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Zeros of ideals of C^r functions

By

Shuzo IZUMI

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Introduction

In this article we treat formal aspects of zeros of ideals of C^r functions including the case $r=0$. Usually the zero of an ideal is defined to be the intersection of the zero sets of its functions. However there occurs a difficulty in treating the zeros of ideals of germs unless they are finitely generated. The reason is that we have no natural definition of the germ of intersection of an infinite number of germs of sets (cf. [8, (‡)]). Thus, as the zero of an ideal of germs, we introduce a new object $\tilde{\mathscr{C}}\ell_a$ which may be expressed by an abusive term "filter of germs of closed sets". Mainly we study lattice-theoretic properties of the correspondence of ideals and "filters" using Tougeron's lemma (3) and the results in [8]. In the last three sections we show miscellaneous results related to cardinal numbers. Special regard is paid for finitely generated C^∞ radicals since they seems to have zero sets of simple figures (cf. [7]). It should be noted that Bochnak [2] has treated a related theme. See [14] also.

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(This article was given out in 1974 as a preprint (cited in [13]: (12) of the present article). But publication has been delayed for correction. This correction needed another article [8]. [7] is originally a development of this study.)

Moore families (cf. [8])

Let $\tilde{S}$ be a *Moore family* on a set S i.e. $\tilde{S}$ is a subfamily of the family $P(S)$ of all subsets of S such that $S\in\tilde{S}$ and $\bigcap_{\lambda\in\Lambda} X_\lambda\in\tilde{S}$ for any $\{X\}_{\lambda\in\Lambda}\subset\tilde{S}$. $c=c_{\tilde{S}}\colon P(S)\to S$ denotes the associated *closure operation*; $c(A)=\bigcap_{X\supset A,X\in\tilde{S}} X$. We say that $X\subset S$ *generates* $Y\in\tilde{S}$ if $Y=c(X)$. If $X\in\tilde{S}$ is generated by one element it is called *principal*. A Moore family has a natural structure of a *complete lattice* with respect to the order of inclusion. Let $\tilde{S}$ and $\tilde{T}$ be Moore families on S and T respectively, $\Phi: \tilde{S} \to \tilde{T}$ a map and $\mathfrak{m}$, $\mathfrak{n}$ be cardinal numbers greater than 1. We call Φ an $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-*morphism* (or $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-*continuous*) when

$$\Phi(\bigvee_{\lambda\in\Lambda} X_\lambda) = \bigvee_{\lambda\in\Lambda} \Phi(X_\lambda), \qquad \Phi(\bigwedge_{\mu\in M} Y_\mu) = \bigwedge_{\mu\in M} \Phi(Y_\mu)$$

hold for any $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_\mu\}_{\mu\in M} \subset \tilde{S}$ such that $\# \Lambda \leqq \mathfrak{m}$, $\# M \leqq \mathfrak{n}$. If Φ is $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-continuous for any $\mathfrak{m}$, we call it $(\forall\vee, \mathfrak{n}\wedge)$-*continuous*, etc. $(\mathfrak{m}\vee)$-*morphisms* and $(\mathfrak{n}\wedge)$-*morphisms* are defined similarly. If $\varphi: S \to T$ is a map of sets with Moore families, we can define the *direct induced map* $\varphi_*: \tilde{S} \to \tilde{T}$ and the *inverse one* $\varphi^*: \tilde{T} \to \tilde{S}$ by

$$\varphi_*(X) = c(\varphi(X)), \qquad \varphi^*(Y) = c(\varphi^{-1}(X)).$$

In the previous paper [8] the author has studied lattice-theoretic properties of these maps. There we paid attention to the following conditions:

(a) $\varphi^*(Y) = \varphi^{-1}(Y)$ *for any* $Y \in \tilde{T}$;

(a′) $\varphi_*(X) = \varphi(X)$ *for any* $X \in \tilde{S}$;

(b) $\varphi^* \circ \varphi_*(X) = X \vee \varphi^*(0)$ *for any* $X \in \tilde{S}$, *where* 0 *denotes the minimal element of* $\tilde{T}$.

Let's call a Moore family $\tilde{S}$ on S (or $c_{\tilde{S}}$) *finitary* when the following two conditions are mutually equivalent for any subset $X \subset S$ (cf. [1; VIII]):

(i) $X = c(X)$;

(ii) $c(Y) \subset X$ *for any finite subset* $Y \subset X$.

All Moore families are finitary in this article. As for an inductive system $\{S_\lambda, \varphi_{\mu\lambda}\}_{\lambda,\mu\in\Lambda}$ of sets with Moore families $\tilde{S}_\lambda$, we have assumed that Λ *is a directed set* and that:

(A) *All* $\varphi_{\mu\lambda}$ *satisfy* (a).

Then there exists a unique Moore family $(\lim S_\mu)^\sim$ on the set-theoretical inductive $\lim S_\mu$ such that:

(i) The inductive maps $\varphi_\lambda: S_\lambda \to \lim S_\mu$ satisfy (a).

(ii) If $\{\psi_\lambda: S_\lambda \to T\}$ is a system of maps satisfying (a) and $\psi_\nu \circ \varphi_{\nu\lambda} = \psi_\nu$, the canonical maps $\lim \psi_\mu: \lim S_\mu \to T$ satisfy (a).

We have also considered the following conditions:

(B) *All $\varphi_{\mu\lambda}$ satisfy* (b).

(C) (i) $\varphi_{\mu\lambda}{}^*(S_\lambda)=S_\mu$.

