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# On a cancellation problem for Dedekind domains

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## §0. Introduction

The following problem is called a cancellation problem ([2]).

Let $A$ and $B$ be rings such that $A[X_1,\dots, X_n]=B[Y_1,\dots, Y_n]$ with $X_1,\dots, X_n$ and $Y_1,\dots, Y_n$ algebraically independent over $A$ and $B$ respectively. Is $A$ isomorphic to $B$? Or, more strongly, does it follow that $A=B$? We say, following [1], $A$ is invariant if for each $B$ as above, there exists an isomorphism between $A$ and $B$; we say $A$ is strongly invariant if it follows that $A=B$.

The following question was raised in connection with the cancellation problem for Dedekind domains ([2]).

**Question.** Suppose $V$ is a D.V.R. (rank one discrete valuation ring) of a field $K$. Let $u$ and $w$ be algebraically independent elements over $K$ and let $U$ be a D.V.R.-extension of $V$ to $K(u, w)$. Suppose that both $V_u=U\cap K(u)$ and $V_w=U\cap K(w)$ are residually algebraic over $V$ (i.e. residue class fields of $V_u$ and $V_w$ are algebraic over that of $V$). Is $U$ residually algebraic over $V$?

In §1, we first prove that the question is affirmative if either $[V_u/\mathfrak{m}_u: V/\mathfrak{m}]_i<\infty$ or $[V_w/\mathfrak{m}_w: V/\mathfrak{m}]_i<\infty$ (Theorem 1).

On the contrary let us assume that there exists a residually transcendental element $Z\in U$ over $V$, then we have $Z=f(u, w)/g(u, w)$ with $f(X, Y), g(X, Y) \in V[X, Y]$. Then, we can reduce the question to the case where $V$ and $U$ have a common uniformizing parameter $t$ (Proposition 1). Then, we can choose $n$ such that $f(u, w)/t^n$ and $t^n/g(u, w)$ are units and at least one of them is residually transcendental over $V$. So we may assume that $Z=f(u, w)/t^n$. Our Theorem 2 asserts that if $Z$ can be chosen so that $f(u, w)$ does not contain any term of the form $u^i w^j$ with $ij>0$, that is, $f(u, w)=f_1(u)+f_2(w)$ ($f_i(X)\in V[X]$), then both $V_u/tV_u$ and $V_w/tV_w$ must have infinite $p$-independent elements over $V/tV$. This result plays an important role in §2.

Finally we give an example which shows that the question is negative in general.

In §2 we study a cancellation problem for Dedekind domains. The following two results are well known.

**Theorem 0.1** ([1], Th. 3.3) *Suppose that $A$ is an integral domain of transcendence degree one over a subfield, then $A$ is invariant. If furthermore $A$ is not a polynomial ring, then $A$ is strongly invariant.*

**Proposition 0.2** ([1], (5.4), (5.5), (5.6), (5.7))

*Let $A$ be a Dedekind domain and suppose that $A[X_1,\dots,X_n]=B[Y_1,\dots,Y_n]$ and that $A\neq B$. Let $K$ be the quotient field of $A\cap B$. Then:*
1) *If $A\cap B=K$, then $A$ is a polynomial ring, say $K[T]$, over $K$.*
2) *If $A\cap B\subsetneq K$, then $A=K[T]\cap V_1\cap\cdots\cap V_r\ (r\geq 1)$ where $T$ is a suitable transcendental element over $K$ and every $V_i$ is a D.V.R. of $K(T)$ such that $V_i\not\supseteq K$ and $V_i$ is residually algebraic over $V_i\cap K$.*

We prove that, in case 2) of Proposition 0.2, each $V_i/\mathfrak{n}_i$ has infinite $p_i$-independent elements over $V_i\cap K/(\mathfrak{n}_i\cap K)$ with $p_i=ch(V_i/\mathfrak{n}_i)>0$ (Theorem 3).

As a corollary, we see that a Dedekind domain $A$ whose quotient field has a finite transcendence degree over a "residually perfect" field is invariant.

On the other hand it has been an open question if the case 2) of Proposition 0.2 exists really ([1] (5.6), [2] (5.8)), and we show it by constructing an example.

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## §1. Residually algebraic extensions

**Theorem 1.**[1] *Under the condition of the question, if either $[V_u/\mathfrak{m}_u: V/\mathfrak{m}]_i<\infty$ or $[V_w/\mathfrak{m}_w: V/\mathfrak{m}]_i<\infty$, then $U$ is residually algebraic over $V$, where $\mathfrak{m}_u$, $\mathfrak{m}_w$ and $\mathfrak{m}$ are maximal ideals of $V_u$, $V_w$ and $V$ respectively and $[\ :\ ]_i$ stands for inseparable degree.*

In order to prove this, we show a lemma.

