar{\xi}) \end{pmatrix}.$$

Here we remark that

$$|\mathscr{J}_-(X)| = c(X)\,|\tilde{B}_+(X)|, \qquad c(X) \neq 0,$$

because

$$\begin{pmatrix} B \\ \tilde{J}_- \end{pmatrix} = \left\langle \begin{pmatrix} B \\ \tilde{J}_- \end{pmatrix}, \begin{pmatrix} \tilde{J}_{+0} \\ \tilde{J}_{-0} \end{pmatrix} \right\rangle_A \left\langle \begin{pmatrix} B \\ C \end{pmatrix}, \begin{pmatrix} \tilde{J}_{+0} \\ \tilde{J}_{-0} \end{pmatrix} \right\rangle_A^{-1} \begin{pmatrix} B \\ C \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{B}_+ & \tilde{B}_- \\ 0 & \langle \tilde{J}_-, \tilde{J}_{-0} \rangle_A \end{pmatrix} \left\langle \begin{pmatrix} B \\ C \end{pmatrix}, \begin{pmatrix} \tilde{J}_{+0} \\ \tilde{J}_{-0} \end{pmatrix} \right\rangle_A^{-1} \begin{pmatrix} B \\ C \end{pmatrix}.$$

Now we denote

$$\tilde{J}'_{+}(\bar{X},\bar{\xi})=\begin{pmatrix}\hat{P}(\bar{X},\bar{\xi})\\ \check{P}'(\bar{X},\bar{\xi})\\ Q^{-}(\bar{X},\bar{\xi})\\ R^{+}(\bar{X},\bar{\xi})\end{pmatrix}=\overline{\begin{pmatrix}\hat{P}(X,\xi)\\ \check{P}'(X,\xi)\\ Q^{-}(X,\xi)\\ R^{-}(X,\xi)\end{pmatrix}},\quad \tilde{J}'_{+0}(\bar{X},\bar{\xi})=\begin{pmatrix}\hat{P}'_{0}(\bar{X},\bar{\xi})\\ \check{P}_{0}(\bar{X},\bar{\xi})\\ Q^{-}(\bar{X},\bar{\xi})\\ R^{+}(\bar{X},\bar{\xi})\end{pmatrix},$$

$$\tilde{J}'_{-}(\bar{X},\bar{\xi})=\begin{pmatrix}\hat{P}'(\bar{X},\bar{\xi})\\ \check{P}(\bar{X},\bar{\xi})\\ Q^{+}(\bar{X},\bar{\xi})\\ R^{-}(\bar{X},\bar{\xi})\end{pmatrix}=\overline{\begin{pmatrix}\hat{P}'(\bar{X},\xi)\\ \check{P}(X,\xi)\\ Q^{+}(X,\xi)\\ R^{+}(X,\xi)\end{pmatrix}}.\quad \tilde{J}'_{-0}(\bar{X},\bar{\xi})=\begin{pmatrix}\hat{P}_{0}(\bar{X},\bar{\xi})\\ \check{P}'_{0}(\bar{X},\bar{\xi})\\ Q^{+}(\bar{X},\bar{\xi})\\ R^{-}(\bar{X},\bar{\xi})\end{pmatrix},$$

and

$$\tilde{J}'_{\pm}(\bar{X})=<\tilde{J}'_{\pm}(\bar{X},\bar{\xi}),\ \tilde{J}'_{\pm 0}(\bar{X},\bar{\xi})>_{A(\bar{X},\bar{\xi})}.$$

Since

$$\begin{pmatrix}\tilde{J}_{+0}(X,\xi)\\ \\ \tilde{J}_{-0}(X,\xi)\end{pmatrix}=\left(\begin{array}{cc|cc}-C_0(X) & & I & \\ & I & & 0\\ \hline I & & -C_1(X) & \\ & 0 & & I\end{array}\right)\begin{pmatrix}\tilde{J}_{+}(X,\xi)\\ \\ \tilde{J}_{-}(X,\xi)\end{pmatrix},$$

where

$$C_0(X)=\begin{pmatrix}I & \\ & \check{\beta}(X)\end{pmatrix}C(X),\qquad C_1(X)=\begin{pmatrix}\hat{\beta}(X) & \\ & I\end{pmatrix}C(X),$$

