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A refinement of explosion condition for branching Lévy processes

By

Michio SHIMURA

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§1. Preliminary and results

The purpose of the present article is to refine the explosion conditions for branching Lévy processes which were obtained in [6][1].

Let $\boldsymbol{X}=(\Omega, \boldsymbol{X}_t, \boldsymbol{P}_{\boldsymbol{x}})$ be branching Markov process on the state space $\boldsymbol{R}$, where the base process $X=(W, X_t, P_x)$ is a Lévy process on the real line R, $\boldsymbol{R}$ is the topological sum of product spaces R^n, $n=0, 1,\dots, \infty$, of R with $R^0=\{\partial\}$ and $R^\infty=\{\Delta\}$, and the branching law is the delta measure $\delta_{(x,x)}(d\boldsymbol{y})$ ($x\in R$, $d\boldsymbol{y}\subset\boldsymbol{R}$) on $\boldsymbol{R}$ having a unit mass only on $(x, x)\in R^2$. Following [6] we call the branching Markov process $((X, k(x), \delta_{(x,x)}(d\boldsymbol{y}))-)$ *branching Lévy process.* Throughout this article we consider branching Lévy processes which satisfy the conditions ($\boldsymbol{X}$-1) and ($\boldsymbol{X}$-2).

($\boldsymbol{X}$-1) *The base process* X *satisfies* $P(\sup_{0\leq t<+\infty} X_t=+\infty)=1$[2].

($\boldsymbol{X}$-2) *The killing rate* $k(x)$ *is non-negative continuous function on* R *such that* $\lim_{x\to+\infty} k(x)=+\infty$.

Let us prepare the several sequences of real numbers.

(S-1) H_n, $n=1, 2,\dots,$ such that $H_n\nearrow+\infty$ as $n\nearrow\infty$. $h_n=H_{n+1}-H_n$, $n=1, 2,\dots.$

(S-2) $l_n\geqq 0$, $n=1, 2,\dots,$ such that $\lim_{n\to\infty}(l_n/H_n)=0$.

(S-3) $t_n>0$, $n=1, 2,\dots,$ such that $\sum_{n\geqq 1} t_n<+\infty$.

1) Notations and terminologies related to our explosion problem are taken from [5] and [6]. For general theory of branching Markov processes we refer, e.g., [2].

2) For abbreviation we denote $P_0(\cdot)$ and $E_0(\cdot)$ related to *a* Lévy process by $P(\cdot)$ and $E(\cdot)$, respectively.

(S–4) M_n, N_n, $n=1, 2,\dots$, positive integers such that $\lim_{n\to\infty} M_n=+\infty$, $\lim_{n\to\infty} N_n=+\infty$.

First we consider branching Lévy process whose base process is a subordinator, that is, a Lévy process with non-decreasing sample functions almost surely. For such a process define the next two quantities I_2 and I_3 by

$$I_2=\sum_n N_n \exp(-k_n t_n), \quad I_3=\sum_n \{P(X(t_n)\leqq h_n)\}^{N_n}$$

where $k_n=\inf_{x\geqq H_n} k(x)$. and the summations $\sum_n$ are taken over all suffciently large n. Then we have

Proposition 1. *Consider branching Lévy process whose base process is a subordinator. If we can find H_n, t_n and N_n in (S–1), (S–3) and (S–4) so that both I_2 and I_3 are finite, then the process is explosive with probability one.*

Next we consider branching Lévy process whose base process may not be a subordinator. For such a process define the next three quantities J_1, J_2 and J_3 by

$$J_1=\sum_n P(\inf_{t\leqq t_n/M_n} X_t<-l_n/M_n), \quad J_2=\sum_n \exp(-k_n t_n/M_n),$$

$$J_3=\sum_n \int_0^{+\infty} \{P(\sup_{s\leqq t} X_s<2l_n+h_n)\}^{A^{M_n}} dt,$$

where $k_n=\inf_{x\geqq H_n-2l_n} k(x)$ and A is any constant such that $1<A<2$. Then we have

Proposition 2. *Consider branching Lévy process whose base process may not be a subordinator. If we can find H_n, l_n, t_n and M_n in (S–1)–(S–4) so that all of J_1, J_2 and J_3 are finite, then the process is explosive with probability one.*

Now let us apply the Propositions to the branching stable processes and the branching Poisson processes which were defined and considered in [6]. In order to simplify the situation, we make the following additional condition (**X**–3) on branching Lévy processes to be considered.

