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# The explosion problem of branching Lévy processes

By

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## §1. Introduction and Preliminary

In the paper [9] the author discussed the explosion problem of branching stable processes in connection with the problem of uniqueness and non-uniqueness of solutions for a class of non-linear integral equations (the $S$-equations of branching stable processes). The present paper is an extension of [9], and is devoted to strengthening of the conditions for explosion. First we shall give two sufficient conditions for explosion for a class of branching Lévy processes (Propositions 1 and 2 of §2). Then we shall apply the conditions to branching stable processes and branching Poisson processes, and explicitly distinguish explosion case from non-explosion case (Theorems 1 and 2 of §3, and Theorem 3 of §4). Finally we shall prove two comparison theorems for explosion of branching Lévy processes, and give some application of them (Theorems 4 and 5 of §5).

**1.** Let $X=(W, X_t, P_x, x\in R)$ be a Lévy process on the real line $R$, that is, a standard Markov process on $R$, homogeneous in space and time and characterized by the representation

$$E(\exp(i\xi X_t))=\exp\{t\Psi(\xi)\}\ ^{1)},$$

1) We denote $P_0(\cdot)$ and $E_0(\cdot)$ related to a Lévy process by $P(\cdot)$ and $E(\cdot)$, respectively.

$$\Psi(\xi)=i\gamma\xi-\frac{1}{2}\sigma^2\xi^2$$
$$+\int_{R\setminus\{0\}}\left(e^{i\xi y}-1-\frac{i\xi y}{1+y^2}\right)n(\mathrm{d}y),$$

where $\xi\in R$, $\gamma$ and $\sigma$ are real constants, and $n(\cdot)$ is the Lévy measure. The function $\Psi(\xi)$ is called the *Fourier exponent* of the Lévy process.

We call a Lévy process the *stable process of indices* $\{\alpha, \beta\}$ whenever its Fourier exponent is of the form

$$\Psi(\xi)=-|\xi|^\alpha\left(1-i\beta\operatorname{sgn}\xi\tan\frac{\pi\alpha}{2}\right)$$
$$\text{if}\quad \alpha\in(0,1)\cup(1,2)\quad\text{and}\quad -1\leqslant\beta\leqslant 1,$$

or

$$\Psi(\xi)=-|\xi|\quad\text{if}\quad\alpha=1\quad\text{and}\quad\beta=0.$$

We call a Lévy process with $\Psi(\xi)=-(1/2)\xi^2$ the *Brownian process* and a Lévy process with $\Psi(\xi)=e^{i\xi}-1$ the *Poisson process.*

**2.** Let $\{p_n; 2\leqslant n<\infty\}$ be a probability sequence, that is, $p_n\geqslant 0$ and $\sum_{n=2}^{\infty}p_n=1$, and define a stochastic kernel $\pi(x, dy)$ on $R\times\boldsymbol{R}$ by

$$\pi(x, dy)=\sum_{n=2}^{\infty}p_n\delta_{(\underbrace{x,\dots,x}_{n})}(dy), \tag{1}$$

where $\boldsymbol{R}=\bigcup_{0\leqslant n\leqslant\infty}R^n$ is the topological sum of product spaces of $R$ with $R^0=\{\partial\}$ and $R^\infty=\{\Delta\}$. Let $k(x)$ be a locally bounded non-negative measurable function on $R$.

Consider $(X, k, \pi)$-branching Markov process on the state space $\boldsymbol{R}$[2]. We call the process ($(X, k, \pi)$-) *branching Lévy process* and denote it by $\boldsymbol{X}=(\Omega, \boldsymbol{X}_t, \boldsymbol{P}_{\boldsymbol{x}}; \boldsymbol{x}\in\boldsymbol{R})$. The Lévy process $X$, the function $k(x)$ and the stochastic kernel $\pi$ are called *base* (*Lévy*) *process*, *killing rate* and *branching law* of $\boldsymbol{X}$, respectively. We call a branching Lévy process *branching stable process of indices* $\{\alpha, \beta\}$ whenever the base

2) Notations and terminologies on branching Markov processes are found in [3] and [9].

process is the stable process of indices $\{\alpha, \beta\}$. We call it *branching Poisson process* (*branching Brownian process*) whenever the base process is the Poisson process (resp. Brownian process).

Branching Lévy process is a standard Markov process, and the points $\partial$ and $\Delta$ are traps of it. $\partial$ represents the state that no Lévy particles exist (extinction), and $\Delta$ represents the state that infinitely many Lévy particles exist (explosion). Let $e_\Delta$ be the *explosion time* of a branching Lévy process, that is, $e_\Delta = \inf\{t;\ \boldsymbol{X}_t = \Delta\}$ [3]. Following [9] *a branching Lévy process is said to be non-explosive if* $\boldsymbol{P}_x(e_\Delta < \infty) = 0$ *for* $x \in R$, *explosive if* $\boldsymbol{P}_x(e_\Delta < \infty) > 0$ *for* $x \in R$, *and explosive with probability one if* $\boldsymbol{P}_x(e_\Delta < \infty) = 1$ *for* $x \in R$.

**3.** Let $\{p_n;\ 2 \leqslant n < \infty\}$ be the probability sequence in the previous subsection, and set

$$F(x;\ u) = \sum_{n=2}^{\infty} p_n u(x)^n \tag{2}$$

and

$$G(x;\ u) = 1 - F(x;\ 1-u), \qquad x \in R \tag{3}$$

for $u \in B_1^+(R)$ [4]. Consider the non-linear integral equation which is derived from the $S$-equation (of initial date 1) of $(X, k, \pi)$-branching Lévy process;

$$\begin{cases} u(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k(X_r)dr\Big\} k(X_s) G(X_s;\ u_{t-s})\Big) ds \ ^{5)} \\ 0 \leqslant u(t, x) \leqslant 1, \qquad (t, x) \in [0, \infty) \times R. \end{cases} \tag{4}$$

It is easy to see that (4) always has the trivial solution $u \equiv 0$. Moreover from [3; III, §4.3] or [9], we have

**Proposition 0.** *The maximal solution* $\bar{u}$ *of* (4) *is given by* $\bar{u}(t, x) = \boldsymbol{P}_x(e_\Delta \leqslant t)$. *Hence we obtain for the equation* (4) *the following*

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3) The infimum of an empty set is taken to be $\infty$.

4) For a measurable space $S$, $B_1^+(S) = \{f;$ measurable function on $S$ with $0 \leqslant f \leqslant 1\}$, where $u \leqslant v$ denotes $u(x) \leqslant v(x)$ for all $x \in S$.

5) $u(t, \cdot) = u_t$ by convention.

*assertion*:

**(i)** *If* $\boldsymbol{X}$ *is non-explosive, then the uniqueness of solutions holds, that is,* $\bar{u}(t, x)=0$ *for* $(t, x)\in[0, \infty)\times R$.

**(ii)** *If* $\boldsymbol{X}$ *is explosive, then the non-uniqueness of solutions holds, that is,* $\bar{u}(t, x)>0$ *for* $(t, x)\in(0, \infty)\times R$. *Moreover if* $\boldsymbol{X}$ *is explosive with probability one, then* $\lim_{t\to\infty}\bar{u}(t, x)=1$ *for* $x\in R$.

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## §2. Sufficient conditions for explosion

**1.** In §§2, 3 and 4 we restrict our consideration to branching Lévy processes which, with the branching law $\pi(x, d\boldsymbol{y})=\delta_{(x,x)}(d\boldsymbol{y})$, satisfy the following conditions:

($\boldsymbol{X}$-1) *The base Lévy process* $X$ *satisfies* $P(\sup_{0<t<\infty} X_t=\infty)=1$.

($\boldsymbol{X}$-2) *The killing rate* $k(x)$ *satisfies* $\lim_{x\to\infty} k(x)=\infty$.

**Remark 1.** A necessary and sufficient condition for Lévy processes to satisfy the condition ($\boldsymbol{X}$-1) is found in Rogozin [6], where he proved that all stable processes except those of indices $\{\alpha, -1\}$ with $0<\alpha<1$ satisfy ($\boldsymbol{X}$-1). The Brownian process and the Poisson process also satisfy ($\boldsymbol{X}$-1).

For each real valued function $f$ on $R, \hat{f}$ is a function on $\boldsymbol{R}$ defined by

$$\hat{f}(\boldsymbol{x})=\begin{cases}1 & \text{if } \boldsymbol{x}=\partial \\ f(x_1)\cdots f(x_n) & \text{if } \boldsymbol{x}=(x_1,\dots, x_n)\in R^n, \qquad 1\leqslant n<\infty \\ 0 & \text{if } \boldsymbol{x}=\Delta.\end{cases}$$

Following [9], let us define for $y\in R$, Markov times $j_y$ and $j^y$ for $X$, and $\boldsymbol{j}_y$ and $\boldsymbol{j}^y$ for $\boldsymbol{X}$ by

$$j_y = \inf\{t: X_t > y\}, \qquad j^y = \inf\{t: X_t < y\},$$

$$\boldsymbol{j}_y = \inf\{t: \hat{I}_{(-\infty, y]}(\boldsymbol{X}_t) = 0\}\ ^{6)},$$

$$\boldsymbol{j}^y = \inf\{t: \hat{I}_{[y,\infty)}(\boldsymbol{X}_t) = 0\},$$

where $I_E$ is the indicator function of a set $E$. Let $Z(t, \omega)$ $(t \geq 0, \omega \in \Omega)$ be the number of particles at $t$, that is, $Z(t, \omega) = n$ if $\boldsymbol{X}_t(\omega) \in R^n$, $0 \leq n \leq \infty$. The next lemma will play essential roles in later discussion.

**Lemma 1.** **(i)** *Let $x$ and $y$ be reals with $x \leq y$, then* $P_x(e_\Delta = \infty, j_y = \infty) = 0$.
**(ii)** $\boldsymbol{P}_x(Z(\boldsymbol{j}_y) \nearrow \infty$ *as* $y \nearrow \infty) = 1$.

**Comments for the proof.** The first part (i) is obtained from [9; Lemma 11] and ($\boldsymbol{X}$-1). The second part (ii) is obtained from ($\boldsymbol{X}$-1), ($\boldsymbol{X}$-2) and the conservativity of the base process.