$$C(X)=\begin{pmatrix}c_1(X) & & \\ & \ddots & \\ & & c_d(X)\end{pmatrix},\qquad \overline{C(X)}=C(\bar{X}),$$

we have

$$\overline{\begin{pmatrix}\tilde{J}_{+0}(X,\xi)\\ \\ \tilde{J}_{-0}(X,\xi)\end{pmatrix}}=\left(\begin{array}{cc|cc} & I & -C_0(\bar{X}) & \\ & & & 0\\ \hline -C_1(\bar{X}) & & & I \\ & 0 & & \end{array}\right)\begin{pmatrix}\tilde{J}'_{-}(\bar{X},\bar{\xi})\\ \\ \tilde{J}'_{+}(\bar{X},\bar{\xi})\end{pmatrix}.$$

Moreover, since

$$\tilde{J}_{\pm}(X)=\begin{pmatrix} J_1(X) & & \\ & J_2^{\pm}(X) & \\ & & J_3^{\pm}(X) \end{pmatrix}, \quad \tilde{J}'_{\pm}(\bar{X})=\begin{pmatrix} J_1(\bar{X}) & & \\ & J_2^{\mp}(\bar{X}) & \\ & & J_3^{\pm}(\bar{X}) \end{pmatrix},$$

where

$$\begin{cases} J_1(X)=<P(X,\xi),\ P'_0(X,\xi)>_{A(X,\xi)}, \\ J_2^{\pm}(X)=<Q^{\pm}(X,\xi),\ Q^{\pm}(X,\xi)>_{A(X,\xi)}, \\ J_3^{\pm}(X)=<R^{\pm}(X,\xi),\ R^{\pm}(X,\xi)>_{A(X,\xi)}, \end{cases}$$

we have

$$\tilde{J}^{*}_{\pm}(X)=\tilde{J}'_{\mp}(\bar{X}).$$

Hence we have

$$\begin{pmatrix} \tilde{C}'(\bar{X},\xi) \\ \tilde{B}'(\bar{X},\xi) \end{pmatrix}=\begin{pmatrix} \tilde{T}^{*}(X) & 0 \\ \tilde{S}^{*}(X) & I \end{pmatrix}\begin{pmatrix} \tilde{J}'^{-1}_{-}(\bar{X}) & \\ & \tilde{J}'^{-1}_{+}(\bar{X}) \end{pmatrix}$$

$$\times\left(\begin{array}{c:c} I & \begin{matrix} -C_0(\bar{X}) & \\ & 0 \end{matrix} \\ \hdashline \begin{matrix} -C_1(\bar{X}) & \\ & 0 \end{matrix} & I \end{array}\right)\begin{pmatrix} \tilde{J}'_{-}(\bar{X},\xi) \\ \\ \tilde{J}'_{+}(\bar{X},\xi) \end{pmatrix}$$

$$=\begin{pmatrix} \tilde{T}^{*}(X) & 0 \\ & \\ \tilde{S}^{*}(X) & I \end{pmatrix}\left(\begin{array}{c:c} I & \begin{matrix} -C_0(\bar{X}) & \\ & 0 \end{matrix} \\ \hdashline \begin{matrix} -C_1(\bar{X}) & \\ & 0 \end{matrix} & I \end{array}\right)\begin{pmatrix} \tilde{J}'^{-1}_{-}(\bar{X}) & \\ & \tilde{J}'^{-1}_{+}(\bar{X}) \end{pmatrix}\begin{pmatrix} \tilde{J}'_{-}(\bar{X},\xi) \\ \\ \tilde{J}'_{+}(\bar{X},\xi) \end{pmatrix},$$

therefore we have

$$(\tilde{B}'_{-}(\bar{X}),\ \tilde{B}'_{+}(\bar{X}))=<\tilde{B}'(\bar{X},\xi),\begin{pmatrix} \tilde{J}'_{-0}(\bar{X},\xi) \\ \tilde{J}'_{+0}(\bar{X},\xi) \end{pmatrix}>_{A(\bar{X},\xi)}$$

$$=(\tilde{S}^{*}(X),\ I)\left(\begin{array}{c:c} I & \begin{matrix} -C_0(\bar{X}) & \\ & 0 \end{matrix} \\ \hdashline \begin{matrix} -C_1(\bar{X}) & \\ & 0 \end{matrix} & I \end{array}\right)$$

