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Convergence theorems of Abelian differentials with applications to conformal mappings II

By

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Introduction

The generalizations of the canonical slit mapping theorems for a planar Riemann surface to open Riemann surfaces with finite genus were studied, at first, by Kusunoki [7] and, afterwards, along his method by many authors, for example, Watanabe [15] [16], Mizumoto [12], Shiba [14] and Matsui [11]. All of these conformal mappings can be classified to the same type in the sense that the image of each boundary component of R is, roughly speaking, a line segment. In this paper we shall consider somewhat different type of conformal mappings of R such that the image of each boundary component of R is, roughly speaking, exactly two segments with only one common point. Such a mapping will be called here a conformal mapping of X-type (concerning the strict definition, see Definition 2). This paper consists of two parts. The aim of the first part (§2) is to show that on a finite Riemann surface R with genus g there exists a meromorphic function f of Kusunoki's type which satisfies the following conditions (cf. Theorem 1): (i) the image of each boundary component of R is exactly two segments with only one common point and each segment of the image has an arbitrary prescribed direction, (ii) the divisor of f is a multiple of $(P_1P_2\cdots P_{g+1})^{-1}$ where $P_1, P_2, \ldots, P_{g+1}$ are suitable points of R, (iii) residue of f at P_1 is equal to 1 (or i), (iv) $f(R)$, the image of R under f, is of at most $g+1$ sheets over the extended plane $\bar{C}$. Further, we shall show a sufficient condition for the existence of a function of Schwarz-Christoffel's type on R (cf. Theorem 2).

The aim of the second part is to extend above mapping theorem to an open surface. For this purpose, in §3 we will introduce the notion of a Kuramochi's boundary point of border type and show several convergence theorems for real harmonic differentials on an open Riemann surface R. Next, in

§4 we shall show the existence theorems of the behavior spaces Λ_x of X-type and the (Shiba's) Riemann-Roch theorem for Λ_x on R (cf. Theorem 3). From these results we can show that there exists, on an arbitrary open Riemann surface with finite genus, the meromorphic function of Kusunoki's type which has Λ_x behavior and satisfies the conditions (ii), (iii) and (iv) as stated above (cf. Theorem 4).

The content of this paper is as follows:

§1 Fundamental results on the behavior spaces.

§2 A conformal mapping of X-type of a finitely bordered Riemann surface and the functions of Schwarz-Christoffel's type.

§3 Orthogonal decompositions on open Riemann surfaces and convergence theorems for real harmonic differentials.

§4 Convergence theorems for the behavior spaces of X-type and its applications to conformal mappings.

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§1. Fundamental results of the behavior spaces

1.1. Behavior space. Let R be an arbitrary Riemann surface. The totality of square integrable complex (resp. real) differentials on R forms a real Hilbert space $\Lambda=\Lambda(R)$ (resp. $\Gamma=\Gamma(R)$) over the real number field with the inner product defined by

$$<\lambda_1, \lambda_2> = \mathrm{Re}\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)\mathrm{d}x\mathrm{d}y,$$

where $\lambda_j=a_j(z)dx+b_j(z)dy$ for a local parameter $z=x+iy$. It should be noticed that the meanings of the letters Λ and Γ are different from those in Ahlfors and Sario [2]. With these exceptions, we inherit the terminologies and the notations of [2] and [11], if not mentioned further.

Suppose R is an open Riemann surface of genus g (may be infinity) and $\{R_n\}$ is a canonical exhaustion of R, then we can choose a canonical homology basis $\{A_j, B_j\}_{j=1}^g$ of R modulo dividing curves such that $\{A_j, B_j\}\cap D_n^k$ is also a canonical homology basis of D_n^k modulo ∂D_n^k for each k and n, where D_n^k denotes a component of $R_{n+1}-\bar{R}_n$ (cf. [2]). Further, suppose $\mathscr{L}=\{L_j\}_{j=1}^g$ is a family of lines on the complex plane each of which passes through the origin. We consider a linear space Λ_0 of Λ_{hse} which satisfies the following conditions:

(a) there exists a subspace Λ_1 of $\Lambda_{hse}(R)$ such that $\Lambda_1+i\Lambda_1^{*\perp}\subset\Lambda_0$, where $\Lambda_1+i\Lambda_1^{*\perp}$ means the vector sum of Λ_1 and $i\Lambda_1^{*\perp}$,

(b) $<\lambda, i\lambda^*>=0$ for any $\lambda\in\Lambda_0$,

(c) $\int_{A_j}\lambda\in L_j$, $\int_{B_j}\lambda\in L_j$ for any $\lambda\in\Lambda_0$ and $j=1, 2,\ldots, g$.

Such a space $\Lambda_0=\Lambda_0(\mathscr{L}, \Lambda_1, R)$ is called *the behavior space on R associated with $\mathscr{L}$ and Λ_1*, and $\bar{\Lambda}_0=\{\lambda: \bar{\lambda}\in\Lambda_0$ where $\bar{\lambda}$ denotes the complex conjugate of $\lambda\}$ *the dual behavior space of* Λ_0 (cf. Shiba [14]). Here we note that the definition of the behavior spaces is simplified as follows:

Lemma 1.1. *A behavior space $\Lambda_0=\Lambda_0(\mathscr{L}, \Lambda_1, R)$ satisfies always the following conditions*:

(a) $\Lambda_0=i\Lambda_0^{*\perp}$ *(therefore, Λ_1 with which Λ_0 associates reduces to Λ_0)*,

(b) $\int_{A_j, B_j}\lambda\in L_j$ *for any* $\lambda\in\Lambda_0$ *and* $j=1, 2,\dots, g$.

Conversely, the subspace Λ_0 of Λ_{hse} satisfying (a), (b) *is a behavior space associated with $\mathscr{L}$ (and $\Lambda_1=\Lambda_0$).*

Proof. Since $<\lambda_1+\lambda_2, i\lambda_1^*+i\lambda_2^*>=0$ for any pair of $\lambda_1, \lambda_2\in\Lambda_0$, we can get $<\lambda_1, i\lambda_2^*>=0$, hence $\Lambda_0\subset i\Lambda_0^{*\perp}$. On the other hand, from $\Lambda_1\subset\Lambda_0$ we have $i\Lambda_0^{*\perp}\subset i\Lambda_1^{*\perp}\subset\Lambda_1+i\Lambda_1^{*\perp}\subset\Lambda_0$, hence we have $\Lambda_1=\Lambda_0=i\Lambda_0^{*\perp}$. The converse is evident.

From Lemma 1.1 we denote a behavior space associated with $\mathscr{L}$ and Λ_0 simply by $\Lambda_0=\Lambda_0(\mathscr{L}, R)$.

Lemma 1.2. *If a subspace Λ_0' of Λ_{hse} satisfies the conditions*:

(a) $\Lambda_0'\supset i\Lambda_0'^{*\perp}$,

(b) $\int_{A_j B_j}\lambda\in L_j$ *for any* $\lambda\in\Lambda_0'$ *and* $j=1, 2,\dots, g$,

(c) $\Lambda_0'\cap\Lambda_a=\{0\}$, $\Lambda_0'\cap\bar{\Lambda}_a=\{0\}$ *where Λ_a denotes the subspace of all analytic differentials,*

then Λ_0' is a behavior space associated with $\mathscr{L}$, and the converse is also true.

Proof. From (a) we have $\Lambda_0'=i\Lambda_0'^{*\perp}+\Lambda_0'\cap i\Lambda_0'^*$, and from (c) $\Lambda_0'\cap i\Lambda_0'^*=\Lambda_0'\cap\Lambda_a+\Lambda_0'\cap\bar{\Lambda}_a=\{0\}$. Therefore, $\Lambda_0'=i\Lambda_0'^{*\perp}$. q.e.d.

Corollary. *A behavior space $\Lambda_0(\mathscr{L}, R)$ and its dual behavior space always exist for any $\mathscr{L}$.*

Proof. Cf. Lemma 1.2 and Theorem 2 in Matsui [11].

Now we consider the following linear subspaces of functions and differentials on R:

$S(\Lambda_0, 1/\delta)=\{f$: (i) f is a meromorphic function which has Λ_0 behavior,

(ii) the divisor of f is a multiple of $1/\delta\}$,

$D(\Lambda_0, \delta)=\{\phi$: (i) ϕ is a meromorphic differentials which has Λ_0 behavior, (ii) the divisor of ϕ is a multiple of $\delta\}$.

Then the following proposition is well known (cf. Shiba [14]).

Proposition 1. *Suppose* $\delta=\delta_P/\delta_Q$ $(\delta_P \cap \delta_Q=\{0\})$ *is a finite divisor on* R, *then the following Riemann-Roch theorem for* Λ_0 *holds*:

$$\dim S(\Lambda_0, 1/\delta)=2[\operatorname{ord}\delta_P+1-\min(\operatorname{ord}\delta_Q, 1)]-\dim[D(\bar{\Lambda}_0, 1/\delta_Q)/D(\bar{\Lambda}_0, \delta)].$$

1.2. Convergence theorems for the behavior spaces. Let $\{G_n\}$ be an exhaustion of R by regions which may be $G_n=R$ for all n. We assume the existence of a canonical homology basis $\{A_j, B_j\}$ of R modulo dividing curves on R such that, for each n, $\{A_j, B_j\}_{j=1}^{g_n}$ is also a canonical homology basis of G_n modulo dividing curves on G_n where g_n denotes the genus of G_n which may be infinity. Let $\Lambda_n=\Lambda_n(\mathscr{L}_n, G_n)$ be a behavior space on G_n associated with $\mathscr{L}_n$. We consider the following subspaces of the differentials:

$$\Lambda'_0=\Lambda'_0(R)=\{\lambda\colon \text{there exists a sequence } \{\lambda_n\} \text{ with } \lambda_n\in\Lambda_n \text{ such that}$$
$$\|\lambda-\lambda_n\|_{G_n}\longrightarrow 0 \text{ as } n\longrightarrow\infty\},$$
$$\tilde{\Lambda}_0=\tilde{\Lambda}_0(R)=i\Lambda'_0(R)^{*\perp}.$$

Lemma 1.3. *If* $\{\lambda_n\}$ *with* $\lambda_n\in\Lambda_n$ *is a sequence such that* $\sup_n\|\lambda_n\|_{G_n}<\infty$, *the limit of each locally uniformly convergent subsequence* $\{\lambda_{n_k}\}$ *belongs to* $\tilde{\Lambda}_0$.

Proof. For each $\sigma'\in\Lambda'_0$ and $\varepsilon>0$ we can find a regular region D on R such that $\|\sigma'\|_{R-D}<\varepsilon$ and a sequence $\{\sigma_n\}$ with $\sigma_n\in\Lambda_n$ such that $\|\sigma'-\sigma_n\|_{G_n}\to 0$ as $n\to\infty$. Let $K=\sup_n\|\lambda_n\|_{G_n}$. For the limit λ of a locally uniformly convergent subsequence $\{\lambda_{n_k}\}$ (which we write simply $\{\lambda_k\}$) we have

$$|<\lambda, i\sigma'^*>|\leqq|<\lambda, i\sigma'^*>_D|+|<\lambda, i\sigma'^*>_{R-D}|\leqq|\lim_{k\to\infty}<\lambda_k, i\sigma'^*>_D|+K\varepsilon$$
$$\leqq|\lim_{k\to\infty}<\lambda_k, i\sigma_k'^*>_{G_k}|+2K\varepsilon=2K\varepsilon,$$

hence we can conclude $\lambda\in i\Lambda_0'^{*\perp}=\tilde{\Lambda}_0$. q.e.d.

In section 4, we shall consider some metric conditions for $\Lambda'_0=i\Lambda_0'^{*\perp}$ (cf. Lemmta 4.2 and 4.3).

Definition 1. Suppose $\mathscr{L}=\{L_j\}_{j=1}^{g}$ and $\mathscr{L}_n=\{L_j\colon j$ such that $A_j, B_j\subset G_n\}$. *We say that a sequence* $\{\Lambda_0^n(\mathscr{L}_n, G_n)\}_{n=1}^{\infty}$ *of behavior spaces converges to* $\hat{\Lambda}(R)$ (a subspace of $\Lambda'_0(R)$) if the following condition $*$ is fulfilled:

* if $\{\lambda_n\}$ with $\lambda_n \in \Lambda_0^n(\mathscr{L}_n, G_n)$ is a sequence such $\sup_n \|\lambda_n\|_{G_n} < \infty$, then the limit of each locally uniformly convergent subsequence $\{\lambda_{n_k}\}$ belongs to $\hat{\Lambda}(R)$.

In this case we write $\Lambda_0^n(\mathscr{L}_n, G_n) \Rightarrow \hat{\Lambda}(R)$.

Note. Since $\hat{\Lambda}(R) \subset \Lambda'_0(R)$, there exists, for any $\lambda \in \hat{\Lambda}(R)$, a sequence $\{\lambda_n\}$ with $\lambda_n \in \Lambda_0^n(L_n, G_n)$ such that $\|\lambda - \lambda_n\|_{G_n} \to 0$ as $n \to \infty$.

Lemma 1.4. *Suppose* $\Lambda_0^n(\mathscr{L}_n, G_n) \Rightarrow \Lambda_0(R)$, *then* $\Lambda_0(R)$ *is a behavior space associated with* $\mathscr{L}$. *Moreover, there exists a sequence* $T = \{n_k\}$ *of positive integers such that the sequence* $\{\phi_{\alpha_j}(A_j, \Lambda_0^k, G_k)\}_{k \in T}$ *(resp.* $\{\phi_{\alpha_j}(B_j, \Lambda_0^k, G_k)\}_{k \in T}$ *and* $\{\phi(0, \Lambda_0^k, G_k)\}_{k \in T}$*) converges locally uniformly on* R *to* $\phi_{\alpha_j}(A_j, \Lambda_0, R)$ *(resp.* $\phi_{\alpha_j}(B_j, \Lambda_0, R)$ *and* $\phi(0, \Lambda_0, R)$*).*

For the definition of $\phi_{\alpha_j}(A_j, \Lambda_0, R)$ etc., see [14] or [11] p. 81.

Proof. See Lemmata 3.1 and 3.3 in [11].

1.3. Extremal length. Suppose $\bar{\Omega}$ is a compact bordered Riemann surface such that each component of $\partial\bar{\Omega}$ is a closed analytic curve or may be a piecewise analytic Jordan curve. We call hereafter such a surface $\bar{\Omega}$ *a finitely bordered Riemann surface*, and an endpoint of an analytic arc on $\partial\bar{\Omega}$ *a vertex of* $\bar{\Omega}$. Let $\hat{\Omega}$ be the double of $\bar{\Omega}$ with respect to $\partial\bar{\Omega} - \{\text{vertex}\}$. For a point P of $\partial\bar{\Omega}$, we consider the following family $\{C\}_P$ of curves on $\bar{\Omega}$:

$\{C\}_P = \{C$: (i) there exists a neighbourhood U_P on $\bar{\Omega}$ of P such that $C = \partial U_P - \partial\bar{\Omega}$, (ii) $U' \cap \partial(\bar{\Omega} - U_P)$ is smooth, where U' is a region (on $\bar{\Omega}$) that contains the closure of U_P, (iii) $C \subset \bar{\Omega} - \hat{K}$ where $\hat{K}$ is a (fixed) compact region on $\hat{\Omega}\}$.