The next corollary is a direct consequence of Lemma 1.

**Corollary 1.** *For each real $x$,* $\boldsymbol{P}_x(\lim_{y \to \infty} \boldsymbol{j}_y = e_\Delta) = 1$.

Consider the following sequences of numbers:

($S$-1) $h_n > 0$ $(n \geq 1)$ satisfying $\sum_n h_n = \infty$, and $H_n = \sum_{m=1}^{n} h_m$,

($S$-2) $l_n \geq 0$ $(n \geq 1)$ satisfying $l_n / H_n \longrightarrow 0$ as $n \longrightarrow \infty$,

($S$-3) $t_n > 0$ $(n \geq 1)$ satisfying $\sum_n t_n < \infty$,

($S$-4) positive integer $N_n$ $(n \geq 1)$ satisfying $N_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

For a positive integer $b$, set $k_n = \inf_{y \geq H_n - b l_n} k(y)$ and set

$$I = \sum_n (N_n)^{b-1} P(\inf_{t < t_n} X_t < -l_n)^{7)}$$

$$II = \sum_n \frac{(N_n)^{b-1}}{N_n!} \int_{k_n t_n}^{\infty} e^{-x} x^{N_n} \mathrm{d}x$$

6) Lemma 11 of [9] is not correct if $j_y$ is defined as [9; p. 45]. In order to make the lemma correct we should modify the definition of $j_y$ and $j^y$ as given here.

7) $\sum_n a_n$ denotes the sum of $a_n$ taken over all sufficiently large $n$.

$$III=\sum_{n}\int_{0}^{\infty}\{P(\sup_{s\leq t} X_s<bl_n+h_{n+1})\}^{(N_n)^b}\mathrm{d}t.$$

Then we have

**Proposition 1.** *If the sequences in (S-1)-(S-4) and positive integer b can be chosen so that I, II and III are finite, then the branching Lévy process is explosive with probability one.*

In order to give another sufficient condition for explosion, set $k'_n=\inf_{y>H_n-l_n} k(y)$ and for a positive integer $b$ set

$$IV=\sum_{n}\{P(\inf_{t<t_n} X_t<-l_n)+\frac{1}{(bN_n)!}\int_{k'_n t_n}^{\infty} e^{-x}x^{bN_n}\mathrm{d}x\}^{b}$$

$$V=\sum_{n}\int_{0}^{\infty}\{P(\sup_{s<t} X_s<l_n+h_{n+1})\}^{N_n}\mathrm{d}t.$$

Then we have

**Proposition 2.** *If the sequences in (S-1)-(S-4) and positive integer b can be chosen so that IV and V are finite, then the branching Lévy process is explosive with probability one.*

**Remark 2.** *When* $\mathbf{X}$ *is a branching stable process of indices* $\{\alpha,\beta\}$, *then I and III can be expressed in the following form by the space-time transformation of stable processes*:

$$I=\sum_{n}P(\inf_{t<1} X_t<-t_n^{-1/\alpha}l_n)$$

$$III=\sum_{n}\alpha(bl_n+h_{n+1})^{\alpha}\int_{0}^{\infty}\{P(\sup_{t<1} X_t<x)\}^{(N_n)^b}\frac{\mathrm{d}x}{x^{\alpha+1}}.$$

*IV and V also can be expressed in a similar way.*

**2.** Let us prove Proposition 1. First we fix an $\omega\in\Omega$ with $Z(0,\omega)\geq 1$ and $e_{\Delta}(\omega)=\infty$, and give some definitions (for simplicity, symbol $\omega$ will be omitted often).

Let $\Phi$ be a mapping from $\bigcup_{1\leq n<\infty} R_n$ to $R$ defined by $\Phi(\boldsymbol{x})=\max\{x_1,\ldots,x_n\}$ for $\boldsymbol{x}=(x_1,\ldots,x_n)\in R^n$. Define random variables $\boldsymbol{j}_{(n)}(\omega)$ and

$y_n(\omega)$ $(n \geqslant 1)$ by

(1) $$\boldsymbol{j}_{(n)}=\boldsymbol{j}_{H_n}, \qquad y_n=\Phi(\boldsymbol{X}(\boldsymbol{j}_{(n)}))\,.$$

Consider the branch of $\omega$ starting from $y_n$ at $\boldsymbol{j}_{(n)}$, and after shifting the time scale by $\boldsymbol{j}_{(n)}$, denote it by $\Theta_n\omega$. Define

$$\boldsymbol{i}_n=t_n \wedge \inf\{t:\ [\boldsymbol{X}_t(\Theta_n\omega)]_1<y_n-l_n\}\ ^{8)},$$

where $[\boldsymbol{X}_t(\cdot)]_1$ denotes the first coordinate of the vector $\boldsymbol{X}_t(\cdot)$. Let $z_n$ be the number of occurrences of splitting of $[\boldsymbol{X}_t(\Theta_n\omega)]_1$ in the time interval $[0, \boldsymbol{i}_n]$. For $k \geqslant 1$, let $\boldsymbol{j}_{n,k}$ and $y_{n,k}$ be the time and place, respectively, of the $k$-th splitting of $[\boldsymbol{X}_t(\Theta_n\omega)]_1$. Denote by $\Theta_{n,k}\omega$ the branch of $\Theta_n\omega$ starting from $y_{n,k}$ at $\boldsymbol{j}_{n,k}$ with the time scale shifted by $\boldsymbol{j}_{n,k}$. For the branch $\Theta_{n,k}\omega$, define

$$\boldsymbol{i}_{n,k}=t_n \wedge \inf\{t:\ [\boldsymbol{X}_t(\Theta_{n,k}\omega)]_1<y_{n,k}-l_n\}\,.$$

Let $z_{n,k}$ be the number of occurrences of splitting of $[\boldsymbol{X}_t(\Theta_{n,k}\omega)]_1$ in the time interval $[0, \boldsymbol{i}_{n,k}]$. For $\boldsymbol{k}_2=(k_1, k_2)$, $k_1, k_2 \geqslant 1$, let $\boldsymbol{j}_{n,\boldsymbol{k}_2}$ and $y_{n,\boldsymbol{k}_2}$ be the time and place, respectively, of the $k_2$-th splitting of $[\boldsymbol{X}_t(\Theta_{n,k_1}\omega)]_1$. Denote by $\Theta_{n,\boldsymbol{k}_2}\omega$ the branch of $\Theta_{n,k_1}\omega$ starting from $y_{n,\boldsymbol{k}_2}$ at $\boldsymbol{j}_{n,\boldsymbol{k}_2}$ with the time scale shifted by $\boldsymbol{j}_{n,\boldsymbol{k}_2}$.

Repeating similar procedures $b$ times, we can define random variables $\boldsymbol{i}_{n,\boldsymbol{k}_m}$, $z_{n,\boldsymbol{k}_m}$, $y_{n,\boldsymbol{k}_{m+1}}$ and the branch $\Theta_{n,\boldsymbol{k}_{m+1}}\omega$ $(1 \leqslant m \leqslant b-1$, $\boldsymbol{k}_m=(k_1,\dots, k_m)$ and $k_1,\dots, k_b \geqslant 1)$. For the branch $\Theta_{n,\boldsymbol{k}_b}\omega$, define

$$\boldsymbol{j}_{n,\boldsymbol{k}_b}=\inf\{t:\ \hat{I}_{(-\infty,H_{n+1}]}(\boldsymbol{X}_t(\Theta_{n,\boldsymbol{k}_b}\omega))=0\}\,.$$

Finally, setting

$$\bar{\boldsymbol{j}}_n=\min\{\boldsymbol{j}_{n,\boldsymbol{k}_b}:\ \boldsymbol{k}_b=(k_1,\dots, k_b),\ 1 \leqslant k_1 \leqslant z_n+1,\ 1 \leqslant k_2 \leqslant z_{n,k_1}+1,\ \dots,\ 1 \leqslant k_b \leqslant z_{n,\boldsymbol{k}_{b-1}}+1\}\,,$$

then we have

(2) $$\boldsymbol{j}_{(n+1)}-\boldsymbol{j}_{(n)} \leqslant bt_n+\bar{\boldsymbol{j}}_n.$$

Next we prove two lemmas, which are important in the proof of

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8) $a \wedge b=\min\{a, b\}$.

Proposition 1.

**Lemma 2.** *If the sequences in (S–1)–(S–4) and positive integer b can be chosen so that I and II are finite, then there exists a random variable* $\boldsymbol{n}$ *taking finite values and satisfying the following property a.s.* $(\boldsymbol{P}_x)$ *on* $\{e_\Delta=\infty\}$: *If* $n\geqslant\boldsymbol{n}$ *and* $1\leqslant k_1,\dots,k_{b-1}\leqslant N_n$, *then*

$$z_n>N_n,\ z_{n,k_1}>N_n,\dots,\ z_{n,(k_1,\dots,k_{b-1})}>N_n.$$

**Proof.** First of all we give several definitions. $\{Q(t);\,t\geqslant0\}$ is a Poisson process with $Q(0)\equiv0$, independent of $X$ and $\boldsymbol{X}$; $\varphi(t)=\int_0^t k(X_s)\mathrm{d}s$ and $D_{n,y}=\{Q(\varphi(j^{y-l_n}\wedge t_n))\leqslant N_n\}$. Consider a sequence of events $A_{n,0}=\{e_\Delta=\infty,\ z_n\leqslant N_n\}$, $n\geqslant1$. Let us estimate $\boldsymbol{P}_x(A_{n,0})$, using the strong Markov property and the branching property of $\boldsymbol{X}$ (see [3; I § 1.2.]).

$$(3)\quad \boldsymbol{P}_x(A_{n,0})\leqslant\boldsymbol{E}_x(\boldsymbol{j}_{(n)}<e_\Delta:\ P_{y_n}(D_{n,y})_{|y=y_n})$$

$$\leqslant\sup_{y>H_n}P_y(D_{n,y})=\sup_{y>H_n}E_y\Big(\sum_{m=0}^{N_n}\exp\{-\varphi(j^{y-l_n}\wedge t_n)\}$$

$$\times\{\varphi(j^{y-l_n}\wedge t_n)\}^m/(m!)\Big)=\frac{1}{N_n!}\sup_{y>H_n}E_y\Big(\int_{\varphi(j^{y-l_n}\wedge t_n)}^{\infty}e^{-x}x^{N_n}\mathrm{d}x\Big).$$

In the first inequality of (3) we employed the fact; $\boldsymbol{j}_{(n)}<e_\Delta$ a.s. $(\boldsymbol{P}_x)$ on $\{e_\Delta=\infty\}$ for $n\geqslant1$, which is a consequence of Lemma 1, (i). In the last equality we employed the next identity

$$(4)\qquad \sum_{m=0}^{n}e^{-a}\frac{a^m}{m!}=\frac{1}{n!}\int_a^\infty e^{-x}x^n\mathrm{d}x.$$

Set $k_{n,1}=\inf_{y>H_n-l_n}k(y)\ (\geqslant k_n)$. Since the last member in (3) is not greater than

$$P(j^{-l_n}\leqslant t_n)+\frac{1}{N_n!}\int_{k_{n,1}t_n}^{\infty}e^{-x}x^{N_n}\mathrm{d}x,$$

the finiteness of I and II gives $\sum_n\boldsymbol{P}_x(A_{n,0})<\infty$. Now let us apply the Borel-Cantelli lemma to $\{A_{n,0}\}$, then we obtain the following assertion: There exists a random variable $\boldsymbol{n}_1$ taking finite values such that, a.s. $(\boldsymbol{P}_x)$ on $\{e_\Delta=\infty\}$, $z_n>N_n$ for $n\geqslant\boldsymbol{n}_1$.