**Lemma 1.** *If $U$ is a* D.V.R.*-extension of a* D.V.R. *$V$, then there exists $W$ such that*
*i) $W$ is a* D.V.R. *and is an unramified extension of $U$, and*
*ii) denoting by $\bar{V}$ the integral closure of $V$ in $W$ and by $\mathfrak{n}$ the maximal*

1) This theorem was suggested by Professor M. Nagata.

*ideal of W, we see that* $\bar{V}/(\mathfrak{n}\cap\bar{V})$ *is separably closed (in its algebraic closure).*

*Proof.* Let $K$ be the quotient field of $U$ and $L$ the algebraic closure of $K$. Consider the set $\Gamma$ of all pairs $(U_\lambda, \mathfrak{n}_\lambda)$ which satisfy the condition;

i′) $U_\lambda$ is a D.V.R. which is an unramified extension of $U$, $U_\lambda\subset L$ and $\mathfrak{n}_\lambda$ is the maximal ideal of $U_\lambda$.

Define an order in $\Gamma$ by the inclusion relation. It is easy to see that $\Gamma$ is an inductive set, and $\Gamma$ has a maximal element, say, $U_{\lambda_0}$. We show that this $U_{\lambda_0}$ is the required one.

In fact i) is obvious. Suppose ii) does not hold for $W=U_{\lambda_0}$, then there exists and element $\alpha$ in the separable closure of $V/\mathfrak{m}$ and not in $V_{\lambda_0}/(V_{\lambda_0}\cap n_{\lambda_0})$, where $\mathfrak{m}$ is the maximal ideal of $V$ and $V_{\lambda_0}$ is the integral closure of $V$ in $U_{\lambda_0}$. Let $f'(X)$ be a monic minimal polynomial over $V_{\lambda_0}/(V_{\lambda_0}\cap n_{\lambda_0})$ which has $\alpha$ as a root. Lift $f'(X)$ to a monic polynomial $f(X)$ over $V_{\lambda_0}$. Then $U_{\lambda_0}[X]/f(X)U_{\lambda_0}[X]$ is in general a semilocal ring, and if we localize it to $U'_{\lambda_0}$ at a maximal ideal, it follows that $U'_{\lambda_0}$ contains $U_{\lambda_0}$ properly and satisfies i′), a contradiction. q.e.d.

*Proof of Theorem* 1. Take $W$ in Lemma 1 with respect to $V\subset U$, and let $U^*$ be the completion of $W$. Take integral closures of $V$, $V_u$ and $V_w$ in $U^*$ and let $(V^*, \mathfrak{m}^*)$, $(V_u^*, \mathfrak{m}_u^*)$ and $(V_w^*, \mathfrak{m}_w^*)$ be their completions in $U^*$ respectively. We may assume $[V_u/\mathfrak{m}_u: V/\mathfrak{m}]_i<\infty$. Then, by our construction, we have $[V_u^*/\mathfrak{m}_u^*: V^*/\mathfrak{m}^*]<\infty$. Since $V_u^*$ is a D.V.R., there exists an integer $n$ such that $(\mathfrak{m}_u^*)^n\subseteq\mathfrak{m}^*V_u^*\neq 0$ and we have $[V_u^*/\mathfrak{m}^*V_u^*: V^*/\mathfrak{m}^*]\leq n[V_u^*/\mathfrak{m}_u^*: V^*/\mathfrak{m}^*]<\infty$. Since $V^*$ is complete, $V_u^*$ is a finite $V^*$-module ([3] (30.6)). Therefore $V_w^*[V_u^*]$ is a finite $V_w^*$-module. Take derived normal ring of $V_w^*[V_u^*]$ and denote by $V'$ its localization at the prime ideal lying under the maximal ideal of $U^*$. Then $V'$ is a D.V.R. and, since $V_u$, $V_w\subset V'\subset U^*$, we have $U\subset V'$. Since $V'$ is algebraic over the D.V.R. $V_w^*$ and since $V_w^*$ is residually algebraic over $V$, $V'$ and $U$ are residually algebraic over $V$. q.e.d.

The following proposition shows that the question can be reduced to the case where $U$ and $V$ have a common uniformizing parameter.