Proposition 2. (Kusunoki [6]). (i) *The extremal length of* $\{C\}_P$ *is zero, independent of* $\hat{K}$.

(ii) *Let* ϕ_1, ϕ_2 *be any two non-negative covariants square integrable over* $\bar{\Omega} - \bar{\Omega} \cap \hat{K}$, *then there exists a sequence of curves* $\gamma_n \in \{C\}_P$ $(\gamma_n \cap \hat{K} = \{0\})$ *tending to* P *such that* $\int_{\gamma_n} \phi_1 |dz| + \phi_2 |dz| \to 0$ *as* $n \to \infty$.

Further, we shall use frequently the following results:

Proposition 3. (Ohtsuka [13]). *Every Dirichlet function* f *on an open Riemann surface has finite limits along almost all curves tending to the ideal boundary.*

Proposition 4. (Fuji-i-e [4]). *Let* $\{f_n\}$ *be a sequence of Dirichlet functions converging to a Dirichlet function* f *in Dirichlet norm such that* $f_n = f$ *on a compact region* K_0 *for all* n, *and* L *be the family of all curves which start* *from points of* K_0 *and tend to the ideal boundary. Then, there exists a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that the limit* $l_{n_k}=\lim_c f_{n_k}$ *along a curve* $c\in L$ *converges to* $l=\lim_c f$ *for* $n_k\to\infty$, *for almost all curves of* L.

§2. Conformal mappings of X-type of a finitely bordered Riemann surface and the meromorphic functions of Schwarz-Christofell's type

In order to use the Green formula for the harmonic differentials, we shall study in 2.1 the properties of differentials in some subspaces and certain orthogonal decompositions. In 2.2, the notion of the behavior spaces of X-type is introduced. Under this notion, we shall consider, in 2.3, the conformal mappings of X-type of a finitely bordered Riemann surface. In 2.4, after defining the meromorphic functions of Schwarz-Christoffel's, type, we shall show a sufficient condition for the existence of such functions.

2.1. The properties of Γ_{he} and the orthpgonal decompositions. Suppose R is the interior of a finitely bordered Riemann surface $\bar{R}$ and $\partial R=\bigcup_{k=1}^{K}\Delta_k$ where each Δ_k is a boundary component of R. We set

$$\Gamma_{he}(\bar{R})=\{df\in\Gamma_{he}(R): f \text{ is harmonic on } \bar{R}-\{\text{vertex}\} \text{ and continuous on } \bar{R}).$$

The following lemma was suggested by Y. Kusunoki and F. Maitani.

Lemma 2.1. $\Gamma_{he}(R)=Cl\{\Gamma_{he}(\bar{R})\}$, *where* Cl *stands for the closure in* $\Gamma_h(R)$.

Proof. At first we prove that $du\in Cl\{\Gamma_{he}(\bar{G}_k)\}$ for $du\in\Gamma_{he}(R)$ where G_k denotes an end towards Δ_k (cf. [11], p. 76) which is conformally equivalent to a ring domain $r<|z|<1$. We write

$$g\,dz=du+idu^*-\frac{1}{2\pi}\left(\int_\gamma du^*\right)\frac{dz}{z}\ ,$$

where γ denotes an analytic closed curve on G_k homologous to $\partial G_k\cap R$. The function g being analytic in $r<|z|<1$, g has the Laurent expansion $g=\sum_{-\infty}^{\infty}a_kz^k$. Next, we set

$$g_n=\sum_{-n}^{n}a_kz^k,\quad a=\int_\gamma du^*,$$

$$du_n^k=\mathrm{Re}\left(g_ndz+\frac{a}{2\pi}\frac{dz}{z}\right).$$

We normalize so that $u_n^k(Q_k)=u(Q_k)$ for a fixed point Q_k on G_k, then we have $du_n^k\in\Gamma_{he}(\bar{G}_k)$ and $\|du-du_n^k\|_{G_k}\to0$ as $n\to\infty$, hence $du\in Cl\{\Gamma_{he}(\bar{G}_k)\}$. Now we show $Cl\{\Gamma_{he}(\bar{R})\}=\Gamma_{he}(R)$. Let G_k' be another end towards Δ_k which is conformally equivalent to a domain $1>|z|>r'\ (>r)$, and ψ_k a function $\in C^\infty(R)$ such that $\psi_k=1$ on G_k', $\psi_k=0$ on $R-G_k$. We write for $du\in\Gamma_{he}(R)$

$$f_m=\sum_{k=1}^{K}\psi_k u_m^k,\qquad f=\sum_{k=1}^{K}\psi_k u\,,$$

$$G=\bigcup_{k=1}^{K}G_k,\qquad G'=\bigcup_{k=1}^{K}G_k'\,.$$

The function f_m (resp. f) on R has the Royden decomposition of the form $f_m=v_m+f_{0m}$ (resp. $f=u+f_0$) where $v_m\in HD(R)$ and f_{0m}, f_0 are Dirichlet potentials. Since f_{0m} is harmonic on G', we can set $f_{0m}=0$ on ∂R from the regularity of each point on ∂R and Lemma 3 in Ohtsuka [13]. Consequently, we have $dv_m\in\Gamma_{he}(\bar{R})$ and

$$\|dv_m-du\|^2\leqq\|dv_m-du\|^2+\|df_{0m}-df_0\|^2=\|df_m-df\|^2$$
$$\leqq\Big(\sum_k\|du_m^k-du\|+\sum_k\operatorname*{Max}_{\bar{G}-G'}|u_m^k-u|\cdot\|d\psi_k\|\Big)^2\,.$$

Therefore, $\|dv_m-du\|\to0$ as $m\to\infty$, hence $du\in Cl\{\Gamma_{he}(\bar{R})\}$. q.e.d.

We consider the following spaces of differentials on R:

$\Gamma_h(\bar{R})=\{\omega\in\Gamma_h(R)$: (a) ω is harmonic on $\bar{R}-\{\text{vertex}\}$, (b) u is continuous at each vertex P, where $\omega=du$ near $P\}$,

$$\Gamma_{hse}(\bar{R})=\Gamma_{hse}(R)\cap\Gamma_h(\bar{R}),$$

$$\Gamma_{ho}(\bar{R})=\Gamma_{ho}(R)\cap\Gamma_h(\bar{R}).$$

Corollary. $\Gamma_{ho}(R)=Cl\{\Gamma_{ho}(\bar{R})\}$, $\Gamma_{hse}(R)=Cl\{\Gamma_{hse}(\bar{R})\}$, $\Gamma_h(R)=Cl\{\Gamma_h(\bar{R})\}$.

Proof. From Lemma 3 in Accola [1] we have $\omega=0$ along $\partial R-\{\text{vertex}\}$ for $\omega\in\Gamma_{ho}(R)$ and from Proposition 2 u is continuous at each vertex P where $\omega=du$ near P, hence $\Gamma_{ho}(R)\subset\Gamma_{ho}(\bar{R})$. On the other hand, it holds $\Gamma_{ho}(\bar{R})\perp\Gamma_{he}(\bar{R})^*$ from Lemma 2.1 and Proposition 2, hence $Cl\{\Gamma_{ho}(\bar{R})\}\subset\Gamma_{ho}(R)$, so we have $\Gamma_{ho}(R)=Cl\{\Gamma_{ho}(\bar{R})\}$. The other are evident by Lemma 2.1. q.e.d.

Now we introduce some subspaces of $\Gamma_h(R)$. We divide each Δ_k into a union of α_k and $\bar{\beta}_k$ where α_k, β_k are open sets on Δ_k, $\bar{\beta}_k$=closure of β_k on Δ_k and $\alpha_k=\Delta_k-\bar{\beta}_k$. Let $\alpha=\bigcup_{k=1}^{K}\alpha_k$, $\beta=\bigcup_{k=1}^{K}\beta_k$. Further, we set

$\Gamma_{eo}^1(\bar{\alpha}, R)=\{df\in\Gamma_c^1(R)$: there exists a neighbourhood of $\bar{\alpha}$ on $\bar{R}$ which is disjoint with the support of $f\}$,

$$\Gamma_{eo}(\bar{\alpha}, R)=Cl\{\Gamma_{eo}^1(\bar{\alpha}, R)\},\ \Gamma_{heo}(\bar{\alpha}, R)=\Gamma_h(R)\cap\Gamma_{eo}(\bar{\alpha}, R)\,,$$

$\Gamma_{heo}(\bar{\alpha}, \bar{R}) = \{df \in \Gamma_{he}(\bar{R}) : f=0 \text{ on } \bar{\alpha}\}$,

$\Gamma_{ho}(\beta, R) = \Gamma_{heo}(\bar{\alpha}, R)^{*\perp} \cap \Gamma_h$,

$\Gamma_{ho}(\beta, \bar{R}) = \{\omega \in \Gamma_h(\bar{R}) : \omega = 0 \text{ along } \beta - \{\text{vertex}\}\}$,

D_0 = the family of Dirichlet potentials on R.

We shall identify hereafter all constant functions with zero.

Lemma 2.2. $\Gamma_{heo}(\bar{\alpha}, R) = Cl\{\Gamma_{heo}(\bar{\alpha}, \bar{R})\}$, $\Gamma_{ho}(\beta, R) = Cl\{\Gamma_{ho}(\beta, \bar{R})\}$.

Proof. The proof consists of the following four steps:
(a) $\Gamma_{heo}(\bar{\alpha}, \bar{R})^* \perp Cl\{\Gamma_{ho}(\beta, \bar{R})\}$
(b) $Cl\{\Gamma_{heo}(\bar{\alpha}, \bar{R})\} \supset \Gamma_{heo}(\bar{\alpha}, R)$, i.e. $Cl\{\Gamma_{ho}(\beta, \bar{R})\} \subset \Gamma_{ho}(\beta, R)$,
(c) $\Gamma_{ho}(\beta, R) \subset Cl\{\Gamma_{ho}(\beta, \bar{R})\}$,
(d) $\Gamma_{ho}(\beta, R) = Cl\{\Gamma_{ho}(\beta, \bar{R})\}$ and $\Gamma_{heo}(\bar{\alpha}, R) = Cl\{\Gamma_{heo}(\bar{\alpha}, \bar{R})\}$.

(a) This is evident from Proposition 2 and the Green formula.
(b) We show at first that we may set $f=0$ on $\alpha - \{\text{vertex}\}$ for $df \in \Gamma_{heo}(\bar{\alpha}, R)$. From the definition of $\Gamma_{eo}(\bar{\alpha}, R)$ there exists a sequence $\{df_n\}$ with $df_n \in \Gamma^1_{eo}(\bar{\alpha}, R)$ and neighbourhoods U_n of $\bar{\alpha}$ on $\bar{R}$ such that $f_n = 0$ on U_n and $\|df_n - df\| \to 0$ as $n \to \infty$. Let $f_n = u_n + f_{on}$ be the Royden decomposition of f_n where $f_{on} \in D_0$ and $u_n \in HD(R)$. Because $f_n = 0$ on U_n, we have $u_n = -f_{on} = H^{U_n}_{-f_{on}}$ on U_n. From the regularity of each point on α and Lemma 3 in Ohtsuka [13] we can set $u_n = 0$ on α. Consequently, if we put $\hat{u}_n(p) = u_n(p)$ for $p \in R$ and $u_n(p) = -u_n(jp)$ for $p \in \hat{R}_\alpha - R$ where $\hat{R}_\alpha$ denotes the double of $\bar{R}$ with respect to $\alpha - \{\text{vertex}\}$ and j is the involutory mapping of $\hat{R}_\alpha$, then $\{d\hat{u}_n\}$ is a Cauchy sequence on $\hat{R}_\alpha$ and $\hat{u}_n = 0$ on α. Accordingly, there exists a subsequence $\{n_k\}$, $f_0 \in D_0$ and $u \in HD(R)$ such that $\|df_{on_k} - df_0\| \to 0$ and $\|du_n - du\| \to 0$ as $n \to \infty$, where $u=0$ on $\alpha - \{\text{vertex}\}$. Therefore, $f = u + f_0 + \text{const.}$, while $f_0 = 0$ as $df \in \Gamma_{he}(R)$, hence $f=0$ on $\alpha - \{\text{vertex}\}$. Next, we show $\Gamma_{heo}\{\bar{\alpha}, R) \subset Cl\{\Gamma_{heo}(\bar{\alpha}, \bar{R})\}$. For $df \in \Gamma_{heo}(\bar{\alpha}, R)$ we set $F(p) = f(p)$ for $p \in R \cup \alpha - \{\text{vertex}\}$ and $F(p) = -f(jp)$ for $p \in \hat{R}_\alpha - \bar{R}$, then dF is odd for j and $dF \in \Gamma_{he}(\hat{R}_\alpha)$. Therefore, from Lemma 2.1 there exists a sequence $\{dF_n\}$ with $dF_n \in \Gamma_{he}(\hat{R}_\alpha - \{\text{vertex}\})$ such that $\|dF_n - dF\| \hat{R}_\alpha \to 0$. Denote $\frac{1}{2}\{F_n(p) - F_n(jp)\}$ by $G_n(p)$, then $dG_n|_{\bar{R}} \in \Gamma_{heo}(\bar{\alpha}, \bar{R})$. Further, we can get $\|dG_n - dF\|_{\hat{R}_\alpha} \leqq \frac{1}{2}\|dF - dF_n\| + \frac{1}{2}\|d(Fj) - d(F_n j)\| \to 0$ as $n \to \infty$. Consequently, we have $df = dF|_R \in Cl\{\Gamma_{heo}(\bar{\alpha}, \bar{R})\}$. By the same way as in (b) we can prove (c) easily and so (d) is evident. q.e.d.

Corollary. *If a differential $\omega \in \Gamma_h(R)$ is in $\Gamma_{ho}(\beta, R)$, $\omega = 0$ along $\beta - \{\text{vertex}\}$, and the converse is also true. If a harmonic Dirichlet function f can be written in a form $f = H^R_{f'}$ where $f' = 0$ on α, then $df \in \Gamma_{heo}(\bar{\alpha}, R)$.*

For a harmonic Dirichlet function f there exists a function f' on ∂R such that $f=H_{f'}^{R}$ (cf. Constantinescu und Cornea [3]). Hereafter, f' will be called the resolution of f.

Lemma 2.3. $\Gamma(R)=\Gamma_{eo}^{*}\dot{+}\Gamma_{eo}(\alpha, R)\dot{+}\Gamma_{ho}^{*}(\beta, R)$,

$$\Gamma_{eo}(\bar{\alpha}, R)=\Gamma_{heo}(\bar{\alpha}, R)\dot{+}\Gamma_{eo}.$$

Proof. Omitted.

Lemma 2.4. $Cl\{\Gamma_{eo}^{0}(\bar{\alpha}, R)\}=Cl\{\Gamma_{eo}^{1}(\bar{\alpha}, R)\}=\Gamma_{eo}(\bar{\alpha}, R)$, *where* $\Gamma_{eo}^{0}(\bar{\alpha}, R)=\{df$: (a) *$f$ is a continuous Dirichlet function*, (b) *there exists on $\bar{R}$ a neighbourhood U of $\bar{\alpha}$ which is disjoint with the support of f*$\}$.

