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Some remarks on the Cauchy problem for weakly hyperbolic equations

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1. Introduction

In this paper we consider the well-posedness of the Cauchy problem for the weakly hyperbolic equation

$$P\left(x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)u(x, t) = f(x, t) \tag{1.1}$$

$$\left(\frac{\partial}{\partial t}\right)^j u(x, 0) = u_j(x) \qquad j=0, 1,\dots, m-1\,. \tag{1.2}$$

Recently A. Menikoff [7] gave a sufficient condition for the well-posedness in the case where the characteristic roots are distinct for $t>0$, but some roots become double when $t=0$.

Here using the same reduction of the problem to a first order system as used by Mizohata and Ohya in [6], we want to point out that the assumption of the distinctness of the characteristic roots for $t>0$ can be removed and that we shall give the proof concerning the existence of the dependence domain.

Let $P\left(x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$ be a differential polynomial of order m defined in $\Omega=\{(x, t); x\in R^l, t\in[0, T]\}$ and we assume that the coefficients are infinitely differentiable and bounded as well as all their derivatives.

We denote

$$D=\left(\frac{1}{i}\frac{\partial}{\partial x_1},\dots, \frac{1}{i}\frac{\partial}{\partial x_l}\right) \qquad D_t=\frac{1}{i}\frac{\partial}{\partial t}$$

and rewrite the principal part P_m of P in the form

$$P_m\left(x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)=i^m\{D_t^m+h_1(x, t; D)D_t^{m-1}+\cdots+h_m(x, t; D)\}\,.$$

Let $(s \leq m-s)$

$$\tau^m + h_1(x, t; \xi)\tau^{m-1} + \cdots + h_m(x, t; \xi) = \prod_{i=1}^{s} (\tau - \mu_i(x, t; \xi)) \times \prod_{i=1}^{m-s} (\tau - \lambda_i(x, t; \xi)), \tag{1.3}$$

where we assume for λ_i, μ_j the following conditions through this paper except for §6,

$$\inf_{i \neq j, (x,t) \in \Omega, |\xi|=1} |\mu_i(x, t; \xi) - \mu_j(x, t; \xi)| \geq c_1(>0). \tag{1.4}$$

$$\inf_{i \neq j, (x,t) \in \Omega, |\xi|=1} |\lambda_i(x, t; \xi) - \lambda_j(x, t; \xi)| \geq c_2(>0).$$

$$\inf_{i \neq j, (x,t) \in \Omega, |\xi|=1} |\mu_i(x, t; \xi) - \lambda_j(x, t; \xi)| \geq c_3(>0). \tag{1.5}$$

For pairs (λ_i, μ_i), we assume the condition A or B which will be indicated later, then under these conditions we have theorems 1, 2, and 4.

Let

$$\partial_i = D_t - \lambda_i(x, t; D), \qquad i = 1, 2, \dots, m-s. \tag{1.6}$$

$$\Delta_i = D_t - \mu_i(x, t; D), \qquad i = 1, 2, \dots, s,$$

where $\lambda_i(x, t; D)$ is the pseudo-differential operator with symbol $\lambda_i(x, t; \xi)$.

Consider

$$\Pi_m = (\partial_{m-s} \cdots \partial_s \cdots \partial_1)(\Delta_s \cdots \Delta_1) \tag{1.7}$$

and put

$$i^{m-1} C_{m-1}(x, t; D, D_t) = i^m \{P_m(x, t; D, D_t) - \Pi_m(x, t; D, D_t)\} + i^{m-1} P_{m-1}(x, t; D, D_t). \tag{1.8}$$

Now we take for basis m operators,

$$1, \Delta_1, \Delta_2\Delta_1, \dots, \Delta_s\Delta_{s-1}\cdots\Delta_1, \tag{1.9}$$

$$\partial_1 \Delta_s \cdots \Delta_1, \dots, \partial_{m-s-1} \cdots \partial_1 \Delta_s \cdots \Delta_1.$$

Then we can represent C_{m-1} in the form

$$(1.10)\quad C_{m-1}(x, t; D, D_t)=c_{m-1}(x, t; D)+c_{m-2}(x, t; D)\Delta_1+\cdots+c_{m-s}\Delta_{s-1}\cdots\Delta_1+\cdots+c_0\partial_{m-s-1}\cdots\partial_1\Delta_s\cdots\Delta_1+Q_{m-2}(x, t; D, D_t),$$

where $c_{m-i}(x, t; D)$ $(i=1,\dots, m)$ is the pseudo-differential operator with symbol $c_{m-i}(x, t; \xi)$ which is homogeneous of degree $m-i$ in ξ, and Q_{m-2} is of order $m-2$.

