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Z_2-homology submanifolds and homology classes of a Z_2-homology manifold

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This note is concerned with the problem of the realization of homology classes mod 2 of a $\mathbf{Z}_2$-homology manifold by $\mathbf{Z}_2$-homology submanifolds.

First the C^∞-case of this problem was studied by R. Thom [10]. Next the *PL*-case, *TOP*-case and the case of homology manifolds were studied in [1], [2], [3], [4].

The present study is based on the Williamson's transversality theorem [11]. We shall apply R. Thom's method [10] to $\mathbf{Z}_2$-homology manifolds.

1. Statement of the result.

We shall obtain the following result:

Theorem 1. *Let V^n be a $\mathbf{Z}_2$-homology manifold of dimension n ($n \geqq 2$). For $1 \leqq k \leqq n/2$, all homology classes of $H_k(V^n, \mathbf{Z}_2)$ can be realized by $\mathbf{Z}_2$-homology submanifolds which have normal* PL-*microbundles.*

Theorem 2. *Let V^n be a $\mathbf{Z}_2$-homology manifold of dimension n ($n \geqq 2$). Then all homology classes of $H_{n-1}(V^n, \mathbf{Z}_2)$ can be realized by $\mathbf{Z}_2$-homology submanifolds which have normal PL-microbundles.*

These results are quite in parallel with those of the case of homology manifolds in [4].

2. Preliminaries.

A compact polyhedron M is called a $\mathbf{Z}_2$*-homology n-manifold*, if there exists a triangulation K of M such that for all $x \in |K|$ and for all r $H_r(LK(x, K), \mathbf{Z}_2)$ are isomorphic to $H_r(S^{n-1}, \mathbf{Z}_2)$. Here $LK(x, K)$ denotes the boundary of the star $\mathrm{St}(x, K)$ of x in K.

It can be seen that this definition is independent of the triangulation chosen. Homology n-manifolds are $\mathbf{Z}_2$-homology n-manifolds. $\mathbf{Z}_2$-homology manifolds were studied in Borel [6], [7].

We have many examples of $\mathbf{Z}_2$-homology manifolds.

Proposition 1. *Let X be a compact n-dimensional generalized manifold over* $\mathbf{Z}_2$ (in the sense of Borel [7]). *If there exists a triangulation K of X, then X is a $\mathbf{Z}_2$-homology n-manifold.*

Proof. Let x be a point of $|K|$. By the definition of an n-dimensional generalized manifold over $\mathbf{Z}_2$, for an open neighborhood U of x, there exist open neighborhoods W, V of x such that

i) $W \subset \overline{W} \subset V \subset \overline{V} \subset U$,

ii) for any open neighborhood W' of x in W, the image of the canonical homomorphism

$$j^i_{V,W'}: H^i_C(W', \mathbf{Z}_2) \longrightarrow H^i_C(V, \mathbf{Z}_2)$$

is 0 for $i \neq n$ and $\mathbf{Z}_2$ for $i = n$ (where H_C denotes the cohomology group with compact support; see Chapter I in Borel [7]). Let $U = \mathrm{int}(\mathrm{St}(x, K))$. Then, for a sufficiently large number k, $\mathrm{int}(\mathrm{St}(x, Sd^k K))$ is contained in W, where $Sd^k K$ is k-th barycentric subdivision of K. Let $W' = \mathrm{int}(\mathrm{St}(x, Sd^k K))$. Since $j^i_{U,W'}$ is isomorphic and $j^i_{U,V} \circ j^i_{V,W'} = j^i_{U,W'}$, $H^i_C(U, \mathbf{Z}_2)$ is 0 for $i \neq n$ and $\mathbf{Z}_2$ for $i = n$. Thus we have obtained that $H_*(Lk(x, K), \mathbf{Z}_2) = H_*(S^{n-1}, \mathbf{Z}_2)$.

Proposition 2. *Let M^n be a closed C^∞-manifold of dimension n and p be an odd prime. Let $\varphi: \mathbf{Z}_p \times M^n \rightarrow M^n$ be an effective C^∞-action. Then the orbit space $M^n/\mathbf{Z}_p$ of φ is a $\mathbf{Z}_2$-homology n-*

manifold.

Proof. First by Yang [12] we can triangulate the orbit space $M^n/\mathbf{Z}_p$. By Proposition 4.8, in Chapter I of Borel [7], M^n is an orientable n-dimensional generalized manifold over $\mathbf{Z}_2$. Note that $\mathbf{Z}_p$ acts trivially on $H^n_C(M^n, \mathbf{Z}_2)=\mathbf{Z}_2$. Then, by Theorem 1 in Raymond [9], $M^n/\mathbf{Z}_p$ is an n-dimensional generalized manifold over $\mathbf{Z}_2$. Applying Proposition 1, we obtain that $M^n/\mathbf{Z}_p$ is a $\mathbf{Z}_2$-homology n-manifold.

