J. Math. Kyoto Univ. (JMKYAZ)
17-2 (1977) 299-307

On submodules of a Verma module The case of $\mathfrak{sl}(4, \mathbf{C})$

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(Received May 27, 1976)

Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ the root system of $(\mathfrak{g}, \mathfrak{h})$. Denote by $\mathfrak{g}^{\alpha}$ the root space corresponding to a root α, then $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}^{\alpha}$. We fix a positive system of roots Δ_+ and denote by Δ_0 the set of simple roots. Put

$$\mathfrak{n}^+=\sum_{\alpha\in\Delta_+}\mathfrak{g}^{\alpha},\ \mathfrak{n}=\sum_{\alpha\in-\Delta_+}\mathfrak{g}^{\alpha},\ \rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha.$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. For any $\chi\in\mathfrak{h}^*=\mathrm{Hom}(\mathfrak{h}, \mathbf{C})$, we consider the factor space $M(\chi)=U(\mathfrak{g})/I_{\chi}$, where I_{χ} is the left ideal of $U(\mathfrak{g})$ generated by $\mathfrak{n}^+$ and $\{H-\chi(H)+\rho(H);\ H\in\mathfrak{h}\}$. Then $M(\chi)$ has the natural structure of $U(\mathfrak{g})$-module and is called the Verma module induced by χ. A nonzero element of a $U(\mathfrak{g})$-module is called extreme if it is annihilated by $\mathfrak{n}^+$.

D.-N. Verma proved in [1] that a submodule of $M(\chi)$ generated by its extreme vector is isomorphic to another Verma module $M(\chi')$. The submodules of this type are called here Verma submodules. He also got a sufficient condition on a pair (χ, χ') for $M(\chi)$ to contain a Verma submodule isomorphic to $M(\chi')$.

After that, I. N. Bernstein and others proved that this condition is also necessary [2]. So all the Verma submodules are already known.

In that work [2], they also constructed an example of submodules which are not generated by their extreme vectors. They treat there the case $\mathfrak{g}=\mathfrak{sl}(4, \mathbf{C})$ and χ is a certain weight ω (see §2). J. Dixmier and N. Conze gave a fundamental necessary condition for the existence of submodules which are not of Verma's type.

It is an interesting problem to determine the structure of the Verma module $M(\chi)$, and especially to find the submodules of $M(\chi)$, not of Verma's type.

But even for algebras of lower ranks, the complete solution is yet unknown.

In this note, we construct the submodules of non-Verma's type when $\mathfrak{g} = \mathfrak{sl}(4, \mathbf{C})$ and $\chi = n\omega$ for any positive integer n by a certain general method.

The author expresses his hearty thanks to Professors T. Hirai and N. Tatsuuma for their kind advices.

§1. Preliminaries

Let X_α, $X_{-\alpha}$ $(\alpha \in \Delta_+)$, H_{α_i} $(\alpha_i \in \Delta_0)$ be a Weyl basis normalized as follows:

1) $\alpha([X_\alpha, X_{-\alpha}]) = 2$,

2) $\alpha_i(H_{\alpha_j}) = 2<\alpha_i, \alpha_j>/<\alpha_j, \alpha_j>$, where $< , >$ is the inner product of $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$.

Denote by s_α the reflection corresponding to a root α and by W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. In the following we denote by $U(\mathfrak{p})$ the universal enveloping algebra of a Lie algebra $\mathfrak{p}$. For $\mu \in \mathfrak{h}^*$ $(= \mathrm{Hom}(\mathfrak{h}, \mathbf{C}))$, we put

$$M(\chi|\mu) = \{v \in M(\chi);\ Hv = \mu(H)v\ (H \in \mathfrak{h})\},$$

$$U(\mathfrak{n}|\mu) = \{v \in U(\mathfrak{n});\ \mathrm{ad}(H)v = \mu(H)v\ (H \in \mathfrak{h})\}.$$

They are called weight subspaces corresponding to a weight μ.