Let $C^0_{m-1}(x, t; \xi, \tau)$ denote the principal symbol of $C_{m-1}(x, t; D, D_t)$ and introduce the following two conditions;

Condition A. For each i $(1\le i\le s)$, $C^0_{m-1}(x, t; \xi, \mu_i)$ can be represented in the form

$$C^0_{m-1}(x, t; \xi, \mu_i(x, t; \xi))=(\lambda_i(x, t; \xi)-\mu_i(x, t; \xi))T_i(x, t; \xi),$$

where $T_i(x, t; \xi)$ is a symbol of some pseudo-differential operator.

Condition B. For each i $(1\le i\le s)$, $tC^0_{m-1}(x, t; \xi, \mu_i)$ can be represented in the form $tC^0_{m-1}(x, t; \xi, \mu_i(x, t; \xi))=(\lambda_i(x, t; \xi)-\mu_i(x, t; \xi))S_i(x, t; \xi)$, where $S_i(x, t; \xi)$ is a symbol of some pseudo-differential operator.

Then it follows

Theorem 1. *Under the condition A, for any given initial data and the second term such that $(u_0(x),\dots, u_{m-1}(x))\in\mathscr{D}^{m+p}_{L^2}\times\mathscr{D}^{m+p-1}_{L^2}\times\cdots\times\mathscr{D}^{p+1}_{L^2}$, $f(x, t)\in\mathscr{E}^0_t(\mathscr{D}^{p+1}_{L^2})$, there exists a unique solution $u(x, t)$ of (1.1)–(1.2) such that*

$$\left(u(x, t),\dots, \left(\frac{\partial}{\partial t}\right)^{m-1}u(x, t)\right)\in\mathscr{E}^0_t(\mathscr{D}^{m+p-1}_{L^2}\times\cdots\times\mathscr{D}^p_{L^2})\quad (p=0, 1,\dots).$$

Theorem 2. *Suppose the condition B, then for any given initial data $(u_0(x),\dots, u_{m-1}(x))\in\Pi\mathscr{D}$ and the second term $f(x, t)\in\mathscr{D}_{x,t}$, the Cauchy problem (1.1)–(1.2) has a unique solution $u(x, t)\in\mathscr{E}_{x,t}$.*

2. Proof of theorem 1

Let

$$(2.1)\quad \tilde{C}_{m-1}(x, t; D, D_t)=c_{m-1}(x, t; D)+c_{m-2}(x, t; D)\Delta_1+\cdots+c_0(x, t; D)\partial_{m-s-1}\cdots\partial_1\Delta_s\cdots\Delta_1$$

then taking account of (1.8), the equation

$$P\left(x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)u(x, t)=f(x, t)$$

becomes

$$(2.2)\qquad (i^m\Pi_m - i^{m-1}\tilde{C}_{m-1})u + R_{m-2}u = f,$$

where R_{m-2} is of order $m-2$. At first we consider the equation

$$(2.3)\qquad (\Pi_m - i\tilde{C}_{m-1})u(x, t) = i^{-m}f(x, t).$$

Put

$$(2.4)\qquad (u, \Delta_1 u, \Delta_2\Delta_1 u,\dots, \partial_{m-s-1}\cdots\partial_1\Delta_s\cdots\Delta_1 u) = (u_0, u_1,\dots, u_{m-1})$$

and denote $U = {}^t(u_0, u_1,\dots, u_{m-1})$, then the equation (2.3) can be expressed as the following matrix form

$$(3.5)\qquad D_t U(x, t) = H(x, t; D)U(x, t) + F(x, t)$$

$$H(x, t; \xi) = \left(\begin{array}{cccc:cccc} \mu_1 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \mu_s & 1 & & & \\ \hdashline & & & & \lambda_1 & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & \lambda_s & \ddots & \\ & & & & & & \ddots & 1 \\ ic_{m-1} & \cdots & & ic_{m-s} & ic_{m-s-1} & \cdots & & ic_0 + \lambda_{m-s} \end{array}\right)$$

where $\lambda_i = \lambda_i(x, t; \xi)$ (homogeneous of degree i in ξ), $c_{m-i} = c_{m-i}(x, t; \xi)$ (homogeneous of degree $m-i$ in ξ) and $F = {}^t(0, 0,\dots, 0, i^{-m}f)$.