$$=(\tilde{S}^*(X)-\begin{pmatrix} C_1(\bar{X}) & \\ & 0\end{pmatrix},\ I-\tilde{S}^*(X)\begin{pmatrix} C_0(\bar{X}) & \\ & 0\end{pmatrix}),$$

and

$$\tilde{J}'_+(\bar{X},\xi)=\tilde{S}'(\bar{X})\tilde{J}'_-(\bar{X},\xi)+\tilde{T}'(\bar{X})\tilde{B}'(\bar{X},\xi),$$

where

$$\begin{cases} \tilde{S}'(\bar{X})=-\tilde{J}'_+(\bar{X})\tilde{B}'^{-1}_+(\bar{X})\tilde{B}'_-(\bar{X})\tilde{J}'^{-1}_-(\bar{X}), \\ \tilde{T}'(\bar{X})=\tilde{J}'_+(\bar{X})\tilde{B}'^{-1}_+(\bar{X}). \end{cases}$$

In the following, we consider of properties of $\tilde{S}'(\bar{X})$, following from those of $\tilde{S}(X)$. For simplicity, we denote

$$\tilde{S}=\left(\begin{array}{ccc:c} \tilde{S}_{00} & \tilde{S}_{01} & \tilde{S}_{02} & \tilde{S}_{03} \\ \tilde{S}_{10} & \tilde{S}_{11} & \tilde{S}_{12} & \tilde{S}_{13} \\ \tilde{S}_{20} & \tilde{S}_{21} & \tilde{S}_{22} & \tilde{S}_{23} \\ \hdashline \tilde{S}_{30} & \tilde{S}_{31} & \tilde{S}_{32} & \tilde{S}_{33} \end{array}\right)=\begin{pmatrix} \Sigma_0 & \Sigma_1 \\ \Sigma_2 & \Sigma_3 \end{pmatrix},$$

$$\tilde{S}'=\left(\begin{array}{ccc:c} \tilde{S}'_{00} & \tilde{S}'_{01} & \tilde{S}'_{02} & \tilde{S}'_{03} \\ \tilde{S}'_{12} & \tilde{S}'_{11} & \tilde{S}'_{12} & \tilde{S}'_{13} \\ \tilde{S}'_{20} & \tilde{S}'_{21} & \tilde{S}'_{22} & \tilde{S}'_{23} \\ \hdashline \tilde{S}'_{30} & \tilde{S}'_{31} & \tilde{S}'_{32} & \tilde{S}'_{33} \end{array}\right)=\begin{pmatrix} \Sigma'_0 & \Sigma'_1 \\ \Sigma'_2 & \Sigma'_3 \end{pmatrix},$$

and

$$H=(\tilde{S}-\begin{pmatrix} C_1 & \\ & 0\end{pmatrix})(I-\begin{pmatrix} C_0 & \\ & 0\end{pmatrix}\tilde{S})^{-1}$$

$$=\begin{pmatrix} \Sigma_0-\begin{pmatrix} C_1 & \\ & 0\end{pmatrix} & \Sigma_1 \\ \Sigma_2 & \Sigma_3 \end{pmatrix}\begin{pmatrix} I-\begin{pmatrix} C_0 & \\ & 0\end{pmatrix}\Sigma_0 & -\begin{pmatrix} C_0 & \\ & 0\end{pmatrix}\Sigma_1 \\ 0 & I \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} \Sigma_0-\begin{pmatrix} C_1 & \\ & 0\end{pmatrix} & \Sigma_1 \\ \Sigma_2 & \Sigma_3 \end{pmatrix}\begin{pmatrix} (I-\begin{pmatrix} C_0 & \\ & 0\end{pmatrix}\Sigma_0)^{-1} & (I-\begin{pmatrix} C_0 & \\ & 0\end{pmatrix})^{-1}\begin{pmatrix} C_0 & \\ & 0\end{pmatrix}\Sigma_1 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} (\Sigma_0 - \begin{pmatrix} C_1 & \\ & 0 \end{pmatrix}) & (\Sigma_0 - \begin{pmatrix} C_1 & \\ & 0 \end{pmatrix})(I - \begin{pmatrix} C_0 & \\ & 0 \end{pmatrix}\Sigma_0)^{-1} \\ \times (I - \begin{pmatrix} C_0 & \\ & 0 \end{pmatrix}\Sigma_0)^{-1} & \times \begin{pmatrix} C_0 & \\ & 0 \end{pmatrix}\Sigma_1 + \Sigma_1 \\ \Sigma_2(I - \begin{pmatrix} C_0 & \\ & 0 \end{pmatrix}\Sigma_0)^{-1} & \Sigma_2(I - \begin{pmatrix} C_0 & \\ & 0 \end{pmatrix}\Sigma_0)^{-1} \times \begin{pmatrix} C_0 & \\ & 0 \end{pmatrix}\Sigma_1 + \Sigma_3 \end{pmatrix}$$