Then $M(\chi)$ (resp. $U(\mathfrak{n})$) is expressed as a direct sum of weight subspaces as

$$M(\chi) = \textstyle\sum_{\mu \in \chi - \Gamma_+} M(\chi|\mu - \rho)$$

$$(\text{resp. } U(\mathfrak{n}) = \textstyle\sum_{\mu \in \Gamma_+} U(\mathfrak{n}|-\mu)),$$

where Γ_+ is the set of all non-negative integral linear combinations of Δ_0. Further $M(\chi)$ is isomorphic to $U(\mathfrak{n})$ as a vector space, and as a $U(\mathfrak{h})$-module $M(\chi|\mu-\rho)$ is isomorphic to $U(\mathfrak{n}|-\chi+\mu)$.

For our later use, we quote here the following known facts in the form of two theorems (see [1], [2], [3] and [4]).

Theorem A ([1], [2], [4]). *Let χ and ψ be two elements of $\mathfrak{h}^*$, then the following properties hold.*

1) $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{g}}(M(\chi), M(\psi)) = 0$ *or* 1.

2) *Every non-zero element of* $\mathrm{Hom}_{\mathfrak{g}}(M(\chi), M(\psi))$ *is an embedding.*

3) $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{g}}(M(\chi), M(\psi)) = 1$ *if and only if there exists a sequence $\gamma_1, \dots, \gamma_k$ of positive roots satisfying the following conditions: put $\psi^{(0)} = \psi$ and $\psi^{(j)} = s_{\gamma_j} \cdots s_{\gamma_1}\psi$ $(1 \leqq j \leqq k)$, then*

a) $\chi=\psi^{(k)}$, *b)* $\psi^{(i-1)}(H_{\gamma_i})$ *is a positive integer for any* i.

4) *Each irreducible sub-quotient module of* $M(\chi)$ *has a highest weight* $\mu-\rho$ *with* $\mu\in W\chi\cap\chi-\Gamma_+$.

Theorem B [3]. *If* $M(\chi)$ *possesses a submodule which is not generated by its Verma submodules, then there exists three different elements* ξ, η, ζ *of* $W\chi$ *such that*

1) $M(\chi)\supsetneqq M(\xi)\supsetneqq M(\zeta)$, $M(\chi)\supsetneqq M(\eta)\supsetneqq M(\zeta)$.

2) $\eta-\xi\in\Gamma_+\backslash\{0\}$

(*Let* M *be such a submodule and* N *be the submodule generated by all its Verma submodules. Then the element* ξ *in the above theorem was chosen in* [3] *in such a way that putting* $\Xi=\{\mu\in\mathfrak{h}^*;\ M(\chi|\mu-\rho)\cap M\supsetneqq M(\chi|\mu-\rho)\cap N\}$, *we have* $\xi\in\Xi$ *and* $\mu-\xi\notin\Gamma_+$ *for any* $\mu\in\Xi$.)

§2. The Verma module $M(\chi_n)$ of $\mathfrak{sl}(4, \mathbf{C})$

Put $\mathfrak{g}=\mathfrak{sl}(4, \mathbf{C})$ and denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices. Then the set Δ_0 consists of three roots $\alpha_1, \alpha_2, \alpha_3$, and with suitable numbering $\Delta_+=\{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$ and its Dynkin diagram is given by

$$\underset{\alpha_1}{\circ}\!-\!\!-\!\!-\underset{\alpha_2}{\circ}\!-\!\!-\!\!-\underset{\alpha_3}{\circ}\qquad .$$