Next we set (see Mizohata and Ohya [6])

$$(2.6)\qquad V(x, t) = {}^t((\Lambda+1)^{m-2}u_0,\dots, (\Lambda+1)^{m-s-1}u_{s-1}, (\Lambda+1)^{m-s-1}u_s,\dots, u_{m-1}),$$

then (2.5) becomes

$$(2.7)\qquad D_t V(x, t) = H_0(x, t; D)\Lambda V(x, t) + B(x, t; D)V(x, t) + F(x, t)$$

$$H_0(x, t; \xi) = \left(\begin{array}{cccc:cccc} \mu_1 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \mu_s & 0 & & & \\ \hdashline & & & & \lambda_1 & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & \lambda_s & \ddots & \\ & & & & & & \ddots & 1 \\ ic_{m-1} & \cdots & & ic_{m-s} & & & & \lambda_{m-s} \end{array}\right) = \begin{bmatrix} A_1 & 0 \\ M & A_2 \end{bmatrix}$$

where $\lambda_i=\lambda_i(x, t; \xi/|\xi|)$, $\mu_i=\mu_i(x, t; \xi/|\xi|)$, $c_i=c_i(x, t; \xi/|\xi|)$, and B is a bounded operator in $(L^2(R^l))^m$.

Construct N, a matrix which diagonalizes $H_0(x, t; \xi)$, namely

$$N=\begin{bmatrix} N_1 & 0 \\ Q & N_2 \end{bmatrix} \qquad N^{-1}=\begin{bmatrix} N_1^{-1} & 0 \\ R & N_2^{-1} \end{bmatrix}$$

where N_1, N_2 is the diagonalizer of A_1, A_2 respectively, and $\det N_1=\det N_2=1$.

Note $Q=(q_{ij})_{\substack{1\leq i\leq m-s \\ 1\leq j\leq s}}$, then q_{ij} must satisfy

$$(2.8)\qquad (\lambda_i-\mu_1)q_{i1}=\frac{ic_{m-1}}{(\mu_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_{m-s})}\qquad i=1, 2,\dots, m-s\,.$$

$$(2.9)\qquad (\lambda_i-\mu_j)q_{ij}=\frac{ic_{m-j}}{(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_{m-s})}+q_{ij-1}\qquad \begin{array}{l} i=1, 2,\dots, m-s\,. \\ j=2, 3,\dots, s\,. \end{array}$$

By using the condition A and the equality

$$C^0_{m-1}(x, t; \xi, \mu_i)=c_{m-1}+c_{m-2}(\mu_i-\mu_1)+\cdots+c_{m-i}(\mu_i-\mu_{i-1})\cdots(\mu_i-\mu_1)$$

we can construct q_{ij} smoothly and accordingly $N(x, t; \xi)$ in such a way that $\det N(x, t; \xi)=1$.

Since $H_0(x, t; D)$ is diagonalizable, then, as well-known, for any given $V(x, 0)\in\mathscr{D}^k_{L^2}$ and the second term $F(x, t)\in\mathscr{E}^0_t(\mathscr{D}^k_{L^2})$, there exists a unique solution $V(x, t)\in\mathscr{E}^0_t(\mathscr{D}^k_{L^2})\cap\mathscr{E}^1_t(\mathscr{D}^{k-1}_{L^2})$ of (2.7) and we have

$$(2.10)\qquad \|V(t)\|_k\leq C(T, k)\Big[\|V(0)\|_k+\int_0^t\|F(s)\|_k ds\Big]\qquad (k=1, 2,\dots).$$

Introduce the following norm

$$(2.11)\quad |||u(x, t)|||_l^2=\|u(x, t)\|^2_{m-1+l}+\Big\|\frac{\partial}{\partial t}u(x, t)\Big\|^2_{m-2+l}+\cdots+\Big\|\Big(\frac{\partial}{\partial t}\Big)^{m-1}u(x, t)\Big\|^2_l$$

and consider the case $k=1$, for instance. Then by (2.11), the inequality

$$|||u(x, t)|||_0\leq\tilde{C}(T)\Big[|||(\Lambda+1)u(x, 0)|||_0+\int_0^t\|f(x, s)\|_1 ds\Big]$$

holds, and this inequality enables us to solve the Cauchy problem for (2.2) by successive approximation, for we have