Proof. Let $f=u+f_0$ be the Royden decomposition of f where $u\in HD(R)$ and $f_0\in D_0$. Analogously as in Lemma 2.2, u has a resolution f' such that $f'=0$ on α. Therefore, by Corollary of Lemma 2.2 we have $du\in\Gamma_{heo}(\bar{\alpha}, R)$ and so $df\in\Gamma_{heo}(\bar{\alpha}, R)\dot{+}\Gamma_{eo}\subset Cl\{\Gamma_{eo}^{1}(\bar{\alpha}, R)\}$. $Cl\{\Gamma_{eo}^{1}(\bar{\alpha}, R)\}\subset Cl\{\Gamma_{eo}^{0}(\bar{\alpha}, R)\}$ is evident.

q.e.d.

Corollary. *If df_1 and df_2 are in $\Gamma_{eo}(\bar{\alpha}, R)$, then $d\{\mathrm{Max}(f_1, f_2)\}\in\Gamma_{eo}(\bar{\alpha}, R)$ and $d\{\mathrm{Min}(f_1, f_2)\}\in\Gamma_{eo}(\bar{\alpha}, R)$. If $df\in\Gamma_{eo}(\bar{\alpha}, R)$, then $d[\mathrm{Max}\{\mathrm{Min}(f, k), -k\}]\in\Gamma_{eo}(\bar{\alpha}, R)$, provided k is a positive constant.*

2.2. Behavior spaces of X-type on a finitely bordered Riemann surface Let $\partial R=\bigcup_{k=1}^{K}\Delta_k$ (resp. $\mathscr{L}$) be the same as in 2.1 (resp. § 1). We divide each boundary component Δ_k into a union of α_k and $\bar{\beta}_k$ where each of α_k, β_k is an open arc on Δ_k and $\alpha_k\cap\beta_k=0$. The set $S=\{\alpha_1, \bar{\beta}_1, \alpha_2, \bar{\beta}_2,\dots, \alpha_K, \bar{\beta}_K\}$ will be called hereafter *a partition of ∂R*. Next, we associate each α_k (resp. β_k) with a complex number z_k (resp. z'_k) such that $|z_k|=|z|'_k=1$ and denote the set $\{z_1, z'_1, z_2, z'_2,\dots, z_K, z'_K\}$ by Z. For S and Z we consider the following subspace of differentials:

$$\Lambda_x(R)=\Lambda_x(\mathscr{L}, R, S, Z)=\{\lambda\in\Lambda_{hse}\text{: (i)}\int_{A_j, B_j}\lambda\in L_j \text{ for any } \lambda\in\Lambda_x \text{ and each } j, \text{ (ii) } \mathrm{Im}(\bar{z}_k\lambda)\in\Gamma_{ho}(\alpha_k, R), \mathrm{Im}(\bar{z}'_k\lambda)\in\Gamma_{ho}(\beta_k, R) \text{ for each } k\}.$$

Lemma 2.5. *Assume that* $\arg z_k-\arg z'_k\not\equiv 0 \pmod{\pi}$ *and* $\phi\in\Lambda_{ase}$ *satisfies the conditions*: $\mathrm{Im}(\bar{z}_k\phi)\in\Gamma_{ho}(\alpha_k, R)$ *and* $\mathrm{Im}(\bar{z}'_k\lambda)\in\Gamma_{ho}(\beta_k, R)$ *for each k. Then there exists a sequence of arcs* $\{\alpha_k^n\}_{n=1}^{\infty}$ *(resp.* $\{\beta_k^n\}_{n=1}^{\infty}$*) such that*

(a) α_k^n *(resp.* β_k^n*) is an arc on* α_k *(resp.* β_k*) and* $\alpha_k^n\uparrow\alpha_k$ *(resp.* $\beta_k^n\uparrow\beta_k$*)*,

(b) $\lim_{n\to\infty}\int_{\alpha_k^n}\phi=0$ *(resp.* $\lim_{n\to\infty}\int_{\beta_k^n}\phi=0$*)*.

Proof. Let P_k, Q_k be the endpoints of β_k and $\{c_k^n\}_{n=1}^\infty$ (resp. $\{c_k'^n\}_{n=1}^\infty$) a sequence of arcs in Proposition 2 that separates P_k (resp. Q_k) from a compact region on $\bar{R}-\{P_k, Q_k\}_{k=1}^K$. Then there exists for each c_k^n (resp. $c_k'^n$) a neighbourhood U_k^n (resp. $U_k'^n$) of P_k (resp. Q_k) on $\bar{R}$ such that $c_k^n=\partial U_k^n \cap R$ (resp. $c_k'^n=\partial U_k'^n \cap R$). Next we write

$$\bar{\alpha}_k^n=\alpha_k-U_k^n \cup U_k'^n,\ \bar{\beta}_k^n=\beta_k-U_k^n \cup U_k'^n,\ l_k^n=\alpha_k^n+\beta_k^n+c_k^n+c_k'^n .$$

Note that each α_k^n (or β_k^n) is an arc. From Proposition 2 and the semiexactness of ϕ we have $\int_{l_k^n}\phi=0$ and $\int_{c_n^n}\phi+\int_{c_k'^n}\phi\to 0$ as $n\to\infty$, hence $\int_{\alpha_k^n}\phi+\int_{\beta_k^n}\phi\to 0$ as $n\to\infty$. On the other hand, from the conditions on ϕ we can get

$$0=\int_{\alpha_k^n}\mathrm{Im}\,(\bar{z}_k\phi)=\int_{\alpha_k^n}(\omega^*\cos\theta-\omega\sin\theta),$$

$$0=\int_{\beta_k^n}\mathrm{Im}\,(\bar{z}_k'\phi)=\int_{\beta_k^n}(\omega^*\cos\theta'-\omega\sin\theta'),$$

where $\mathrm{Re}(\phi)=\omega$, $z_k=e^{i\theta}$ and $z_k'=e^{i\theta'}$. Consequently, we can conclude that $\int_{\alpha_k^n}\phi$ and $\int_{\beta_k^n}\phi$ converge to 0 as $n\to\infty$. q.e.d.

In order to prove that $\Lambda_x(R)$ is a behavior space, we consider the following auxiliary subspaces:

$\Lambda'(R)$ = the space spanned by the set $\{\zeta_j\sigma(A_j), \zeta_j\sigma(B_j)\}_{j=1}^g$ where $\zeta_j\in L_j$, $\zeta_j\neq 0$ for each j, g means the genus of R and $\sigma(\gamma)$ is the γ-reproducer in Γ_c,

$$\Lambda''(R)=\Gamma_{hm}(R)+i\Gamma_{hm}(R),$$

$$\Lambda'''(R)=Cl\{\sum_{k=1}^K(z_k\Gamma_{heo}(\Delta-\alpha_k, R)+z_k'\Gamma_{heo}(\Delta-\beta_k, R))\},$$

$$\Lambda_x'(R)=Cl\{\Lambda'(R)+\Lambda''(R)+\Lambda'''(R)\},$$

where $z_x\Gamma_x$ denotes the subspace $\{z_k df: df\in\Gamma_x\}$ and $A+B$ (resp. ΣA_n) means the vector sum of A and B (resp. $\{A_n\}$).

Lemma 2.6. *$\Lambda_x(R)$ is a behavior space,* i.e. $\Lambda_x(R)=i\Lambda_x(R)^{*\perp}$.

Proof. By the same method as in Lemmata 4.1 and 4.2 in Matsui [11] we can get easily the relations $i\Lambda_x(R)^{*\perp}=\Lambda_x'(R)\subset i\Lambda_x'(R)^{*\perp}=\Lambda_x(R)$. Therefore, we have only to prove that $i\Lambda_x^*\cap\Lambda_x\cap\Lambda_a=\{0\}$ and $i\Lambda_x^*\cap\Lambda_x\cap\bar{\Lambda}_a=\{0\}$ (cf. Lemma 1.2). Let P_k, Q_k be the endpoints of β_k. From Lemma 2.5 there exists on

$\bar{R}$ $\{\alpha_k^n\}_{n=1}^{\infty}$, $\{\beta_k^n\}_{n=1}^{\infty}$, $\{U_k^n\}_{n=1}^{\infty}$ and $\{U_k'^n\}_{n=1}^{\infty}$ such that each of $\int_{\alpha_k^n}\phi$, $\int_{\beta_k^n}\varphi$, $\int_{c_k^n}\phi$ and $\int_{c_k'^n}\phi$ converges to 0 as $n\to\infty$, where $c_k^n=\partial U_k^n\cap R$, $c_k'^n=\partial U_k'^n\cap R$, $\bar{\alpha}_k^n=\alpha_k-U_n^k\cup U_k'^n$ and $\bar{\beta}_k^n=\beta_k-U_k^n\cup U_k'^n$. On the other hand, from the semiexactness of ϕ we can take a function f_k separately near each Δ_k such that $df_k=\phi$. Now we denote $V_n=\bigcup_{k=1}^{K}U_k^n$ and $V'_n=\bigcup_{k=1}^{K}U_k'^n$, then we have by the Green formula

$$\|\phi\|^2=\lim_{n\to\infty}<\phi,\, i\phi^*>_{R-V_n-V'_n}$$

$$=\sum_j\mathrm{Re}\Big(\int_{A_j}\phi\int_{B_j}\overline{i\phi}-\int_{B_j}\phi\int_{A_j}\overline{i\phi}\Big)-\lim_{n\to\infty}\mathrm{Re}\Big[\sum_k\Big(\int_{\alpha_k^n}f_k\overline{i\phi}+\int_{\beta_k^n}f_k\,\overline{i\phi}\Big)\Big].$$

While from $\phi\in\Lambda_x$ we know $\mathrm{Im}(\bar{z}_k\phi)=0$ along α_n^n and $\mathrm{Im}(\bar{z}_kf_k)=$const. on α_k^n. Consequently, from Lemma 2.5 we obtain

$$\mathrm{Re}\Big(\int_{\alpha_k^n}f_k\,\overline{i\phi}\Big)=-\int_{\alpha_k^n}[\mathrm{Re}\,(\bar{z}_kf_k)\cdot\mathrm{Im}(\bar{z}_k\phi)$$

$$+\mathrm{Im}(\bar{z}_kf_k)\cdot\mathrm{Re}(\bar{z}_k\phi)]\longrightarrow 0 \text{ as } n\longrightarrow\infty\,.$$

Similarly we know $\mathrm{Re}\Big(\Big(\int_{\beta_k^n}f_ki\phi\Big)\to 0$ as $n\to\infty$, hence $\Lambda_x\cap i\Lambda_x^*\cap\Lambda_a=\{0\}$. By the same way, we can prove $\Lambda_x\cap i\Lambda_x^*\cap\bar{\Lambda}_a=\{0\}$ and $\Lambda_x=i\Lambda_x^{*\perp}$. q.e.d.

Hereafter, such a space $\Lambda_x(R)$ will be called *a behavior space of X-type associated with* (S, Z).

2.3. Conformal mappings X-type. At first we show the existence of a meromorphic function with Λ_x behavior. Let $S=\{\alpha_1, \beta_1, \alpha_2, \beta_2,\dots, \alpha_K, \beta_K\}$ and $Z=\{z_1, z_1',\dots, z_K, z_K'\}$ be the same as in 2.2. From Lemma 2.6 and Proposition 1 *the Riemann-Roch theorem holds for* $\Lambda_x(\mathscr{L}, R, S, Z)$, hence by the same method as in Kusunoki [7] or Matsui [11] we can prove the following lemma:

Lemma 2.7. *For each pair* (S, Z), *there exists a meromorphic function* f *on a finitely bordered Riemann surface* R *such that* (i) f *has* Λ_x *behavior,* (ii) *the divisor of* f *is a multiple of* $(P_1P_2\cdots P_{g+1})^{-1}$ *where* P_1 *is an arbitrary point and* $P_2, P_3,\dots, P_{g+1}$ *are suitably chosen* g *points of* R, (iii) *residue of* f *at* P_1 *is equal to* 1 (*or* i).

Lemma 2.8. *The meromorphic function* f *in Lemma* 2.7 *is continuous on* ∂R.

Proof. We have only to prove the continuity of f at the point P belonging

to the set {vertex of $\bar{R}$} $\cup$ {endpoint of α_k}. At first we prove it in case $P \in$ {vertex of $\bar{R}$} $\cap$ {endpoint of α_k}. Let U be a neighbourhood of P on $\bar{R}$ which is mapped homeomorphically by $\xi = h(Q)$ onto $D_\xi = \{\xi: |\xi| < 1, \operatorname{Im}\xi \geqq 0\}$ such that $h(P) = 0$ and $U - P$ is conformally equivalent to $D_\xi - \{0\}$. Further, suppose D_w is the image of U under the function $F(Q) = \{f(Q)\}^{\frac{\pi}{\theta}} e^{-\gamma i}$ where $\theta = \theta_k = \arg z_k - \arg z'_k$, $0 < \theta < 2\pi$ and γ is a suitably chosen real constant. Then $G(\xi) = F(h^{-1}) = u(\xi) + iv(\xi)$ is finite Dirichlet integrable over D_ξ and $v(\xi) = 0$ on real axis. Therefore, if we set $\overline{G(\bar{\xi})} = G(\xi)$, then $G(\xi)$ is analytic in $\{\xi: 0 < |\xi| < 1\}$. Consequently, the Laurent expansion of $G(\xi)$ in $0 < |\xi| < 1$ has the singular part with finite terms and $G(\xi)$ may have a pole at $\xi = 0$. But, from Proposition 2 there exists a sequence $\{\gamma_n\}$ of curves on $\bar{R}$ tending to P such that $\int_{s_n} dG \to 0$ as $n \to \infty$ where $s_n = h(\gamma_n)$. Accordingly, $G(\xi)$ has no singularities at $\xi = 0$, hence f is continuous at P. By the same way, we can prove the continuity of f at each point belonging to the set {vertex of $\bar{R}$} $\cup$ {endpoint of α_k}.
q. e. d.

At last, we show the existence of a special kind of conformal mappings of R. Suppose R is the interior of a finitely bordered Riemann surface with genus g and $\partial R = \bigcup_{k=1}^{K} \Delta_k$ where each Δ_k is a component of ∂R. Let $S = \{\alpha_k, \beta_k\}_{k=1}^{K}$ be a partition of ∂R where each of α_k, β_k is an open arc on Δ_k and $\alpha_k = \Delta_k - \bar{\beta}_k$. Further, let $Z = \{z_1, z'_1, \dots, z_K, z'_K\}$ be a set of complex numbers.

Definition 2. A conformal mapping f of R into a Riemann sphere $\bar{C}$ is said to be *of X-type* if f is a meromorphic function with $\Lambda_x(R)$ behavior.

According to Lemmata 2.6 and 2.8, we can get the following theorem.

Theorem 1. *For each pair (S, Z), there exists a meromorphic function f on R uniquely except additive constants such that:*

(i) *f is a conformal mapping of X-type, that is, each image of Δ_k under f is exactly two segments with only one common point. Moreover, the direction of each segment on $f(\Delta_k)$ is arbitrarily prescribed,*

(ii) *the divisor of f is a multiple of $(P_1P_2 \cdots P_{g+1})^{-1}$ where P_1 is an arbitrary point of R and $P_2, \dots, P_{g+1}$ are suitably chosen g points of R,*

(iii) *residue of f at P_1 is equal to* 1 (*or* i),

(iv) *$f(R)$, the image of R under f, is at most $g+1$ sheeted over $\bar{C}$.*

Proof. From Lemma 2.7 we have only to prove (iv). For $w \notin f(\partial R)$ we have by the argument principle

$$\frac{1}{2\pi i}\int_{\partial R}\frac{df}{f-w}=\frac{1}{2\pi i}\sum_{k=1}^{K}\int_{\Delta_k} d\arg(f-w)=0 \text{ (cf. Lemma 2.8)}.$$

Consequently, we obtain $N(f, w, R)=N(f, \infty, R)\leqq g+1$. q.e.d.