Define a weight ω by

$$\omega(H_{\alpha_1})=\omega(H_{\alpha_3})=0, \quad \omega(H_{\alpha_2})=1$$

and put $\chi_n=n\omega$ $(n=1, 2,\dots)$. We study the Verma module $M(\chi_n)$. By Theorem A, all the Verma submodules are given as follows:

$$M(s_{\alpha_2}\chi_n)=M(\chi_n-n\alpha_2)=U(\mathfrak{g})X^n_{-\alpha_2},$$

$$M(s_{\alpha_1}s_{\alpha_2}\chi_n)=M(\chi_n-n\alpha_1-n\alpha_2)=U(\mathfrak{g})X^n_{-\alpha_1}X^n_{-\alpha_2},$$

$$M(s_{\alpha_3}s_{\alpha_2}\chi_n)=M(\chi_n-n\alpha_3-n\alpha_2)=U(\mathfrak{g})X^n_{-\alpha_3}X^n_{-\alpha_2},$$

$$M(s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\chi_n)=M(\chi_n-n\alpha_1-n\alpha_2-n\alpha_3)$$

$$=U(\mathfrak{g})X^n_{-\alpha_1}X^n_{-\alpha_3}X^n_{-\alpha_2},$$

$$M(s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\chi_n)=M(\chi_n-n\alpha_1-2n\alpha_2-n\alpha_3)$$
$$=U(\mathfrak{g})X^n_{-\alpha_2}X^n_{-\alpha_1}X^n_{-\alpha_3}X^n_{-\alpha_2}.$$

I. N. Bernstein and others constructed a submodule of $M(\chi_1)$ which is not generated by its Verma submodules [2]. In this note we construct such a submodule of $M(\chi_n)$.

Lemma 1. *In $M(\chi_n)$, if the situation in Theorem B occurs, the element ξ must be equal to $\chi_n-n\alpha_1-n\alpha_2-n\alpha_3$.*

Proof. Let M be such a submodule of $M(\chi_n)$ and N be the submodule generated by all its Verma submodules. We see easily that ξ is one of the following three elements:

$$\chi_n-n\alpha_1-n\alpha_2,\quad \chi_n-n\alpha_2-n\alpha_3,\quad \chi_n-n\alpha_1-n\alpha_2-n\alpha_3.$$

Choose a weight vector x of weight $\xi-\rho$ in M which does not belong to N.

Suppose $\xi=\chi_n-n\alpha_1-n\alpha_2$. Let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$ generated by $X_{\pm\alpha_1}$, $X_{\pm\alpha_2}$, and W' a subgroup of W generated by s_{α_1} and s_{α_2}. Then $\mathfrak{g}'$ is isomorphic to $\mathfrak{sl}(3,\mathbf{C})$ and W' is the Weyl group of $\mathfrak{g}'$. Since $X_{\alpha_3}X_{-\alpha_i}=X_{-\alpha_i}X_{\alpha_3}$ $(i=1, 2)$, $X_{\alpha_3}x=0$ and $U(\mathfrak{g}')X^n_{-\alpha_2}\subsetneqq U(\mathfrak{g}')x\subsetneqq U(\mathfrak{g}')1$. This fact means that $\mathfrak{sl}(3,\mathbf{C})$ has a submodule which is not of Verma's type. But the number of the elements $W'\chi_n$ is three. This contradicts the assertion of Theorem B. Similarly we have $\xi\neq\chi_n-n\alpha_2-n\alpha_3$. So we get our assertion. Q.E.D.

Put $\tilde{M}=M(\chi_n)/M(s_{\alpha_2}\chi_n)$, and denote by $\tilde{x}$ the image of $x\in M(\chi_n)$ under the canonical map.