(**X**–3) *The killing rate $k(x)$ is bounded for $x<0$ if the base process is not a subordinator.*

Theorem 1. *Consider branching stable process of indices $\{\alpha, \beta\}$ with $\alpha\in(0, 1)\cup(1, 2)$ and $-1<\beta\leqq 1$ or of indices $\{1, 0\}$. Let the killing rate*

$k(x) \asymp (\log x)^{\gamma}$ as $x \to +\infty$[3]. *Then the process is explosive with probability one or non-explosive according as the constant $\gamma > 1$ or $\gamma \leqq 1$, respectively.*

Theorem 2. ([6; Theorem 2]) *Consider branching stable process of indices $\{\alpha, -1\}$ with $1 < \alpha < 2$, and let the killing rate $k(x) \asymp x^{\gamma}$ as $x \to +\infty$. Then the process is explosive with probability one or non-explosive according as the constant $\gamma > \alpha/(\alpha-1)$ or $\gamma \leqq \alpha/(\alpha-1)$, respectively.*

Theorem 3. *Consider branching Poisson process, and let the killing rate $k(x) \asymp x(\log x)^{\gamma}$ as $x \to +\infty$. Then the process is explosive with probability one or non-explosive according as the constant $\gamma > 2$ or $\gamma < 1$, respectively.*

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§2. Proof of results

1. Proof of Proposition 1. First define the sequence of events A_n, $n = 1, 2, \ldots$, of branching Lévy process $\boldsymbol{X}$ by $A_n = \{\boldsymbol{j}_{H_{n+1}} - \boldsymbol{j}_{H_n} > 2t_n\}$, where $\boldsymbol{j}_H$ $(H \in R)$ is the Markov time of $\boldsymbol{X}$ defined by $\boldsymbol{j}_H = \inf\{t;\ \hat{I}_{(-\infty, H]}(\boldsymbol{X}_t) = 0\}$[4]. Here I_E $(E \subset R)$ is the indicator function on the set E, that is, $I_E(x) = 1$ if $x \in E$ and $I_E(x) = 0$ if $x \in R \backslash E$, and for each real valued function f on R $\hat{f}$ is a function on $\boldsymbol{R}$ such that $\hat{f}(\partial) = 1, \hat{f}(\Delta) = 0$ and $\hat{f}(\boldsymbol{x}) = f(x_1) \cdots f(x_n)$ if $\boldsymbol{x} = (x_1, \ldots, x_n) \in R^n$, $n = 1, 2, \ldots$.

By the Borel-Cantelli lemma and [6; Corollary 1], the finiteness of $\sum_n \boldsymbol{P}_x(A_n)$ for each $x \in R$ implies that $\boldsymbol{X}$ is explosive with probability one. Let us estimate $\boldsymbol{P}_x(A_n)$, using the strong Markov property and the branching property.

$$
\begin{aligned}
(1) \qquad \boldsymbol{P}_x(A_n) &\leqq \boldsymbol{E}_x(H_n \leqq y_n = \Phi(\boldsymbol{X}(\boldsymbol{j}_{H_n})) \leqq H_{n+1};\ \boldsymbol{P}_{y_n}(\boldsymbol{j}_{H_{n+1}} > 2t_n)) \\
&\leqq \boldsymbol{E}_x(H_n \leqq y_n \leqq H_{n+1};\ \boldsymbol{P}_{y_n}(Z(t_n) \leqq N_n) + \boldsymbol{E}_{y_n}(\boldsymbol{j}_{H_{n+1}} > t_n,\ Z(t_n) > N_n; \\
&\qquad \boldsymbol{P}_{\boldsymbol{X}(t_n)}(\boldsymbol{j}_{H_{n+1}} > t_n))).
\end{aligned}
$$

where $\Phi(\boldsymbol{x}) = \max\{x_1, \ldots, x_n\}$ if $\boldsymbol{x} = (x_1, \ldots, x_n) \in R^n$, $n = 1, 2, \ldots$, and $Z(t)$ is the number of particles at $t \geqq 0$, that is, $Z(t) = n$ if $\boldsymbol{X}_t \in R^n$, $n = 0, 1, \ldots, \infty$. Let us estimate the integrands of the right hand side of (1). For the first term, consider the simple Galton-Watson process $\{\tilde{Z}(t);\ t \geqq 0, \tilde{P}\}$ such that $\tilde{P}(\tilde{\tau} > t \mid \tilde{Z}(0) = 1) = e^{-kt}$ and $\tilde{P}(\tilde{Z}(\tilde{\tau}) = i + 1 \mid \tilde{Z}(0) = i) = 1$ $(i = 1, 2, \ldots)$, where k is a positive constant and $\tilde{\tau}$ is the first branching time, that is, $\tilde{\tau} = \inf\{t;\ \tilde{Z}(t) \neq \tilde{Z}(0)\}$. It is