Let $n_1$ be an arbitrary fixed positive integer, and consider events $B_1=B_{(1;n_1)}=\{e_\Delta=\infty\}\cap\{z_n>N_n$ for all $n\geqslant n_1\}$ and $A_{n,1}=B_1\cap\{z_{n,k}\leqslant N_n$ for some $k=1,\dots,N_n\}$, $n\geqslant n_1$. Let us estimate $\boldsymbol{P}_x(A_{n,1})$ in a similar way as we did in (3).

$$\boldsymbol{P}_x(A_{n,1})\leqslant\sum_{k=1}^{N_n}\boldsymbol{P}_x(B_1\cap\{z_{n,k}\leqslant N_n\})\leqslant N_n\sup_{y>H_n-l_n}P_y(D_{n,y}). \tag{5}$$

Set $k_{n,2}=\inf_{y>H_n-2l_n}k(y)\ (\geqslant k_n)$. Since the last member in (5) is not greater than

$$N_nP(j^{-l_n}\leqslant t_n)+\frac{N_n}{N_n!}\int_{k_{n,2}t_n}^{\infty}e^{-x}x^{N_n}\mathrm{d}x,$$

the finiteness of $I$ and $II$ gives $\sum_n\boldsymbol{P}_x(A_{n,1})<\infty$. Again applying the Borel-Cantelli lemma to $\{A_{n,1}\}$, we obtain the assertion: There exists a random variable $\boldsymbol{n}_2$ taking finite values ($\boldsymbol{n}_2\geqslant n_1$ and $\boldsymbol{n}_2$ may depend on the choice of $n_1$) such that, a.s. ($\boldsymbol{P}_x$) on $B_1$, $z_{n,k}>N_n$ for all $n$, $k$ such that $n\geqslant\boldsymbol{n}_2$ and $1\leqslant k\leqslant N_n$.

Repeating similar procedures $b-1$ times and setting $B_{b-1}=B_{(b-1;n_1,\dots,n_{b-1})}=B_{(b-2;n_1,\dots,n_{b-2})}\cap\{z_{n,(k_1,\dots,k_{b-2})}>N_n$ for all $n$, $k$ such that $n\geqslant n_{b-1}$ and $1\leqslant k_1,\dots,k_{b-2}\leqslant N_n\}$ for an arbitrary fixed integer $n_{b-1}$ with $n_{b-1}\geqslant n_{b-2}\ (\geqslant\cdots\geqslant n_1)$, then from the finiteness of $I$ and $II$ we obtain the assertion: There exists a random variable $\boldsymbol{n}_b$ taking finite values ($\boldsymbol{n}_b\geqslant n_{b-1}$ and $\boldsymbol{n}_b$ may depend on the choice of $n_1,\dots,n_{b-1}$), such that, a.s. ($\boldsymbol{P}_x$) on $B_{b-1}$, $z_{n,(k_1,\dots,k_{b-1})}>N_n$ for all $n,k_1,\dots,k_{b-1}$ such that $n\geqslant\boldsymbol{n}_b$ and $1\leqslant k_1,\dots,k_{b-1}\leqslant N_n$.

Finally, setting $B_{(b;n_1,\dots,n_b)}=B_{(b;n_1,\dots,n_{b-1})}\cap\{z_{n,(k_1,\dots,k_{b-1})}>N_n$ for all $n,k_1,\dots,k_{b-1}$ such that $n\geqslant n_b$ and $1\leqslant k_1,\dots,k_{b-1}\leqslant N_n\}$ for an arbitrary fixed integer $n_b$ with $n_b\geqslant n_{b-1}$ and summarizing the above results, we have

$$\begin{aligned}\boldsymbol{P}_x(\{e_\Delta=\infty\}\setminus\bigcup_{n_1=1}^{\infty}B_{(1;n_1)})&=\boldsymbol{P}_x(B_{(1;n_1)}\setminus\bigcup_{n_2=n_1}^{\infty}B_{(2;n_1,n_2)})\\&=\cdots\\&=\boldsymbol{P}_x(B_{(b-1;n_1,\dots,n_{b-1})}\setminus\bigcup_{n_b=n_{b-1}}^{\infty}B_{(b;n_1,\dots,n_b)})=0\end{aligned}$$

$(1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_b)$, from which we conclude

$$P_x(\{e_\Delta = \infty\} \setminus \bigcup_{1 \leqslant n_1 \leqslant \cdots \leqslant n_b} B_{(b; n_1, \ldots, n_b)}) = 0.$$

This completes the proof.

Let $C_n$, $n \geqslant 1$, be a sequence of events defined by $C_n = \{e_\Delta = \infty\} \cap \{z_n > N_n, z_{n,k_1} > N_n, \ldots, z_{n,(k_1,\ldots,k_{b-1})} > N_n$ for all $k_1, \ldots, k_{b-1}$ such that $1 \leqslant k_1, \ldots, k_{b-1} \leqslant N_n\}$. It follows from Lemma 2 that

$$P_x(\{e_\Delta = \infty\} \setminus \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} C_m) = 0. \tag{6}$$

**Lemma 3.** *For each positive integer* $m$,

$$E_x(\bigcap_{n=m}^{\infty} C_n : \sum_{n=m}^{\infty} (j_{(n+1)} - j_{(n)})) \leqslant b \sum_{n=m}^{\infty} t_n + \tag{7}$$
$$\sum_{n=m}^{\infty} \int_0^{\infty} \{P(\sup_{s \leqslant t} X_s < b l_n + h_{n+1})\}^{(N_n)^b} dt.$$

**Proof.** First using the inequality (2), we have

(8) the left hand side of (7)
$$\leqslant \sum_{n=m}^{\infty} E_x(C_n : b t_n + \bar{j}_n) \leqslant b \sum_{n=m}^{\infty} t_n + \sum_{n=m}^{\infty} E_x(C_n : \bar{j}_n).$$

In order to estimate $E_x(C_n : \bar{j}_n)$, set $\zeta_n = \inf\{t : Q(\varphi(t)) = n\}$, where $Q(t)$ and $\varphi(t)$ are given in the proof of Lemma 2, $x_k = X(\zeta_k)$ and $\bar{D}(n, y) = \{Q(\varphi(j^{y-l_n})) > N_n\}$. Then by the strong Markov property and the branching property of $X$, we have

$$E_x(C_n : \bar{j}_n) \leqslant E_x(j_{(n)} < e_\Delta : \int_{\bar{D}(n, y_n)} P_{y_n}(dw) \times \tag{9}$$
$$\prod_{k_1=1}^{N_n} \int_{\bar{D}(n, x_{k_1}(w))} P_{x_{k_1}(w)}(dw_{k_1}) \cdots \prod_{k_{b-1}=1}^{N_n} \int_{\bar{D}(n, x_{k_{b-1}}(w_{k_{b-2}}))}$$
$$P_{x_{k_{b-1}}(w_{k_{b-2}})}(dw_{k_{b-1}}) \int_0^{\infty} \cdots \int_0^{\infty} \min\{t_{k_b} : k_b = (k_1, \ldots, k_b),$$

$$1\leqslant k_1,\dots,k_b\leqslant N_n\} \prod_{1\leqslant k_1,\dots,k_b\leqslant N_n} \boldsymbol{P}_{x_{k_b}(w_{k_{b-1}})}(\boldsymbol{j}_{y+h_{n+1}}\in \mathrm{d}t_{k_b})_{|y=y_n})$$ [9]

$$=\boldsymbol{E}_x(\boldsymbol{j}_{(n)}<e_\Delta:\int_{\bar{D}(n,y_n)} P_{y_n}(\mathrm{d}w)\prod_{k_1=1}^{N_n}\int_{\bar{D}(n,x_{k_1}(w))} P_{x_{k_1}(w)}(\mathrm{d}w_{k_1})$$

$$\cdots\prod_{k_{b-1}=1}^{N_n}\int_{\bar{D}(n,x_{k_{1-1}}(w_{k_{b-2}}))} P_{x_{k_{b-1}}(w_{k_{b-2}})}(\mathrm{d}w_{k_{b-1}})$$

$$\int_0^\infty \prod_{1\leqslant k_1,\dots,k_b\leqslant N_n} \boldsymbol{P}_{x_{k_b}(w_{k_{b-1}})}(\boldsymbol{j}_{y+h_{n+1}}>t)_{|y=y_n}\mathrm{d}t).$$

Now, using [9; Lemma 11], we have

$$\boldsymbol{P}_{x_{k_b}(w_{k_{b-1}})}(\boldsymbol{j}_{y+h_{n+1}}>t)_{|y=y_n}\leqslant P_{x_{k_b}(w_{k_{b-1}})}(j_{y+h_{n+1}}>t)_{|y=y_n}$$

$$\leqslant P_{y_n-bl_n}(j_{y+h_{n+1}}>t)_{|y=y_n}=P(j_{bl_n+h_{n+1}}>t)$$

for $w_{k_{b-1}}$ in the last expression of (9). Hence the last member of (9) does not exceed $\int_0^\infty\{P(j_{bl_n+h_{n+1}}>t)\}^{(N_n)^b}\mathrm{d}t$. This, combined with (8) and (9), gives the inequality (7).