(ii) $\varphi_\lambda(X)=\varphi_\mu(\varphi_{\mu\lambda}{}^*(X))$ *for any* $\mu\geqq\lambda$, *i.e. if* $X\in\tilde{S}_\lambda$ *and* $b\in\varphi_{\mu\lambda}{}^*(X)$, *there exist* $\nu\geqq\mu$ *and* $a\in x$ *such that* $\varphi_{\nu\lambda}(a)=\varphi_{\nu\mu}(b)$.

Rings of C^r functions

Let Ω be an open subset of $\mathbf{R}^n$ and $\mathscr{G}$ be the family of all nonempty open subsets of Ω. If $a\in\Omega$, $\mathscr{G}_a$ denotes the subfamily $\{U\in\mathscr{G}: U\ni a\}$. $\mathscr{G}_a$ is a directed set with respect to the order dual to inclusion. $\mathscr{E}^r(U)$ $(0\leqq r\leqq\infty)$ denotes the ring of real valued C^r functions on $U\in\mathscr{G}$. If $V\subset U$ the map of restriction $\varphi_{VU}: \mathscr{E}^r(U)\ni f\to f|V\in\mathscr{E}^r(V)$ is a unitary ring homomorphism. $\{\mathscr{E}^r(U), \varphi_{VU}\}_{U,V\in\mathscr{G}_a}$ is an inductive system of rings with limit $\mathscr{E}^r_a$, the ring of germs of C^r-functions at a. $\varphi_{aU}: \mathscr{E}^r(U)\to\mathscr{E}^r_a$ denotes the inductive map.

The following is well known.

1. Lemma. *If $\{U_i\}_{i\in I}\subset\mathscr{G}$ is a locally finite covering of Ω, there exists $\{\varphi_i\}_{i\in I}\subset\mathscr{E}^r(\Omega)$ such that $\varphi_i>0$ on U_i, $\varphi_i=0$ on $\Omega-U$ and ${}_i\Sigma\varphi_i=1$ on Ω.*

2. Lemma. *If $V\subset U, f_i\in\mathscr{E}^0(U), f_i(x)=0$ on $U-V$, $\varphi_{VU}(f_i)\in\mathscr{E}^r(V)$ and $g_j\in\mathscr{E}^r(V)$ for $i,j=1,2,..$, then there exists a function $\theta\in\mathscr{E}^r(\mathbf{R})$ such that $\theta'(x)>0$ $(x\neq 0)$, θ is ∞-flat at 0 and the functions $h_{ij}\equiv(\theta\circ f_i)\times g_j$ (defined on V) have extensions $h_{ij}\in\mathscr{E}^r(U)$ which are r-flat on $U-V$.*

Proof. (cf. [4; (3.3)]) We treat the essential case $r=\infty$. Let $V_1\subset V_2\subset\cdots$ be a sequence of relatively compact open subsets of U such that $\cup V_k=U$. For $I_0=(-\infty,\infty)$ and $I_k=(-1/k, 1/k)$ $(k\geqq 1)$ we take $\theta_k\in\mathscr{E}^\infty(\mathbf{R})$ such that $\theta_k'(x)>0$ on $I_{k-1}-\bar{I}_{k+1}$, $\theta_k'(x)=0$ elsewhere, $\theta_k(-x)=-\theta_k(x)$ and $|\theta_k|^{\mathbf{R}}_{k+1}\leqq 1$, where $|\theta_k|^{\mathbf{R}}_{k+1}\equiv\sup_{\substack{x\in\mathbf{R}\\ \nu\leqq k+1}}|\theta_k^{(\nu)}(x)|$. Putting $A^i_k=f_i^{-1}(I_k)\cap V_k$ and $\theta=\sum_{k=1}^\infty\varepsilon_k\theta_k$ for a positive sequence $\{\varepsilon_k\}$, we have

$$|h_{ij}|_k^{A^i_{k-1}-A^i_k}\leqq a_k|\theta|_k^{I_{k-1}-I_k}(1+|f_i|_k^{A^i_{k-1}-A^i_k})^k|g_j|_k^{A^i_{k-1}-A^i_k}$$

$$\leqq b^{ij}_k\sum_{l=k-1}^\infty\varepsilon_l\qquad(b^{ij}_k>0)\,.$$

Choosing $\varepsilon_l\leqq\min\{1/(2^l\sup_{\substack{i+j\leqq 1+l\\ m\leqq 1+l}}b^{ij}_m), 1/2^l\}$, we have

$$|\theta|_k^{I_{k-1}-I_k} \leqq 1/2^{k-2}, \quad |h_{ij}|_k^{A^i_{k-1}-A^i_k} \leqq 1/2^{k-2}$$

for $i+j \leqq k$. θ is clearly C^∞ and $\theta'(x)>0$ on $(-\infty, 0) \cup (0, \infty)$. Since I_k are neighbourhoods of 0, all the derivatives of θ approach 0 when x does. Then, by Hestence's lemma (cf. [12; p. 80]), $\theta \in \mathscr{E}^\infty(\mathbf{R})$ and θ is ∞-flat at 0. Hence $h_{ij} \in \mathscr{E}^\infty(V)$.
If $x_0 \in U-V$, A^i_l $(l \geqq k_0)$ are neighbourhoods of x_0 for some k_0. Then by the inequalities $|h_{ij}|_k^{A^i_{k-1}} \leqq 1/2^{k-2}$ $(i+j \leqq k)$ and by Hestece's lemma, h_{ij} have the required properties, *q.e.d.*

3. Corollary. (Tougeron [11] or [12; p. 113]). *If $V \subset U$ and $f_i \in \mathscr{E}^r(V)$ $(i=1, 2,\dots)$ then there exists an invertible $g \in \mathscr{E}^r(V)$ such that $f_i g$ have extensions $h_i \in \mathscr{E}^r(U)$ which are r-flat on $U-V$.*

4. Corollary. (i) *If $V \subset U$, $\mathscr{E}^r(V)$ is flat over $\mathscr{E}^r(U)$ and any ideal of $\mathscr{E}^r(V)$ is generated by the image of an ideal of $\mathscr{E}^r(U)$.*
(ii) *If $a \in U$, $\mathscr{E}^r_a$ is flat over $\mathscr{E}^r(U)$ and φ_{aU} is surjective.*

Proof. Flatness of $\mathscr{E}^r(V)$ is proved in Tougeron [12; p. 113]. Flatness of $\mathscr{E}^r_a$ follows from the fact that flatness over a fixed ring is preserved for inductive limits (cf. [3]). The rest are easy to prove.