3) $f(x) \asymp g(x)$ as $x \longrightarrow c \Longleftrightarrow 0 < \varliminf_{x \to c} \{f(x)/g(x)\} \leqq \varlimsup_{x \to c} \{f(x)/g(x)\} < +\infty$.

4) The infimum of an empty set is taken to be $+\infty$.

easy to see

$$\tilde{P}(\tilde{Z}(t) \leqq N | \tilde{Z}(0)=1) = 1-(1-e^{-kt})^N. \tag{2}$$

Let us apply (2) for the estimate, noting that sample paths of the base process is non-decreasing. Then

$$\boldsymbol{P}_{y_n}(Z(t_n) \leqq N_n) \leqq 1-(1-e^{-k_n t_n})^{N_n} \leqq N_n e^{-k_n t_n}, \tag{3}$$

where $k_n = \inf_{x \geqq H_n} k(x)$. For an estimate of the remainder part, [5; Lemma 11] is applicable. Then

$$\begin{aligned} &\boldsymbol{E}_{y_n}(\boldsymbol{j}_{H_{n+1}} > t_n,\ Z(t_n) > N_n;\ \boldsymbol{P}_{\boldsymbol{X}(t_n)}(\boldsymbol{j}_{H_{n+1}} > t_n)) \\ &\leqq \{P(j_{h_n} > t_n)\}^{N_n} = \{P(X(t_n) \leqq h_n)\}^{N_n}, \end{aligned} \tag{4}$$

where j_h ($h \in R$) is the Markov time of the base process defined by $j_h = \inf\{t; X_t > h\}$. By (1), (3) and (4), the finiteness of $\sum_n \boldsymbol{P}_x(A_n)$ follows from the finiteness of I_2 and I_3. This completes the proof.

2. Proof of Proposition 2. Let us begin with the following comment. Noting [6; Theorem 4] and the definition of J_1, J_2 and J_3, we see that we may prove, without loss of generality, Proposition 2 under the auxiliary condition that the killing rate is non-decreasing function. Hence we adopt the condition throughout our proof.

Our proof is divided into three steps. In the first and second steps we assume that the branching Lévy process is non-explosive, that is, $\boldsymbol{P}_x(e_\Delta = +\infty) = 1$ for all $x \in R$, where e_Δ is the explosion time of $\boldsymbol{X}$ defined by $e_\Delta = \inf\{t; \boldsymbol{X}_t = \Delta\}$.

First step. We give several definitions related to the branching Lévy process $\boldsymbol{X}$. Let $\mathfrak{t}$ be a Markov time of $\boldsymbol{X}$. For an $\omega \in \Omega$ with $\mathfrak{t}(\omega) < +\infty$ and $e_\Delta(\omega) = +\infty$, consider the Lévy particle which is a component of ω at $\mathfrak{t}(\omega)$, and let the position at $\mathfrak{t}(\omega)$ be $\mathfrak{x}$. Let $\mathfrak{z}$ be the time of creation of the first descendant of the Lévy particle scaling from $\mathfrak{t}(\omega)$, and let $\mathfrak{y}$ be the place of the creation. We call ε_n occurs for the Lévy particle, when the following circumstance occurs: "*The Lévy particle wanders in the half line* $[\mathfrak{x} - l_n/M_n, +\infty)$ *throughout the time interval* $[\mathfrak{t}, \mathfrak{t}+\mathfrak{z})$, *where* $\mathfrak{z} < t_n/M_n$ *takes place. Each of the two Lévy particles, the original particle and the first descendant, wanders in the half line* $[\mathfrak{y} - l_n/M_n, +\infty)$ *throughout the time interval* $[\mathfrak{t}+\mathfrak{z}, \mathfrak{t}+t_n/M_n]$." If ε_n occurs, then