Remark. Suppose R is plamar and f is a conformal mapping in Theorem 1, then $f(\Delta_k)$ is a figure like Roman capital letter V or T.

2.4. The meromorphic function of Schwarz-Christoffel's type. Let R be the interior of a compact bordered Riemann surface with genus g and $\partial R = \bigcup_{k=1}^{K} \Delta_k$ where each Δ_k is a contour. Suppose $\Delta_k = \bigcup_{r=1}^{n_k} \bar{\alpha}_k^r$ where each α_k^r is an open arc such that $\alpha_k^r \cap \alpha_k^m = \{0\}$ for $r \neq m$. The set $\{\alpha_1^1, \alpha_1^2, \ldots, \alpha_1^{n_1}, \ldots, \alpha_K^{n_K}\}$ is denoted by S_G. Next, we associate each α_k^r with a complex number z_k^r such that $|z_k^r|=1$ and denote the set $\{z_1^1, \ldots, z_K^{n_K}\}$ by Z_G.

Definition 3. *We say that a meromorphic function f on R is of Schwarz-Christoffel's type associated with (S_G, Z_G) if $\mathrm{Im}(\bar{z}_k^r df)=0$ along α_k^r, $1\leqq k\leqq K$, $1\leqq r\leqq n_k$. Hereafter, we call such a function simply a* G.S.C. *function associated with (S_G, Z_G).*

Now our next problem is whether we can construct a G.S.C. function associated with given (S_G, Z_G) or not. In order to study this problem, we consider the following subspace of differentials:

$$\Lambda_s=\Lambda_s(R)=\Lambda_s(\mathscr{L}, R, S_G, Z_G)=\{\lambda\in\Lambda_{hse}\colon \text{(a)} \int_{A_j, B_j}\lambda\in L_j \text{ for each } \lambda\in\Lambda_s$$

$$\text{and } j=1, 2, \ldots, g, \text{ (b) } \mathrm{Im}(\bar{z}_k^r\lambda)\in\Gamma_{ho}(\alpha_k^r, R) \text{ for each pair } (k, r)\}.$$

Lemma 2.9. $\Lambda_s\supset i\Lambda_s^{*\perp}$.

Proof. This is proved analogously as in [11], hence omitted.

If $\Lambda_s=i\Lambda_s^{*\perp}$ for a pair (S_G, Z_G), we can construct a G.S.C. function associated with (S_G, Z_G) according to the Riemann-Roch theorem for Λ_s (cf. [7] or [11]). However, the next example shows that there exists a pair (S_G, Z_G) such that $i\Lambda_s^{*\perp}\neq\Lambda_s$.

Example. Let $R=\{|z|<1\}$ and R' be the interior of triangle $A_1A_2A_3$. Then there exists an analytic function f (classical Schwarz-Christoffel's function) which maps R conformally onto R'. Now we set $S_G=\{\alpha_1, \alpha_2, \alpha_3\}$ with $\alpha_i=f^{-1}$ (segment A_iA_{i+1}) where $A_4=A_1$, and $Z_G=\{z_1, z_2, z_3\}$ with $z_i=\overrightarrow{A_iA_{i+1}}$. The existence of above stated function f means $\Lambda_s\cap i\Lambda_s^*\cap\Lambda_{ae}\neq\{0\}$. Therefore, we can conclude $\Lambda_s \neq i\Lambda_s^{*\perp}$ (cf. Lemma 1.2).

This example shows that so long as we restrict our method to the behavior space's theory only, it is difficult to construct a G.S.C. function.

Next, we show a sufficient condition on (S_G, Z_G) for the existence of a G.S.C. function. We divide each component Δ_k of ∂R into a union of consecutive arcs $\bar{\alpha}_k^1, \bar{\beta}_k^1, \ldots, \bar{\alpha}_k^{m_K}, \bar{\beta}_k^{m_K}$ so that each of α_k^r, β_k^r is an open arc and $\bar{\alpha}_k^m \cap \bar{\alpha}_k^r = \bar{\beta}_k^m \cap \bar{\beta}_k^r = \{0\}$, $\alpha_k^r \cap \beta_k^r = \{0\}$ for each $k, r, m, r \neq m$ ($\bar{\alpha}_k^r$ is the closure of α_k^r on Δ_k as in 2.1). Next, we set $\bigcup_{k,r} \alpha_k^r = \alpha$ and $\bigcup_{k,r} \beta_k^r = \beta$. Let $\hat{R}$ be the interior of the double of $\bar{R}$ with respect to β. Then $\hat{R}$ is a finitely bordered Riemann surface and each component $\hat{\Delta}_k^r$ of $\partial\hat{R}$ can be written by $\hat{\Delta}_k^r = \bar{\alpha}_k^r \cup j\bar{\alpha}_k^r$ where j is the involutory mapping of $\hat{R}$. Further, we associate each α_k^r with a complex number z_k^r and denote the set $\{z_1^1, \bar{z}_1^1, \ldots, z_K^{m_K}, \bar{z}_K^{m_K}\}$ (resp. $\{\alpha_1^1, j\alpha_1^1, \ldots, \alpha_K^{m_K}, j\alpha_K^{m_K}\}$) by $\hat{Z}_0$ (resp. $\hat{S}_0$). From Lemma 2.6, $\Lambda_x(\hat{\mathscr{L}}, \hat{R}, \hat{S}_0, \hat{Z}_0)$ is a behavior space of X-type on $\hat{R}$ and so by Theorem 1 there exists a meromorphic function $\hat{f}$ on $\hat{R}$ such that, for each pair (k, r), $\mathrm{Im}(\bar{z}_k^r d\hat{f}) = 0$ along α_k^r, $\mathrm{Im}(z_k^r d\hat{f}) = 0$ along $j\alpha_k^r$ and the divisor of $\hat{f}$ is a multiple of $(P_1, \ldots, P_{2g+K})^{-1}$ where $P_1, \ldots, P_{2g+K}$ are suitably chosen $2g+K$ points on R (not on $\hat{R}$). Denote $\mathrm{Re}(d\hat{f})$ by ω, then, for each (k, r), we can get

$$\mathrm{Im}[\bar{z}_k^r(\omega - \omega^\sim) + i\bar{z}_k^r(\omega - \omega^\sim)^*] = 0 \text{ along } \alpha_k^r,$$

$$\mathrm{Im}[z_k^r(\omega - \omega^\sim) + iz_k^r(\omega - \omega^\sim)^*] = 0 \text{ along } j\alpha_k^r,$$

$$\mathrm{Im}[i(\omega - \omega^\sim) + (\omega^\sim - \omega)^*] = 0 \text{ along } \beta_k^r.$$

Now we write $\psi = (\omega - \omega^\sim) + i(\omega - \omega^\sim)^*$, then ψ is a differential of a function f on R whose divisor is a multiple of $(P_1 P_2 \cdots P_{2g+K} jP_1 \cdots jP_{2g+K})^{-1}$. By the argument principle we have for $w \notin f(\partial\hat{R})$

$$\frac{1}{2\pi i}\int_{\partial\hat{R}} \frac{df}{f-w} = \sum_{k,r}\int_{\hat{\Delta}_k^r} \frac{df}{f-w} = 0 = N(f, w, \hat{R}) - N(f, \infty, \hat{R}),$$

$$N(f, w, \hat{R}) \leqq 2(2g+K).$$

Thus we obtain the next theorem.

Theorem 2. *Let R be the interior of a compact bordered Riemann surface with genus g and $\partial R = \sum_{k=1}^{K} \Delta_k$ where each Δ_k is a contour. Suppose $\Delta_k = \sum_{r=1}^{m_k} (\bar{\alpha}_k^r \cup \bar{\beta}_k^r)$ $(k = 1, 2, \ldots, K)$ where each α_k^r, β_k^r is an open arc and $\bar{\alpha}_k^r \cap \bar{\alpha}_k^m = \bar{\beta}_k^r \cap \bar{\beta}_k^m = \{0\}$, $\alpha_k^r \cap \beta_k^r = \{0\}$ for each $k, r, r \neq m$. Let z_k^r be complex numbers and $S_0 = \{\alpha_1^1, \beta_1^1, \ldots, \alpha_K^{m_K}, \beta_K^{m_K}\}$ $Z_0 = \{z_1^1, i, z_1^2, i, \ldots, z_k^{m_K}, i\}$, then there exists a* G.S.C.

function f associated with (S_0, Z_0) on R such that

(i) $\mathrm{Im}(\bar{z}_k^r df)=0$ *along* α_k^r *and* $\mathrm{Im}(idf)=0$ *along* β_k^r *for each pair* (k, r),

(ii) *the divisor of* f *is a multiple of* $(P_1 P_2 \cdots P_{2g+K})^{-1}$ *where* $P_1, \dots, P_{2g+K}$ *are suitably chosen* $2g+K$ *points of* R,

(iii) *the residue of* f *at* P_1 *is equal to* 1 (*or* i),

(iv) $f(R)$, *the image of* R *under* f, *is at most* $2(2g+K)$ *sheeted over* $\bar{C}$.

§3. Orthogonal decompositions on open Riemann surfaces and convergence theorems for real harmonic differentials

In this section we consider some orthogonal decompositions of $\Gamma_h(R)$ and the convergence theorems of a sequence $\{\omega_n\}$ with $\omega_n \in \Gamma_x^n(R_n)$ where $\{R_n\}$ is a properly given exhaustion of R (R_n is not necessarily relative compact) and $\Gamma_x^n(R_n)$ is a subspace of $\Gamma_h(R_n)$. The results of this section is very useful when we consider the existence theorem of a meromorphic function with $\Lambda_x(R)$ behavior in §4.

3.1 Elementary convergence theorems. Let $\Gamma_x(R)$ and $\Gamma_{xn}(R)$, $n=1,2,\dots$ be subspaces of $\Gamma_h(R)$ and $\Gamma_x^\perp$ the orthogonal complement in Γ_h of Γ_x.

Lemma 3.1. *Suppose the following conditions are fulfilled*:

(a) $\Gamma_x(R) \supset \Gamma_{xm}(R) \supset \Gamma_{xn}(R)$ *for each* m *and* n $(m>n)$,

(b) $\bigcap_{n=1}^{\infty} \{\Gamma_{xn}(R)^\perp\} = \Gamma_x(R)^\perp$.

Then for any $\sigma \in \Gamma_x(R)$ *there exists a sequence* $\{\sigma_n\}$ *with* $\sigma_n \in \Gamma_{xn}(R)$ *such that* $\|\sigma-\sigma_n\| \to 0$ *as* $n \to \infty$. *The same conclusion as above holds, if* $\Gamma_x(R) \subset \Gamma_{xn}(R)$ *for each* n.

Proof. A differential $\sigma \in \Gamma_x(R)$ has a decomposition of the form $\sigma=\sigma_n+\omega_n$ where $\sigma_n \in \Gamma_{xn}(R)$ and $\omega_n \in \Gamma_{xn}(R)^\perp$. For $m>n$, we get $\sigma_m-\sigma_n=\omega_n-\omega_m \in \Gamma_{xn}(R)^\perp$, hence $\|\sigma_m-\sigma_n\|^2 = \langle\sigma_m-\sigma_n, \sigma_m\rangle = \|\sigma_m\|^2 - \|\sigma_n\|^2$, that is, $\|\sigma_n\| \leqq \|\sigma_m\| \leqq \|\sigma\|$. Therefore, we have $\|\sigma_m-\sigma_n\| \to 0$ as $m>n\to\infty$, hence by the triangle inequality we have a harmonic differential σ_0 such that $\|\sigma_n-\sigma_0\| \to 0$ as $n\to\infty$. On the other hand, for each $\omega \in \Gamma_x(R)^\perp$ $(\subset \Gamma_{xn}(R)^\perp)$ we have $\langle\sigma_0, \omega\rangle = \lim_{n\to\infty} \langle\sigma_n, \omega\rangle = 0$, so $\sigma_0 \in \Gamma_x(R)$. Consequently, we have $\sigma=\sigma_0+\omega_0$ where $\|\omega_0-\omega_n\| \to 0$ as $n\to\infty$, while $\omega_m \in \Gamma_{xn}(R)^\perp$ for each $m>n$, hence $\omega_0 \in \bigcap_{n=1}^{\infty} \{\Gamma_{xn}(R)^\perp\} = \Gamma_x(R)^\perp$. Therefore, $\sigma-\sigma_0=\omega_0 \in \Gamma_x(R) \cap \Gamma_x(R)^\perp = \{0\}$, so $\sigma=\sigma_0$. The last statement is evident if we set $\sigma_n=\sigma$. q.e.d.

Lemma 3.2. *The limit of each locally uniformly convergent subsequence of* $\{\sigma_n\}$ *with* $\sigma_n \in \Gamma_{xn}(R)$ *and* $\sup_n \|\sigma_n\| < \infty$ *belongs to* $\Gamma_x(R)$ *if the following conditions* (i) *or* (ii) *holds*:

(i) $\Gamma_{xn}(R)\subset\Gamma_x(R)$ *for any* n,

(ii) *there exists, for any* $\omega\in\Gamma_x(R)^{\perp}$, *a sequence* $\{\omega_n\}$ *with* $\omega_n\in\Gamma_{xn}(R)^{\perp}$ *such that* $\|\omega-\omega_n\|\to 0$ *as* $n\to\infty$.

Proof. Suppose (i) holds. For $\varepsilon>0$ and each $\omega\in\Gamma_x(R)^{\perp}$ there exists a regular region D such that $\|\omega\|_{R-D}<\varepsilon$. Suppose σ_0 is the limit of a locally uniformly convergent subsequence $\{\sigma_{n_k}\}$, then we have

$$|<\sigma, \omega>|\leqq|<\sigma, \omega>_D|+|<\sigma, \omega>_{R-D}|\leqq K\varepsilon+|\lim_{k\to\infty}<\sigma_{n_k}, \omega>_D|$$

$$\leqq K\varepsilon+|\lim_{k\to\infty}<\sigma_{n_k}, \omega>|+|\lim_{k\to\infty}<\sigma_{n_k}, \omega>_{R-D}|$$

$$\leqq 2K\varepsilon+|\lim_{k\to\infty}<\sigma_{n_k}, \omega>|\leqq 2K\varepsilon,$$

where K is a constant$>\sup_n\|\sigma_n\|_{R_n}$. Therefore, $<\sigma, \omega>=0$, hence $\sigma\in\Gamma_x(R)$. The case (ii) is evident. q.e.d.

Suppose, for each n, $\Gamma_x^n(R_n)$ is a subspace of $\Gamma_h(R_n)$. By the same way as in Lemmata 3.1 and 3.2 we can prove simply the following Lemmata:

Lemma 3.3. *Assume the following conditions are fulfilled*:

(a) $\Gamma_x^m(R_m)^{\perp}|_{R_n}\subset\Gamma_x^n(R_n)^{\perp}$, $\Gamma_x(R)^{\perp}|_{R_n}\subset\Gamma_x^n(R_n)^{\perp}$ *for* m *and* n $(m>n)$,

(b) *if* $\{\omega_n\}$ *with* $\omega_n\in\Gamma_x^n(R_n)^{\perp}$ *is a sequence such that* $\|\omega-\omega_n\|_{R_n}\to 0$ *as* $n\to\infty$, *then* $\omega\in\Gamma_x(R)^{\perp}$,

then, for any $\sigma\in\Gamma_x(R)$, *there exists a sequence* $\{\sigma_n\}$ *with* $\sigma_n\in\Gamma_x^n(R_n)$ *such that* $\|\sigma-\sigma_n\|_{R_n}\to 0$ *as* $n\to\infty$. *The same conclusion as above holds if* $\Gamma_x(R)|_{R_n}\subset\Gamma_x^n(R_n)$ *for each* n.