The family of all ideals of $\mathscr{E}^r(U)$ forms a finitary Moore family $\tilde{\mathscr{E}}^r(U)$. Then $\{\mathscr{E}^r(U), \varphi_{VU}\}_{U,V \in \mathscr{G}_a}$ is an inductive system of sets with Moore families satisfying (A) and (C). (C) follows from (1). Hence the canonical Moore family $\tilde{\mathscr{E}}^r_a$ on $\mathscr{E}^r_a$ is characterized by the fact that φ_{aU} satisfies (a) and (a′) by [8; (7)]. By [8; (1, iv), (Ex. 1, ii)], $\tilde{\mathscr{E}}^r_a$ coincides with the family of all ideals of $\mathscr{E}^r_a$ and φ_{aU} satisfy (b).

5. Theorem. *If $a \in V \subset U$ we have the following;*
(i) *φ_{VU*}: $\tilde{\mathscr{E}}^r(U) \to \tilde{\mathscr{E}}^r(V)$ is $(\forall \vee, \mathfrak{a} \wedge)$-continuous, where $\mathfrak{a}$ denotes the cardinal of the set of natural numbers.*
(ii) *φ^*_{VU}: $\tilde{\mathscr{E}}^r(V) \to \tilde{\mathscr{E}}^r(U)$ is $(\forall \wedge)$-continuous.*
(iii) *$\varphi_{VU*} \circ \varphi^*_{VU}$ is the identity.*
(iv) *φ_{aU*}: $\tilde{\mathscr{E}}^r(U) \to \tilde{\mathscr{E}}^r_a$ is $(\forall \vee, 2 \wedge)$-continuous.*
(v) *φ^*_{aU}: $\tilde{\mathscr{E}}^r_a \to \tilde{\mathscr{E}}^r(U)$ is $(\forall \vee, \forall \wedge)$-continuous.*
(vi) *$\varphi_{aU*} \circ \varphi^*_{aU}$ is the identity.*

Proof. Excepting the fact that φ_{VU*} is $(\mathfrak{a} \wedge)$-continuous, all the assertions follow from [8; (1), (2), (Ex. 1)] and (4). Let $\mathfrak{a}_1, \mathfrak{a}_2,\dots$ be a countable number of ideals of $\mathscr{E}^r(U)$. The inclusion $\varphi_{VU*}(\wedge \mathfrak{a}_i) \subset \wedge \varphi_{VU*}(\mathfrak{a}_i)$ is obvious. Take $f \in \wedge \varphi_{VU*}(\mathfrak{a}_i)$. Then for any i there are $g_{i1},\dots, g_{ip_i} \in \mathscr{E}^r(V)$ and $h_{i1},\dots, h_{ip_i}$

$\in \mathfrak{a}_i$ such that $f=\sum_{j=1}^{p_i} g_{ij}\varphi_{VU}(h_{ij})$. By (3) there exists an invertible $k\in\mathscr{E}^r(V)$ such that any $g_{ij}k$ has an extension $l_{ij}\in\mathscr{E}^r(U)$ which vanishes on $U-V$. Then the expression $\sum_{j=1}^{p_j} l_{ij}h_{ij}\in\mathfrak{a}_i$ does not depend upon i and $f=k^{-1}\varphi_{VU}(\sum_{j=1}^{p_i} l_{ij}h_{ij})$ $\in\varphi_{VU*}(\wedge\mathfrak{a}_i)$, *q.e.d.*

6. *Example.* We put $U=\mathbf{R}$, $V=\{x\in\mathbf{R}: x>0\}$ and $M=\{f\in\mathscr{E}^r(U): f=0$ on $(-\infty, 0], f'>0$ on $(0, \infty)\}$. Let (f) be the principal ideal of $\mathscr{E}^r(U)$ generated by f. Obviously $\bigwedge_{f\in M}\varphi_{VU*}((f))=\mathscr{E}^r(V)$. If $g\in\bigwedge_{f\in M}(f)$ we put $a_i=\max_{0\leqq x\leqq 1/i}|g|$. Let x_i be the point where $|g|$ attains its maximum a_i on $[0, 1/i]$. If all a_i are positive, there exists $f\in M$ such that $f(x_i)\leqq a_i/i$ or $|g(x_i)|/f(x_i)\geqq i$. This contradicts to the inclusion $g\in(f)$. Thus $a_i=0$ for some i and g vanishes in a neighbourhood of 0. Hence $\varphi_{VU*}(\wedge(f))\neq\mathscr{E}^r(V)$ and φ_{VU*} is not $(\forall\wedge)$-continuous (not $(\mathfrak{c}\wedge)$-continuous by (20)).

7. *Example.* We put $U=\mathbf{R}^2=\{(x, y)\}$, $V=\{(x, y): x>0\}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the principal ideals of $\mathscr{E}^r(V)$ generated by $y-\sin 1/x$ and $y+\sin 1/x$ respectively. Then $y\in\varphi_{VU}^*(\mathfrak{a}\vee\mathfrak{b})$ and $y\notin\varphi_{VU}^*(\mathfrak{a})\vee\varphi_{VU}^*(\mathfrak{b})$. For

$$\varphi_{VU}(y)=\{(y-\sin 1/x)+(y+\sin 1/x)\}/2$$

and $f\in\varphi_{VU}^*(\mathfrak{a})\vee\varphi_{VU}^*(\mathfrak{b})$ vanishes on $\{(0, y): |y|\leqq 1\}$. Thus φ_{VU}^* is not a $(2\vee)$-morphism.