$$\mathfrak{y} \geqq \mathfrak{x} - l_n/M_n. \tag{5}$$

Let $\mathfrak{n}'_{n,0}$ be the number of occurrence of ε_n for the Lévy particle of ω which enters into $(H_n, +\infty)$ for the first time, where we take $\mathfrak{t}=j_{H_n}$ and $\mathfrak{x}=\Phi(\boldsymbol{X}(j_{H_n}))$. Further $\mathfrak{n}_{n,m}$ and $\mathfrak{n}'_{n,m}$ $(m=1, 2,\dots, M_n)$ are the random variables defined as follows: $\mathfrak{n}_{n,m}$ is the number of Lévy particles of ω which are in the half line $(H_n-2ml_n/M_n, +\infty)$ at time $j_{H_n}+mt_n/M_n$. $\mathfrak{n}'_{n,m}$ is the number of occurrences of ε_n for the $\mathfrak{n}_{n,m}$ Lévy particles, where we take $\mathfrak{t}=j_{H_n}+mt_n/M_n$ for each m.

Choose a constant a such that $1/2<a<1$ and take p_n by

$$p_n=\{1-P(\inf_{t\leqq t_n/M_n} X_t<-l_n/M_n)\}^3\{1-\exp(-k_nt_n/M_n)\}. \tag{6}$$

Define the sequence of events A_n, $n=1, 2,\dots$, by

$$A_n=\{e_\Delta=+\infty \text{ and } j_{H_n}<+\infty\}\cap\{\mathfrak{n}'_{n,0}=1 \text{ and } \mathfrak{n}'_{n,m}\geqq ap_n\mathfrak{n}_{n,m},\quad m=1, 2,\dots, M_n-1\}.$$

Then by the branching law of $\boldsymbol{X}$ we have

$$\mathfrak{n}_n(\omega)\overset{\text{def.}}{=}\mathfrak{n}_{n,M_n}(\omega)\geqq(2ap_n)^{M_n} \quad \text{for} \quad \omega\in A_n. \tag{7}$$

Second step. In order to estimate $\boldsymbol{P}_x(A_n)$ from below we need the next two estimates.

The first estimate. *For any* x, $y\in R$,

$$\begin{aligned}&\boldsymbol{P}_x(\varepsilon_n \textit{ occurs for the Lévy particle with } \mathfrak{x}>y)\\ &\geqq \boldsymbol{P}_x(\mathfrak{x}>y)\{P(\inf_{0\leqq t\leqq t_n/M_n} X_t\geqq -l_n/M_n)\}^3\{1-\exp(-k(y-l_n/M_n)\}.\end{aligned} \tag{8}$$

To prove (8) set $\varphi(t)=\int_0^t k(X_s)ds$ and $\zeta=\inf\{t;\, Q(\varphi(t))=1\}$, where $Q(t)$; $t\geqq 0$ $(Q(0)=0)$ is a Poisson process of step one and parameter one, and is independent of X. Then using the strong Markov property and the branching property of $\boldsymbol{X}$, and using the fact that the non-branching part of $\boldsymbol{X}$ in R is equivalent to the $e^{-\varphi(t)}$-subprocess of X, we obtain

$$\begin{aligned}&\text{the left hand side of (8)}\\ &\geqq \boldsymbol{E}_x(\mathfrak{x}>y\,;\, E_{\mathfrak{z}}(\zeta\geqq t_n/M_n,\ \inf_{0\leqq t\leqq\zeta} X_t\geqq z-l_n/M_n\,;\\ &\quad \{P_{X(\zeta)}(\inf_{0\leqq t\leqq t_n/M_n-s} X_t\geqq z'-l_n/M_n)\,|_{z'=X(\zeta)}^{s=\zeta}\}^2)_{|z=\mathfrak{x}})\\ &\geqq \text{the right hand side of (8)}.\end{aligned}$$

The second estimate. *Let X_i, $i=1,2,\dots,N$, be mutually independent random variables such that $X_i=1$ with probability p_i and $X_i=0$ with probability $1-p_i$. Suppose that $p_i \geqq p>0$, $i=1,2,\dots,N$, and $0<a<1$. Then*

$$P\Big(\sum_1^N X_i \geqq apN\Big) \geqq 1-\frac{1-p}{(1-a)^2p^2N} \tag{9}$$

Using the Chebyshev's inequality, we have a proof of (9) as follows.