Lemma 3.4. *The limit of each locally uniformly convergent subsequence of* $\{\sigma_n\}$ *with* $\sigma_n\in\Gamma_x^n(R_n)$ *and* $\sup_n\|\sigma_n\|_{R_n}<\infty$ *belongs to* $\Gamma_x(R)$, *if there exists, for each* $\omega\in\Gamma_x(R)^{\perp}$, *a sequence* $\{\omega_n\}$ *with* $\omega_n\in\Gamma_x^n(R_n)^{\perp}$ *such that* $\|\omega_n-\omega\|_{R_n}\to 0$ *as* $n\to\infty$.

Definition 4. We say that *a sequence* $\{\Gamma_x^n(R_n)\}_{n=1}^{\infty}$ *(resp.* $\{\Gamma_{xn}(R)\}_{n=1}^{\infty}$*) converges to* $\Gamma_x(R)$ if the following conditions are fulfilled:

(i) for each $\omega_n\in\Gamma_x(R)$ there exists a sequence $\{\omega_n\}$ with $\omega_n\in\Gamma_x^n(R_n)$ (resp. $\omega_n\in\Gamma_{xn}(R)$) such that $\|\omega_n-\omega\|_{R_n}\to 0$ (resp. $\|\omega_n-\omega\|\to 0$) as $n\to\infty$,

(ii) if $\{\omega_n\}$ with $\omega_n\in\Gamma_x^n(R_n)$ (resp. $\omega_n\in\Gamma_{xn}(R)$) is a sequence such that $\sup\|\omega_n\|_{R_n}<\infty$ (resp. $\sup\|\omega_n\|<\infty$), then the limit of each locally uniformly convergent subsequence $\{\omega_{n_k}\}$ belongs to $\Gamma_x(R)$.

In this case we write simply $\Gamma_x^n(R_n)\Rightarrow\Gamma_x(R)$ *(resp.* $\Gamma_{xn}(R)\Rightarrow\Gamma_x(R)$*).*

From Lemmata 3.1 and 3.2 we can prove $\Gamma_{xn}(R) \Rightarrow \Gamma_x(R)$ if $\Gamma_{xn}(R) \subset \Gamma_{xm}(R) \subset \Gamma_x(R)$ for each m, n $(m>n)$ and $\bigcap_{n=1}^{\infty} \{\Gamma_{xn}(R)^{\perp}\} = \Gamma_x(R)^{\perp}$. Further, from Lemmata 3.3 and 3.4, we can obtain $\Gamma_x^n(R_n) \Rightarrow \Gamma_x(R)$ if the following conditions are fulfilled: (a) $\Gamma_x^n(R_n)^{\perp} \supset \Gamma_x^m(R_m)^{\perp}|_{R_n}$ and $\Gamma_x^n(R_n)^{\perp} \supset \Gamma_x(R)^{\perp}|_{R_n}$ for each m, n $(m>n)$, (b) if $\{\omega_n\}$ with $\omega_n \in \Gamma_x^n(R_n)$ satisfies $\|\omega - \omega_n\|_{R_n} \to 0$ as $n \to \infty$, then $\omega \in \Gamma_x(R)$.

In the following, we shall consider, for specific spaces $\Gamma_x(R)$, the conditions in order that $\Gamma_x^n(R_n) \Rightarrow \Gamma_x(R)$ or $\Gamma_{xn}(R) \Rightarrow \Gamma_x(R)$.

3.2. Orthogonal decompositions on an open Riemann surface. Let R be an open Riemann surface and R^* (resp. $\Delta = R^* - R$) its Kuramochi compactification (resp. Kuramochi ideal boundary). Suppose Ω is a region on R whose relative boundary $\partial\Omega$ consists of at most countable number of analytic arc clustering nowhere on R. *Note that the topology on $\bar{\Omega}$ (closure of Ω) is the relative topology induced by the topology on R^*.* We set

$$\Delta_\Omega = \{\bar{\Omega} \cap \Delta\} \cup \overline{\partial\Omega}.$$

Let α be non-empty open set on Δ_Ω and $\beta = \Delta_\Omega - \bar{\alpha} + \{0\}$ where $\bar{\alpha}$ denotes the closure of α on Δ_Ω. We consider the following linear subspaces of differentials:

$\Gamma_{eo}^0(\bar{\alpha}, \Omega) = \{df \in \Gamma_e(\Omega)$: (a) f is continuous on Ω, (b) there exists on $\bar{\Omega}$ a neighbourhood U_f of $\bar{\alpha}$ which is disjoint with the support of $f\}$,

$$\Gamma_{eo}(\bar{\alpha}, \Omega) = Cl\{\Gamma_{eo}^0(\bar{\alpha}, \Omega)\}, \quad \Gamma_{heo}(\bar{\alpha}, \Omega) = \Gamma_h(\Omega) \cap \Gamma_{eo}(\bar{\alpha}, \Omega),$$

$$\Gamma_{eo}(\bar{\alpha}, \Omega)^{*\perp} = \Gamma_{co}(\beta, \Omega), \quad \Gamma_{ho}(\beta, \Omega) = \Gamma_h(\Omega) \cap \Gamma_{co}(\beta, \Omega).$$

Hereafter we identify all constant functions with zero.

Lemma 3.5. (i) $\Gamma(\Omega) = \Gamma_{eo}(\Omega)^* \dotplus \Gamma_{eo}(\bar{\alpha}, \Omega) \dotplus \Gamma_{ho}(\beta, \Omega)^*$

(ii) If $\bar{\alpha}' \supset \bar{\alpha}$, then we have

$$\Gamma_{heo}(\bar{\alpha}, \Omega) \supset \Gamma_{heo}(\bar{\alpha}', \Omega), \quad \Gamma_{ho}(\alpha, \Omega) \supset \Gamma_{ho}(\alpha', \Omega),$$

(iii) $\Gamma_{heo}(\bar{\alpha}, \Omega) \subset \Gamma_{ho}(\alpha, \Omega)$.

Proof. (i) and (ii) are evident and so omitted. To prove the case (iii), we have only to prove $\Gamma_{eo}^0(\bar{\alpha}, \Omega) \perp \Gamma_{eo}^0(\bar{\beta}, \Omega)^*$. Let $\{\Omega_n\}$ be a canonical exhaustion of Ω. For any $df \in \Gamma_{eo}^0(\bar{\alpha}, \Omega)$ and any $dg \in \Gamma_{eo}^0(\bar{\beta}, \Omega)$ there exists a large number n such that $\partial\Omega_n \subset U_f \cup U_g$. Consequently, it holds

$$<df, dg^*> = <df, dg^*>_{\Omega_n} + <df, dg^*>_{\Omega - \Omega_n} = <df, dg^*>_{\Omega_n}.$$

On the other hand, the restriction of f (resp. g) to Ω_n has the Royden decomposition of the form

$$f|_{\Omega_n}=u_n+f_{on} \text{ (resp. } g|_{\Omega_n}=v_n+g_{on}),$$

where f_{on} (resp. g_{on}) is a Dirichlet potential on Ω_n and $du_n, dv_n \in \Gamma_{he}(\Omega_n)$. But, from Lemma 2.4 and Corollary of Lemma 2.2 we can get $du_n \in \Gamma_{heo}(\bar{\alpha}_n, \Omega_n)$ and $dv_n \in \Gamma_{heo}(\bar{\beta}_n, \Omega_n)$ where $\bar{\alpha}_n$=closure of $\{\partial\Omega_n \cap U_f\}$ on $\bar{\Omega}$ and $\bar{\beta}_n$= closure of $\{\partial\Omega_n \cap U_g\}$ on $\bar{\Omega}$. By Green formula we have $<du_n, dv_n^*>_{\Omega_n}=0$, hence $<df, dg^*>=\lim_{n\to\infty}<df, dg^*>_{\Omega_n}=0$. Therefore $\Gamma_{eo}^0(\bar{\alpha}, \Omega)\perp\Gamma_{eo}^0(\bar{\beta}, \Omega)^*$ which implies $\Gamma_{heo}(\bar{\alpha}, \Omega)\subset\Gamma_{ho}(\alpha, \Omega)$. q.e.d.

3.3. Kuramochi's local capacity and the boundary point of border type. Let P be a point of $\Delta=R^*-R$ and W a region on R such that $\bar{W}$ is a compact neighbourhood on R^* of P and the relative boundary ∂W of W consists of at most a countable number of analytic arcs clustering nowhere on R. Suppose $F_m=\left\{Q: Q\in R^*, d(P, Q)\leq\frac{1}{m}\right\}$ where $d(P, Q)$ denotes the distance between P and Q (note R^* is a metric space, cf. Constantinescu und Cornea [3]) and $\{R_n\}$ is a canonical exhaustion of R. We consider a function ω_m^n on R_n such that $\omega_m^n=1$ on F_m, $\omega_m^n=0$ on R_n-W, ω_m^n is harmonic in $R_n\cap(W-F_m)$ and the inner normal derivative of ω_m^n on $\{\partial R_n\cap(W-F_m)\}$ is zero. If $D_{W\cap R_n}(\omega_m^n)<K$ for any n, there exists a function ω_m on R such that $D_{W\cap R_n}(\omega_m^n-\omega_m)\to 0$ as $n\to\infty$, and moreover there exists a function ω on R such that $D_W(\omega_m-\omega)\to 0$ as $m\to\infty$. *Such a function ω is called the local capacitary potential of P with respect to W and $D_W(\omega)=C_W(P)$ is called the local capacity of P with respect to W* (cf. Kuramochi [8]). Hereafter, we express simply the boundary behavior on Δ of a function f like above ω_m as $\frac{\partial f}{\partial N}|_{\Delta\cap(W-F_m)}=0$.

Let Δ' be a component of Δ. If there exists a connected neighbourhood U of Δ' such that $U\cap(\Delta-\Delta')=\{0\}$, then *$\Delta'$ is called an isolated boundary component of Δ.*

Definition 5. We say that *a point P of an isolated boundary component is a boundary point of border type* if the following condition (*) is fulfilled:

(*) there exists a sequence $\{W_n\}$ of regions on R such that

(a) each $\bar{W}_n$ is a compact neighbourhood of P on R^* and $\bar{W}_n\downarrow P$ as $n\to\infty$,

(b) $C_{W_n}(P)=0$ for each n.

For a boundary point P of border type, there exists a sequence $\{W_n\}$ of regions on R which satisfies $\bar{W}_{m+1}\subset F_m\subset\bar{W}_m$. Let ω_{mn} be a harmonic function on W_n-W_m such that $\omega_{mn}=1$ on W_m, $\omega_{mn}=0$ on ∂W_n and further, $\frac{\partial\omega_{mn}}{\partial N}|_{\Delta\cap\overline{(W_n-W_m)}}$

$=0$. Then, by Dirichlet principle, we have easily $\lim_{m\to\infty} D(\omega_{mn})=0$. *We call the function* ω_{mn} *a local capacitary potential of* W_m *with respect to* W_n.

Lemma 3.6. *Assume* Δ' *is an isolated component of* Δ *and* α *is an open connected set on* Δ' *such that the set* $\bar{\alpha}-\alpha$ *is exactly two points of border type, then we have*

$$\bigcap_{n=1}^{\infty} \Gamma_{heo}(\bar{\alpha}_n, R)=\Gamma_{heo}(\bar{\alpha}, R)=Cl\{\bigcup_{n=1}^{\infty} \Gamma_{heo}(\bar{\gamma}_n, R)\},$$

for any sequence $\{\alpha_n\}$ *(resp.* $\{\gamma_n\}$*) of open sets on* Δ' *such that* $\alpha_n\uparrow\alpha$ *(resp.* $\gamma_n\downarrow\alpha$*) as* $n\to\infty$.

Proof. We have only to prove $df\in\Gamma_{heo}(\bar{\alpha}, R)$, if $df\in\bigcap_{n=1}^{\infty}\Gamma_{heo}(\bar{\alpha}_n, R)$ for any fixed $\{\alpha_n\}$ (cf. Lemma 3.5). At first, we may assume $|f|<K$ on R. Let the set $\bar{\alpha}-\alpha$ be $\{P, Q\}$ and $\{W_n\}$, $\{W'_n\}$ be the sequences of regions satisfying the conditions (*) of Definition 5 such that $\overline{W}_n\to P$ (resp. $\overline{W}'_n\to Q$) as $n\to\infty$. Next, denote by ω_{mn} (resp. ω'_{mn}) the local capacitary potential of W_m (resp. W'_n) with respect to W_n (resp. W'_n). Suppose $\{\varepsilon_n\}$ is a sequence of positive numbers such that $\varepsilon_n\downarrow 0$ as $n\to\infty$. Then, we can choose a sequence $\{m_n\}$ of positive integers such that $\|d\omega_{m_n n}\|+\|d\omega'_{m_n n}\|<\varepsilon_n$ (therefore, $m_n>n$) and $m_n\uparrow\infty$ as $n\to\infty$. Further, we can choose another sequence $\{k_n\}$ of positive integers which satisfies $\alpha_{k_n}\supset\alpha-W_{m_n}-W'_{m_n}$, $k_n>m_n$ and $k_n\uparrow\infty$ as $n\to\infty$. While, from the condition $df\in\bigcap_{n=1}^{\infty}\Gamma_{heo}(\bar{\alpha}_n, R)$, we can find a sequence $\{dg_n\}$ with $dg_n\in\Gamma^0_{eo}(\bar{\alpha}_n, R)$ such that $\|df-dg_n\|<\varepsilon_n$. For simplicity, we denote $k_n=k$ and $m_n=m$. Next, denote $(1-\omega_{mn})(1-\omega'_{mn})g'_k$ by f_n where $g'_n=\mathrm{Min}[\mathrm{Max}(g_n, -2K), 2K]$, then we have $df_n\in\Gamma^0_{eo}(\bar{\alpha}, R)$ and moreover

$$\|df-df_n\|^2<\|df-dg_k\|^2+\|df\|^2_{W_n\cup W'_n}+\|df-df_n\|^2_{W_n-W_m}+\|df-df_n\|^2_{W'_n-W'_m}$$

$$<25\|df-dg_k\|^2+4\|df\|^2_{W_n\cup W'_n}+12K^2(\|d\omega_{mn}\|^2+\|d\omega'_{mn}\|^2).$$

Thus we have a sequence $\{df_n\}$ with $df_n\in\Gamma^0_{eo}(\bar{\alpha}, R)$ such that $\|df-df_n\|\to 0$ as $n\to\infty$, hence $df\in\Gamma_{heo}(\bar{\alpha}, R)$, so $\Gamma_{heo}(\bar{\alpha}, R)=\bigcap_{n=1}^{\infty}\Gamma_{heo}(\bar{\alpha}_n, R)$. q.e.d.

Now we consider an example of a Riemann surface with a boundary point of border type which has arbitrary small neighbourhoods of infinite genus.