**Proposition 1.** *Under the condition of the question, there exist* $L$, $V'$ *and* $U'$ *such that;*

(1) *$L$ is a field extension of $K$ and, $u$ and $w$ are algebraically independent over $L$,*

(2) *$V'$ is a* D.V.R. *of $L$ and $U'$ is a* D.V.R. *extension of $V'$ to $L(u, w)$ and they have a common uniformizing parameter,*

(3) *both* $V'_u=U'\cap L(u)$ *and* $V'_w=U'\cap L(w)$ *are residually algebraic over* $V'$ *and*

(4) *if U is residually algebraic (or transcendental) over V, then U' is residually algebraic (or transcendental, respectively) over V'.*

*Proof.* Let $X$ be an indeterminate over $U$, then, since $K(X)(\alpha)\cap U(X) = V_\alpha(X)$ with $\alpha=u$ or $w$, (1), (2), (3) and (4) hold for $V'=V(X)$, $U'=U(X)$ and $L=K(X)$ except that $V(X)$ and $U(X)$ have a common uniformizing parameter.

Take uniformizing parameters $q$ of $V$ and $t$ of $U$ respectively, then we have $q=t^r s$ with an integer $r>0$ and a unit $s$ of $U$. Define $U'=U(X)[F]/(F^r-sX)$, $V'=V(X)[tf]=V(X)[Y]/(Y^r-qX)$ and $L=Q^{-1}V'$, then they are required ones.

In fact, since $L$ is algebraic over $K(X)$, (1) is obvious. Since $U'/tU' = k^*(X)[F]/(F^r-\bar{s}X)$ and $V'/tfV'=V'/yV'=k(X)$ are fields and since $V'/qV' = k(X)[Y]/(Y^r)$ is local (where $k$ and $k^*$ are residue fields of $V$ and $U$ respectively and $\bar{s}$ is the class of $s$ in $k^*$), $U'$ and $V'$ are D.V.R.s which have a common uniformizing parameter $y=tf$. (2) holds because $Q^{-1}U'=\frac{K(u, w)(X)[F]}{(F^r-sX)} = \frac{K(u, w, X)[F]}{t^r(F^r-sX)} = \frac{K(u, w, X)[Y]}{(Y^r-qX)} = L(u, w)$. Finally (3) and (4) holds because $V'$, $V'_\alpha = L(\alpha)\cap U'$ and $U'$ are algebraic over $V(X)$, $V_\alpha(X)$ and $U(X)$ respectively. q.e.d.

In order to prove Theorem 2, we show a lemma in fields theory and one more lemma.

**Lemma 2.** *Let $k$ be a separably closed field of positive characteristic $p$ and $K$ an algebraic extension of $k$. Let $\{k_z\}_{z\in\Lambda}$ be a set of fields such that $k\subseteq k_z\subseteq K$ and $K$ is finite over $k_z$ for each $z\in\Lambda$. If $K$ has a finite $p$-base over $k$, then $K$ is a finite $\bigcap_{z\in\Lambda} k_z$-module.*

*Proof.* Let $a_1,\dots, a_r$ be a finite $p$-base of $K$ over $k$. Then there exists an integer $n$ such that $a_i^{p^n}\in k$ $(i=1,\dots, r)$ because $K$ is purely inseparable over $k$. So we have

$$k(K^{p^n}) = k(K^{p^{n+1}}) = k(K^{p^{n+2}}) = \cdots$$

Since $K$ is finite and purely inseparable over $k_z$, there exists an integer $m_z$ such that $k_z\supseteq K^{p^{m_z}}$. Then we have

$$k_z\supseteq k(K^{p^{m_z}})\supseteq k(K^{p^n}).$$

Therefore $K$ is finite $\bigcap_{z\in\Lambda} k_z$-module because $K=k(K^{p^n}, a_1,\dots, a_r)$ is a finite $k(K^{p^n})$-module.

**Lemma 3.** *Let $(V, t)$ be a* D.V.R. *of a field $K$ with separably closed* *residue class field k of positive characteristic p. Let x be a transcendental element over K and assume that* $(V_x, t)$ *is a residually algebraic* D.V.R.-*extension of V to K(x) such that* $V_x/tV_x$ *has finite p-base over k. Then there exists a transcendental element* $y \in K[x]$ *over K such that*;

*) *if* $\alpha \in K[y] \cap V_x$, $\alpha^{p^e} - a = t\beta$ *with* $a \in V$ *and* $\beta \in V_x$, *there exists* $\gamma \in V_x \cap K[\beta]$ *such that* $\alpha - \gamma \in tV_x$.