Lemma 2. *Let $\xi=\chi_n-n\alpha_1-n\alpha_2-n\alpha_3-\rho$. For any $x\in M(\chi_n|\xi)$, $\notin M(s_{\alpha_2}\chi_n)$, there exist only the following two possibilities:*

1) *$\tilde{x}$ is an extreme vector,*

2) *there exists a sequence of simple roots $\beta_1,\dots,\beta_{3n}$ such that $X_{\beta_{3n}}\cdots X_{\beta_1}\tilde{x}$ $=\text{const.}\,\tilde{1}$.*

Proof. Assume that $\tilde{x}$ is not an extreme vector in $\tilde{M}$. Then there exists a sequence $\beta_1,\dots,\beta_k$ of simple roots such that $X_{\beta_k}\cdots X_{\beta_1}\tilde{x}$ is an extreme one. By Theorem A, $\beta_1+\cdots+\beta_k$ must be equal to one of the followings:

$$n\alpha_1,\ n\alpha_3,\ n\alpha_1+n\alpha_3,\ n\alpha_1+n\alpha_2+n\alpha_3.$$

First suppose $\beta_1+\cdots+\beta_k=n\alpha_3$. Put $y=X^n_{\alpha_3}x$. Then both $X_{\alpha_1}y$ and $X_{\alpha_2}y$ belong to $M(s_{\alpha_2}\chi_n)$. By the same way as in the proof of Lemma 1, the $U(\mathfrak{g}')$-

module generated by $X^n_{-\alpha_2}$ and y cannot be generated by its Verma submodules. This contradicts Theorem B. Therefore $\beta_1+\cdots+\beta_k \neq n\alpha_3$. Similarly $\beta_1+\cdots+\beta_k \neq n\alpha_1$.

Next suppose $\beta_1+\cdots+\beta_k=n\alpha_1+n\alpha_3$. Put $y=X_{\beta_k}\cdots X_{\beta_1}x=X^n_{\alpha_1}X^n_{\alpha_3}x$. Then $y\in M(\chi_n|\chi_n-n\alpha_2-\rho)$, $\notin M(s_{\alpha_2}\chi_n)$. But $M(\chi_n|\chi_n-n\alpha_2-\rho)=\mathbf{C}X^n_{-\alpha_2}\subset M(s_{\alpha_2}\chi_n)$. This is a contradiction. Q.E.D.

As a result of this lemma, our problem is reduced to finding an element x such that $\tilde{x}$ is extreme.

§3. Basic relations in an enveloping algebra

In this section, we prepare two lemmas. We consider the mapping of $\mathfrak{g}$ given by $\iota(X_{\pm\alpha})=X_{\mp w\alpha}$, $\iota(H_\alpha)=w(H_\alpha)$ $(\alpha\in\Delta)$, where $w=s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$. Then ι can be uniquely extended to an anti-automorphism of $U(\mathfrak{g})$ which is denoted by ι again. Note that ι maps $\sum_{n_1,n_2\in\mathbf{N}}U(\mathfrak{n}|-n_1\alpha_1-n_2\alpha_2)$ into itself.

Lemma 3. *The map ι is an identity on $U(\mathfrak{n}|-n\alpha_1-n\alpha_2)$ for any positive integer n.*

Proof. This is proved by induction on n. The assertion holds for $n=1$, because a basis of $U(\mathfrak{n}|-\alpha_1-\alpha_2)$ is given by $X_{-\alpha_1}X_{-\alpha_2}$, $X_{-\alpha_2}X_{-\alpha_1}$ and $\iota(X_{-\alpha_1})=X_{-\alpha_2}$, $\iota(X_{-\alpha_2})=X_{-\alpha_1}$.

Suppose $n>1$. Let $x=X_{-\beta_1}\cdots X_{-\beta_{2n}}$ where β_i's are simple roots and $\sum\beta_i=n(\alpha_1+\alpha_2)$. If $\beta_1\neq\beta_{2n}$, then

$$\begin{aligned}\iota(X_{-\beta_1}X_{-\beta_2}\cdots X_{-\beta_{2n-1}}X_{-\beta_{2n}})&=\iota(X_{-\beta_{2n}})\iota(X_{-\beta_2}\cdots X_{-\beta_{2n-1}})\iota(X_{-\beta_1})\\&=X_{-\beta_1}(X_{-\beta_2}\cdots X_{-\beta_{2n-1}})X_{-\beta_{2n}}\\&=x\end{aligned}$$