Proposition 3. *Let M be a $\mathbf{Z}_2$-homology manifold of dimension n. Then M satisfies the Poincaré duality with coefficient $\mathbf{Z}_2$:*

$$D: H_k(M, \mathbf{Z}_2) \cong H^{n-k}(M, \mathbf{Z}_2).$$

Proof. We can show this proposition in quite a parallel way as the proof of Poincaré duality for homology manifolds (cf. Maunder [8]).

Otherwise, we can prove that $\mathbf{Z}_2$-homology manifolds are generalized cohomology manifolds over $\mathbf{Z}_2$. However, we know that generalized cohomology manifolds over $\mathbf{Z}_2$ satisfy the Poincaré duality with coefficient $\mathbf{Z}_2$ (cf. Borel [6]).

Proposition 4. *Let (M, K) be a $\mathbf{Z}_2$-homology manifold of dimension n, $n \geqq 2$. Then for any $x \in K$, $Lk(x, K)$ is a $\mathbf{Z}_2$-homology $(n-1)$-manifold.*

Proof. This proposition can be proved in quite a parallel way as the proof for homology manifolds (cf. Alexandroff [5]).

Let M be a $\mathbf{Z}_2$-homology m-manifolds, PL-embedded in a $\mathbf{Z}_2$-homology q-manifold Q. Then we shall say M is a *$\mathbf{Z}_2$-homology submanifold* of Q.

Let V^n be a $\mathbf{Z}_2$-homology n-manifold and W^p be a $\mathbf{Z}_2$-homology submanifold of dimension p of V^n. The inclusion map
$i: W^p \to V^n$ induces the homomorphism

$$i_*: H_p(W^p, \mathbf{Z}_2) \longrightarrow H_p(V^n, \mathbf{Z}_2).$$

Let $z \in H_p(V^n, \mathbf{Z}_2)$ be the image by i_* of the fundamental class w of the $\mathbf{Z}_2$-homology p-manifold W^p. Then we say that the homology class z is *realized* by the $\mathbf{Z}_2$-homology submanifold W^p.

Here the following question is considered: Let a homology class z mod 2 of a $\mathbf{Z}_2$-homology n-manifold V^n be given. Is it realizable by a $\mathbf{Z}_2$-homology submanifold?

3. Williamson's transversality theorem.

In this section we shall recall Williamson's transversality theorem (cf. Williamson [11]).

Let ξ be a PL-microbundle:

$$\xi: B(\xi) \xrightarrow{i} E(\xi) \xrightarrow{j} B(\xi),$$

X be a complex, and suppose $E(\xi)$ is contained in X so that $B(\xi)$ is a closed PL-subspace of X. Then we say X *contains the PL-microbundle* ξ. If $E(\xi)$ is a neighborhood of $B(\xi)$, then we say ξ is a *normal PL-microbundle* for $B(\xi)$ in X.

Difinition. Let S and T be locally finite simplicial complexes and ξ be a normal PL-microbundle for $B=B(\xi)$ in T. Let $f: S\to T$ be a PL-map. If $A=f^{-1}(B)$ has a normal PL-microbundle η in S such that η is isomorphic to $(f|A)^*\xi$, then we shall say f is *transverse regular* for (η, ξ), or briefly, f is *t-regular*.

R. Williamson Jr. obtained the following theorem.

Theorem 3. *Let S and T be locally finite simplicial complexes and let $f: S\to T$ be a PL-map. Suppose that T contains a PL-microbundle ξ. Then there is a* PL-*homotopy H_t of f such that H_1 is t-regular for (η, ξ).*

4. A lemma on $\mathbf{Z}_2$-homology manifolds.

In this section we shall prove a lemma on $\mathbf{Z}_2$-homology manifolds which will be used in the next section.

Lemma. *Suppose V is a $\mathbf{Z}_2$-homology $(n+q)$-manifold and M is a PL-subspace of V which has a normal PL-microbundle of dimension q in $V(n, q \geqq 1)$. Then M is a $\mathbf{Z}_2$-homology n-manifold.*

Proof. Given any $x \in M$ there is an open neighborhood U of x in M and a neighborhood W of x in V, also open, such that $U \times \mathbf{R}^q$ is PL-homeomorphic to W, by the definition of normal PL-microbundles. So it suffices to prove the lemma for the special case $M=U$, $V=W$, and W itself is $U \times \mathbf{R}^q$.

If the lemma is true for $q=1$, it follows that $U \times \mathbf{R}^{q-1}$ is a $\mathbf{Z}_2$-homology $(n+q-1)$-manifold, therefore by induction that U is a $\mathbf{Z}_2$-homology n-manifold. So it suffices to consider $q=1$. We also need only to show that U is a $\mathbf{Z}_2$-homology manifold.