8. *Example.* Let $U=\mathbf{R}^n$ and $\mathfrak{a}_\rho$ $(\rho>0)$ be the ideal of all $f\in\mathscr{E}^r(U)$ vanishing on $\{|x|\geqq\rho\}$. Then

$$\bigwedge_{i=1}^{\infty}\varphi_{OU*}(\mathfrak{a}_{1/i})=\mathscr{E}_n^r\neq\{0\}=\varphi_{OU*}(\bigwedge_{i=1}^{\infty}\mathfrak{a}_{1/i})$$

and φ_{OU*} is not $(\mathfrak{a}\wedge)$-continuous.

Closed subsets (cf. [8; (Ex. 3)])

Let $\mathscr{C}\ell(U)$ be the set of all closed subsets of $U\in\mathscr{G}$. The order of inclusion defines a distributive lattice structure on $\mathscr{C}\ell(U)$. The dual ideals[†] of $\mathscr{C}\ell(U)$ form a finitary Moore family $\tilde{\mathscr{C}}\ell(U)$. Elements of $\tilde{\mathscr{C}}\ell(U)$ are just 1-1 correspondent to filters generated by closed subsets of U except the maximal element $\mathscr{C}\ell(U)\in\tilde{\mathscr{C}}\ell(U)$. If $V\subset U$, ψ_{VU}: $\mathscr{C}\ell(U)\ni A\rightarrow A|V\in\mathscr{C}\ell(V)$ denotes the restriction map. This is obviously surjective and satisfies (a), (a') and (b). It is easy to

[†] A dual ideal of $\mathscr{C}\ell(U)$ means a subset $\mathfrak{U}\subset\mathscr{C}\ell(U)$ such that (i) $A\cap B\in\mathfrak{U}$ for any A and $B\in\mathfrak{U}$ and (ii) $A\cup C\in\mathfrak{U}$ for any $A\in\mathfrak{U}$ and $C\in\mathscr{C}\ell(U)$.

see that $\{\mathscr{C}\ell(U), \psi_{VU}\}_{V,U\in\mathscr{G}_a}$ is an inductive system of sets with finitary Moore families satisfying (A), (B) and (C). Let $\tilde{\mathscr{C}\ell}_a$ denote the canonical Moore family on the limit $\mathscr{C}\ell_a = \lim_{U\in\mathscr{G}_a} \mathscr{C}\ell(U)$ and $\psi_{aU}: \mathscr{C}\ell(U)\to\mathscr{C}\ell_a$, the inductive map. Then ψ_{aU} is a surjection satisfying (a), (a') and (b) by [8; (6), (7)]. On the other hand, since ψ_{UV} is $(2\vee, 2\wedge)$-continuous, $\mathscr{C}\ell_a$ is a lattice and ψ_{aU} is $(2\vee, 2\wedge)$-continuous by [8; (‡)]. Let $\hat{\mathscr{C}\ell}_a$ denotes the Moore family of dual ideals of $\mathscr{C}\ell_a$ in this sense. Then $\psi_{aU}: \mathscr{C}\ell(U)\to\mathscr{C}\ell_a$ (with $\hat{\mathscr{C}\ell}_a$) satisfies (a) and (a') by [8; (Ex. 2)]. Hence we have the following:

9. Proposition. $\tilde{\mathscr{C}\ell}_a = \hat{\mathscr{C}\ell}_a$.

10. Theorem. *If $U, V\in\mathscr{G}$, $V\subset U$ and $a\in U$ we have the following:*

(i) $\psi_{VU*}: \tilde{\mathscr{C}\ell}(U)\to\tilde{\mathscr{C}\ell}(V)$ *is $(\forall\vee, \forall\wedge)$-continuous.*

(ii) $\psi^*_{VU}: \tilde{\mathscr{C}\ell}(V)\to\tilde{\mathscr{C}\ell}(U)$ *is $(\forall\vee, \forall\wedge)$-continuous.*

(iii) $\psi_{VU*}\circ\psi^*_{VU}$ *is the identity.*

(iv) $\psi_{aU*}: \tilde{\mathscr{C}\ell}(U)\to\tilde{\mathscr{C}\ell}_a$ *is $(\forall\vee, 2\wedge)$-continuous.*

(v) $\psi^*_{aU}: \tilde{\mathscr{C}\ell}_a\to\tilde{\mathscr{C}\ell}(U)$ *is $(\forall\vee, \forall\wedge)$-continuous.*

(vi) $\psi_{aU*}\circ\psi^*_{aU}$ *is the identity.*

Proof. Excepting (i') $(\forall\wedge)$-continuity of ψ_{VU*} and (ii') $(2\wedge)$-continuity of ψ_{aU*}, all assertions follow from [8; (1), (2)]. (i') The minimal element of $\tilde{\mathscr{C}\ell}(V)$ is the principal one, (V), and $\psi^*_{VU}((V)) = (\bar{V}\cap U)$. If $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}\subset\tilde{\mathscr{C}\ell}(U)$,

$$\psi^*_{VU}((V))\vee(\wedge\mathfrak{A}_\lambda) = \{B\in\mathscr{C}\ell(U); \exists A_\lambda\in\mathfrak{A}_\lambda \text{ for any } \lambda,$$

$$B\supset\bar{V}\cap U\cap A_\lambda\} = \cap\{\psi^*_{VU}((V))\vee\mathfrak{A}_\lambda\}.$$

Hence ψ_{VU*} is $(\forall\wedge)$-continuous by [8; (2, iii)].
(ii') Since $\mathscr{C}\ell(U)$ is distributive, $\tilde{\mathscr{C}\ell}(U)$ is Browerian by Stone's theorem (cf. [1; (V, 10)]) and hence distributive. Then $\psi_{aU}{}^*$ is $(2\wedge)$-continuous by [8; (2, iii)], *q.e.d.*

ψ_{aU*} is not $(\mathfrak{a}\wedge)$-continuous (cf. (8)).