$$= \begin{pmatrix} H_0 & H_1 \\ H_2 & H_3 \end{pmatrix},$$

then

$$\tilde{S}'(\bar{X}) = -\tilde{J}'_+(\bar{X}) H^*(X) \tilde{J}'^{-1}_-(\bar{X}),$$

that is,

$$\begin{cases} \Sigma'_0(\bar{X}) = -\begin{pmatrix} J_1(\bar{X}) & \\ & J_2^-(\bar{X}) \end{pmatrix} H_0^*(X) \begin{pmatrix} J_1(\bar{X}) & \\ & J_2^+(\bar{X}) \end{pmatrix}^{-1}, \\ \Sigma'_1(\bar{X}) = -\begin{pmatrix} J_1(\bar{X}) & \\ & J_2^-(\bar{X}) \end{pmatrix} H_2^*(X) J_3^-(\bar{X})^{-1}, \\ \Sigma'_2(\bar{X}) = -J_3^+(\bar{X}) H_1^*(X) \begin{pmatrix} J_1(\bar{X}) & \\ & J_2^+(\bar{X}) \end{pmatrix}^{-1}, \\ \Sigma'_3(\bar{X}) = -J_3^+(\bar{X}) H_3^* J_3^-(\bar{X}). \end{cases}$$

Now we calculate H_0, in order to see the properties of Σ'_0. Let us denote

$$\Sigma_0^{(0)} = \Sigma_0 - \begin{pmatrix} \delta & \\ & 0 \end{pmatrix} = \begin{pmatrix} \tilde{S}_{00}^{(0)} & \tilde{S}_{01} & \tilde{S}_{02} \\ \tilde{S}_{10} & \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{20} & \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix},$$

then $\Sigma_0^{(0)}$ is bounded with respect to (ν, ρ). Therefore we have

$$\left\{I-\begin{pmatrix} C_0 & \\ & 0\end{pmatrix}\Sigma_0\right\}^{-1}=\left\{I-\begin{pmatrix}\hat{C} & \check{\beta}\check{C} \\ & 0\end{pmatrix}\left\{\begin{pmatrix}\delta & \\ & 0\end{pmatrix}+\Sigma_0^{(0)}\right\}\right\}^{-1}$$

$$=\left\{\begin{pmatrix} I-\hat{C}\delta & \\ & I\end{pmatrix}-\begin{pmatrix}\hat{C} & \\ & 0\end{pmatrix}\Sigma_0^{(0)}+\{\text{small}\}\right\}^{-1}$$

$$=\begin{pmatrix}(I-\hat{C}\delta)^{-1} & \\ & I\end{pmatrix}\left\{I-\begin{pmatrix}\hat{C} & \\ & 0\end{pmatrix}\Sigma_0^{(0)}+\{\text{small}\}\right\}^{-1}$$

$$=\begin{pmatrix}(I-\hat{C}\delta)^{-1} & \\ & I\end{pmatrix}\left\{I+\begin{pmatrix}\hat{C} & \\ & 0\end{pmatrix}\Sigma_0^{(0)}+\{\text{small}\}\right\}$$

and

$$H_0=\left\{\Sigma_0-\begin{pmatrix}C_1 & \\ & 0\end{pmatrix}\right\}\left\{I-\begin{pmatrix}C_0 & \\ & 0\end{pmatrix}\Sigma_0\right\}^{-1}$$

$$=\left\{\Sigma_0^{(0)}+\begin{pmatrix}\delta-\hat{\beta}\hat{C} & -\check{C} \\ & 0\end{pmatrix}\right\}\begin{pmatrix}(I-\hat{C}\delta)^{-1} & \\ & I\end{pmatrix}\left\{I+\begin{pmatrix}\hat{C} & \\ & 0\end{pmatrix}\Sigma_0^{(0)}\right\}$$