$$P\Big(\sum_1^N X_i \geqq apN\Big)=1-P\Big(\sum_1^N (X_i-p_i)<apN-\sum_1^N p_i\Big)$$

$$\geqq 1-\frac{\sum_1^N E(X_i-p_i)^2}{(apN-\sum_1^N p_i)^2} \geqq 1-\frac{1-p}{(1-a)^2p^2N}\,.$$

Estimate of $\boldsymbol{P}_x(A_n)$.

$$\boldsymbol{P}_x(A_n)=\boldsymbol{E}_x\Big(j_{H_n}+\frac{M_n-1}{M_n}t_n<e_\Delta,\ \mathfrak{n}'_{n,0}=1,\ \mathfrak{n}'_{n,m}\geqq ap_n\mathfrak{n}_{n,m} \quad \text{for} \quad m=1,\dots,M_n-2;\ \boldsymbol{P}_y((\text{the number of occurrences of } \varepsilon_n) \geqq ap_ns, \text{ where we take } t=0)_{|s=\mathfrak{n}_{n,M_n-1}}\Big), \tag{10}$$

where $\boldsymbol{y}$ is the random vector in $R^{\mathfrak{n}_{n,M_n-1}}$ obtained from $\boldsymbol{X}\Big(j_{H_n}+\frac{M_n-1}{M_n}t_n\Big)$ by omitting the components which take the values in $\Big(-\infty,\ H_n-\frac{2(M_n-1)}{M_n}l_n\Big]$. We can apply (9) to the integrand of the right hand side of (10) if we mind the branching property and (8). Then

$$\boldsymbol{P}_x(A_n)\geqq \boldsymbol{E}_x\Big(j_{H_n}+\frac{M_n-1}{M_n}t_n<e_\Delta,\ \mathfrak{n}'_{n,0}=1,\ \mathfrak{n}'_{n,m}\geqq ap_n\mathfrak{n}_{n,m} \quad \text{for}$$

$$m=1,\dots,M_n-2;\ 1-\frac{1-p_n}{(1-a)^2p_n^2\mathfrak{n}_{n,M_n-1}}\Big)\geqq \Big\{1-\frac{1-p_n}{(1-a)^2p_n^2(2ap_n)^{M_n-1}}\Big\}$$

$$\boldsymbol{P}_x\Big(j_{H_n}+\frac{M_n-1}{M_n}t_n<e_\Delta,\ \mathfrak{n}'_{n,0}=1,\ \mathfrak{n}'_{n,m}\geqq ap_n\mathfrak{n}_{n,m} \quad \text{for} \quad m=1,\dots,M_n-2\Big).$$

Repeating the similar estimates, we obtain

$$\boldsymbol{P}_x(A_n) \geqq \boldsymbol{P}_x(\boldsymbol{j}_{H_n}<e_\Delta)\prod_{m=0}^{M_n-1}\left\{1-\frac{1-p_n}{(1-a)^2p_n^2(2ap_n)^m}\right\}$$

$$\geqq 1-\sum_{m=0}^{\infty}\frac{1-p_n}{(1-a)^2p_n^2(2ap_n)^m} \geqq 1-\frac{2a}{(1-a)^2p(2ap-1)}(1-p_n),$$

for all $n \geqq N$, where N is taken such that $1-\frac{1-p_n}{(1-a)^2p_n^2}>0$ and $p_n \geqq p > 1/2a$ for all $n \geqq N$. In the above estimate we used the fact;

$$\text{(11)} \qquad \boldsymbol{P}_x(\boldsymbol{j}_{H_n}<e_\Delta=+\infty)=1 \qquad \text{for} \quad x \in R.$$

In conjunction with [6; Lemma 1, (i)], (11) follows from the assumption of non-explosion and the condition ($\boldsymbol{X}$-1) on $\boldsymbol{X}$.