**Proof of Proposition 1.** Choose the sequences in ($S$-1)-($S$-4) and and positive integer $b$ so that $I$, $II$ and $III$ are finite. By Corollary 1,

$$\sum_n(\boldsymbol{j}_{(n+1)}-\boldsymbol{j}_{(n)})=\infty \quad \text{a.s.} \quad (\boldsymbol{P}_x) \quad \text{on} \quad \{e_\Delta=\infty\}.$$

Suppose that $\boldsymbol{P}_x(e_\Delta=\infty)>0$ for some $x\in R$, then $\boldsymbol{P}_x(\{e_\Delta=\infty\}\cap\bigcap_{n=m}^\infty C_n)>0$ for some $m$ by (6). Hence we have

$$\boldsymbol{E}_x(\bigcap_{n=m}^\infty C_n:\sum_{n=m}^\infty(\boldsymbol{j}_{(n+1)}-\boldsymbol{j}_{(n)}))=\infty,$$

which contradicts the finiteness of $III$. This proves $\boldsymbol{P}_x(e_\Delta=\infty)=0$ for all $x\in R$, completing the proof.

**3.** Let us prove Proposition 2. First of all we introduce some notations. For a Markov time $T$ of $\boldsymbol{X}$, $\theta_T$ is the mapping from $\Omega$ into itself such that $\boldsymbol{X}_t(\theta_{T(\omega)}\omega)=\boldsymbol{X}_{t+T(\omega)}(\omega)$ for all $t\geqslant 0$. For an $\omega\in\Omega$

9) $\int_D f(w)P_x(\mathrm{d}w)=E_x(D;f(w))$.

consider a subgroup of the branching Lévy particles that exist at $t=0$. When we denote the history of the subgroup for $t\geqslant 0$ by $\omega'$ $(\in\Omega)$, let us denote the history of the complementary subgroup by $\omega\backslash\omega'$ $(\in\Omega)$.

Now we give some definitions related to an $\omega\in\Omega$ with $Z(0,\omega)\geqslant 1$ and $e_\Delta(\omega)=\infty$ as we did in the previous subsection. Let $\boldsymbol{j}_{(n)}(\omega)$, $y_n(\omega)$ and $\Theta_n\omega$ $(n\geqslant 1)$ be those defined in the previous subsection. For arbitrary fixed positive integers $b$ and $m$, $B_m=\{e_\Delta=\infty,\ Z(\boldsymbol{j}_{(m)})\geqslant b\}$. For $\omega\in B_m$ and $n\geqslant m$, let us define $\boldsymbol{j}_{n,k}$, $\boldsymbol{i}_{n,k}$ and $z_{n,k}$ by induction of $k=1,2,\dots,b$.

$$\Theta_{n,1}\omega=\Theta_n\omega,\quad \Theta'_{n,1}\omega=\theta_{\boldsymbol{j}_{(n)}(\omega)}\omega\backslash\Theta_{n,1}\omega,$$

$$\Theta_{n,k+1}\omega=\Theta_n\Theta'_{n,k}\omega,\quad \Theta'_{n,k+1}\omega=\theta_{\boldsymbol{j}_{(n)}(\Theta'_{n,k}\omega)}\Theta'_{n,k}\omega\backslash\Theta_{n,k+1}\omega$$

$$(1\leqslant k\leqslant b-1).$$

$\boldsymbol{j}_{n,1}=\boldsymbol{j}_{(n)}$ and $\boldsymbol{j}_{n,k+1}=\boldsymbol{j}_{n,k}+\boldsymbol{j}'_{n,k+1}$ $(1\leqslant k\leqslant b-1)$, where

$$\boldsymbol{j}'_{n,k}=\inf\{t\colon \hat{I}_{(-\infty,H_n]}(\boldsymbol{X}_t(\Theta'_{n,k}\omega))=0\}.$$

$y_{n,1}=y_n$ and $y_{n,k+1}=\Phi(\boldsymbol{X}(\boldsymbol{j}'_{n,k},\ \Theta'_{n,k}\omega))$ $(1\leqslant k\leqslant b-1)$. For $1\leqslant k\leqslant b$, define

$$\boldsymbol{i}_{n,k}=t_n\wedge\inf\{t\colon [\boldsymbol{X}_t(\Theta_{n,k}\omega)]_1<y_{n,k}-l_n\}.$$

Let $z_{n,k}$ be the number of occurrences of splitting of $[\boldsymbol{X}_t(\Theta_{n,k}\omega)]_1$ in the time interval $[0,\boldsymbol{i}_{n,k}]$.

Following two lemmas will play similar role in the proof of Proposition 2 as Lemmas 2 and 3 did in the proof of Proposition 1.

**Lemma 4.** *If the sequences in (S-1)-(S-4) and positive integer $b$ can be chosen so that IV are finite, then there exists a random variable $\boldsymbol{n}'$ $(\geqslant\min\{n\colon Z(\boldsymbol{j}_{(n)})\geqslant b\})$ taking finite values and satisfying the following property a.s. $(\boldsymbol{P}_x)$ on $\{e_\Delta=\infty\}$: For each $n\geqslant\boldsymbol{n}'$, there exists an integer $k$ with $1\leqslant k\leqslant b$ for which $z_{n,k}>bN_n$ holds.*

**Proof.** Let $D_{n,y}$ be the event defined in the proof of Lemma 2 (of course we should replace $N_n$ in the previous definition with $bN_n$

here). Consider a sequence of events $A_{n,m}=B_m\cap(\bigcap_{k=1}^{b}\{z_{n,k}\leqslant bN_n\})$, $n\geqslant m$. Then by the strong Markov property and the branching property of $\boldsymbol{X}$ (refer to the estimate (3) of Lemma 2),

$$\begin{aligned}\boldsymbol{P}_x(A_{n,m})&\leqslant\{\sup_{y>H_n}P_y(D_{n,y})\}^b\\&\leqslant\left\{P(j^{-l_n}<t_n)+\frac{1}{(bN_n)!}\int_{k'_nt_n}^{\infty}e^{-x}x^{bN_n}dx\right\}^b,\end{aligned}$$

where $k'_n=\inf_{y>H_n-l_n}k(y)$. Therefore if $IV$ is finite, then $\sum_n\boldsymbol{P}_x(A_{n,m})<\infty$. Hence the Borel-Cantelli lemma implies that the assertion of Lemma 4 is valid if we replace the phrase "on $\{e_\Delta=\infty\}$" in it with "on $B_m$". Moreover, since $\boldsymbol{P}_x(\{e_\Delta=\infty\}\setminus\bigcup_{m=1}^{\infty}B_m)=0$ because of Lemma 1, (ii), it follows that Lemma 4 is valid in itself. This completes the proof.

Now set $\bar{A}_{n,m}=B_m\cap(\bigcup_{k=1}^{b}\{z_{n,k}>bN_n\})$ for $n\geqslant m$. It follows from Lemma 4 that

$$\boldsymbol{P}_x(\{e_\Delta=\infty\}\setminus\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\bar{A}_{n,m})=0. \tag{10}$$

For $\omega\in\bar{A}_{n,m}$ $(n\geqslant m)$ define

$$k_0=k_0(n,\omega)=\min\{k\colon 1\leqslant k\leqslant b\text{ with }z_{n,k}>bN_n\}.$$

Consider the first $bN_n$ branches split off from $[\boldsymbol{X}_t(\Theta_{n,k_0}\omega)]_1$ in the time interval $[0,\boldsymbol{i}_{n,k_0}]$ and classify them into $b$ groups as follows: From the first split branch to the $N_n$-th one to the first group $G_1$, from the $(N_n+1)$-th one to the $2N_n$-th one to the second group $G_2,\ldots,$ and from the $((b-1)N_n+1)$-th one to the $bN_n$-th one to the $b$-th group $G_b$. We note that each group $G_k$ $(1\leqslant k\leqslant b)$ consists of just $N_n$ branches of $\Theta_{n,k_0}\omega$. Let $\boldsymbol{j}_{(n,k_0,k)}$ $(1\leqslant k\leqslant b)$ be a random variable defined by

$$\boldsymbol{j}_{(n,k_0,k)}=\min\{\boldsymbol{j}_{H_{n+1}}(\omega)\colon \omega\in G_k\}.$$

Then by the definitions given in this subsection, we have

$$\begin{aligned}\boldsymbol{j}_{n+1,b}-\boldsymbol{j}_{n,b}&\leqslant t_n+\max\{\boldsymbol{j}_{(n,k_0,k)}\colon 1\leqslant k\leqslant b\}\\&\leqslant t_n+\sum_{k=1}^{b}\boldsymbol{j}_{(n,k_0,k)}\quad\text{on}\quad\bar{A}_{n,m}.\end{aligned} \tag{11}$$

Using the inequality (11), we have

**Lemma 5.** *For each positive integer $m$,*

$$E_x\left(\bigcap_{n=m}^{\infty} \bar{A}_{n,m}: \sum_{n=m}^{\infty} (j_{n+1,b} - j_{n,b})\right) \tag{12}$$

$$\leqslant \sum_{n=m}^{\infty} t_n + b^2 \sum_{n=m}^{\infty} \int_0^{\infty} \{P(\sup_{s<t} X_s < l_n + h_{n+1})\}^{N_n} dt.$$

Proof of Lemma 5 is similar to that of Lemma 3 and we omit the detail here.

Now we are at the final step of proof of Proposition 2. It is similar to that of Proposition 1, and we obtain Proposition 2.

## §3. Explosion of branching stable processes

In order to simplify the situation, we make the following additional assumption on branching stable processes to be considered in this section.

($\boldsymbol{X}$-3) *The killing rate $k(x)$ is bounded for $x<0$.*

Main results of this section are the following two theorems.