*Proof.* If $\alpha$ is algebraic over $K$ then $\alpha \in K \cap V_x = V$ and the assertion is obvious. Therefore we assume that $\alpha$ is transcendental over $K$. Note that in this case, $\beta$ is also transcendental over $K$. For each transcendental element $z \in K[x]$ over $K$, we define $V_z = K(z) \cap V_x$. Then $V_x/tV_x$ is finite over $V_z/tV_z$ because $K(x)$ is finite over $K(z)$ and $V_x$ is a D.V.R.-extension of $V_z$ to $K(x)$. So $V_x/tV_x$ is finite $k' = \bigcap_{\substack{z \in K[x] \\ \text{trans.}/K}} V_x/tV_z$-module by Lemma 2. Choose a transcendental element $y \in K[x]$ over $K$ which makes $[V_z/tV_z : k']$ least with $z = y$ and we want to show that this $y$ is the required one. To see this, it is sufficient to show that $V_y/tV_y = V_\beta/tV_\beta$ if $\alpha \in K[y] \cap V_x$, $\alpha^{p^e} - a = t\beta$ with $a \in V$ and $\beta \in V_x$. In fact, if we can take $\gamma' = f(\beta)/g(\beta)$ with $f(X), g(X) \in K[X]$ such that $\alpha - \gamma' \in tV_x$, then since $V$ is residually algebraic over $V$, we can take $g'(\beta) \in K[\beta]$ and $h \in V$ such that $\gamma - \gamma' \in tV_x$ with $\gamma = f(\beta)g'(\beta)/h$. Now $[V_y/tV_y : k'] = [V_y/tV_y : V_\beta/tV_\beta][V_\beta/tV_\beta : k']$. By our construction $[V_y/tV_y : k'] \leq [V_\beta/tV_\beta : k']$. Therefore it holds that $[V_y/tV_y : V_\beta/tV_\beta] = 1$ and $V_y/tV_y = V_\beta/tV_\beta$. q.e.d.

**Theorem 2.** *Let V, K, U, u, w,* $V_u$ *and* $V_w$ *be the same as in the question and t a uniformizing parameter for* $V_u$. *Suppose that* $V_u$ *and* $V_w$ *are residually algebraic over V and that there exists a residually transcendental element* $Z \in U$ *over V such that*

$$v = f(u) + t^n Z \in K(w) \quad \text{with} \quad f(X) \in K(X),$$

*then the characteristic p of* $V_u/tV_u$ *is positive and* $V_u/tV_u$ *has infinite p-independent elements over* $V/(tV_u \cap V)$.

*Proof.* The first assertion is obvious by Theorem 1. To see the last assertion, we show a contradiction assuming that $V_u/tV_u$ has finite $p$-base over $V/(tV_u \cap V)$. We may assume that $t$ is a uniformizing parameter for both $V$ and $U$ by Proposition 1; note that the finiteness of $p$-base is preserved. By Lemma 1, we may assume that $k = V/(tV_u \cap V)$ is separably closed. If $f(u) \in K$, then we have $Z \in K(w) \cap U = V_w$, contradicting that $V_w$ is residually algebraic over $V$. So we use Lemma 3 in the case $x = f(u)$, then there exists $y \in K[x]$ which satisfies *). If $y = g(x)$ with $g(x) \in K[x]$, then we have

$$g(v) = g(x + t^n Z) = g(x) + t^m Z' = y + t^m Z' \in K(w)$$

with a residually transcendental element $Z' \in U$ over $V$ and with an integer $m$. By multiplying $t^r$ if necessary, we may assume $y \in V_u$.

Now replacing $y$ by $u$ and $y+t^m Z'$ by $w$, we want to show a contradiction (i.e. that $V_w$ is residually transcendental over $V$) under the assumptions that $u \in V_u$, $w=u+t^n Z$ and that $*$) of Lemma 3 holds for $y=u$.

If $n \leq 0$, $t^{-n}w=t^{-n}u+Z \in V_w$ is residually transcendental over $V$. So we show the case of $n>0$ by induction on $n$. Since $k$ is separably closed, $u$ satisfies a relation

$$u^{p^e}-a \in tV_u, \quad \text{with} \quad a \in V.$$

We write $u^{p^e}-a=tu'$ with $u' \in V_u$. If $e=0$, then we have

$$w-a=u+t^n Z-a=t(u'+t^{n-1}Z) \in K[w].$$

Therefore by our induction hypothesis applied to $u'$ and $w'=u'+t^{n-1}Z$, we see that $V_w=V_{w'}$ is residually transcendental. If $e>0$, we may assume that $(a \bmod tV) \notin (V/tV)^p$. By $*$) there exists $\gamma \in K[u'] \cap V_u$ such that $u-\gamma \in tV_u$. We may assume that $\gamma=f'(u')$ $(f'(X) \in K[X])$ is the one whose degree is the least in the polynomials that have these properties. Then (i) in the case where $ch.\,V=p$