Example. Suppose G_z (resp. G_0) be a disk $|z|<1$ (resp. $1>|z|>r$), and maps G_0 one to one conformally onto a region G_w on w-plane by the function

$$w(z)=u(z)+iv(z)$$

$$=\log\frac{|z|}{|z-\rho|\,|z-1/\rho|}+i[\operatorname{Arg} z-\operatorname{Arg}(z-\rho)-\operatorname{Arg}(z-1/\rho)+2\pi]\,,$$

where $\operatorname{Arg} z$ denotes the principal value (i.e. $0\leqq\operatorname{Arg} z<2\pi$) and $0<\rho<r$. (cf. Ishida [5]). ∂G_w consists of a Jordan curve and a horizontal slit $\left\{(u, v)\colon \log\frac{\rho}{(1+\rho)^2}\leqq u\leqq\log\frac{\rho}{(1-\rho)^2},\ v=\pi\right\}$ which is the image of $|z|=1$ under $w(z)$. The point $z=i$ is mapped to the point $w_0=\log\left[\frac{\rho}{1+\rho^2}\right]+\pi i$ which lies on the lower side of above slit. Suppose $\{a_n\}$ be a sequence of positive numbers such that $a_{n+1}=\frac{a_n}{1+1/n}$ and a_1 is a suitable positive number. Let $\{J_n\}$ be a sequence of horizontal segments on G_w satisfying the following conditions:

(i) each of J_{2n} and J_{2n+1} lies in a domain $\left\{w\colon \frac{4}{3}\pi<\operatorname{Arg}\ (w-w_0)<\frac{5}{3}\pi,\ a_{2n+1}<|w-w_0|<a_{2n}\right\}$,

(ii) J_{2n}, J_{2n+1} have the same projection to the real axies.

Denote the inverse image $w^{-1}(J_n)$ of J_n by I_n, and cut G_w (resp. G_0) along each J_n (resp. I_n). Denoting the lower side of J_n (resp. I_n) by J_n^- (resp. I_n^-) and the upper side of J_n (resp. I_n) by J_n^+ (resp. I_n^+), we identify J_{2n}^+ (resp. I_{2n}^+) with J_{2n+1}^- (resp. I_{2n+1}^-) and J_{2n}^- (resp. I_{2n}^-) with J_{2n+1}^+ (resp. I_{2n+1}^+). This gives a surface D_w (resp. D_z). A surface $D_z\cup\{|z|\leqq r\}$ is denoted by R. Further, let g be a function $\in C^\infty(R)$ such that $g=1$ on D_z, $g=0$ on $|z|<\frac{r+\rho}{2}$. Then, the function $U_\rho(P)=g(P)u(P)$ belongs to the family N of functions where $N=\{F\in CD(R)\colon F$ has continuous extension on Δ and points of Δ are separated by the above extended functions$\}$, that is to say, N-compactification of R is equivalent to R^* (cf. [3]). Let $\hat{R}$ be the double of R with respect to $\{z\colon |z|=1,\ z\neq i\}$. Then, $\hat{R}\in O_g$, hence from Proposition 7 in Kusunoki [7], Δ is a quotient space of $\{z\colon |z|=1\}$. On the other hand, we know the class $N_0=\{U_{\rho e^{i\theta}}\}_{0\leqq\theta\leqq\pi}$ separates the points on $\{z\colon |z|=1\}$ and so Δ is equivalent to $\{z\colon |z|=1\}$. Thus we know the boundary point $z=i$ is of border type and has arbitrary small neighbourhoods of infinite genus.

3.4. Region D^* of $\mathscr{D}^*$-type and normalized exhaustion associated with D^*. Let Δ_1 be an isolated boundary component and $\Delta_0=\Delta-\Delta_1$.

Definition 6. *A region D^* on R^* is called to be $\mathscr{D}_1^*$-type associated with Δ_1 or simply $\mathscr{D}_1^*$-type if the following conditions are fulfilled:*

(i) *each of $D^*\cap\Delta_1$ and Δ_1-D^* is connected nonempty and $\bar{D}^*\cap\Delta_0=\{0\}$.*

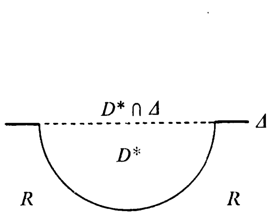
(ii) *$D^*\cap R=D$ is connected on R and ∂D is an analytic curve,*

(iii) *$\overline{D^*\cap\Delta}=D^*\cap\Delta$.*

For D^* *of* $\mathscr{D}_1^*$*-type, we call* $D=D^*\cap R$ *a region (on* R*) of* $\mathscr{D}_1$*-type.*

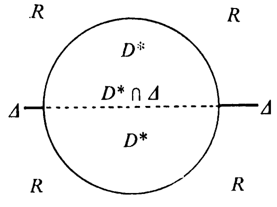
Some example of regions of $\mathscr{D}_1^*$-type (resp. non $\mathscr{D}_1^*$-type) are shown in Fig. 1–3.

Fig. 1



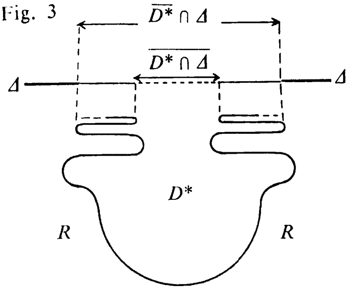
(D^*: D_1^*-type)

Fig. 2



(D^*: non D_1^*-type)

Fig. 3



(D^*: non D_1^*-type)

Lemma 3.7. *Suppose* D^* *is of* $\mathscr{D}_1^*$*-type. Then we have* (i) $\bigcap_{n=1}^{\infty}\overline{D\cap(R-R_n)}=\overline{D^*}\cap\Delta$ *for each regular exhaustion* $\{R_n\}$ *of* R*, and* (ii) *if* D^* *satisfies* $\overline{D^*}-D^*=\overline{\partial D}$*, it holds* $(\overline{R-D-\partial D})\cap\Delta=\Delta-\overline{D^*\cap\Delta}$*. Here bar stands for the closure in* R^**.*

Proof. Case (ii). $R^*-\overline{D}^*\subset\overline{R-D}\subset R^*-D^*$ is evident. On the other hand, from $\partial D\subset\overline{R-D}$ we have $R^*-D^*=R^*-\overline{D^*}+\overline{\partial D}\subset\overline{R-D}$, hence $\overline{R-D}=R^*-D^*$. Consequently, we have

$$\overline{R-D}-\overline{\partial D}=R^*-D^*-\overline{\partial D}=R^*-\overline{D^*}+\overline{D^*}-D^*-\overline{\partial D}=R^*-\overline{D^*},$$

and so we obtain $(\overline{R-D-\partial D})\cap\Delta=\Delta-\overline{D^*\cap\Delta}$. The proof of (i) is omitted.

Let D be of $\mathscr{D}_1$-type. Then ∂D is weakly homologous zero on R which we write $\partial D\approx 0$ on R. Recall that $\partial D\approx 0$ if, for any regular region G on R, we have $C\cup j(-C)\sim 0$ on $\hat{G}$ where $\hat{G}$ denotes the double of G, $C=(\partial D)\cap G$ and j the involutory mapping of $\hat{G}$ (cf. Marden [9]). From the assumption of D we have

Lemma 3.8. *For a given* D *of* $\mathscr{D}_1$*-type, there exists a canonical exhaustion* $\{R_n\}$ *of* R *such that* (i) ∂R_n *is piecewise analytic,* (ii) $D\cap(\partial R_n)$ *is an arc which divides* D *into two regions of* D*.*

Proof. Denote by $\hat{D}$ (resp. $\hat{\Omega}$) the double of $D\cup\partial D$ (resp. $\Omega=R-D$) with respect to ∂D, and $\{\hat{D}_n\}$ (resp. $\{\hat{\Omega}_n\}$) a regular canonical exhaustion of $\hat{D}$ (resp. $\hat{\Omega}$). We set $\hat{D}_n'=\hat{D}_n\cup j\hat{D}_n$ (resp. $\hat{\Omega}_n'=\hat{\Omega}_n\cup j\hat{\Omega}_n$) where j means the involutory mapping of $\hat{D}$ (resp. $\hat{\Omega}$), then $\{\hat{D}_n'\}$ (resp. $\{\hat{\Omega}_n'\}$) is an exhaustion of

$\hat{D}$ (resp. $\hat{\Omega}$) by relative compact regions. Further, by setting $G_n=(\hat{D}'_n \cap D) \cup (\hat{\Omega}'_n \cap \Omega)$, we get an exhaustion $\{G_n\}$ of R. Next, by suitable method, we can take three sets $\{S_n\}$, $\{S'_n\}$ and $\{S''_n\}$ of strips on R such that, (a) each component of $\{S_n\}$ contains a subarc on ∂D, (b) $\{S'_n\}$ (resp. $\{S''_n\}$) is contained in D (resp. Ω), so that the set $R_n=(\hat{D}'_n \cap D-\{S_n\}-\{S''_n\}) \cup (\hat{\Omega}'_n \cap \Omega-\{S'_n\}-\{S''_n\})$ is a canonical region on R. Thus we can obtain the exhaustion $\{R_n\}$ of R satisfying the conditions of this Lemma (cf. Ahlfors, L. and Sario, L. [2], p. 61–63).

Hereafter, we call *such a canonical exhaustion* $\{R_n\}$ *a normalized exhaustion associated with* D^* *of* $\mathscr{D}_1^*$*-type.*

3.5. Convergence theorems of harmonic differentials I. Let R, D be the same as in 3.4. We set $\Omega=R-D \cup \partial D$ and $\Delta_\Omega=\bar{\Omega}-\Omega$ where bar stands for the closure in R^*, and consider the following sets:

$$\mathscr{A}=\text{a relative compact (on } R\text{) open arc on } \partial\Omega,$$

$$\mathscr{B}=\{\text{the boundary component of } \Delta_\Omega \text{ including } \partial\Omega\}-\mathscr{A}.$$

Lemma 3.9. (i) *For* $df \in \Gamma_{heo}(\Delta_\Omega-\mathscr{B}, \Omega)$ (*resp.* $\Gamma_{heo}(\Delta_\Omega-\mathscr{A}, \Omega)$) *we may set, on* $\mathscr{A}$ (*resp.* $\partial\Omega \cap \mathscr{B}$), $f=0$. (ii) *For* $\omega \in \Gamma_{ho}(\mathscr{A}, \Omega)$ (*resp.* $\Gamma_{ho}(\mathscr{B}, \Omega)$) *it holds that* $\omega=0$ *along* $\mathscr{A}$ (*resp.* $\mathscr{B} \cap \partial\Omega$).

Proof. Omitted (cf. Lemma 2.2 and its Corollary).

At first, we consider convergence theorems of real harmonic differentials in case where $R_n \cap \Omega=\Omega_n \to \Omega$. Suppose $\{R_n\}$ is a normalized exhaustion associated with D^* of $\mathscr{D}_1^*$-type, and we set

$$l_n=\{\text{the boundary component of } \partial\Omega_n \text{ including } \mathscr{A}\}-\bar{\mathscr{A}},$$

$$\Delta_{On}=\partial\Omega_n-\bar{\mathscr{A}} \cup l_n,$$

and show the following lemma.

Lemma 3.10. (i) $\Gamma_{heo}(\partial\Omega_n-\mathscr{A}, \Omega_n) \Rightarrow \Gamma_{heo}(\Delta_\Omega-\mathscr{A}, \Omega)$, $\Gamma_{ho}(\mathscr{A}, \Omega_n) \Rightarrow \Gamma_{ho}(\mathscr{A}, \Omega)$,

(ii) $\Gamma_{heo}(\partial\Omega_n-l_n, \Omega_n) \Rightarrow \Gamma_{heo}(\Delta_\Omega-\mathscr{B}, \Omega)$, $\Gamma_{ho}(l_n, \Omega_n) \Rightarrow \Gamma_{ho}(\mathscr{B}, \Omega)$,

(iii) $\Gamma_{heo}(\partial\Omega_n-\Delta_{On}, \Omega_n) \Rightarrow \Gamma_{heo}(\Delta_\Omega-\Delta_0, \Omega)$, $\Gamma_{ho}(\Delta_{On}, \Omega_n) \Rightarrow {}_{ho}(\Delta_0, \Omega)$.

Proof. We must show only that for $\omega \in \Gamma_x(\Omega)$ there exists $\{\omega_n\}_{n=1}^\infty$ with $\omega_n \in \Gamma_x^n(\Omega)$ such that $\|\omega-\omega_n\|_{\Omega_n} \to 0$ as $n \to \infty$, where $\Gamma_x(\Omega)$ (resp. $\Gamma_x^n(\Omega_n)$ means $\Gamma_{heo}(\Delta_\Omega-\mathscr{A}, \Omega)$ etc., (resp. $\Gamma_{heo}(\partial\Omega_n-\mathscr{A}, \Omega_n)$ etc.,) (cf. Lemmata 3.3 and 3.4). Concerning the case (iii), cf. Lemma 2.5 in [11].

Case (i). From Lemma 3.9 and Lemma 2.2, it holds that $\Gamma_{ho}(\mathscr{A}, \Omega)|_{\Omega_n} \subset \Gamma_{ho}(\mathscr{A}, \Omega_n)$, hence we get the conclusion in case $\Gamma_{ho}(\mathscr{A}, *)$ by Lemma 3.3. Next, we prove the case $\Gamma_{heo}(*-\mathscr{A}, *)$. From Corollary of Lemma 2.2 we have, for $m>n$, the relations: $\Gamma_{ho}(\mathscr{A}, \Omega_n) \supset \Gamma_{ho}(\mathscr{A}, \Omega_m)|_{\Omega_n}$ and $\Gamma_{ho}(\mathscr{A}, \Omega)|_{\Omega_n} \subset \Gamma_{ho}(\mathscr{A}, \Omega_n)$. While, suppose that $\{\omega_n\}$ with $\omega_n \in \Gamma_{ho}(\mathscr{A}, \Omega_n)$ is a sequence such that $\|\omega_n - \omega\|_{\Omega_n} \to 0$ as $n \to \infty$. For each $dg \in \Gamma_{eo}^0(\Delta_\Omega - \mathscr{A}, \Omega)$, the restriction of dg to Ω_n has a decomposition of the form $dg|_{\Omega_n} = du_n + dg_{on}$ where $dg_{on} \in \Gamma_{eo}(\Omega_n)$ and $du_n \in \Gamma_{heo}(\partial\Omega_n - \mathscr{A}, \Omega_n)$ for sufficiently large n. Consequently, we get $\langle \omega, dg^* \rangle_\Omega = \lim_{n\to\infty} \langle \omega_n, du_n \rangle_{\Omega_n} = 0$. Hence, $\omega \in \Gamma_{ho}(\mathscr{A}, \Omega)$ and from Lemma 3.3 we have the conclusion.