**Theorem 1.** *Consider a branching stable process of indices $\{\alpha, \beta\}$ with $\alpha \in (0, 1) \cup (1, 2)$, $-1<\beta\leqslant 1$ or indices $\{1, 0\}$.*
(i) *Let the killing rate be such that $k(x) \asymp x^{\gamma}$* [10] *as $x \to \infty$. Then for any constant $\gamma>0$, the process is explosive with probability one.*
(ii) *Let the killing rate be such that $k(x) \asymp \log x$ as $x \to \infty$, then the process is non-explosive.*

**Theorem 2.** *Consider a branching stable process of indices $\{\alpha, -1\}$ with $1<\alpha<2$, and let the killing rate be such that $k(x) \asymp x^{\gamma}$ as $x \to \infty$. Then the process is explosive with probability one or non-explosive according as the constant $\gamma>\alpha/(\alpha-1)$ or $\gamma \leqslant \alpha/(\alpha-1)$, respectively.*

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10) $f(x) \asymp g(x)$ as $x \to c \Leftrightarrow 0 < \varliminf_{x\to c} \{f(x)/g(x)\} \leqslant \varlimsup_{x\to c} \{f(x)/g(x)\} < \infty$.

## 1. Proof of the case of explosion in Theorems 1 and 2

First we choose the sequences of numbers in $(S\text{-}1)$-$(S\text{-}4)$ as follows;

(1) $$h_n=l_n=n^{\delta},\quad t_n=n^{-\mu}\quad\text{and}\quad N_n=[n^{\nu}]$$ [11]

where $\delta$, $\mu$ and $\nu$ are constants satisfying the condition;

(2) $$\delta>-1,\quad \mu>1\quad\text{and}\quad \nu>0.$$

Let $M_{+}=\sup_{t\leq 1}X_t$ and $M_{-}=\inf_{t\leq 1}X_t$. Recalling Remark 2, we see that $I$, $II$ and $III$ of Proposition 1 are

(3) $$I=\sum_n([n^{\nu}])^{b-1}P(M_{-}<-n^{\delta+\mu/\alpha})$$

(4) $$II\lesssim\sum_n\{([n^{\nu}])^{b-1}/[n^{\nu}]!\}\int_{C_1 n^{\gamma(\delta+1)-\mu}}^{\infty}e^{-x}x^{[n^{\nu}]}\mathrm{d}x$$ [12]

(5) $$III\lesssim C_2\sum_n n^{\alpha\delta}\int_0^{\infty}\{P(M_{+}<x)\}^{[n^{\nu}]^b}\frac{\mathrm{d}x}{x^{\alpha+1}}$$

and $IV$ and $V$ of Proposition 2 are

(6) $$IV\lesssim\sum_n\{P(M_{-}<-n^{\delta+\mu/\alpha})+\{1/(b[n^{\nu}])!\}\int_{C_2 n^{\gamma(\delta+1)-\mu}}^{\infty}e^{-x}x^{b[n^{\nu}]}\mathrm{d}x\}^b$$

(7) $$V\lesssim C_4\sum_n n^{\alpha\delta}\int_0^{\infty}\{P(M_{+}<x)\}^{[n^{\nu}]}\frac{\mathrm{d}x}{x^{\alpha+1}},$$

where $b$ is a positive integer, and $C_i$ $(1\leqslant i\leqslant 4)$ are positive constants. Next we list the conditions for the above quantities to be finite.

**Lemma 6.** (i) *Let the indices be either* $\{\alpha,\beta\}$ *with* $\alpha\in(0,1)\cup(1,2)$, $-1<\beta<1$ *or* $\{1,0\}$. *Then* $I$ *is finite if* $\alpha\delta+\mu-(b-1)\nu>1$.
(ii) *Let the indices be* $\{\alpha,1\}$ *with* $1<\alpha<2$. *Then* $I$ *is finite if* $\alpha\delta+\mu>0$.

**Lemma 7.** *II is finite if* $\gamma(\delta+1)-\mu>\nu$.

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11) $[a]$ is the greatest integer not exceeding $a$.
12) For two positive quantities $A$ and $B$, we write $A\lesssim B$ whenever the following two conditions hold:
(i) $A\leqslant B$. (ii) $A<\infty$ if and only if $B<\infty$.

**Lemma 8.** *Let the indices be either* $\{\alpha, \beta\}$ *with* $\alpha \in (0, 1) \cup (1, 2)$, $-1<\beta \leqslant 1$ *or* $\{1, 0\}$. *Then III is finite if* $\alpha\delta - bv < -1$.

**Lemma 9.** *Let the indices be* $\{\alpha, -1\}$ *with* $1<\alpha<2$. *Then IV is finite if both* $b(\alpha\delta+\mu)>1$ *and* $\gamma(\delta+1)-\mu>v$ *hold.*

**Lemma 10.** *Let the indices be* $\{\alpha, -1\}$ *with* $1<\alpha<2$. *Then V is finite if* $\alpha\delta<-1$.

Proof of Lemmas 6 to 10 follows from Lemmas 11 and 12 below. Define $\psi(x)=P(M_+<x)$ and $J(n)=\int_0^\infty \psi^n(x)x^{-\alpha-1}\mathrm{d}x$, then we have

**Lemma 11.** **(i)** *Let the indices be either* $\{\alpha, \beta\}$ *with* $\alpha \in (0, 1) \cup (1, 2)$, $-1<\beta \leqslant 1$ *or* $\{1, 0\}$, *then* $J(n) \asymp n^{-1}$ *as* $n \to \infty$.
**(ii)** *Let the indices be* $\{\alpha, -1\}$ *with* $1<\alpha<2$, *then* $J(n) \asymp (\log n)^{1-\alpha}$ *as* $n \to \infty$.

**Proof.** For $0<\varepsilon<\kappa$, put

$$J(n)=\left\{\int_0^\varepsilon+\int_\varepsilon^\kappa+\int_\kappa^\infty\right\}\psi^n(x)\frac{\mathrm{d}x}{x^{\alpha+1}}=J_1(n)+J_2(n)+J_3(n). \tag{8}$$

Let us estimate $J_i(n)$ $(i=1, 2, 3)$. For $J_1(n)$, because of [1; Theorem 3a], we have

$$J_1(n) \leqslant \int_0^\varepsilon (d_1 x^{\alpha\rho})^n \frac{\mathrm{d}x}{x^{\alpha+1}} = (d_1\varepsilon^{\alpha\rho})^n/\{\alpha\varepsilon^\alpha(\rho n-1)\},$$

where $\rho=P(X_1>0)$, and $\varepsilon$ is chosen sufficiently small so that $d_1\varepsilon^{\alpha\rho}<1$.

For $J_2(n)$, put $q=\psi(\kappa)$, then $0<\psi(x)\leqslant q<1$ for $\varepsilon \leqslant x \leqslant \kappa$. Hence

$$J_2(n) \leqslant \alpha^{-1}(\varepsilon^{-\alpha}-\kappa^{-\alpha})q^n.$$

Estimation of $J_3(n)$ for the case (i). Because of [1; Theorem 4a], we have for some choice of a positive constant $d_2$

$$J_3(n) \leqslant \int_\kappa^\infty (1-d_2x^{-\alpha})^n \frac{\mathrm{d}x}{x^{\alpha+1}}$$

$$\left(or \geqslant \int_{\kappa}^{\infty} (1-d_2 x^{-\alpha})^n \frac{dx}{x^{\alpha+1}}\right) \asymp n^{-1} \quad as \quad n \longrightarrow \infty.$$

Hence it follows that $J_3(n) \asymp n^{-1}$ as $n \to \infty$.

Estimation of $J_3(n)$ for the case (ii). Because of [1; Proposition 3b], we have for some choice of positive constants $d_3$ and $d_4$

$$(9) \qquad J_3(n) \leqslant \int_{\kappa}^{\infty} \{1-d_3 \exp(-d_4 x^{\eta})\}^n \frac{dx}{x^{\alpha+1}}$$

$$\left(or \geqslant \int_{\kappa}^{\infty} \{1-d_3 \exp(-d_4 x^{\eta})\}^n \frac{dx}{x^{\alpha+1}}\right)$$

$$= \frac{\alpha-1}{\alpha} (d_4)^{\alpha-1} \int_0^{\kappa'} e^{-(n+1)y} \frac{dy}{(1-e^{-y})\{\log d_3 - \log(1-e^{-y})\}^{\alpha}}$$

where $\eta = \alpha/(\alpha-1)$, $y = -\log\{1-d_3 \exp(-d_4 x^{\eta})\}$ and $\kappa' = -\log\{1-d_3 \exp(-d_4 \kappa^{\eta})\}$. Since $1/\{(1-e^{-y})\{\log d_3 - \log(1-e^{-y})\}^{\alpha}\} \sim 1/\{y(-\log y)^{\alpha}\}$ $(y \to 0+)$ and since $(-\log y)$ is a slowly varying function at 0, we have from the Abelian Theorem [2], the last member of (9)$\asymp (\log n)^{1-\alpha}$ as $n \to \infty$[13]. Hence it follows that $J_3(n) \asymp (\log n)^{1-\alpha}$ as $n \to \infty$.

Finally applying these estimates for $J_i(n)$ $(i=1, 2, 3)$ to (8), we complete the proof.

**Lemma 12.** **(i)** *Let $C$ be a constant with $C>1$, then for every positive integer $n$ large enough and for $r \geqslant Cn$,*

$$\frac{1}{n!} \int_r^{\infty} e^{-x} x^n dx \leqslant C_1 \exp(-C_2 n),$$

*where $C_1$ and $C_2$ are positive constants given by $C_1 = C/\{\sqrt{\pi}(C-1)\}$ and $C_2 = C - \log C - 1$.*

**(ii)** *Let $C$ be a constant with $0 \leqslant C < 1$, then*

$$\frac{1}{n!} \int_r^{\infty} e^{-x} x^n dx \longrightarrow 1 \quad as \quad n \longrightarrow \infty (r \longrightarrow \infty)$$

*under $r \leqslant Cn$.*

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13) $f(x) \sim g(x)$ as $x \to c \Leftrightarrow \lim_{x \to c} \{f(x)/g(x)\} = 1$.