$$\delta=f'((w^{p^e}-a)/t)=f'((u^{p^e}-a+t^{np^e}Z^{p^e})/t)$$

$$=\gamma+Z^{p^e}g'(u', Z^{p^e}) \in K[w]$$

with $g'(X, Y) \in K[X, Y]$ and (ii) in the case where $ch.\,V=0$

$$\delta=f'((w^{p^e}-a)/t)=f'((u^{p^e}+p^e u^{p^e-1}t^n Z+Z^2 h(u, Z)-a)/t)$$

$$=f'(u'+p^e u^{p^e-1}t^{n-1}Z+Z^2 h(u, Z)/t)=f'(u')+$$

$$p^e u^{p^e-1}t^{n-1}Z\frac{\partial f'}{\partial u'}+Z^2 h'(u, Z)=\gamma+p^e t^{n-1}u^{p^e-1}\frac{\partial f'}{\partial u'}Z$$

$$+Z^2 h'(u, Z) \in K[w]$$

with $h(X, Y)$, $h'(X, Y)$, $\in K[X, Y]$. Note that in case (ii), if we choose integer $r$ such that $b=t^r u^{p^e-1}\frac{\partial f'}{\partial u'}$ is a unit in $V_u$, then $(b \bmod tV_u) \notin V/tV$, because $\deg\frac{\partial f'}{\partial u'}<\deg f'$ and $t^r c\frac{\partial f'}{\partial u'} \not\equiv u \bmod tV_u$ for any $c \in V$. Therefore, if we write $u-\gamma=tu''$ with $u'' \in V_u$, then in each case we have $w-\delta=tu''+t^m Z' \in K[w]$ with $m \leq n$ and a residually transcendental element $Z' \in U$ over $V$. Then use induction hypothesis applied to $u''$ and $w''=u''+t^{m-1}Z'$. q.e.d.

**Remark 1.** We do not know whether the question is affirmative even if we assume only the finiteness of $p$-base of $V_u/tV_u$ (or $V_w/tV_w$) over $k$. On the other hand, we see that finiteness of $p$-base does not imply $[V_u/\mathfrak{m}_u: V/\mathfrak{m}]_i<\infty$ in the following case:

$$V=k(x)[t]_{(t)}, \qquad K=Q^{-1}V,$$

$$W=k(x, x^{p^{-1}},\dots, x^{p^{-n}},\dots)[[t]],$$

$$u=x^{p^{-1}}t+x^{p^{-(1+2)}}t^2+\cdots+x^{p^{\frac{-n(n+1)}{2}}}t^n+\cdots \text{ and}$$

$$V_u=W\cap K(u)$$

where $k$ is a field of positive characteristic $p$ and $x$, $t$ are algebraically independent over $k$.

Now we show that the question is negative in general.

**Example 1.** Let $p$ be a prime integer and let $t$, $Z$, $a_1, a_2,\dots, a_n,\dots$ be algebraically independent elements over $\mathbf{F}_p=\mathbf{Z}/p\mathbf{Z}$. Define

$$K=\mathbf{F}_p(t, a_1, a_2,\dots, a_n,\dots) \quad V=\mathbf{F}_p(a_1, a_2,\dots, a_n,\dots)[t]_{(t)}$$

$$W=\frac{V(Z)[X_1, X_2,\dots, X_n,\dots]}{(X_1^p-a_1-tX_2, X_2^p-a_2-tX_3,\dots, X_n^p-a_n-tX_{n+1},\dots)}$$

$$S=W-tW \quad\text{and}\quad U=W_S, \text{ then}$$

1) *$U$ is a* D.V.R.

Since $tU$ is a unique maximal ideal of $U$, it is sufficient to show that $\bigcap_n t^nU=0$. Since $W/tW$ is a field, $tW$ is a maximal ideal of $W$ and is a unique prime ideal containing $t^nW$ for any positive integer $n$. So it follows that $t^nW_S\cap W=t^nW$ and $\bigcap_n t^nW_S=(\bigcap_n t^nW)W_S$. Therefore it is sufficient to show $\bigcap_n t^nW=0$. For $g\in\bigcap_n t^nW$, take a representative $G(X_1,\dots, X_m)$ of $g$ in $V(Z)[X_1,\dots, X_n,\dots]$. Since we may replace $X_i^p$ in $G(X)$ by $a_i+tX_{i+1}$, we may assume that $G$ is of the form

$$G(X_1,\dots, X_r)=\sum b_\lambda M_\lambda \quad\text{with}\quad b_\lambda\in V(Z),\quad M_\lambda=X_1^{e_1}\cdots X_r^{e_r}$$

$$(0\le e_i<p, \qquad i=1, 2,\dots, r).$$

On the other hand, denoting the class of $X_i$ in $W$ by $x_i$, $W$ is a free $V(Z)$-module having basis

$\{\prod_{i=1}^{\infty} x_i^{e_i} | 0 \leq e_i < p$ and $e_i = 0$ except only for a finite number of $i\}$.