The case (ii). At first, we prove the case of $\Gamma_{heo}(*-\mathscr{B}, *)$. Let γ be an analytic closed curve on Ω_1 which separates the component of Δ_Ω including $\mathscr{A}$ from Δ_0. We write $\Omega_n - \gamma = \Omega'_n \cup \Omega''_n$ where $\mathscr{A} \subset \partial\Omega'_n$. For a function g with $dg \in \Gamma_{heo}(\Delta_\Omega - \mathscr{B}, \Omega)$ we set $f_n = g$ on Ω'_n and $f_n = H_{g'}^{\Omega''_n}$ where $g' = g$ on γ and $g' = 0$ on $\partial\Omega''_n - \gamma$. Then df_n has a decomposition of the form $df_n = du_n + df_{on}$ where $du_n \in \Gamma_{heo}(\partial\Omega_n - \ell_n, \Omega_n)$ (cf. Corollary of Lemma 2.2) and $df_{on} \in \Gamma_{eo}(\Omega_n)$. Thus we have a sequence $\{du_n\}$ with $du_n \in \Gamma_{heo}(\partial\Omega_n - \ell_n, \Omega_n)$ such that $\|du_n - df\|_{\Omega_n} \leqq \|df - df_n\|_{\Omega_n} \to 0$. Next, we show the case $\Gamma_{ho}(\mathscr{B}, \Omega)$. In order to use Lemma 3.3, we consider a special exhaustion. Let G be an end towards $\Delta - \Delta_1$. We set $G_n = \Omega_n \cup G$ and $\tilde{\Delta}_n = \partial G_n \cup (\Delta - \Delta_1)$, then for each m, n $(m>n)$, we have $\Gamma_{heo}(\tilde{\Delta}_m - \ell_m, G_m)|_{G_n} \subset \Gamma_{heo}(\tilde{\Delta}_n - \ell_n, G_n)$ and $\Gamma_{heo}(\Delta_\Omega - \mathscr{B}, \Omega)|_{G_n} \subset \Gamma_{heo}(\tilde{\Delta}_n - \ell_n, G_n)$ (cf. Lemma 3.9). The restriction of ω to G_n with $\omega \in \Gamma_{ho}(\mathscr{B}, \Omega)$ has a decomposition of the form $\omega = \sigma_n + \omega_n$ where $\sigma_n^* \in \Gamma_{heo}(\tilde{\Delta}_n - \ell_n, G_n)$ and $\omega_n \in \Gamma_{ho}(\ell_n, G_n)$. From this form we know $\sigma_n^* = 0$ along $\ell_n \cap \partial\Omega$, hence $\sigma_n = df_n^* \in \Gamma_{ho}(\ell_n \cap \partial\Omega, G_n)$. By the same way as in Lemma 3.3, we have $\omega = \omega_0 + df^*$ where $\omega_0 \in \Gamma_{ho}(\mathscr{B}, \Omega)$ and $\|df - df_n\|_{G_n} \to 0$ as $n \to \infty$. From $df_n^* \in \Gamma_{ho}(\ell_n \cap \partial\Omega, G_n)$ we have $df \in \Gamma_{heo}(\Delta_\Omega - \mathscr{B}, \Omega)$, so $\omega = \omega_0$. Writing $\omega'_n = \omega_n|_{\Omega_n}$ we can get $\{\omega'_n\}$ with $\omega'_n \in \Gamma_{ho}(\ell_n, \Omega_n)$ so that $\|\omega - \omega'_n\|_{\Omega_n} \to 0$ as $n \to \infty$. Thus from Lemma 3.3 we have the conclusion.

q.e.d.

3.6. Convergence theorems of harmonic differentials II. We consider the convergence theorems of real harmonic differentials on $R_n \cup \Omega$ where n is fixed. Let R, D^* and $\{R_n\}$ be the same as in 3.5. We set $W_n = R_n \cup \Omega$, $\Delta'_n = \overline{W}_n - W_n$ and

$$\alpha = D^* \cap \Delta,\ \mathscr{A}_n = \partial W_n,\ \mathscr{A}_{nm} = R_m \cap \mathscr{A}_n$$

$$\beta = \{\text{the component of } \Delta'_n \text{ including } \mathscr{A}_n\} - \bar{\mathscr{A}}_n,$$

$$\mathscr{B}_{nm} = \{\text{the component of } \Delta'_n \text{ including } \mathscr{A}_n\} - \bar{\mathscr{A}}_{nm}.$$

Lemma 311. *Assume that $\bar{\alpha} - \alpha$ consists of exactly two boundary points of border type and is equal to $\overline{\partial D} \cap \Delta$, then we have the following relations* *for* $m\to\infty$ (n: *fixed*):

(i) $\Gamma_{heo}(\Delta'_n-\mathscr{A}_{nm}, W_n)\Rightarrow\Gamma_{heo}(\Delta'_n-\mathscr{A}_n, W_n)$, $\Gamma_{ho}(\mathscr{A}_{nm}, W_n)\Rightarrow\Gamma_{ho}(\mathscr{A}_n, W_n)$,

(ii) $\Gamma_{heo}(\Delta'_n-\mathscr{B}_{nm}, W_n)\Rightarrow\Gamma_{heo}(\Delta'_n-\beta, W_n)$, $\Gamma_{ho}(\mathscr{B}_{nm}, W_n)\Rightarrow\Gamma_{ho}(\beta, W_n)$.

Proof. This lemma can be proved by the analogous method as in Lemma 3.13 and so omitted (cf. Lemma 3.6 and 3.7).

Lemma 3.12. (i) *Suppose* L_β *is the family of all curves on* W_n (n: *fixed*) *which start from a parametric disk and tend to* β, *then we may set, for each* $dg\in\Gamma_{heo}(\Delta'_n-\mathscr{A}_n, W_n)$, $\lim_c g=0$ *for almost all* c *of* L_β *where* $\lim_c g$ *means the limit* g *along* c.

(ii) *Assume that* $\bar{\alpha}-\alpha$ *consists of exactly two points of border type. If* $\{df_n\}_{n=1}^{\infty}$ *with* $df_n\in\Gamma_{heo}(\Delta'_n-\mathscr{A}_n, W_n)$ *is a sequence such that* $\|df-df_n\|_{W_n}\to 0$ *as* $n\to\infty$, *then* $df\in\Gamma_{heo}(\Delta-\alpha, R)$.

Proof. Case (i). By use of Propositions 3, 4 and Lemma 1 in Yamaguchi [17] we can prove (i) easily and so omitted.
Case (ii). Because $\partial D\subset\bar{\Omega}$, we have $\overline{W}_n\cap\Delta=\Delta\cap\bar{\Omega}=\Delta\cap(\overline{R-D})\supset(R^*-\overline{D^*})\cap\Delta$, hence from the definition of D^* of $\mathscr{D}_1^*$-type we have $\beta=\overline{W}_n\cap\Delta_1-\partial\overline{W_n}\cap\Delta_1\supset\Delta_1-\bar{\alpha}-(\bar{\alpha}-\alpha)=\Delta_1-\bar{\alpha}$, and so we have $\bar{\alpha}\cup\beta=\Delta_1$. Next, we may set $|f|<K$, $|f_k|<2K$ and $\lim_{k\to\infty}f_k(P)=f(P)$ for fixed $P\in R$. From Lemma 3.6 we have only to prove $df\in\Gamma_{heo}(\gamma_m\cup\Delta_0, R)$ where $\gamma_m=\{Q\in\Delta: d(Q,\alpha)\geqq 1/m\}$ and $\Delta_0=\Delta-\Delta_1$. Denote the set $\{Q\in R^*: d(Q, D^*)\leq 1/m\}$ by S_m^*. Then there exists a Dirichlet function ψ_m on R which is continuous on R^* and $\psi_m=1$ on S_{3m}^*, $\psi_m=0$ on $R^*-S_{2m}^*$ and $0\leqq\psi_m\leqq 1$. From Lemma 1 in [17], the differential $dG_{m,k}=d[(1-\psi_m)\cdot f_k]$ belongs to $\Gamma_{eo}(R)$ because $\bar{\alpha}\cup\beta=\Delta_1$ and $\lim_c G_{m,k}=0$ for almost all $c\in L$ (concerning L, see Proposition 4), and we have $d[f\psi_m+(1-\psi_m)f_k]=dF_{m,k}\in\Gamma_{eo}(\gamma_m\cup\Delta_0, R)$ because $\bar{\alpha}\cap\gamma_m=\{0\}$ and $d(f\psi_m)\in\Gamma_{oe}^0(\gamma_m\cup\Delta_0, R)$. While we have

$$\|dF_{m,k}-df\|\leqq 2\|df_k-df\|_{W_k}+\operatorname*{Max}_S|f_k-f|\,\|d\psi_m\|+2K\varepsilon,$$

where S denotes a compact set on R such that $\|d\psi_m\|_{R-S}<\varepsilon$. Note m is fixed. Therefore, from $|f_k-f|\to 0$ on S as $k\to\infty$, we get $\|dF_{m,k}-df\|\to 0$ as $k\to\infty$, we obtain $df\ \Gamma_{heo}(\gamma_m\cup\Delta_0, R)$. q.e.d.

From now on, we consider the convergence theorems of real harmonic differentials in case $W_n=R_n\cup\Omega\to R$. Note $\Delta'_n=\overline{W}_n-W_n$.

Lemma 3.13. *Assume that* $\bar{\alpha}-\alpha$ *consists of exactly two boundary points of border type and is equal to* $\overline{\partial D}\cap\Delta$, *then we have the following relations*:

(i) $\Gamma_{ho}(\beta, W_n) \Rightarrow \Gamma_{ho}(\beta, R)$, $\Gamma_{heo}(\Delta'_n - \beta, W_n) \Rightarrow \Gamma_{heo}(\Delta - \beta, R)$,

(ii) $\Gamma_{ho}(\mathscr{A}_n, W_n) \Rightarrow \Gamma_{ho}(\alpha, R)$, $\Gamma_{heo}(\Delta'_n - \mathscr{A}_n, W_n) \Rightarrow \Gamma_{heo}(\Delta - \alpha, R)$.

Proof. Case (i). At first we show the case $\Gamma_{ho}(\beta, R)$. We can extend $df \in \Gamma^0_{eo}(\Delta'_n - \beta, W_n)$ to R such that $df \in \Gamma^0_{eo}(\Delta - \beta, R)$ since $\beta = \Delta - \bar{\alpha}$ (cf. Lemma 3.7). Therefore, for each $\omega \in \Gamma_{ho}(\beta, R)$ we have $<\omega, df^*>_{W_n} = 0$ where $df \in \Gamma^0_{eo}$ $(\Delta'_n - \beta, W_n)$, hence $\omega_n = \omega|_{W_n} \in \Gamma_{ho}(\beta, W_n)$ and from Lemma 3.3, we have the conclusion. Next, we prove the case $\Gamma_{heo}(\Delta - \beta, R)$. We have already known that $\Gamma_{ho}(\beta, R) \supset \Gamma_{ho}(\beta, W_m)|_{W_n}$ $(m > n)$. While, suppose $\{\omega_n\}$ with $\omega_n \in \Gamma_{ho}(\beta, W_n)$ is a sequence such that $\|\omega_n - \omega\|_{W_n} \to 0$ as $n \to \infty$, then by Lemma 3.7 we have, for each $df \in \Gamma^0_{eo}(\Delta - \beta, R)$, $<\omega, df^*> = \lim_{n \to \infty} <df^*, \omega_n>_{W_n} = 0$. Hence, we have $\omega \in \Gamma_{ho}(\beta, R)$ and from Lemma 3.3 we can get the result of this lemma since $df \in \Gamma^0_{eo}(\Delta'_n - \beta, W_n)$ for sufficiently large n (cf. Lemma 3.4).
Case (ii). For the case $\Gamma_{heo}(\Delta - \alpha, R)$, we have $df|_{W_n} = df_n \in \Gamma_{heo}(\Delta'_n - \mathscr{A}_n, W_n)$ with each $df \in \Gamma_{heo}(\Delta - \alpha, R)$ (cf. Lemma 3.7), hence from Lemma 3.3 we get the result. For the case $\Gamma_{ho}(\alpha, R)$, we already know that $\Gamma_{heo}(\Delta - \alpha, R)|_{W_n}$ $\subset \Gamma_{heo}(\Delta'_n - \mathscr{A}_n, W_n)$, $\Gamma_{heo}(\Delta'_m - \mathscr{A}_m, W_m)|_{W_n} \subset \Gamma_{heo}(\Delta'_n - \mathscr{A}_n, W_n)$ for $m > n$. On the other hand, if $\{df_n\}$ with $df_n \in \Gamma_{heo}(\Delta'_n - \mathscr{A}_n, W_n)$ is a sequence such that $\|df_n - df\|_{W_n} \to 0$ as $n \to \infty$, then from Lemma 3.12 we have $df \in \Gamma_{heo}(\Delta - \alpha, R)$, hence we can get the conclusion. q.e.d.

§4. Convergence theorems of behavior spaces of X-type and its applications to conformal mappings

Let R be an open Riemann surface of genus g $(g \leqq \infty)$ and R^* (resp. Δ) its Kuramochi's compactification (resp. Kuramochi ideal boundary). Suppose D^* (resp. $D = D^* \cap R$) is a region of $\mathscr{D}^*_1$-(resp. $\mathscr{D}_1$-)type and $\{R_n\}$ is a normalized exhaustion associated with D^*. As in §3, we set $\Omega = R - \bar{D}$, $\Delta_\Omega = \bar{\Omega} - \Omega$, $\Omega_n = R_n \cap \Omega$ and $W_n = R_n \cup \Omega$, then there exists a canonical homology basis $\{A_j, B_j\}_{j=1}^\infty$ of R (mod Δ) such that (i) $\{A_j, B_j\}$ with $A_j, B_j \subset \Omega_n$ is also a canonical homology basis of Ω_n (mod dividing curves on Ω_n) for each n, (ii) $\{A_j, B_j\}$ with $A_j, B_j \subset W_n$ is also a canonical homology basis of W_n (mod dividing curves on W_n) (cf. Ahlfors and Sario [2]). Further, as in §3, we set

$\mathscr{L} = \{L_j, j = 1, 2, \ldots, g$: each L_j is a straight line on the complex plane which passes through the origin$\}$,

$\mathscr{L}_\Omega = \{L_j : f$ for $A_j, B_j \subset \Omega\}$,

$\mathscr{A}$ = a relative compact (on R) open arc on ∂D,

$\mathscr{B} = \{$the component of Δ_Ω including $\partial D\} - \bar{\mathscr{A}}$,

$\ell_n = \{$the component of $\partial\Omega_n$ including $\mathscr{A}\} - \bar{\mathscr{A}}$,

$$\Delta_{On} = \partial\Omega_n - \mathscr{A} \cup \ell_n .$$

Besides these sets we consider a set of complex numbers and spaces of differentials on R and Ω_n such that

$$Z = \{z_0, z_1, z'_1 : |z_0| = |z_1| = |z'_n| = 1\},$$

$$\Lambda_x(\Omega) = \Lambda_x(\mathscr{L}_\Omega, \Omega, Z)$$

$$= \{\lambda \in Cl\{\Lambda_{ho}(\Omega) + \Lambda_{he}(\Omega)\} : \text{(i)} \int_{A_j, B_j} \lambda \in L_j \text{ for } j \text{ with } A_j, B_j \subset \Omega, \text{(ii)}$$

$$\mathrm{Im}(\bar{z}_1\lambda) \in \Gamma_{ho}(\mathscr{A}, \Omega), \mathrm{Im}(\bar{z}'_1\lambda) \in \Gamma_{ho}(\mathscr{B}, \Omega) \text{ and } \mathrm{Im}(\bar{z}_0\lambda) \in \Gamma_{ho}(\Delta 0, \Omega)\},$$

$$\Lambda_x^n(\Omega_n) = \Lambda_x^n = \Lambda_x(\mathscr{L}_{\Omega_n}, \Omega_n, Z) = \{\lambda \in \Lambda_{hse}(\Omega_n) : \text{(i)} \int_{A_j, B_j} \lambda \in L_j \text{ with } A_j, B_j \subset \Omega, \text{(ii)} \mathrm{Im}(\bar{z}_1\lambda) \in \Gamma_{ho}(\mathscr{A}, \Omega_n), \mathrm{Im}(\bar{z}'_1\lambda) \in \Gamma_{ho}(\ell_n, \Omega_n) \text{ and } \mathrm{Im}(\bar{z}_0\lambda) \in \Gamma_{ho}(\Delta_{On}, \Omega_n)\}.$$