Now set $B_n=\{e_\Delta=+\infty, \boldsymbol{j}_{H_n}<+\infty \text{ for all } n=1, 2,\dots\}\backslash A_n$. Since

$$\boldsymbol{P}_x(B_n) \leqq \boldsymbol{P}_x(\boldsymbol{j}_{H_n}<e_\Delta=+\infty)-\boldsymbol{P}_x(A_n)=1-\boldsymbol{P}_x(A_n)$$

$$\leqq \frac{2a}{(1-a)^2p(2ap-1)}\{3P(\inf_{t \leqq t_n/M_n} X_t<-l_n/M_n)+\exp(-k_nt_n/M_n)\},$$

the finiteness of J_1 and J_2 implies $\sum_n \boldsymbol{P}_x(B_n)<+\infty$. Hence by the Borel-Cantelli lemma we have the following assertion: There exists a random variable $\boldsymbol{n}$ taking finite values such that for all $n \geqq \boldsymbol{n}$, $\mathfrak{n}'_{n,0}=1$, $\mathfrak{n}'_{n,m} \geqq ap_n\mathfrak{n}_{n,m}$, $m=1,\dots, M_n-1$, a.s. ($\boldsymbol{P}_x$) on $\{e_\Delta=+\infty, \boldsymbol{j}_{H_n}<+\infty$ for all $n=1, 2,\dots\}$. This, combined with (7), implies

$$\text{(12)} \quad \boldsymbol{P}_x(\{e_\Delta=+\infty, \boldsymbol{j}_{H_n}<+\infty \text{ for all } n=1, 2,\dots\}\backslash \bigcup_{m=N}^{\infty}\bigcap_{n=m}^{\infty}\{\mathfrak{n}_n \geqq (2ap)^{M_n}\})=0,$$

$$\text{for all} \quad x \in R.$$

Third step. First we show that the assumption of non-explosion leads to the contradiction. (Refer the discussion in [6; Proof of Proposition 1].) Choose the sequences in (S-1)-(S-4) so that all of J_1, J_2 and J_3 are finite. By [6; Corollary 1]

$$\text{(13)} \qquad \sum_n(\boldsymbol{j}_{H_{n+1}}-\boldsymbol{j}_{H_n})=+\infty \text{ a.s. } (\boldsymbol{P}_x) \text{ on } \{e_\Delta=+\infty\}.$$

Suppose that $\boldsymbol{P}_x(e_\Delta=+\infty)=1$ for all $x \in R$, then

$$\boldsymbol{P}_x(\{e_\Delta=+\infty, \boldsymbol{j}_{H_n}<+\infty \text{ for all } n\} \cap \bigcap_{n=m}^{\infty}\{\mathfrak{n}_n \geqq (2ap)^{M_n}\})>0$$

for some m by (12). Hence by (13)

(14) $E_x(\{e_\Delta=+\infty, j_{H_n}<+\infty \text{ for all } n\} \cap \bigcap_{n=m}^{\infty}\{\text{n}_n \geqq (2ap)^{M_n}\}; \sum_{n=m}^{\infty}(j_{H_{n+1}}-j_{H_n}))$

$=+\infty.$

On the other hand, by a similar estimate to that of [6; Lemma 3] and by the finiteness of J_3, we obtain

the left hand side of (14)

$$\leqq \sum_{n=m}^{\infty}\int_0^{+\infty}\{P(\sup_{s\leqq t} X_s<2l_n+h_n)\}^{(2ap)^{M_n}}dt<+\infty.$$

This contradicts (14). Hence we conclude that there exists an $x_0 \in R$ such that $P_{x_0}(e_\Delta=+\infty)<1$.

Next we prove "explosive with probability one". Noting the spatial homogeneity of Lévy processes and the monotone non-decreasing property of the killing rate, we obtain from [6; Theorem 4] the next inequality

(15) $$P_x(e_\Delta=+\infty) \geqq P_y(e_\Delta=+\infty) \quad \text{for} \quad x \leqq y.$$

On the other hand if $x \leqq y$, we obtain by [6; Lemma 1, (i)] $P_x(e_\Delta=+\infty)=P_x(j_y<+\infty, e_\Delta=+\infty)$. Applying the strong Markov property on the right hand side, we obtain

(16) $$P_x(e_\Delta=+\infty)=E_x(j_y<e_\Delta; P_{X(j_y)}(e_\Delta=+\infty))$$

$$\leqq E_x(j_y<e_\Delta; P_{\Phi(X(j_y))}(e_\Delta=+\infty)) \leqq P_y(e_\Delta=+\infty).$$

Combining (15) and (16), we obtain

$$P_x(e_\Delta=+\infty)=c; \text{ constant for all } x \in R.$$

By a similar discussion as in [5; Lemma 6] we obtain $c=0$ or 1. Since $P_{x_0}(e_\Delta=+\infty)<1$, $c=0$, that is, the process is explosive with probability one. This completes the proof.