Since $g = G(x_1, \dots, x_r) = \sum b_\lambda \overline{M}_\lambda \in \bigcap_n t^n W$; where $\overline{M}_\lambda$ is the class of $M_\lambda$ in $W$, we have $b_\lambda \in \bigcap_n t^n V(Z)$. Therefore we have $b_\lambda = 0$ and $g = 0$, because $V(Z)$ is a D. V. R.

Now we define $u = x_1$ and $w = x_1 + tZ$, then

2) *$u$ and $w$ are algebraically independent over $K$.*

Let $L$ be the quotient field of $U$. Then $L = K(Z, x_1)$ and $Z$ is transcendental over $K$. On the other hand, since $[U/tU: V(Z)/tV(Z)] = \infty$, $x_1$ is transcendental over $K(Z)$. Therefore the assertion is obvious by our construction.

3) *$U$ is residually transcendental over $V$.*

In fact, $Z$ is residually transcendental over $V$.

4) *$V_u = U \cap K(u)$ is residually algebraic over $V$.*

Since $V_u = \left\{\frac{V[X_1, \dots, X_n, \dots]}{(x_1^p - a_1 - tX_2, \dots, X_n^p - a_n - tX_{n+1}, \dots)}\right\}_{S'}$ with $S' = S \cap K(u)$ and each $x_n$ is residually algebraic over $V$, we see that $V_u$ is residually algebraic over $V$.

5) *$V_w = U \cap K(w)$ is residually algebraic over $V$.*

To see this, it is sufficient to show the following.

**Proposition 2.** *$V_w$ is $V$-isomorphic to $V_u$.*

*Proof.* Let $v$ be the valuation defined by $U$ such that $v(t) = 1$ and let $v^*$ be the valuation of $K(\{X_i\}_{i \in \mathbf{N}})$ defined by $v^*(g) = \operatorname{Min}_{\lambda \in \Lambda} v(b_\lambda)$ if $g = \sum_{\lambda \in \Lambda} b_\lambda M_\lambda \in K[\{X_i\}_{i \in \mathbf{N}}]$ where $b_\lambda \in K$ and each $M_\lambda$ is a monomial of $X_i$.

In order to prove Proposition 2, it is sufficient to show $v(f(u)) = v(f(w))$ for every $f(X) \in V[X]$, because the $K$-isomorphism $\varphi: K(u) \to K(w)$ defined by $\varphi(u) = w$ would give an $V$-isomorphism of $V_u$ onto $V_w$.

Now, if $v(f(u)) = \infty$ then $f(u) = 0$ and $f(X) = 0$ because $u$ is transcendental over $K$, which implies $v(f(w)) = \infty$.

Suppose that $v(f(u)) = n \in \mathbf{N}$. We define $f_1(X_1), f_2(X_1, X_2), \dots, f_k(X_1, \dots, X_k)$ with $f_i(X_1, \dots, X_i) \in V[X_1, \dots, X_i]$ $(i = 1, \dots, k)$ inductively as follows.

If $v^*(f(X_1)) = r_0$, then we define $f_1(X_1) = t^{-r_0} f(X_1) \in V[X_1]$. When $f_i(X_1, \dots, X_i) \in V[X_1, \dots, X_i]$ is defined and if $f_i(x_1, \dots, x_i) \notin tU$ then we finish the procedure $(i = k)$. If $f_i(x_1, \dots, x_i) \in tU$, then since $\{(a_i \bmod tV)\}_{i \in \mathbf{N}}$ are $p$-independent over $(V/tV)^p$, we have

$$f_i(X_1, \dots, X_i) \in (X_1^p - a_1, X_2^p - a_2, \dots, X_i^p - a_i, t) V[X_1, \dots, X_i].$$

Substituting $x_j^p - a_j$ by $t x_{j+1}$ $(1 \leq j \leq i)$, we have

$f_i(x_1,\dots, x_i)=f'_i(x_1,\dots, x_{i+1})$ with $f'_i(X_1,\dots, X_{i+1})\in tV[X_1,\dots, X_{i+1}]$.