Then we can get the following lemma:

Lemma 4.1. *$\Lambda_x^n(\Omega_n) \Rightarrow \Lambda_x(\Omega)$, hence $\Lambda_x(\Omega) = i\Lambda_x(\Omega)^{*\perp}$.*

Proof. At first, we consider the following auxiliary subspaces:
Λ' = the space spanned over the real number field by $\{\zeta_j\sigma_\Omega(A_j), \zeta_j\sigma_\Omega(B_j) : A_j, B_j \subset \Omega\}$ where $\sigma_\Omega(\gamma)$ denotes the reproducing differentials in $\Gamma_c(\Omega)$ associated with γ and ζ_j is a complex number such that $\zeta_j \in L_j$ and $|\zeta_j| = 1$,

$$\Lambda'' = Cl\{z_1\Gamma_{heo}(\Delta_\Omega - \mathscr{A}, \Omega) + z'_1\Gamma_{heo}(\Delta_\Omega - \mathscr{B}, \Omega) + z_0\Gamma_{heo}(\Delta_\Omega - \Delta_0, \Omega)\},$$

$$\Lambda''' = \Lambda_{he}(\Omega) \cap \Lambda_{ho}(\Omega), \ \Lambda'_x = Cl\{\Lambda' + \Lambda'' + \Lambda'''\}.$$

Then from Lemma 2.6 we have $\Lambda_x^n = i\Lambda_x^{n*\perp}$. Next, from Lemma 3.10 we can get $i\Lambda_x^{*\perp} = \Lambda'_x \subset i\Lambda'^{*\perp}_x = \Lambda_x$. Since $i\Lambda_x^{*\perp} \supset \Lambda_x$ can be proved analogously as in Lemma 4.3 in [11] (cf. added in proof this paper), we can conclude $\Lambda_x^n(\Omega_n) \Rightarrow \Lambda_x(\Omega)$. q.e.d.

Next, we set $\Delta'_n = \overline{W}_n - W_n$ (where $W_n = R_n \cup \Omega$ as above) and

$$\mathscr{A}_n = \partial W_n, \ \mathscr{A}_{nm} = \mathscr{A}_n \cap R_m, \ \alpha = \Delta \cap D^*,$$

$$\mathscr{B}_n = \{\text{the component of } \Delta'_n \text{ including } \mathscr{A}_n\} - \bar{\mathscr{A}}_n,$$

$$\mathscr{B}_{nm}=\{\text{the component of } \Delta'_n \text{ including } \mathscr{A}_{nm}\}-\bar{\mathscr{A}}_{nm}.$$

Note that $\Delta_0=\Delta-\Delta_1=\Delta'_n-\bar{\mathscr{A}}_n\cup\mathscr{B}_n$ and D^* is $\mathscr{D}_1^*$-type. Further, we consider the following subspaces of differentials on W_n:

$\tilde{\Lambda}_x^n=\tilde{\Lambda}_x^n(W_n)=\tilde{\Lambda}_x(\mathscr{L}_{W_n}, W_n, \overline{W}Z)=\{\lambda\in Cl\{\Lambda_{ho}(W_n)+\Lambda_{he}(W_n)\}$: (i) $\int_{A_j,B_j}\lambda\in L_j$ for $A_j, B_j\subset W_n$, (ii) $\mathrm{Im}(\bar{z}_1\lambda)\in\Gamma_{ho}(\mathscr{A}_n, W_n)$, $\mathrm{Im}(\bar{z}'_1\lambda)\in\Gamma_{ho}(\mathscr{B}_n, W_n)$ and $\mathrm{Im}(\bar{z}_0\lambda)\in\Gamma_{ho}(\Delta_0, W_n)\}$,

$\Lambda_{xm}^n=\Lambda_{xm}^n(\mathscr{L}_{W_n}, W_n, Z)=\Lambda_{xm}^n(W_n)=$ the spaces which are defined analogously by replacing $\mathscr{A}_n$ and $\mathscr{B}_n$ with $\mathscr{A}_{nm}$ and $\mathscr{B}_{nm}$ respectively, where $m>n$. Then, analogously as in Lemma 4.1, we have

Lemma 4.2. *Assume that the set $\bar{\alpha}-\alpha$ consists of exactly two boundary points of border type and is equal to $\partial\bar{D}\cap\Delta$, then we have $\Lambda_{xm}^n(W_n)\Rightarrow\tilde{\Lambda}_x^n(W_n)$, hence we obtain $\tilde{\Lambda}_x(W_n)=i\tilde{\Lambda}_x(W_n)^{*\perp}$.*

At last, we write $\alpha=D^*\cap\Delta$ and $\beta=\Delta-\bar{\alpha}$, and consider the following spaces of differentials:

$\tilde{\Lambda}_x(\mathscr{L}, R, Z)=\tilde{\Lambda}_x(R)=\tilde{\Lambda}_x=\{\lambda\in Cl\{\Lambda_{ho}(R)+\Lambda_{he}(R)\}$: (i) $\int_{A_j,B_j}\lambda\in L_j$, $j=1, 2,\dots, g$, (ii) $\mathrm{Im}(\bar{\alpha}_1\lambda)\in\Gamma_{ho}(\alpha, R)$, $\mathrm{Im}(\bar{z}'_1\lambda)\in\Gamma_{ho}(\beta, R)$ and $\mathrm{Im}(\bar{z}_0\lambda)\in\Gamma_{ho}(\Delta_0, R)\}$. Then, by the same way as in Lemma 4.1, we have the following lemma:

Lemma 4.3. *Assume that Δ_1 has a planar neighbourhood in Stoilow's sense and D a region of $\mathscr{D}_1$-type, then we have $\tilde{\Lambda}_x^n(W_n)\Rightarrow\tilde{\Lambda}_x(R)$, hence we obtain $\tilde{\Lambda}_x(R)=i\tilde{\Lambda}_x(R)^{*\perp}$.*

Hereafter, we call such a behavior space $\tilde{\Lambda}_x(R)$ the behavior space of X-type associated with $\mathscr{L}$, D^* and Z.

Consequently, by use of Lemmata 4.1, 4.2 and 4.3, we obtain the following theorems:

Theorem 3. *Assume that Δ_1 has a planarneighbourhood in Stoilow's sense and D^* is a region on R^* of $\mathscr{D}_1^*$-type. Then we have following* (i), (ii) *and* (iii).

- (i) *There exists the behavior space $\tilde{\Lambda}_x$ of X-type associated with $\mathscr{L}$, D^* and Z.*
- (ii) *There exists a sequence $T=\{n_k\}$ ($n_k\to\infty$ as $k\to\infty$) of positive integers such that the sequence $\{\phi_{\alpha_j}(A_j, \tilde{\Lambda}_x^k, W_k)\}_{k\in T}$ (resp. $\{\phi_{\alpha_j}(B_j, \tilde{\Lambda}_x^k, W_k)\}_{k\in T}$ and $\{\phi(\theta, \tilde{\Lambda}_x^k, W_k)\}_{k\in T}$) converges locally uniformly on R to $\phi_{\alpha_j}(A_j, \tilde{\Lambda}_x, R)$ (resp. $\phi_{\alpha_j}(B_j, \tilde{\Lambda}_x, R)$ and $\phi(\theta, \tilde{\Lambda}_x, R)$).*
- (iii) *Riemann-Roch's theorem for $\tilde{\Lambda}_x$ holds.*

Proof. Case (i). Cf. Lemma 4.3. Case (ii). Cf. Lemmata 1.4 and 4.3. Case (iii). Since $\tilde{\Lambda}_x$ is a behavior space, we have this conclusion by Theorem 4 in Shiba [14].

Theorem 4. *Let R be an open Riemann surface with finite genus g and Δ_1 an isolated boundary component which consists of more than one point. Suppose D^* is a region of $\mathscr{D}_1^*$-type, and that $\tilde{\Lambda}_x(R)$ is a behavior space of X-type associated with $\mathscr{L}$, D^* and Z. Then, for suitable choice of $g+1$ points $P_1, P_2,\dots, P_{g+1}$ on R, there exists a meromorphic function f which satisfies the following conditions*: (i) *f has $\tilde{\Lambda}_x(R)$ behavior and the residue of f at P_1 is equal to* 1 (*or* i), (ii) *the divisor of f is a multiple of $(P_1P_2\cdots P_{g+1})^{-1}$*, (iii) *$f(R)$, the image of R under f, is at most $g+1$ sheeted over the Riemann sphere.*

Let $D^*=\bigcap_{i=1}^{K} D_i^*$ be a open set where D_i^* denotes a region of $\mathscr{D}_i^*$-type, $Z=\{z_0, z_i, z_i', 1\leqq i\leqq k$ and $\alpha_i=\Delta\cap D_i^*$, $\beta_i=\Delta_i-\alpha_i$. Generally speaking, the same conclusion as above holds for $\hat{\Lambda}_x$, where $\hat{\Lambda}_x=\{\lambda\in Cl\{\Lambda_{ko}+\Lambda_{he}\}$; (i) $\int_{A_j,B_j}\lambda\in L_j$ *for each j*, (i) $\mathrm{Im}\bar{z}_i\lambda\in\Gamma_{ho}(\alpha_i, R)$, $\mathrm{Im}\bar{z}_i'\lambda\in\Gamma_{ho}(\beta_i, R)$, $i=1, 2,\dots, K$ *and* $\mathrm{Im}\bar{z}\lambda\in\Gamma_{ho}(\Delta_0)$ *where* $\Delta_0)=\Delta-\bigcup_{i=1}^{K}\Delta_i$.

Proof. Since Kuramochi's compactification has a boundary property, Δ_1 can be considered as a unit circle. Therefore, the conditions in Theorem 3 are satisfied because R is of finite genus, hence by the same way as in Matsui [11] we can prove this theorem.

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Added in Proof

1. Correction in "Convergence theorems of Abelian differentials with applications to conformal mappings. I" This Journal, Vol. 15, No. 1, (1975), pp. 73–100.

We replace the proof of Theorem 2 with the following (concerning the precise proof, cf. next paper). At first, we note ${}_n\Lambda \supset \Lambda_{ho} \cap \Lambda_{he}$, hence ${}_n\Lambda \subset Cl\{\Lambda_{ho} + \Lambda_{he}\}$, therefore, ${}_n\Lambda = Cl\{{}_n\Lambda_1 + \Lambda'_2 + \Lambda_3\}$ where ${}_n\Lambda_1 =$ the space spanned by $\{\zeta_j\sigma(A_j), \zeta_j\sigma(B_j)\}_{j=1}^{\infty}$ (here $\zeta_j \in L_j \subset {}_{R_n}\mathscr{L}$), $\Lambda'_2 = \Lambda_{hm}$ and $\Lambda_3 = Cl\{\sum_{k=1}^{K} z_k \Gamma_{heo}(\Delta - \beta_k)\}$. Consequently, we have $Cl\{\Lambda_1 + \Lambda'_2 + \Lambda_3\} \subset \mathfrak{A} = \bigcap_{n=1}^{\infty} Cl(\sum_{k=n}^{\infty} {}_n\Lambda) = \{\lambda \in Cl(\Lambda_1 + \Lambda'_2 + \Lambda_3): \int_{A_j B_j} \lambda \in L_j \subset \mathscr{L}\}$ where $\Lambda_1 =$ the space spanned by $\{\zeta_j\sigma(A_j), \zeta_j\sigma(B_j)\}_{j=1}^{\infty}$, (here $\zeta_j \in L_j \subset \mathscr{L}$), $\Lambda'_2 = \Lambda_{ho} \cap \Lambda_{he}$. Therefore, we have $i\mathfrak{A}^{*\perp} \supset \mathfrak{A}$, hence we can conclude $i\mathfrak{A}^{*\perp} = \mathfrak{A}$ and ${}_n\Lambda \Rightarrow \mathfrak{A}$ in the sense of Definition 1 of this paper

2. Supplement of this paper. In the proof of Lemma 4.1 (or Lemma 4.2), the fact $i\Lambda^{*\perp}_x \supset \Lambda_x$ can be proved as follows (I found that it is not so analogously as in Lemma 2.6). At first, from Theorem 2 in [11] we can assume $\mathscr{L}_\Omega = \{L_j : L_j \in iz_0$ for $A_j, B_j \subset U_0$ and $L_j \in iz'_1$ for $A_j, B_j \subset U_1$ where U_0 (resp. U_1) denotes a neighbourhood of Δ_0 (resp. $\Delta_\Omega - \Delta_0 = \Delta_{1\Omega}$)$\}$. For $\phi = \lambda + i\lambda^*$ with $\lambda \in \Lambda_x \cap i\Lambda^*_x$, $\bar{z}'\phi = df + i\sigma = df + i{}^*df$ on U_1 where ${}^*df = \sigma \in \Gamma_{ho}(B, \Omega) \cap \Gamma_{hse}(\Omega) \cap \Gamma_{hse}(\Omega)^*$. Thus we have $\|\phi\|^2 < \phi, i\phi^* > = -\mathrm{Re}\int_{\Delta_{1\Omega}} \left(\int \phi\right) \overline{i\phi}$

(see p. 93 in [11]). Concerning the meanings of $\int_{\Delta_1\Omega}\omega$, see [11] p. 77. From $0=\int_{A\cup B}d\left(f\int\sigma\right)$, we get $\|\phi\|^2=-2\int_{A\cup B}f\sigma$. But, since $\int_A\phi=0=\int_B\phi$, we have $\int_A f\sigma=\text{constant}\int_A d(f^2)=0$ in case $\arg z'-\arg z\neq\frac{\pi}{2}$, $\int_A f\sigma=\text{constant}\int_A\sigma=0$ in case $\arg z'-\arg z=\frac{\pi}{2}$, hence $\left|\int_B f\sigma\right|=\left|\int_\gamma f\sigma\right|=\lim_{n\to\infty}\left|\int_r f\sigma\right|<\varepsilon$ where $B-\gamma\sim 0$ and $\sigma_n\in\Gamma_{ho}(\ell_n,\Omega_n)\cap\Gamma_{hse}$ with $\|\sigma_n-\sigma\|_{\Omega_n}\to 0$ as $n\to\infty$. Therefore, we have $\phi=0$, hence $\Lambda_x\subset i\Lambda_x^{*\perp}$ (cf. Lemma 1.3 in this paper). Concerning the relation $\tilde{\Lambda}_x^n(W_n)\subset i\tilde{\Lambda}_x^n(W_n)^{*\perp}$, see Lemma 4.5 in [11] and Theorem 2 in [11]. At last, the relation $\tilde{\Lambda}_x\subset i\tilde{\Lambda}_x^{*\perp}$ in Lemma 4.3 is evident since Δ_1 has a planar neighbourhood.