Duality (cf. [5]).

Here we study the relation between the complete lattices $\tilde{\mathscr{E}}(U)$ and $\tilde{\mathscr{C}\ell}(U)$. Let's put $Z_U(f) = \{x\in U; f(x)=0\}$. This defines a surjective map $Z_U: \mathscr{E}^r(U)\to\mathscr{C}\ell(U)$ satisfying (a) and (a').

11. Theorem. Z_{U*} *is $(\forall\vee, \mathfrak{a}\wedge)$-continuous and Z^*_U is $(\forall\vee)$-continuous. $Z_{U*}\circ Z^*_U$ is the identity.*

By (16), (17), (18) and [8; (8)], Z_{U*} is not $(\forall\wedge)$-continuous and Z_U^{-1} is not $(2\vee)$-continuous.

Proof. Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots \in \tilde{\mathscr{E}}^r(U)$. If $A \in \wedge Z_{U*}(\mathfrak{a}_i)$, there exist $f_i \in \mathfrak{a}_i$ such that $A = Z_U(f_i)$. Then by (3) there exist $g \in Z_U^{-1}(A)$ and $h_i \in \mathscr{E}^r(U)$ such that $g = f_i h_i$ and $A = Z_U(g)$. Thus $\wedge Z_{U*}(\mathfrak{a}_i) \subset Z_{U*}(\wedge \mathfrak{a}_i)$. The converse inclusion is obvious. The rest follow from [8; (2)], *q.e.d.*

Now we define the C^r-radical ${}^{(r)}\sqrt{\mathfrak{a}}$ of $\mathfrak{a} \in \tilde{\mathscr{E}}^r(U)$ as the set of all $f \in \mathscr{E}^r(U)$ such that $\theta \circ f \in \mathfrak{a}$ for some $\theta \in \mathscr{E}^r(R)$ satisfying $\theta(0) = 0$ and $\theta'(x) > 0$ for any $x \neq 0$ (if $r = 0$ we assume only $\theta(0) = 0$ and θ is strictly increasing). Then we can easily prove the following using (2):

12. Proposition. $Z_U^* \circ Z_{U*}(\mathfrak{a}) = {}^{(r)}\sqrt{\mathfrak{a}}$ for any $\mathfrak{a} \in \tilde{\mathscr{E}}(U)$.

13. Lemma. *If* $V \subset U$, $\mathfrak{A} \in \tilde{\mathscr{C}\ell}(U)$, *and* $\mathfrak{b} \in \tilde{\mathscr{E}}^r(V)$, *we have the following*:

(i) $Z_V^{-1} \circ \psi_{VU}(\mathfrak{A}) = \varphi_{VU*}(Z_U^{-1}(\mathfrak{A}))$.

(ii) $Z_U \circ \varphi_{VU}^{-1}(\mathfrak{b}) = \psi_{VU}^{-1} \circ Z_V(\mathfrak{b})$.

Proof. (i) follows from (3). The inclusion $Z_U \circ \varphi_{VU}^{-1}(\mathfrak{b}) \subset \psi_{VU}^{-1} \circ Z_V(\mathfrak{b})$ is obvious. If $A \in \psi_{VU}^{-1} Z_V(\mathfrak{b})$, there exists $f \in \mathfrak{b}$ such that $Z_V(f) = A \cap V$. Then αf has an extension $g \in Z_U^{-1}(A \cup (U - V))$ for some invertible $\alpha \in \mathscr{E}^r(V)$. Let's put $B = \{x \in U : d(x, A) \leqq d(x, U - V)\}$ and take a function $h \in Z_U^{-1}(B)$. Then $\varphi_{VU}(h) \in (f)$ and $A = Z_U(k)$ for $k = g^2 + h^2 \in \varphi_{VU}^{-1}(\mathfrak{b})$. This proves (ii).

14. *Example.* Put $U = \mathbf{R}^2$, $V = \mathbf{R}^2 - \{(x, 0) : x \leqq 0\}$ and $A = \{(x, 0) : x \geqq 0\}$. Then $y \in Z_V^{-1} \circ \psi_{VU}(A) \cap \{\varphi_{VU} \circ Z_U^{-1}(A)\}^c$ and $A \in \psi_{VU}^{-1} \circ Z_V(y) \cap \{Z_U \circ \varphi_{VU}^{-1}(y)\}^c$.

15. Proposition. *If* $U \supset V$ *we have the following*:

(i) $Z_{V*} \circ \varphi_{VU*} = \psi_{VU*} \circ Z_{U*}$.

(ii) $Z_V^* \circ \psi_{VU*} = \varphi_{VU*} \circ Z_U^*$.

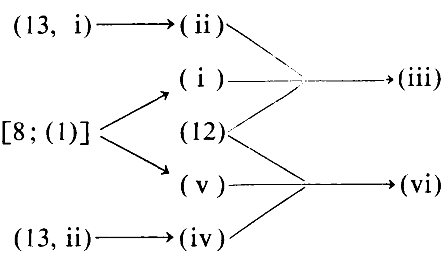
(iii) ${}^{(r)}\sqrt{\varphi_{VU*}(\mathfrak{a})} = \varphi_{VU*}({}^{(r)}\sqrt{\mathfrak{a}})$.