If $v^*(f'_i(X_1,\dots, X_{i+1}))=r_i$, we define

$$f_{i+1}(X_1,\dots, X_{i+1})=t^{-r_i}f'_i(X_1,\dots, X_{i+1}).$$

Since $r_0+r_1+\cdots+r_{k-1}=n$ with $r_i>0\ (i=1, 2,\dots, k-1)$, we finish these procedure by at most $(n+1)$st step.

On the other hand, due to following Lemma 4, we have $y_i\in U\ (i=2, 3,\dots, k+1)$ such that $w^p-a_1=ty_2$, $y_2^p-a_2=ty_3,\dots, y_k^p-a_k=ty_{k+1}$.

So by construction of $f_i(X_1,\dots, X_i)$, we have

$$f(w)=t^n f_k(y_1,\dots, y_k) \quad \text{with} \quad f_k(y_1,\dots, y_k)\notin tU.$$

Therefore $v(f(w))=n$ q.e.d.

**Lemma 4.** *For $w$ of* Example 1, *there exist $y_i\in U\ (i=2,\dots, k+1)$ such that $w^p-a_1=ty_2$, $y_2^p-a_2=ty_3,\dots, y_k^p-a_k=ty_{k+1}$. Moreover $y_i=x_i+tc_i$ wiyh $c_i\in U$.*

*Proof.* We show the existence by induction on $i$. Since $w=x_1+tZ$, we may include the case of $i=1$, defining $y_1=w$. Suppose we have $y_i=x_i+tc_i$ with $i\geq 1$ and $c_i\in U$. Then $y_i^p-a_i=x_i^p+t^pc_i^p-a_i=tx_{i+1}+t^pc_i^p=t(x_{i+1}+t^{p-1}c_i^p)$ and it is sufficient to put $y_{i+1}=x_i+t^{p-1}c_i^p$, $c_{i+1}=t^{p-2}c_i^p$. q.e.d.

**Example 2.** We have similar examples in unequal characteristic case too. For example, let $p$ be a prime integer and $\mathbf{Z}_p$ a localization of $\mathbf{Z}$ at $(p)$. Let $Z, a_1,\dots, a_n,\dots$ be algebraically independent elements over $\mathbf{Z}_p$. Define

$$K=\mathbf{Q}(a_1,\dots, a_n,\dots) \quad V=\mathbf{Z}_p(a_1,\dots, a_n,\dots)$$

$$W=\frac{V(Z)[X_1,\dots, X_n,\dots]}{(X_1^p-a_1-pX_2,\dots, X_n^p-a_n-pX_{n+1},\dots)}$$

$$U=W_{pW},\ u=x_1 \quad \text{and} \quad w=x_1+pZ.$$

The proof is same as that of Example 1.

## §2. Application

In this section we give some sufficient conditions for a Dedekind domain to be (strongly) invariant. For the purpose we define:

**Definition.** We say that a Dedekind domain $A$ is a (D.C.P.) if $A=K[T]$

$\cap V_1 \cap \cdots \cap V_r$ $(r \geq 1)$ where $K$ is a field, $T$ is transcendental over $K$ and each $(V_i, \mathfrak{n}_i)$ is a D.V.R. of $K(T)$ such that $V_i/\mathfrak{n}_i$ is algebraic over $k_i = (V_i \cap K)/(\mathfrak{n}_i \cap K)$ and has positive characteristic $p_i$ and has an infinitely many $p_i$-independent elements over $k_i$.

**Theorem 3.** *If a Dedekind domain $A$ is not a* (D.C.P.), *then $A$ is invariant. If furthermore $A$ is not a polynomial ring, then $A$ is strongly invariant.*

*Proof.* Suppose that $A$ is neither a (D.C.P.) nor a polynomial ring and that $A[X_1,\dots, X_n] = B[Y_1,\dots, Y_n] = R$, $A \neq B$.

By Proposition 0.2, $A = K[u] \cap V_1 \cap \cdots \cap V_r$ $(r \geq 1)$. Since $A$ is not a (D.C.P.), for at least one $i$ $(1 \leq i \leq r)$, say 1, either $V_1/\mathfrak{n}_1$ has zero characteristic or $V_1/\mathfrak{n}_1$ has a positive characteristic $p$ and $V_1/\mathfrak{n}_1$ has finite $p$-base over $V/(\mathfrak{n}_1 \cap V)$ where $V = V_1 \cap K$, and $\mathfrak{n}_1$ is the maximal ideal of $V_1$.