When $\beta_1=\beta_{2n}=\alpha_1$, x can be written as a linear combination of $X_{-\alpha_1}yX^2_{-\alpha_2}X_{-\alpha_1}$'s, where $y\in U(\mathfrak{n}|-(n-2)(\alpha_1+\alpha_2))$. In fact, let P be a Kostant's partition function (see [1]), then

$$\begin{aligned}\dim U(\mathfrak{n}|-(n-2)\alpha_1-n\alpha_2)&=P((n-2)\alpha_1+n\alpha_2)\\&=n-1\\&=P((n-2)(\alpha_1+\alpha_2))\\&=\dim U(\mathfrak{n}|-(n-2)(\alpha_1+\alpha_2)).\end{aligned}$$

Therefore, let $y_1, y_2, \dots, y_{n-1}$ be a basis of $U(\mathfrak{n}|-(n-2)(\alpha_1+\alpha_2))$, then $y_1X^2_{-\alpha_2}$, $y_2X^2_{-\alpha_2}, \dots, y_{n-1}X^2_{-\alpha_2}$ are mutually linearly independent. Hence they form a basis of $U(\mathfrak{n}|-(n-2)\alpha_1-n\alpha_2)$. On the other hand, we have the following equality in $U(\mathfrak{n})$:

$$X^2_{-\alpha_2}X_{-\alpha_1}=-X_{-\alpha_1}X^2_{-\alpha_2}+2X_{-\alpha_2}X_{-\alpha_1}X_{-\alpha_2}.$$

Therefore this case can be reduced to the previous one. Q.E.D.

Let us consider the right ideal I'_χ generated by $\iota(\mathfrak{n}^+)$ and $\{H-w\chi(H)+w\rho(H);\ H\in\mathfrak{h}\}$. Then $I'_\chi=\iota(I_\chi)$ and $U(\mathfrak{n})\oplus I_\chi=U(\mathfrak{g})=I'_\chi\oplus U(\iota(\mathfrak{n}))$. Denote by P_l (resp. P_r) the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{n})$ (resp. onto $U(\iota(n))$) according to the above direct sum decomposition. Then

$$P_l(X_{\alpha_2}x)=\iota(P_r(\iota(x)X_{\alpha_1}) \qquad (x\in U(\mathfrak{g})).$$

Moreover there holds the following useful equality.

Lemma 4. *Let x be an element of $U(\mathfrak{n}|-n\alpha_1-n\alpha_2)$, then*

$$P_l(X_{\alpha_2}x)=-\iota(P_l(X_{\alpha_1}x)).$$

Proof. Let $x=X_{-\beta_1}\cdots X_{-\beta_{2n}}$ $(\beta_k=\alpha_1$ or $\alpha_2)$, then

$$P_l(X_{\alpha_1}x)=P_l(\textstyle\sum_{\beta_k=\alpha_1}X_{-\beta_1}\cdots[X_{\alpha_1}, X_{-\beta_k}]\cdots X_{-\beta_{2n}})$$

$$=P_l(\textstyle\sum X_{-\beta_1}\cdots\check{X}_{-\beta_k}\cdots X_{-\beta_{2n}}(H_{\alpha_1}-(\beta_{k+1}+\cdots+\beta_{2n})(H_{\alpha_1}))$$

$$=\textstyle\sum X_{-\beta_1}\cdots\check{X}_{-\beta_k}\cdots X_{-\beta_{2n}}(-1-(\beta_{k+1}+\cdots+\beta_{2n})(H_{\alpha_1})).$$