**Proof.** (**i**) When $C>1$ and $r\geqslant Cn$, it is easy to see that $\max_{r<x}(e^{-x/C}x^n)=e^{-r/C}r^n$. Therefore for $r\geqslant Cn$,

$$\int_r^\infty e^{-x}x^n\mathrm{d}x=\int_r^\infty e^{-(1-1/C)x}(e^{-x/C}x^n)\mathrm{d}x$$

$$\leqslant e^{-r/C}r^n\int_r^\infty e^{-(1-1/C)x}\mathrm{d}x=\frac{C}{C-1}e^{-r}r^n.$$

Now applying the Stirling's formula $n!\sim\sqrt{2\pi}\,n^{n+1/2}e^{-n}$ $(n\to\infty)$, then for every positive integer $n$ large enough, we have

$$\text{(10)}\qquad \frac{1}{n!}\int_r^\infty e^{-x}x^n\mathrm{d}x\leqslant\frac{C}{\sqrt{\pi}\,(C-1)}(r/n)^n e^{-r+n}n^{-1/2}$$

$$=C_1\exp\{(\log r-\log n+1)n-r-(1/2)\log n\}.$$

Now put $r=C'n$ $(C'\geqslant C)$, then

(11) the exponent of the last member of (10)

$$=-(C'-\log C'-1)n-(1/2)\log n$$

$$\leqslant-(C-\log C-1)n=-C_2 n.$$

In the inequality of (11) we used the fact; $f(C)=C-\log C-1$ is a strictly increasing function for $C\geqslant 1$, so that $f(C')\geqslant f(C)=C_2>f(1)=0$ for $C'\geqslant C>1$. By (10) and (11), we have proved (i).

(**ii**) Note that

$$\frac{1}{n!}\int_r^\infty e^{-x}x^n\mathrm{d}x=\sum_{m=0}^{n}e^{-r}\frac{r^m}{m!}=P(Q_r\leqslant n)=P(Q_r/r\leqslant n/r),$$

where $\{Q_r\colon r\geqslant 0\}$ is a Poisson process with $Q_0\equiv 0$. And the law of large number implies that $P(Q_r/r\leqslant n/r)\geqslant P(Q_r/r\leqslant 1/C)\to 1$ as $r\to\infty$ $(n\to\infty)$ under $n/r\geqslant 1/C>1$. Thus we have proved (ii).

**Comments for the proof of Lemmas 6 to 10.** Since Lemmas 6 to 10 are direct consequences of Lemmas 11, 12 and [1; Proposition 3b, Theorem 4a], we will not give detailed proof, instead, we state a few comments. Lemma 6, (i) (Lemma 6, (ii)) is proved if we apply

[1; Theorem 4a] (resp. [1; Proposition 3b]) to the right hand side of (3). In the application we remark that $P(\inf_{t\leq 1} X_t < -x) = P(\sup_{t\leq 1}(-X_t) > x)$ and that $(-X_t)$ is a stable process of indices $\{\alpha, -\beta\}$ whenever $X_t$ is that of indices $\{\alpha, \beta\}$: Lemma 7 (Lemma 8) is proved if we apply Lemma 12, (i) (resp. Lemma 11, (i)) to the right hand side of (4) (resp. (5)): Lemma 9 (Lemma 10) is proved if we apply [1; Theorem 4a] and Lemma 12, (i) (resp. Lemma 11, (ii)) to the right hand side of (6) (resp. (7)).

**Proof of Theorem 1, (i).** The proof is based on Proposition 1, and it takes following form on account of Lemmas 6, 7 and 8.
**(i–1)** Let the indices be either $\{\alpha, \beta\}$ with $\alpha \in (0, 1) \cup (1, 2)$, $-1 < \beta < 1$ or $\{1, 0\}$. If we can find $\delta$, $\mu$ and $\nu$ in (1) and (2), and positive integer $b$ satisfying a set of inequalities

$$(12) \qquad \begin{cases} \alpha\delta + \mu - (b-1)\nu > 1, \qquad \gamma(\delta+1) - \mu > \nu, \\ \alpha\delta - b\nu < -1, \end{cases}$$

then the process is explosive with probability one. (12) has a solution if $\gamma > 0$.
**(i–2)** Let the indices be $\{\alpha, 1\}$ with $0 < \alpha < 1$ or $\{\alpha, 1\}$ with $1 < \alpha < 2$, then the set of inequalities on $\delta$, $\mu$ and $\nu$ in (1) and (2), and positive integer $b$ are

$$(13) \qquad \gamma(\delta+1) - \mu > \nu, \qquad \alpha\delta - b\nu < -1$$

or

$$(14) \qquad \alpha\delta + \mu > 0, \quad \gamma(\delta+1) - \mu > \nu, \quad \alpha\delta - b\nu < -1,$$

respectively. Both (13) and (14) have solutions if $\gamma > 0$. That establishes Theorem 1, (i).

**Proof of Theorem 2 (the case of explosion).** The proof is based on Proposition 2, and it takes following form on account of Lemmas 9 and 10. If we can find $\delta$, $\mu$ and $\nu$ in (1) and (2), and positive integer $b$ satisfying a set of inequalities

(15) $$b(\alpha\delta+\mu)>1, \quad \gamma(\delta+1)-\mu>\nu, \quad \alpha\delta<-1,$$

then the process is explosive with probability one. (15) has a solution if $\gamma>\alpha/(\alpha-1)$. That establishes Theorem 2 (the case of explosion).

### 2. Proof of the case of non-explosion in Theorems 1 and 2

A useful sufficient condition for non-explosion is given in [5; §5.14.] or [9; Theorem 2], and we adopt it for the proof. Because a similar discussion was precisely given in [9; Theorem 3], we give only a few comments for it here. Note that by the Jensen's inequality, we have

(16) $$E\left(\exp\left\{\int_0^t k(X_s)\mathrm{d}s\right\}\right)\leqslant\frac{1}{t}\int_0^t E(\exp\{k(X_s)\})\mathrm{d}s = \frac{1}{t}\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}F_s(y)\exp\{k(y)\},$$

where $F_s(y)=P(X_s\leqslant y)$. Let us apply on the last member of (16) the asymptotic property of stable distribution function $F_s(y)$ $(s>0)$ for $y\to\infty$ (see Skorohod [10]). Then for each killing rate $k(y)$ considered in this subsection, we see that the last member is finite for some $t>0$, which establishes the result just desired.

## §4. Explosion of branching Poisson processes

In this section we show that Proposition 1 is also valid to another typical branching Lévy processes–branching Poisson processes, and the results are as follows.

**Theorem 3.** *Consider a branching Poisson process, and let the killing rate be such that* $k(x)\asymp x^{\gamma}$ *as* $x\to\infty$. *Then the process is explosive with probability one or non-explosive according as the constant* $\gamma>1$ *or* $\gamma\leqslant 1$, *respectively*

Consider a semi-linear differential-difference equation of the form:

(1) $$\begin{cases} \dfrac{\partial u}{\partial t}(t, x)=u(t, x+1)-u(t, x)+x^{\gamma}u(t, x)(1-u(t, x)) \\ 0\leqslant u(t, x)\leqslant 1, \quad u(0, x)=0, \quad t>0, \quad x\geqslant 0. \end{cases}$$

**Corollary 2.** *For the equation* (1), *the uniqueness or non-uniqueness of solutions holds according as the constant* $\gamma \leqslant 1$ *or* $\gamma > 1$, *respectively. Moreover in the case of non-uniqueness, for the maximal solution* $\bar{u}$, $\lim_{t\to\infty} \bar{u}(t, x) = 1$ *for* $x \geqslant 0$.

**Proof of Theorem 3. The case of explosion.** First we choose the sequences of numbers in $(S-1)$–$(S-4)$ as follows:

$$(2)\qquad \begin{cases} h_n = 1 = \text{the height of a jump of a Poisson process,} \\ (\text{and therefore } H_n = n),\ l_n = 0, \\ t_n = n^{-\mu} \text{ and } N_n = [n^{\nu}], \end{cases}$$

where $\mu$ and $\nu$ are constants satisfying

$$(3)\qquad \mu > 1 \quad \text{and} \quad \nu > 0.$$

Then $I$, $II$ and $III$ of Proposition 1 are $I = 0$ and

$$(4)\qquad II \lesssim \sum_n \frac{[n^{\nu}]^{b-1}}{[n^{\nu}]!} \int_{C_1 n^{\nu-\mu}}^{\infty} e^{-x} x^{[n^{\nu}]} \mathrm{d}x,$$

$$(5)\qquad III = \sum_n [n^{\nu}]^{-b},$$

where $b$ is a positive integer and $C_1$ is a positive constant. Here we note that (5) follows from

$$\int_0^{\infty} \{P(\sup_{s \leqslant t} X_s < 1)\}^{[n^{\nu}]^b} \mathrm{d}t = \int_0^{\infty} \{P(X_t < 1)\}^{[n^{\nu}]^b} \mathrm{d}t$$

$$= \int_0^{\infty} \exp(-[n^{\nu}]^b t) \mathrm{d}t = [n^{\nu}]^{-b}.$$

Now applying Proposition 1 and Lemma 13, (i) in this case, we have the following: If we can find $\mu$ and $\nu$ in (2) and (3), and positive integer $b$ satisfying

$$(6)\qquad \gamma - \mu > \nu \quad \text{and} \quad b\nu > 1,$$

then the process is explosive with probability one. (6) has a solution if the constant $\gamma > 1$. That establishes the result on explosion.

**The case of non-explosion.** The proof is similar to that of non-explosion for Theorems 1 and 2 given in 2 of §3. Hence we omit the detail, instead, give a comment: When $X$ is a Lévy process with a Lévy measure of bounded support including a Poisson process, satisfactory information on the asymptotic property of distribution function $F_s(y)=P(X_s\leqslant y)$ $(s>0)$ for $y\to\infty$ are found in [7].

**Proof of Corollary 2.** Consider the equation (1) in §1 of a branching Poisson process with $k(x)=x^\gamma$ $(x\geqslant 0)$ and $\pi(x, d\boldsymbol{y})=\delta_{(x,x)}(d\boldsymbol{y})$

$$\text{(7)}\qquad \begin{cases} u(t, x)=\displaystyle\int_0^t E_x\Big(\exp\Big\{-\int_0^s Q^\gamma(r)dr\Big\}Q^\gamma(s)u(t-s, Q(s)) \\ \qquad\qquad\qquad\qquad \times\{2-u(t-s, Q(s))\}\Big)ds \\ 0\leqslant u(t, x)\leqslant 1, \quad (t, x)\in[0, \infty)\times[0, \infty), \end{cases}$$

where $(W, Q(t), P_x;\ x\geqslant 0)$ is a Poisson process.