On the other hand by [3] (11.11), $A \cap B = K \cap V_1 \cap \cdots \cap V_r = C$ is a semilocal Dedekind domain. Take $s \neq 0$ in the Jacobson radical of $C$. Then it holds that $A\left[\frac{1}{s}\right] = K[u]$ and $B\left[\frac{1}{s}\right] = K[w]$ with algebraically independent elements $u$ and $w$ over $K$ by Theorem 0.1, by [1] (1.11) and by our assumption $A \neq$ B. Now put $\mathfrak{p} = \mathfrak{n}_1 \cap A$, then since $\mathfrak{P} = \mathfrak{p}R$ is a height one prime ideal of $R$, $R_{\mathfrak{P}}$ is a D.V.R. Define $U = R_{\mathfrak{P}} \cap K(u, w)$, then $V_u = K(u) \cap U = V_1$ is residually algebraic over $V$. $B$ is also a Dedekind domain and we have $B = K[w] \cap V'_1 \cap \cdots \cap V'_{r'}$ by Proposition 0.2. Since $V_w = U \cap K(w) \supseteq B$ and $V_w \not\supseteq K$, it follows that $V_w = V'_i$ for some $i$ $(1 \leq i \leq r')$. Therefore $V_w$ is residually algebra ic over $V_w \cap K = V$. For $s \in C$ above

$$R\left[\frac{1}{s}\right] = K[u, X_1,\dots, X_n] = K[w, Y_1,\dots, Y_n]$$

and we have $w = f(u) + t^m Z$, where $f(u) \in K[u]$, $Z \in U$ and $Z$ is a non-zero polynomial of $X_i$ without constant term (with respect to $X_i$) and with coefficients in $V_u$ such that at least one of the coefficients is a unit of $V_u$. However since the class of $X_i$ in the residue field of $R_{\mathfrak{P}}$ is transcendental over $A/\mathfrak{p}$ and over $V/(\mathfrak{p} \cap V)$, $Z$ in $U$ is residually transcendental over $V$, a contradiction to Theorem 2. q.e.d.

**Corollary 1.** ([1] Th. 6.5) *If $A$ is a Dedekind domain containing a field of characteristic zero, then $A$ is invariant. If furthermore $A$ is not a polynomial ring, then $A$ is strongly invariant.*

**Corollary 2.** *If $A$ is a Dedekind domain whose quotient field $L$ has a finite transcendence degree over either* (*i*) *some perfect subfield $k$ of positive* *characteristic p or (ii) the rational number subfield* $\mathbf{Q}$, *then A is invariant. If furthermore A is not a polynomial ring, then A is strongly invariant.*

*Proof.* It suffices to show that $A$ is not a (D.C.P.), which is obvious in the case (ii). In the case (i), it is sufficient to show that $A \supseteq k$. Suppose that $V = A_{\mathfrak{p}}$ does not contain $k$ for same height one prime ideal $\mathfrak{p}$ of $A$. Then $V \cap k$ is a D.V.R. (not a field). Take its uniformizing parameter $t$. Then $t^{1/p} \in k \cap V$ because $k$ is perfect and we have a contradiction.

Finally, we show that there exists a non-polynomial Dedekind domain which is not strongly invariant.

**Example 3.** Let $K$, $u$, $w$, $Z$ and $U$ be the same as in Example 1. Put $A = K[u] \cap V_u$ and $B = K[w] \cap V_w$, then they are non-polynomial Dedekind domains and $Z \in U$ is transcendental over both $A$ and $B$. We want to show in $U$ that $A[Z] = B[Z]$, which would imply that $A$ is not strongly invariant because $A \neq B$.

We first show that $A[Z] = K[u, Z] \cap U$. In fact, it is obvious that $A[Z] \subseteq K[u, Z] \cap U$. Conversely, if $x \in K[u, Z] \cap U$, we can write $x = \alpha_0 + \alpha_1 Z + \cdots + \alpha_n Z^n$ with $\alpha_i \in K[u]$ $(i = 1, 2, \dots, n)$. Since $Z$ is residually transcendental over $V_u$, it follows that $\alpha_i \in V_u$, that is $\alpha_i \in V_u \cap K[u] = A$ $(i = 1, 2, \dots, n)$. Therefore we have $x \in A[Z]$. Similar holds for $B[Z]$ and we have

$$A[Z] = K[u, Z] \cap U = K[w, Z] \cap U = B[Z].$$

q.e.d.

By the way, we note that the restriction to $A = K[u] \cap U$, of $K$ isomorphism $\varphi: K(u) \to K(w)$ such that $\varphi(u) = w$, gives an isomorphism of $A$ onto $B$ by Proposition 2.

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