Here $\check{X}$ means that X is absent. On the other hand,

$$P_l(X_{\alpha_2}x)=\iota(P_r(\iota(x)X_{\alpha_1}))=\iota(P_r(xX_{\alpha_1}))$$

$$=\iota(P_r(\textstyle\sum_{\beta_k=\alpha_1}X_{-\beta_1}\cdots[X_{-\beta_k}, X_{\alpha_1}]\cdots X_{-\beta_{2n}})$$

$$=\iota(P_r(\textstyle\sum(-H_{\alpha_1}+(\beta_1+\cdots+\beta_{k-1})(-H_{\alpha_1})X_{-\beta_1}\cdots\check{X}_{-\beta_k}\cdots X_{-\beta_{2n}}))$$

$$=\textstyle\sum(n-1-(n(\alpha_1+\alpha_2)-\beta_{k+1}-\cdots-\beta_{2n}-\alpha_1)(H_{\alpha_1}))$$

$$\times\,\iota(X_{-\beta_1}\cdots\check{X}_{-\beta_k}\cdots X_{-\beta_{2n}})$$

$$=\textstyle\sum(1+(\beta_{k+1}+\cdots+\beta_{2n})(H_{\alpha_1}))\iota(X_{-\beta_1}\cdots\check{X}_{-\beta_k}\cdots X_{-\beta_{2n}})$$

$$= -\iota(P_l(X_{\alpha_1}x)).$$

Every element of $U(\mathfrak{n}|-n\alpha_1-n\alpha_2)$ is a linear combination of the above monomials. Therefore we get our assertion. Q.E.D.

§4. The construction of extreme vector in $\tilde{M}$

Let $x \in M(\chi_n|\chi_n-n\alpha_1-n\alpha_2-\rho)$ $(\cong U(\mathfrak{n}|-n\alpha_1-n\alpha_2))$. By Lemma 4 we see that $X_{\alpha_1}x=0 \Leftrightarrow X_{\alpha_2}x=0$. Therefore if $X_{\alpha_1}x=0$, x is an extreme vector. We see in §2 that $X^n_{-\alpha_1}X^n_{-\alpha_2}$ is a unique extreme vector in $M(\chi_n|\chi_n-n\alpha_1-n\alpha_2-\rho)$. On the other hand,

$$\dim U(\mathfrak{n}|-n\alpha_1-n\alpha_2)-1=(n+1)-1$$

$$=\dim U(\mathfrak{n}|-(n-1)\alpha_1-n\alpha_2)$$

$$=\dim M(\chi_n|\chi_n-(n-1)\alpha_1-n\alpha_2-\rho).$$

We can choose x_1 from $M(\chi_n|\chi_n-n\alpha_1-n\alpha_2-\rho)$ in such a way that $X_{\alpha_1}x_1 =X^{n-1}_{-\alpha_1}X^{n}_{-\alpha_2}$. Then by Lemma 4,

$$X_{\alpha_2}x_1=-\iota(X^{n-1}_{-\alpha_1}X^n_{-\alpha_2})=-X^n_{-\alpha_1}X^{n-1}_{-\alpha_2}.$$

Replacing α_1 with α_3, we can take from $M(\chi_n|\chi_n-n\alpha_2-n\alpha_3-\rho)$ an element x_3 such that

$$X_{\alpha_3}x_3=X^{n-1}_{-\alpha_3}X^n_{-\alpha_2}, \quad X_{\alpha_2}x_3=-X^n_{-\alpha_3}X^{n-1}_{-\alpha_2}.$$

For any positive integer n, we define a submodule $\overline{M}$ of $M(\chi_n)$ as follows:

$$\text{put} \quad z=X^n_{-\alpha_1}x_3-X^n_{-\alpha_3}x_1 \quad \text{and} \quad \overline{M}=M(s_{\alpha_2}\chi_n)+U(\mathfrak{g})z.$$

Theorem. *For any positive integer n, the submodule $\overline{M}$ of $M(\chi_n)$ is not generated by its Verma submodules.*