(iv) $Z_{U*} \circ \varphi_{VU}^* = \psi_{VU}^* \circ Z_{V*}$.

(v) $Z_U^* \circ \psi_{VU}^* = \varphi_{VU}^* \circ Z_V^*$.

(vi) ${}^{(r)}\sqrt{\varphi_{VU}^*(\mathfrak{b})} = \varphi_{VU}^*({}^{(r)}\sqrt{\mathfrak{b}})$.

Proof. Obvious from the following implications.



Duality in the case of germs

We have observed that $\{\mathscr{C}\ell(U), \psi_{VU}\}_{U,V\in\mathscr{G}_a}$ and $\{\mathscr{E}^r(U), \varphi_{VU}\}_{U,V\in\mathscr{G}_a}$ are inductive systems of sets with finitary Moore families satisfying (A) and (C). φ_U and ψ_U satisfy (b). $Z_U\colon \mathscr{E}^r(U)\to\mathscr{C}\ell(U)$ satisfies $Z_V\circ\varphi_{VU}=\psi_{VU}\circ Z_U$, $Z_V^*\circ\psi_{VU*}=\varphi_{VU*}\circ Z_U^*$ and (a), (a′). Z_{U*} is $(\forall\vee, \mathfrak{a}\wedge)$-continuous and Z_U^* is $(\forall\wedge)$-continuous. It is obvious that $Z_{U*}\circ\varphi_{VU}^*(0))=\psi_{VU}^*((V))$ and $Z_U^*\cdot\psi_{VU}^*((V))=\varphi_{VU}^*((0))$. Then by [8; (6), (7), (8), (9)] the induced maps of the limit map $\lim\limits_{\mathscr{G}_a} Z_U$ satisfy the following:

16. Theorem. *$Z\equiv\lim Z_U\colon \mathscr{E}_a^r\to\mathscr{C}\ell_a$ is an epimorphism satisfying (a) and (a′). $Z_*\colon \tilde{\mathscr{E}}_a^r\to\tilde{\mathscr{C}}\ell_a$ is $(\forall\vee, \mathfrak{a}\wedge)$-continuous and $Z^*\colon \tilde{\mathscr{C}}\ell_a\to\tilde{\mathscr{E}}_a^r$ is $(\forall\wedge)$-continuous. These satisfy the following:*

$$Z_*\circ\varphi_{aU*}=\psi_{aU*}\circ Z_{U*}. \qquad Z_{U*}\circ\varphi_{aU}=\psi_{aU}^*\circ Z_*. \qquad Z_*=\varphi_{aU*}\circ Z_{U*}\circ\psi_{aU}^*.$$

$$Z^*\circ\psi_{aU*}=\varphi_{aU*}\circ Z_U^*. \qquad Z_U^*\circ\psi_{aU}^*=\varphi_{aU}^*\circ Z^*. \qquad Z^*=\varphi_{aU*}\circ Z_U^*\circ\psi_{aU}^*.$$

$$Z^*\circ Z_*(\mathfrak{a})\geqq\mathfrak{a}. \qquad Z_*\circ Z^*=(\text{identity}).$$

17. *Example.* Let I be the set of all ideals $\mathfrak{a}\in\tilde{\mathscr{E}}_0^r$ such that $Z_*(\mathfrak{a})$ is the principal dual ideal ({the origin 0}) $\in\tilde{\mathscr{C}}\ell_0$. Then we have

$$Z_*(\bigcap_{\mathfrak{a}\in I}\mathfrak{a})=\varphi_{0\mathbf{R}*}(\mathbf{R}^n))\neq(\{0\})=\bigcap_{\mathfrak{a}\in I}Z_*(\mathfrak{a}) \qquad \text{(cf. (6))}.$$

Hence Z_* is not $(\forall\wedge)$-continuous.

It is obvious that $Z^*(A\vee B)\supset Z^*(A)\vee Z^*(B)$. The following proposition gives a necessary condition for the equality $Z^*((A)\vee(B))=Z^*((A))\vee Z^*((B))$ for principal dual ideals $((A)), ((B))\in\tilde{\mathscr{C}}\ell_0$. Using it we can easily show that Z^* is not $(2\vee)$-continuous. Yet the condition is not sufficient for the equality.

18. Proposition. *If $r\geqq 1$ and if $Z^*((A)\vee(B))=Z^*((A))\vee Z^*((B))$, $P_{A,a}^r\cap P_{B,a}^r\subset P_{A\cap B,a}^r$, where $P_{A,a}^r$ means the set of Glaeser's linearized paratingents of order r at a (see [6; p. 55]).*

Proof. If $P^r_{A,a} \cap P^r_{B,a} \not\subset P^r_{A\cap B,a}$, there is a vector $v \in (P^r_{A,a} \cap P^r_{B,a}) \cap (P^r_{A\cap B,a})^c$ and a germ $f \in \mathscr{E}^r_a$ such that $(v, f) \neq 0$ and $f=0$ on $A \cap B$ i.e. $f \in Z^*((A) \vee (B))$. By the assumption, $f=g+h$ for some $g \in Z^*((A))$ and some $h \in Z^*((B))$. Since $(v, g) \neq 0$ or $(v, h) \neq 0$ we may assume $(v, g) \neq 0$. But this contradicts to the fact $v \in P^r_{A,a}$ and $g \in Z^*((A))$. Thus we have $P^r_{A,a} \cap P^r_{B,a} \subset P^r_{A\cap B,a}$, *q.e.d.*