3. By [3] stable processes except those of indices $\{\alpha, -1\}$ with $0<\alpha<1$ satisfy ($\boldsymbol{X}$-1). Hence Proposition 2 is applicable for proof of the explosive part of Theorems 1 and 2.

Proof of Theorem 1. For proof of the explosive part set $H_n=\exp(n^\delta)$, $l_n=\exp(n^{\delta/2})$, $t_n=n^{-\mu}$ and $M_n=[n^\nu]$[5] in (S-1)-(S-4), where δ, μ and ν are constants satisfying

5) $[a]$ is the greatest integer not exceeding a.

(17) $$\delta \geqq 1,\ \mu>1 \quad \text{and} \quad \nu>0.$$

Then by a similar estimate as in [6; Lemma 6] the finiteness of J_1 automatically holds. For the finiteness of J_2 it is sufficient that $\sum_n \exp(-cn^{\gamma\delta-\mu-\nu})$ is finite, where c is a positive constant. For an estimate of J_3 we apply [6; Lemma 11]. Then for the finiteness of J_3 it is sufficient that $\sum_n \exp(2\alpha n^\delta - n^\nu \log A)$ is finite. Hence if we can find δ, μ and ν satisfying (17) and the next inequalities

(18) $$\gamma\delta-\mu-\nu>0 \quad \text{and} \quad \delta<\nu,$$

then the process is explosive with probability one. It has a solution if $\gamma>1$.

Proof of the non-explosive part is given in [6; Theorem 1].

Proof of Theorem 2. For proof of the explosive part set $H_n=n^\delta$, $l_n=h_n=H_{n+1}-H_n$, $t_n=n^{-\mu}$ and $M_n=[n^\nu]$, where δ, μ and ν are constants satisfying

(19) $$\delta>0,\ \mu>1 \quad \text{and} \quad \nu>0.$$

The rest of proof is similar to that of Theorem 1, so we omit it.

Proof of Theorem 3. For proof of the explosive part we apply Proposition 1 as follows. Set $H_n=n^\delta$, $t_n=n^{-1}(\log n)^{-\mu}$ and $N_n=[\exp\{n^{\delta-1}(\log n)^\nu\}]$, where δ, μ and ν are constants satisfying

(20) $$\delta>1 \quad \text{and} \quad \mu>1.$$

For the finiteness of I_2 it is sufficient that $\sum_n \exp(n^{\delta-1}\{(\log n)^\nu - c_1(\log n)^{\gamma-\mu}\})$ is finite, where c_1 is a positive constant.

Next let us estimate I_3.

$$\begin{aligned} P(X(t_n)\leqq h_n) &= 1-\frac{1}{[h_n]!}\int_0^{t_n} e^{-x}x^{[h_n]}dx \\ &\leqq 1-e^{-t_n}\frac{t_n^{([h_n]+1)}}{([h_n]+1)!}. \end{aligned}$$

Applying the Stirling's formula on the right hand side, we have for all sufficiently large n

$$\begin{aligned} P(X(t_n)\leqq h_n) \leqq 1-\frac{1}{3}\exp\{&-t_n+[h_n]+([h_n]+1)\log t_n \\ &-\left([h_n]+\frac{2}{3}\right)\log[h_n]\} \leqq 1-c_2\exp(-c_3 n^{\delta-1}\log n), \end{aligned}$$

where c_2 and c_3 are some positive constants. Then

$$\{P(X(t_n)\leqq h_n)\}^{N_n}\leqq\exp(-c_2\exp\{n^{\delta-1}((\log n)^{\nu}-c_3\log n)\}).$$

Hence if we can find δ, μ and ν satisfying (20) and the next inequalities

(21) $$\gamma-\mu>\nu \quad\text{and}\quad \nu>1,$$

then the process is explosive with probability one. It has a solution if $\gamma>2$.

Proof of the non-explosive part is quite similar to that of Proposition of [5], so we omit it.

Remark. When we apply Proposition 2 to branching Poisson process instead of Proposition 1, we obtain the following weaker result than that of Theorem 3: "*Let the killing rate* $k(x)\asymp x(\log x)^{\gamma}$ *as* $x\to+\infty$, *then the process is explosive with probability one if* $\gamma>3$."

DEPARTMENT OE MATHEMATICS,
UNIVERSITY OF TSUKUBA

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