Proof. We see that,

$$X_{\alpha_1}z=X_{\alpha_1}X^n_{-\alpha_1}x_3-X^n_{-\alpha_3}X_{\alpha_1}x_1$$

$$=nX^{n-1}_{-\alpha_1}(H_{\alpha_1}-(n-1))x_3-X^n_{-\alpha_3}X^{n-1}_{-\alpha_1}X^n_{-\alpha_2}$$

$$=nX^{n-1}_{-\alpha_1}x_3(H_{\alpha_1}-n(\alpha_2+\alpha_3)(H_{\alpha_1})-(n-1))-X^n_{-\alpha_3}X^{n-1}_{-\alpha_1}X^n_{-\alpha_2}$$

$$=-X^n_{-\alpha_3}X^{n-1}_{-\alpha_1}X^n_{-\alpha_2}\in M(s_{\alpha_2}\chi_n).$$

Similarly,

$$X_{\alpha_3}z = X^n_{-\alpha_1}X^{n-1}_{-\alpha_3}X^n_{-\alpha_2} \in M(s_{\alpha_2}\chi_n).$$

Further we get

$$\begin{aligned} X_{\alpha_2}z &= X^n_{-\alpha_1}X_{\alpha_2}x_3 - X^n_{-\alpha_3}X_{\alpha_2}x_1 \\ &= -X^n_{-\alpha_1}X^n_{-\alpha_3}X^{n-1}_{-\alpha_2} + X^n_{-\alpha_3}X^n_{-\alpha_1}X^{n-1}_{-\alpha_2} \\ &= 0 \qquad (\because \quad X_{-\alpha_1}X_{-\alpha_3} = X_{-\alpha_3}X_{-\alpha_1}). \end{aligned}$$

So z is not an extreme vector in $M(\chi_n)$. Since

$$\begin{aligned} &\dim M(s_{\alpha_2}\chi_n|\chi_n - n\alpha_1 - n\alpha_2 - n\alpha_3 - \rho) \\ &= P(n\alpha_1 + n\alpha_3) = 1, \end{aligned}$$

and $X^n_{-\alpha_1}X^n_{-\alpha_3}X^n_{-\alpha_2}$ is an extreme vector in $M(s_{\alpha_2}\chi_n)$, z does not belong to $M(s_{\alpha_2}\chi_n)$.

Note that $\bar{M}$ is a proper submodule of $M(\chi_n)$ and contains $M(s_{\alpha_2}\chi_n)$ as its proper submodule. This submodule cannot be generated by its Verma submodules, because every proper Verma submodule is contained in $M(s_{\alpha_2}\chi_n)$. Thus we get our results. Q. E. D.

Remark 1. The existence of such a submodule means that the generalized Verma modules considered by M. Duflo and N. Conze in [5] are reducible in some cases.

Remark 2. In the case of $n=1$ or 2, z is written explicitly as follows (modulo $X^n_{-\alpha_1}X^n_{-\alpha_3}X^n_{-\alpha_2}$):

for $n=1$, $X_{-\alpha_1}X_{-\alpha_2}X_{-\alpha_3} - X_{-\alpha_3}X_{-\alpha_2}X_{-\alpha_1}$,

for $n=2$, $X^2_{-\alpha_1}(X_{-\alpha_2}X_{-\alpha_3}X_{-\alpha_2}X_{-\alpha_3} - 5X_{-\alpha_3}X_{-\alpha_2}X_{-\alpha_3}X_{-\alpha_2})$

$$-X^2_{-\alpha_3}(X_{-\alpha_2}X_{-\alpha_1}X_{-\alpha_2}X_{-\alpha_1} - 5X_{-\alpha_1}X_{-\alpha_2}X_{-\alpha_1}X_{-\alpha_2}).$$

In these two cases, we can prove by an explicit calculation that $\bar{M}$ and the known Verma submodules give essentially a complete Jordan-Hörder sequence of $M(\chi_n)$.

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