We define the local C^r-radical ${}^{(r)}\sqrt{\mathfrak{a}}$ of $\mathfrak{a} \in \mathscr{E}^r_a$ as the set of $f \in \mathscr{E}^r_a$ such that $\theta_0 \circ f \in \mathfrak{a}$ for the germ θ_0 of some $\theta \in \mathscr{E}^r(\mathbf{R})$ such that $\theta(0)=0$ and $\theta'(x)>0$ for $x \neq 0$ (if $r=0$, we assume θ is strictly increasing). By (12) and (16) we have:

19. Proposition. *If $U \in G_a$ we have the following equalities.*

$$ {}^{(r)}\sqrt{\mathfrak{a}} = \varphi_{aU*}({}^{(r)}\sqrt{\varphi^*_{aU}(\mathfrak{a})}) = Z^* \circ Z_*(\mathfrak{a}). $$

$$ {}^{(r)}\sqrt{\varphi_{aU*}(\mathfrak{a})} = \varphi_{aU*}({}^{(r)}\sqrt{\mathfrak{a}}). \qquad {}^{(r)}\sqrt{\varphi^*_{aU}(\mathfrak{a})} = \varphi^*_{aU}({}^{(r)}\sqrt{\mathfrak{a}}). $$

The cardinals of the familiy of ideals

If X satisfies the second countability axiom the cardinal of the family of all closed subsets is at most $\mathfrak{c} = 2^{a} = \#\mathbf{R}$, where $\#\mathbf{R}$ denotes the cardinal number of $\mathbf{R}$. Since $f \in \mathscr{E}^r(U)$ has a closed graph, $\#\mathscr{E}^r(U) \leqq \mathfrak{c}$. On the other hand, $\mathscr{C}\ell_a$ contains all the germs of lines through a. It is clear that $\#\mathscr{C}\ell_a \leqq \#\mathscr{C}\ell(U)$, $\#\mathscr{E}^r_a \leqq \#\mathscr{E}^r(U)$, $\#\mathscr{C}\ell_a \leqq \#\mathscr{E}^r_a$ and $\#\mathscr{C}\ell(U) \leqq \#\mathscr{E}^r(U)$. Thus we have:

20. Proposition. $\#\mathscr{C}\ell_a = \#\mathscr{C}\ell(U) = \#\mathscr{E}^r_a = \#\mathscr{E}^r(U) = \mathfrak{c}$.

This means $\#\tilde{\mathscr{C}}\ell_a \leqq \#\tilde{\mathscr{C}}\ell(U)$, $\#\tilde{\mathscr{E}}^r_a$, $\#\tilde{\mathscr{E}}^r(U) \leqq 2^{\mathfrak{c}} \equiv \mathfrak{f}$. We show that these are all equal. Let $X = \{x_i\}$ be a sequence of points on $\mathbf{R}^n - \{0\}$ converging to 0 and let Φ be the set of all ultrafilters on X which do not converge in X. Then, if $A \in \mathfrak{A} \in \Phi$, $A_p \equiv \{x_i \in A : i \geqq p\} \in \mathfrak{A}$. We put $\bar{A} = A \cup \{0\} \in \mathscr{C}\ell(\mathbf{R}^n)$ and $\bar{\mathfrak{A}} = \{\bar{A} : A \in \mathfrak{A}\} \in \tilde{\mathscr{C}}\ell(\mathbf{R}^n)$ for $A \in \mathfrak{A} \in \Phi$. Let $\mathfrak{A}, \mathfrak{B} \in \Phi$ and $\psi_{0\mathbf{R}^n*}(\bar{\mathfrak{A}}) = \psi_{0\mathbf{R}^n}(\bar{\mathfrak{B}})$. If $A \in \mathfrak{A}$ there exists a natural number p and $B \in \mathfrak{B}$ such that $A_p = B_p \in \mathfrak{B}$. Hence $A \in \mathfrak{B}$ and $\mathfrak{A} \subset \mathfrak{B}$. By the symmetry $\mathfrak{A} = \mathfrak{B}$. Thus $\#\tilde{\mathscr{C}}\ell_0 \geqq \#\{\varphi_{0\mathbf{R}^n*}(\bar{\mathfrak{A}}) : \mathfrak{A} \in \Phi\} = \#\Phi$. On the other hand, it is known that $\#\Phi = \mathfrak{f}$ (cf. [5; (6.10), (9.2)]. These prove the following:

21. Theorem. $\#\tilde{\mathscr{C}}\ell_0 = \#\tilde{\mathscr{C}}\ell(U) = \#\tilde{\mathscr{E}}^r_0 = \#\tilde{\mathscr{E}}^r(U) = \mathfrak{f}$.

Cardinals of generators

22. *Example.* Let $\mathfrak{A} \in \tilde{\mathscr{C}}\ell(\mathbf{R})$ be generated by the closed subsets of the form

$$\{0\} \cup \left(\bigcup_{i=2} \left[\frac{1}{i}, \frac{1}{i} + a_i \right] \right) \qquad (0 < a_i < 1/(i-1) - 1/i).$$

Then neither $\mathfrak{A}$ nor $\psi_{0\mathbf{R}*}(\mathfrak{A})$ has a countable basis. Hence neither $Z^*_{\mathbf{R}}(\mathfrak{A})$ nor $Z^* \circ \psi_{0\mathbf{R}*}(\mathfrak{A})$ has a countable basis.

23. *Example.* Let A be a closed subset of $\mathbf{R}$ such that A and A^c are adherent to 0 and $\mathfrak{a} \in \tilde{\mathscr{E}}^\infty(\mathbf{R})$ be the ideal consisting of all functions ∞-flat on A. Then $(A) = Z_{\mathbf{R}*}(\mathfrak{a})$ is principal but $\mathfrak{a}$ has no countable basis. To prove this, assume the contrary. Then $\mathfrak{a}$ is principal by [12; p. 93]. If f is the generator, $f/x \in \mathfrak{a}$. This cannot be generated by f, a contradiction.