To prove the corollary, we only need to show the equivalence of equations (1) and (7) because of 4 of §1. For the first step, let us prove the equivalence of (1) and the following equation

$$\text{(8)}\qquad \begin{cases} u(t, x)=\displaystyle\int_0^t E_x(Q^\gamma(s)u(t-s, Q(s))\{1-u(t-s, Q(s))\})ds \\ 0\leqslant u(t, x)\leqslant 1, \quad (t, x)\in[0, \infty)\times[0, \infty). \end{cases}$$

Differentiate on both sides of the equality (8) with respect to $t$, then we obtain the equality (1). (Note that the infinitesimal operator $\mathscr{T}$ of a Poisson process is the following difference operator; $\mathscr{T}f(x)=f(x+1)-f(x)$.) Moreover remark that bounded solution of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x)=u(t, x+1)-u(t, x) \\ u(0, x)=0, \quad t>0, \quad x\geqslant 0 \end{cases}$$

is only the trivial solution $u\equiv 0$. Thus we proved the first step. The second step consists of proving the equivalence of equations (7) and (8). A proof will be given by Lemma 14 in the next section (note also Remark 3), so we omit it. Thus we proved the equivalence of equations (1) and (7). This completes the proof.

## §5. Comparison theorems for explosion

**1.** In this section we consider branching Lévy processes with the general branching law given in §1, (1), and note also we need not assume on them any of the conditions ($\boldsymbol{X}$-1)-($\boldsymbol{X}$-3) in the previous sections. Now let us state our main results.

**Theorem 4.** *Let $\boldsymbol{X}_i$ $(i=1, 2)$ be $(X, k_i(x), \pi)$-branching Lévy processes, and suppose that $k_1(x) \leqslant k_2(x)$ for $x \in R$. Then*

$$\boldsymbol{P}_x^{(1)}(e_\Delta^{(1)} \leqslant t) \leqslant \boldsymbol{P}_x^{(2)}(e_\Delta^{(2)} \leqslant t) \qquad \textit{for} \quad (t, x) \in [0, \infty) \times R,$$

*and*

$$\boldsymbol{P}_x^{(1)}(e_\Delta^{(1)} < \infty) \leqslant \boldsymbol{P}_x^{(2)}(e_\Delta^{(2)} < \infty) \qquad \textit{for} \quad x \in R,$$

*where $\boldsymbol{P}_\cdot^{(i)}$ and $e_\Delta^{(i)}$ are probability measure and explosion time of $\boldsymbol{X}_i$, respectively.*

Let $X$ and $Y$ be Lévy processes such that $Y_t = X_t + Ct$ for $t \geqslant 0$, where $C$ is a real constant. Let $\boldsymbol{Y}$ be $(Y, k(x), \pi)$-branching Lévy process. Let $\boldsymbol{X}_i$ $(i=1, 2)$ be $(X, k_i(x), \pi)$-branching Lévy processes, where $k_i(x)$ are given by

(1) $$k_1(x) = \inf_{0 \leqslant t \leqslant T} k(x - Ct) \quad \text{and} \quad k_2(x) = \sup_{0 \leqslant t \leqslant T} k(x - Ct)$$

($T$ is a positive constant). Then we have

**Theorem 5.** **(i)** *If $\boldsymbol{X}_1$ is explosive, then so is $\boldsymbol{Y}$.*
**(ii)** *If $\boldsymbol{X}_2$ is non-explosive, then so is $\boldsymbol{Y}$.*

Let us state simple application of Theorems 4 and 5. Let $\{p_n; n \geqslant 2\}$ be the probability sequence in the definition of the branching law of $\boldsymbol{X}$, and set $F(\xi) = \sum_{n=2}^{\infty} p_n \xi^n$ $(0 \leqslant \xi \leqslant 1)$. Then we have

**Corollary 3.** (*Savits* [8]). *Let $k(x)$ be the killing rate of a branching Lévy process $\boldsymbol{X}$, and suppose*

$$(2)\qquad 0<k_1=\inf_{x\in R} k(x)\leqslant \sup_{x\in R} k(x)=k_2<\infty.$$

*Then* $\boldsymbol{X}$ *is explosive (with probability one) if and only if*

$$\int_{1-\varepsilon}^{1}(\xi-F(\xi))^{-1}d\xi<\infty \qquad \textit{for some} \quad \varepsilon>0.$$

Let $\boldsymbol{X}$ be one of branching Lévy processes considered in Theorems 1, 2 and 3, and let $(X_t)$ be the base process of $\boldsymbol{X}$. Let $\boldsymbol{Y}$ be the branching Lévy process with the base process $(Y_t=X_t+Ct)$ ($C$ is a real constant) and with the same killing rate and branching law as those of $\boldsymbol{X}$. Then we have

**Corollary 4.** *If we see from [Theorem 1, 2 or 3] that* $\boldsymbol{X}$ *is explosive, then we conclude that* $\boldsymbol{Y}$ *is also explosive.*

## 2. Proof of Theorems

**(i)** Let $S$ be a measurable space, and for $0<T<\infty$ let $S_T$ be the product space $[0, T]\times S$. Let $\Xi$ be a mapping from $B_1^+(S_T)$ into itself which satisfies the conditions ($\Xi$-1)–($\Xi$-3):

($\Xi$-1) $\qquad \Xi 0=0, \quad \Xi 1\leqslant 1.$

($\Xi$-2) (*Monotonicity*) $\quad u, v\in B_1^+(S_T)$ *and* $u\leqslant v$, *then* $\Xi u\leqslant \Xi v$.

($\Xi$-3) (*Continuity*) $\quad u_n, u\in B_1^+(S_T)$ *and* $u_n\nearrow u$ $(u_n\searrow u)$,

*then* $\Xi u_n\nearrow \Xi u$ (*resp.* $\Xi u_n\searrow \Xi u$) [14].

Consider the following equation in $B_1^+(S_T)$

$$(3)\qquad u=\Xi u.$$

Then from [4; §1] (3) always has the unique maximal solution $\bar{u}$, and moreover we have

**Lemma 13.** *Let* $v\in B_1^+(S_T)$ *and* $v\leqslant \Xi v$, *then* $v\leqslant \bar{u}$.

For $u\in B_1^+(R_T)$ let us set

14) $u_n\nearrow u$ $(u_n\searrow u)\Longleftrightarrow u_n(x)\nearrow u(x)$ (resp. $u_n(x)\searrow u(x)$) as $n\nearrow\infty$ for each $x\in S$.

$$(4)\qquad (\Xi_i u)(t, x)=\int_0^t E_x\left(\exp\left\{-\int_0^s k_i(t-r, X_r)\mathrm{d}r\right\} k_i(t-s, X_s)\right.$$
$$\left.\times G(X_s;\, u_{t-s})\right)\mathrm{d}s, \quad (t, x)\in R_T, \qquad i=1, 2,$$

where $k_i(t, x)$ is a locally bounded non-negative measurable function on $R_T$. It is easy to see that from the definition of $G(x; u)$ in § 1, each mapping $\Xi_i$ satisfies the conditions ($\Xi$-1)-($\Xi$-3).

**Proposition 3.** *Let* $\bar{u}_i$ $(i=1, 2)$ *be the maximal solution of* $u=\Xi_i u$ *in* $B_1^+(R_T)$. *Suppose that* $k_1(t, x)\leqslant k_2(t, x)$ *for* $(t, x)\in R_T$, *then* $\bar{u}_1(t, x) \leqslant \bar{u}_2(t, x)$ *for* $(t, x)\in R_T$.

Before the proof of Proposition 3, we show the next lemma.

**Lemma 14.** *Suppose that* $k_1(t, x)\leqslant k_2(t, x)$ *for* $(t, x)\in R_T$. *If* $u\in B_1^+(R_T)$ *satisfies the equation*

$$(5)\qquad u(t, x)=(\Xi_1 u)(t, x)=\int_0^t E_x\left(\exp\left\{-\int_0^s k_1(t-r, X_r)\mathrm{d}r\right\}\right.$$
$$\left.\times k_1(t-s, X_s)G(X_s;\, u_{t-s})\right)\mathrm{d}s, \quad (t, x)\in R_T,$$

*then u also satisfies the equation*

$$(6)\qquad u(t, x)=\int_0^t E_x\left(\exp\left\{-\int_0^s k_2(t-r, X_r)\mathrm{d}r\right\}\{k_1(t-s, X_s)\times G(X_s;\, u_{t-s})\right.$$
$$\left.+(k_2(t-s, X_s)-k_1(t-s, X_s))u(t-s, X_s)\}\right)\mathrm{d}s, \quad (t, x)\in R_T.$$

**Proof.** For simplicity we introduce the notations; $\varphi_i(r, s, t)=\varphi_i(r, s, t; w)=\int_r^s k_i(t-q, X_q(w))\mathrm{d}q$ for $0\leqslant r\leqslant s\leqslant t\leqslant T$, $i=1, 2$, $k_3(t, x)=k_2(t, x)-k_1(t, x)$. Here we note that $k_3(t, x)$ is non-negative in $R_T$ by the assumption. Then