We triangulate $U \times \mathbf{R}$ by the convex product cells of U and a simplicial subdivision of $\mathbf{R}$, and we suppose x is a vertex of U and 0 is a vertex of $\mathbf{R}$. The link of x relative to $U \times \mathbf{R}$, that is the unique cell complex $Lk(x, W)$ such that the closed star $St(x, W)$ is the join $Lk(x, W)*x$, is the same, up to PL-homeomorphism, for any two convex cell subdivision of $U \times \mathbf{R}$. In the product cell triangulation of $U \times \mathbf{R}$,

$$\mathrm{St}((x, 0), W)=\mathrm{St}(x, U) \times \mathrm{St}(0, \mathbf{R}),$$

$$Lk((x, 0), W)=Lk(x, U) \times \mathrm{St}(0, \mathbf{R}) \cup \mathrm{St}(x, U) \times Lk(0, \mathbf{R}).$$

Now $Lk(0, \mathbf{R})$ is just two points, say 1 and -1, while in $Lk((x, 0), W)$,

$$\mathrm{St}((x, 1), Lk((x, 0), W))=\mathrm{St}(x, U) \times 1.$$

It follows that

$$Lk((x, 1), Lk((x, 0), W))=Lk(x, U) \times 1.$$

However, $Lk((x, 0), W)$ is a $\mathbf{Z}_2$-homology n-manifolds (Proposition 3). Therefore, $Lk(x, U)$ has the same homology group mod 2 as the $(n-1)$-

sphere. Thus we have obtained the lemma.

5. Fundamental theorem.

Definition. We say that a cohomology class $u \in H^k(A, Z_2)$ of a space A if *PL_k-realizable*, if there exists a mapping f: $A \to MPL_k$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class U_k of the Thom complex MPL_k of the universal PL-microbundle $\Upsilon(PL_k)$ of dimension k.

Then we have the following fundamental theorem.

Theorem 4. *Let V^n be a $\mathbf{Z}_2$-homology manifold of dimension n $(n \geqq 2)$. Then, in order that a homology class $z \in H_{n-k}(V^n, \mathbf{Z}_2)$, $k > 0$, can be realized by a $\mathbf{Z}_2$-homology submanifold W^{n-k} of dimension $(n-k)$ which has a normal PL-microbundle in V^n, it is necessary and sufficient that the cohomology class $u \in H^k(V^n, \mathbf{Z}_2)$, corresponding to z by the Poincaré duality, is PL_k-realizable.*

Proof. i) *Necessity.* $\mathbf{Z}_2$-homology manifolds satisfy the Poincaré duality with coefficient $\mathbf{Z}_2$ (Proposition 1). Therefore, the proof of the necessity is the same as that of PL-case in [1].

ii) *Sufficiency.* Let

$$\Upsilon(PL_k): BPL_k \xrightarrow{i_k} EPL_k \xrightarrow{j_k} BPL_k$$

be the universal PL-microbundle of dimension k. Suppose that there exists a mapping f of V^n into MPL_k such that $f^*(U_k) = u$. Then the Thom complex MPL_k, deprived the point $*$ at infinity, is considered as a locally finite simplicial complex, and PL-subspace BPL_k has the normal PL-microbundle $\Upsilon(PL_k)$ in $MPL_k - *$. By the Williamson's transversality theorem, we have a mapping $f_1: V^n \to MPL_k - *$, homotopic to f, t-regular for $(\gamma, \Upsilon(PL_k))$, where γ is a normal PL-microbundle of $(f_1)^{-1}(BPL_k)$ in V^n. However, by the lemma in §4, $(f_1)^{-1}(BPL_k)$ is a $\mathbf{Z}_2$-homology submanifold W^{n-k} of dimension $(n-k)$. Moreover, by the definition of t-regularity, the induced PL-microbundle $(f_1)^*\Upsilon(PL_k)$ is isomorphic to γ. We know $(f_1)^*(U_k) = f^*(U_k) = u$. Then, as in the proof of Theorem in [1], we can see that the $\mathbf{Z}_2$-homology submanifold W^{n-k} realized the homology class z, corresponding to u by the Poincaré duality. Thus we have obtained the theorem.

6. Proof of Theorem 1 and Theorem 2.

In [2], §2, we have obtained the following proposition.

Proposition 5. *Let* $n \geqq 2$. *Then there exists a mapping* g *of the* $2n$*-skeleton of* $\prod_i K(\mathbf{Z}_2, n+n_i)$ *to* MPL_n *such that* $h_n \circ g$ *and* $g \circ h_n$ *(restricted to the* $2n$*-skelton of* MPL_n*) are homotopic to the identities.* (h_n) is a mapping of MPL_n into $\prod_i K(\mathbf{Z}_2, n+n_i)$ defined by Browder-Liulevicius-Peterson; for precise see [2], §2).

Moreover, we know that MPL_1 has the homotopy type of $K(\mathbf{Z}_2, 1)$ (cf. [2], §2).

As in §3 of [2], Theorem 1 follows easily the fundamental theorem and Proposition 5. Theorem 2 follows also the fundamental theorem and the fact stated above.

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