24. Proposition. *Let $\mathfrak{a} \in \tilde{\mathscr{E}}^r(U)$. Then $Z_{U*}(\mathfrak{a})$ is principal if and only if ${}^{(r)}\sqrt{\mathfrak{a}}$ is closed with respect to the C^r-topology.*

Proof. Only-if-part is obvious. Assume that ${}^{(r)}\sqrt{\mathfrak{a}}$ is closed. If $a \notin A \equiv \bigcap_{f \in \mathfrak{a}} Z(f)$, there exists $f^a \in \mathfrak{a}$ such that $f^a(a) \neq 0$. Then by the covering theorem of Lindelöf, there exist a point sequence $\{a_i\} \subset U - A$ and a positive sequence $\{\varepsilon_i\}$ such that $\sum \varepsilon_i (f^{a_i})^2 \in {}^{(r)}\sqrt{\mathfrak{a}}$ and $Z_U[\sum \varepsilon_i (f^{a_i})^2] \subset A$. Hence $A \in Z_U{}^*({}^{(r)}\sqrt{\mathfrak{a}}) = Z_{U*}(\mathfrak{a})$ and $Z_{U*}(\mathfrak{a})$ is principal, *q.e.d.*

Let $\mathscr{E}^r$ be the sheaf of germs of C^r functions over U and let M be the sheaf of germs of C^r functions vanishing on a closed subset $A \subset U$. Then by (3) M is a quasi-flasque ideal of $\mathscr{E}^r$. Hence Tougeron's theorem [11; (IV)] implies the following (r may not be ∞.):

25. Proposition. *Let $U \in \mathscr{G}_a$, $\mathfrak{a} \in \tilde{\mathscr{E}}^r(U)$ and $Z_{U*}(\mathfrak{a}) = (A)$, the principal dual ideal generated by $A \in \mathscr{C}\ell(U)$. $\varphi_{aU*}(a)$ is a local C^r-radical if and only if there exists $V \in \mathscr{C}_a$ such that $\varphi_{VU*}(\mathfrak{a})$ is a C^r-radical in $\mathscr{E}^r(V)$.*

Finitely generated C^∞-radicals

In this section we treat C^∞-case only. If the ideal $Z^*_U((A)) \subset \mathscr{E}^\infty(U)$ of $A \in \mathscr{C}\ell(U)$ is finitely generated, it is a Łojasiewicz ideal (cf. [12; p. 102]). By the theorem of Thom [10] A is the closure of a submanifold M. Since $\varphi_{aU*} \circ Z^*_U((A))$ is a C^∞-radical, any system of generators contains the equations of M_a for any $a \in M$. Thus we have the following:

26. Proposition. (cf. Bochnak [2]). *If $A_a \in \mathscr{C}\ell_a$ and $Z^*((A))$ is generated by $p\,(<\infty)$ elements, then A_a is the germ of the closure of a submanifold whose codimension is less than or equal to p everywhere.*

Let $\mathscr{F}=\mathscr{F}_n$ be the ring of formal power series in n cordinate variables of $\mathbf{R}^n$ and $T: \mathscr{E}^{\infty}_{a,n} \to \mathscr{F}_n$ be the ring homomorphism defined by the formal Taylor expansion at a. It is surjective by Borel's theorem.

27. Lemma. *Let $\mathfrak{a} \in \bar{\mathscr{F}}^{\infty}_a$ be a C^∞-radical generated by a finite number of elements $f_1, f_2, \ldots, f_p$. If Tf_p is dispensable i.e. $Tf_1, \ldots, Tf_{p-1}$ generate $T_*\mathfrak{a} \in \bar{\mathscr{F}}$ then f_p is dispensable i.e. $f_1, \ldots, f_{p-1}$ generate $\mathfrak{a}$. Especially a flat generator is dispensable.*

Proof. There are $g_1, \ldots, g_{p-1} \in \mathscr{E}^{\infty}_a$ such that $f_p - \sum_{i=1}^{p-1} g_i f_i$ is ∞-flat. Then $(f_p - \sum_{i=1}^{p-1} g_i f_i)/\sum_{j=1}^{n} x_j^2 \in {}^{(\infty)}\sqrt{\mathfrak{a}} = \mathfrak{a}$ by the theory of multipliers [9; p. 54]. Hence we have

$$(f_p - \sum_{i=1}^{p-1} g_i f_i) \sum_{j=1}^{n} x_j^2 = \sum_{i=1}^{p} h_i f_i$$

for some $h_1, \ldots, h_p \in \mathscr{E}^{\infty}_a$. Then

$$f_p = \sum_{i=1}^{p-1} (g_i + h_i \sum_{j=1}^{n} x_j^2) f_i / (1 - h_p \sum_{j=1}^{n} x_j^2), \qquad q.e.d.$$

28. Proposition. *Let A_a be the germ of a real analytic set. Then $Z^*((A_a))$ is finitely generated if and only if A_a is the germ of a coherent analytic set.*

Proof. If-part follows from the theorem of Malgrange and Tougeron (cf. [9; p. 95] or [12; p. 127]). Let $f_1, \ldots, f_p$ be the generator of ideals of analytic functions vanishing on A_a. Then $Tf_1 = f_1, \ldots, Tf_p = f_p$ generate $T_* \circ Z^*((A_a))$ by [9; IV, (3.5), (3.8)]. Hence if $Z^*((A_a))$ is finitely generated, $f_1, \ldots, f_p$ do so by the previous lemma. Thus A_a is coherent again by the theorem of Malgrange and Tougeron, *q.e.d.*

Faculty of Science and
Technology, Kinki University

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