$$+\{\text{small}\}$$

$$=\Sigma_0^{(0)}+\begin{pmatrix}(\delta-\hat{\beta}\hat{C})(I-\hat{C}\delta)^{-1} & -\check{C} \\ & 0\end{pmatrix}+\{\text{small}\},$$

where "small" means "small" with respect to (ν, ρ). Hence we have

$$\Sigma_0'(\bar{X})=-\begin{pmatrix}J_1(\bar{X}) & \\ & J_2^-(\bar{X})\end{pmatrix}\Sigma_0^{(0)}(X)^*\begin{pmatrix}J_1(\bar{X}) & \\ & J_2^+(\bar{X})\end{pmatrix}^{-1}$$

$$-\begin{pmatrix}(\overline{\delta(X)}-\hat{\beta}(\bar{X})\hat{C}(\bar{X}))(I-\hat{C}(\bar{X})\overline{\delta(X)})^{-1} & -\check{C}(\bar{X}) \\ & 0\end{pmatrix}$$

$$+\{\text{small}\}.$$

Let $\delta'(\bar{X})$ be the diagonal elements of $\tilde{S}'_{00}(\bar{X})$, and let

$$\tilde{S}'_{00}{}^{(0)}(\bar{X}) = \tilde{S}'_{00}(\bar{X}) - \delta'(\bar{X}),$$

$$\Sigma'_0{}^{(0)}(\bar{X}) = \Sigma'_0(\bar{X}) - \begin{pmatrix} \delta'(\bar{X}) & \\ & 0 \end{pmatrix},$$

then we have

Lemma 2.10.

i) $\Sigma'_0{}^{(0)}(X)$ *is bounded with respect to* (ν, ρ),

ii) $\delta'(\bar{X}) = -(\overline{\delta(X)} - \hat{\beta}(\bar{X})\hat{C}(\bar{X}))(I - \hat{C}(\bar{X})\overline{\delta(X)})^{-1} + \{small\}_{(\nu,\rho)}$,

iii) $\tilde{S}'_{00}{}^{(0)}(\bar{X}) = -\hat{J}_1(\bar{X})\tilde{S}^{(0)}_{00}(X)^* \hat{J}_1(\bar{X})^{-1} + \{small\}_{(\nu,\rho)}$,

iv) Σ'_1 *and* Σ'_2 *are smooth*,

v) $\Sigma'_3(\bar{X}) = -J_3^+(\bar{X})\Sigma_3^*(X)J_3^-(\bar{X}) + \{smooth\}$.

Finally, we consider of diagonals of $\tilde{S}'_{00}$ for real X:

$$\delta' = \begin{pmatrix} \delta'_1 & & \\ & \ddots & \\ & & \delta'_{d_0} \end{pmatrix}.$$

Since

$$-\frac{\delta_j - \beta_j c_j}{1 - \delta_j c_j} = -\frac{\mathrm{Re}\,\delta_j + i\,\mathrm{Im}\,\delta_j - (\nu_j - (\mathrm{Im}\delta_j)^2)c_j}{1 - \mathrm{Re}\,\delta_j c_j - i\,\mathrm{Im}\,\delta_j c_j}$$

$$= -\mathrm{Re}\,\delta_j\left(1 + \frac{\delta_j c_j}{1 - \delta_j c_j}\right) + \nu_j c_j\left(1 + \frac{\delta_j c_j}{1 - \delta_j c_j}\right)$$

$$+ i\,\mathrm{Im}\,\delta_j\left(1 + \frac{\mathrm{Re}\,\delta_j c_j}{1 - \delta_j c_j}\right),$$

we have

Lemma 2.11. *Let X be real, then*

$$\delta'_j = \mathrm{Re}\,\delta_j(-1 + a_j) + \nu_j(c_j + b_j) + i\,\mathrm{Im}\,\delta_j,$$

where $a_j(X)$ *and* $b_j(X)$ *are smooth and*

$$a_j(X_0) = b_j(X_0) = 0.$$

Let us define

$$v'_j=\beta_j+(\operatorname{Im}\delta'_j)^2,$$

then

$$\begin{aligned} v'_j&=\beta_j+(\operatorname{Im}\delta_j+\operatorname{Im}a_j\operatorname{Re}\delta_j+\operatorname{Im}b_jv_j)^2\\ &=\tilde a_j\operatorname{Re}\delta_j+v_j(1+\tilde b_j),\end{aligned}$$