(7) the right hand side of (6)

$$=\int_0^t E_x(e^{-\varphi_2(0,s,t)}k_1(t-s, X_s)G(X_s;\, u_{t-s}))\mathrm{d}s$$
$$+\int_0^t E_x(e^{-\varphi_2(0,s,t)}k_3(t-s, X_s)u(t-s, X_s))\mathrm{d}s=I_1+I_2,$$

and then

$$I_1=\int_0^t E_x(e^{-\varphi_1(0,s,t)}k_1(t-s,\ X_s)G(X_s;\ u_{t-s}))\mathrm{d}s$$

$$+\int_0^t E_x(\{e^{-\varphi_2(0,s,t)}-e^{-\varphi_1(0,s,t)}\}k_1(t-s,\ X_s)G(X_s;\ u_{t-s}))\mathrm{d}s$$

$$=I_{11}+I_{12}.$$

Substituting the equality (5) and using the Markov property, we have

$$I_2=\int_0^t \mathrm{d}s\int_0^{t-s}\mathrm{d}rE_x(e^{-\varphi_2(0,s,t)}k_3(t-s,\ X_s)$$

$$\times E_{X_s}(e^{-\varphi_1(0,r,t-s)}k_1(t-s-r,\ X_r)G(X_r;\ u_{t-s-r})))$$

$$=\int_0^t \mathrm{d}s\int_0^{t-s}\mathrm{d}rE_x(\exp\{-\varphi_2(0,\ s,\ t)+\varphi_1(0,\ r,\ t-s;\ \theta_s w)\}k_3(t-s,\ X_s)$$

$$\times k_1(t-s-r,\ X_r(\theta_s w))G(X_r(\theta_s w),\ u_{t-s-r})).$$

By the definition, $\varphi_1(0,\ r,\ t-s;\ \theta_s w)=\varphi_1(s,\ s+r,\ t;\ w)$ and $\varphi_1(s,\ s+r,\ t)=\varphi_1(0,\ s+r,\ t)-\varphi_1(0,\ s,\ t)$ so that we have

$$I_2=\int_0^t \mathrm{d}s\int_0^{t-s}\mathrm{d}rE_x(\exp\{-\varphi_2(0,\ s,\ t)+\varphi_1(0,\ s,\ t)\}k_3(t-s,\ X_s)$$

$$\times e^{-\varphi_1(0,s+r,t)}k_1(t-s-r,\ X_{s+r})G(X_{s+r},\ u_{t-s-r}))$$

$$=\int_0^t \mathrm{d}rE_x\Big(\Big(\int_0^r \mathrm{d}_s\{-\exp(-\varphi_2(0,\ s,\ t)+\varphi_1(0,\ s,\ t))\}e^{-\varphi_1(0,r,t)}$$

$$\times k_1(t-r,\ X_r)G(X_r;\ u_{t-r})\Big)=\int_0^t E_x(\{e^{-\varphi_1(0,r,t)}$$

$$-e^{-\varphi_2(0,r,t)}\}k_1(t-r,\ X_r)G(X_r;\ u_{t-r}))\mathrm{d}r.$$

Finally summing up the above results, we conclude

the right hand side of $(6)=I_{11}+(I_{12}+I_2)=u(t,\ x)$,

which completes the proof.

**Remark 3.** Suppose that $\int_0^t E_x(k_i(t-s,\ X_s))\mathrm{d}s<\infty$ for $0<t\leqslant T$ and $x\in R$ $(i=1,\ 2)$. Then we easily see that Lemma 14 holds without the

assumption "$k_1(t, x) \leqslant k_2(t, x)$ for $(t, x) \in R_T$".

**Remark 4.** As the proof shows, Lemma 14 holds when $X$ is a standard process on a locally compact Hausdorff space with countable base. Proposition 3 and Theorem 4 also hold under the same condition on $X$.

**Proof of Proposition 3.** By Lemma 14

$$\bar{u}_1(t, x) = (\Xi_1 \bar{u}_1)(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k_2(t-r, X_r)\mathrm{d}r\Big\}$$

$$\times \{k_1(t-s, X_s)(G(X_s; \bar{u}_1(t-s, \cdot)) - \bar{u}_1(t-s, X_s))$$

$$+ k_2(t-s, X_s)\bar{u}_1(t-s, X_s)\}\Big)\mathrm{d}s$$

Since $k_1(t, x) \leqslant k_2(t, x)$ for $(t, x) \in R_T$ and $G(x; u) - u(x) \geqslant 0$, $x \in R$ for $u \in B_1^+(R)$ by the definition of $G$, we have $\bar{u}_1(t, x) \leqslant (\Xi_2 \bar{u}_1)(t, x)$ for $(t, x) \in R_T$. Hence Lemma 13 implies $\bar{u}_1(t, x) \leqslant \bar{u}_2(t, x)$ for $(t, x) \in R_T$, which completes the proof.

(ii) **Proof of Theorem 4.** By Proposition 0, $e_i(t, x) = \boldsymbol{P}_x^{(i)}(e_\Delta^{(i)} \leqslant t)$ is the maximal solution of the equation

$$(8) \qquad u(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k_i(X_r)\mathrm{d}r\Big\} k_i(X_s) G(X_s; u_{t-s})\Big)\mathrm{d}s,$$

in $B_1^+([0, \infty) \times R)$ $(i = 1, 2)$. Because $k_1(x) \leqslant k_2(x)$ for $x \in R$ (independent of $t$), we conclude by Proposition 3 that $e_1(t, x) \leqslant e_2(t, x)$ for $(t, x) \in [0, \infty) \times R$, which is just the first conclusion. Now, letting $t \to \infty$ in the first conclusion, we obtain the second conclusion. This completes the proof.

In order to prove Theorem 5, we need the following lemma.

**Lemma 15.** *Suppose that $u \in B_1^+(R_T)$ satisfies the equation*

$$(9) \qquad u(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k(Y_r)\mathrm{d}r\Big\} k(Y_s) G(Y_s; u_{t-s})\Big)\mathrm{d}s,$$

*for* $(t, x) \in R_T$, *and set* $v(t, x) = u(t, x - Ct)$. *Then* $v$ *satisfies the equation*

$$(10)\quad v(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k(X_r - C(t-r))\mathrm{d}r\Big\} k(X_s - C(t-s)) \times \times G(X_s; v_{t-s})\Big)\mathrm{d}s$$

*for* $(t, x) \in R_T$.

**Proof.** By the spatial homogeneity of Lévy processes and the relation $Y_t = X_t + Ct$ for $t \geqslant 0$, (9) is rewritten to

$$(11)\quad u(t, x) = \int_0^t E_0\Big(\exp\Big\{-\int_0^s k(x + X_r + Cr)\mathrm{d}r\Big\} k(x + X_s + Cs) \times G(x + X_s + Cs; u_{t-s})\Big)\mathrm{d}s$$

Applying the change of variables; $t = \tau$, $x = y - C\tau$, and using the relation $G(a+b; u(\cdot)) = G(a; u(b+\cdot))$, (11) is rewritten to

$$(12)\quad u(\tau, y - C\tau) = \int_0^\tau E_0\Big(\exp\Big\{-\int_0^s k(y + X_r - C(\tau - r))\mathrm{d}r\Big\} \times k(y + X_s - C(\tau - s))G(y + X_s; u(\tau - s, \cdot - C(\tau - s))\Big)\mathrm{d}s.$$

Substituting $v(\tau, y) = u(\tau, y - C\tau)$, and again using the spatial homogeneity of Lévy processes, (12) is just rewritten to (10). This completes the proof.

**Proof of Theorem 5.** Let us prove only (ii) because we can prove (i) quite similarly. Set $e_i(t, x) = \boldsymbol{P}_x^{(i)}(e_\Delta^{(i)} \leqslant t)$ $(i = 1, 2)$ and $e_Y(t, x) = \boldsymbol{P}_x^{(Y)}(e_\Delta^{(Y)} \leqslant t)$, where $\boldsymbol{P}_\cdot^{(i)}$ and $e_\Delta^{(i)}$ ($\boldsymbol{P}_\cdot^{(Y)}$ and $e_\Delta^{(Y)}$) are probability measure and explosion time of $\boldsymbol{X}_i$ (resp. $\boldsymbol{Y}$), respectively. Then by Proposition 0, $e_i$ $(e_Y)$ is the maximal solution of the equation

$$u(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k_i(X_r)\mathrm{d}r\Big\} k_i(X_s)G(X_s; u_{t-s})\Big)\mathrm{d}s$$

$$\Big(\text{resp. } u(t, x) = \int_0^t E_x\Big(\exp\Big\{-\int_0^s k(Y_r)\mathrm{d}r\Big\} k(Y_s) \times G(Y_s; u_{t-s})\Big)\mathrm{d}s\Big)$$

in $B_1^{\dagger}(R_T)$. Setting $v(t, x)=e_Y(t, x-Ct)$ and applying Lemma 15, we conclude that $v$ is the maximal solution of the equation

$$u(t, x)=\int_0^t E_x\left(\exp\left\{-\int_0^s k(X_r-C(t-r))\mathrm{d}r\right\}k(X_s-C(t-s))\right.$$

$$\left.\times\, G(X_s;\, u_{t-s})\right)\mathrm{d}s$$

in $B_1^{\dagger}(R_T)$. Because $k(x-Ct)\leqslant k_2(x)$ for $(t, x)\in R_T$ by (1), we conclude by Proposition 3 that $e_Y(t, x-Ct)=v(t, x)\leqslant e_2(t, x)$ for $(t, x)\in R_T$. Since $\boldsymbol{X}_2$ is non-explosive, we have $e_2(t, x)=0$ for $(t, x)\in R_T$, so that we have $e_Y(t, x)=0$ for $(t, x)\in R_T$. Finally by the Markov property of $\boldsymbol{Y}$, we conclude that $e_Y(t, x)=0$ for $(t, x)\in[0, \infty)\times R$. Hence $\boldsymbol{Y}$ is non-explosive.

3. **Proof of Corollary 3.** Let $\boldsymbol{X}_i$ $(i=1, 2)$ be branching Lévy processes with the same base process and branching law as those of $\boldsymbol{X}$, and with the killing rate $k_i$ (=positive constant in (2)). From [3; II, §3.5], we see that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are explosive (with probability one) if and only if $\int_{1-\varepsilon}^1 (\xi-F(\xi))^{-1}\mathrm{d}\xi<\infty$ for some $\varepsilon>0$. Now applying Theorem 4 on $\boldsymbol{X}_i$ and $\boldsymbol{X}$, we easily obtain the conclusion.

**Proof of Corollary 4.** We only remark that $k_1(x)=\inf_{0\leq t\leq T} k(x-Ct)\asymp x^{\gamma}$ and $k_2(x)=\sup_{0\leq t\leq T} k(x-Ct)\asymp x^{\gamma}$ as $x\to\infty$ when $k(x)\asymp x^{\gamma}$ as $x\to\infty$ for some positive constant $\gamma$. Then the rest of proof is obvious from [Theorem 1, 2 or 3] and Theorem 5, so we omit the detail.

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