where

$$\tilde a_j(X_0)=\tilde b_j(X_0)=0.$$

Therefore

$$v_j=-\frac{\tilde a_j}{1+\tilde b_j}\operatorname{Re}\delta_j \qquad \text{on } v'_j=0,$$

and

$$\operatorname{Re}\delta'_j=\operatorname{Re}\delta_j\left\{-1+\operatorname{Re}a_j-\frac{\tilde a_j}{1+\tilde b_j}(c_j+\operatorname{Re}b_j)\right\} \qquad \text{on } v'_j=0.$$

Now we define

$$\rho'_j=-\varepsilon_j(\operatorname{Re}\delta'_j)|_{v'_j=0},$$

then we have

$$c_1\rho_j<\rho'_j<c_2\rho_j,$$

hence

Corollary. *"bounded" (resp. "small") with respect to* (v, ρ) *is equivalent to "bounded" (resp. "small") with respect to* (v', ρ').

Now we define

$$R'_0(\overline{X})=\det\left(\frac{1}{2\pi i}\oint\frac{B'_{j0}(\overline{X},\xi)\xi^{k-1}}{\overline{A_-(X,\xi)}}d\xi\right)_{j,k=1,\dots,m-\mu}$$

$$\overline{(A_-(X,\xi)}=\prod_{j=1}^{m-\mu}(\xi-\overline{\xi_j^-(X)})),$$

then we have ([10])

$$R'_0(\overline{X}) \neq 0 \quad \text{for} \quad X=(t, y; \tau, \eta) \in R^n \times (C^1 \times R^{n-1}), \quad \operatorname{Im} \tau < 0.$$

Here we define condition (S′):

Condition (S'. 1) $R'_0(\overline{X})=0$ for $X=(t, y; \tau, \eta) \in R^n \times (C^1 \times R^{n-1})$, $\operatorname{Im} \tau < 0$.

Condition (S'. 2) Let X_0 be real, then there exists a neighbourhood U of X_0, where

i) $\tilde{S}'_{33}(\overline{X}) = C'_1(\overline{X}) \varphi'^{-1}(\overline{X}) C'_2(\overline{X}) + C'_0(\overline{X})$,

where $C'_0(\overline{X})$, $C'_1(\overline{X})$, $C'_2(\overline{X})$, $\varphi'(\overline{X})$ are smooth, $|C'_1(\overline{X})| \neq 0$, $|C'_2(\overline{X})| \neq 0$, and

$$\varphi'(\overline{X}) = \begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \; M - M_0 & \\ & \begin{matrix} \varphi'_1(\overline{X}) & & \\ & \ddots & \\ & & \varphi'_{M_0}(\underline{X}) \end{matrix} \end{pmatrix},$$

$$\varphi'_j(\overline{X}) = \bar{\tau} - \tau'_j(t, y; \eta), \operatorname{Im} \tau'_j \leqslant 0,$$

ii) $\tilde{S}_{ij}(\overline{X})((i, j) \neq (3, 3))$ are smooth,

iii) $\rho' \geqslant 0$,

iv) there exists a smooth positive diagonal matrix

$$\psi' = \begin{pmatrix} \psi'_1 & & \\ & \ddots & \\ & & \psi'_{d_0} \end{pmatrix}$$

such that the hermitian part of $-\psi' \varepsilon \tilde{S}'_{00}$ is positive with respect to (v', ρ'),

v) $\begin{pmatrix} \tilde{S}'_{00}{}^{(0)} & \tilde{S}'_{01} & \tilde{S}'_{02} \\ \tilde{S}'_{10} & \tilde{S}'_{11} & \tilde{S}'_{12} \\ \tilde{S}'_{20} & \tilde{S}'_{21} & \tilde{S}'_{22} \end{pmatrix}$

is bounded with respect to (v', ρ').

Then we have

Proposition 2.2. *Let problem (P) satisfy condition (S), then its adjoint problem (P') satisfy condition (S').*

Applying proposition 2.1 to the problem (P'), it holds the energy inequality for (P'). Hence we have theorem II by usual techniques.

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