$$\|R_{m-2}u(x, t)\|\leq C'\,|||u(x, t)|||_0\,.$$

In addition, $U\in\mathscr{E}_t^0(\mathscr{D}_{L^2}^{m-1}\times\cdots\times\mathscr{D}_{L^2}^{m-s}\times\mathscr{D}_{L^2}^{m-s}\times\cdots\times\mathscr{D}_{L^2}^1)\cap\mathscr{E}_t^1(\mathscr{D}_{L^2}^{m-2}\times\cdots\times\mathscr{D}_{L^2}^{m-s-1}\times\mathscr{D}_{L^2}^{m-s-1}\times\cdots\times L^2)$ implies at once

$$\left(u(x,t),\ \frac{\partial}{\partial t}u(x,t),\dots,\ \left(\frac{\partial}{\partial t}\right)^{m-1}u(x,t)\right)\in\mathscr{E}_t^0(\mathscr{D}_{L^2}^{m-1}\times\cdots\times L^2)\,.$$

3. Proof of theorem 2

Assume the condition B and consider the Cauchy problem

$$(3.1)\qquad P_\varepsilon\left(x,t;\frac{\partial}{\partial x},\ \frac{\partial}{\partial t}\right)u(x,t)=f(x,t)\ \ (x,t)\in\Omega'=\{(x,t);\,x\in R^l,\ t\in[0,\,T']\}\qquad (T'=T-\varepsilon_0)$$

$$(3.2)\qquad \left(\frac{\partial}{\partial t}\right)^j u(x,0)=u_j(x)\qquad j=0,\ 1.\ \dots.\ m-1\,,$$

where P_ε is a differential operator depending on ε (positive parameter) defined by

$$(3.3)\qquad P_\varepsilon\left(x,t;\frac{\partial}{\partial x},\ \frac{\partial}{\partial t}\right)=P\left(x,t+\varepsilon;\frac{\partial}{\partial x},\ \frac{\partial}{\partial t}\right)\qquad(0<\varepsilon<\varepsilon_0)\,.$$

Instead of the previous transform (2.6), let

$$(3.4)\qquad V_\varepsilon(x,t)={}^t((\Lambda+1)^{m-2}u_0,\dots,(\Lambda+1)^{m-s-1}u_{s-1},(t+\varepsilon)(\Lambda+1)^{m-s-1}u_s,\dots,(t+\varepsilon)u_{m-1})\,,$$

then the system (2.7) becomes

$$(3.5)\qquad D_tV_\varepsilon(x,t)=H_0^{(\varepsilon)}(x,t;D)\Lambda V_\varepsilon(x,t)+B^{(\varepsilon)}(x,t;D)V_\varepsilon(x,t)+(t+\varepsilon)F(x,t)$$

$$H_0^{(\varepsilon)}(x,t;\xi)=\left(\begin{array}{cccc:cccc}
\mu_1^{(\varepsilon)} & 1 & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & \ddots & 1 & & & & \\
 & & & \mu_s^{(\varepsilon)} & 0 & & & \\
\hdashline
 & & & & \lambda_1^{(\varepsilon)} & 1 & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & 1 \\
i(t+\varepsilon)c_{m-1}^{(\varepsilon)} & \cdots & & i(t+\varepsilon)c_{m-s}^{(\varepsilon)} & & & & \lambda_{m-s}^{(\varepsilon)}
\end{array}\right)$$

where $\lambda_i^{(\varepsilon)}=\lambda_i(x, t+\varepsilon; \xi/|\xi|)$, $\mu_i^{(\varepsilon)}=\mu_i(x, t+\varepsilon; \xi/|\xi|)$, $c_i^{(\varepsilon)}=c_i(x, t+\varepsilon; \xi/|\xi|)$, and $B^{(\varepsilon)}$ is a bounded operator in $(L^2(R^l))^m$.

By the condition B, the relation

$$(3.6)\qquad (t+\varepsilon)C^0_{\varepsilon,m-1}(x, t; \xi, \mu_i)=(\lambda_i^{(\varepsilon)}(x, t; \xi)-\mu_i^{(\varepsilon)}(x, t; \xi))S_i(x, t+\varepsilon; \xi)$$

$(1\leq i\leq s)$ is valid, and therefore the same result as theorem 1 holds for the Cauchy problem (3.1)–(3.2).

Let $u_\varepsilon(x, t)$ be a solution of the Cauchy problem (3.1)–(3.2) with the initial data $(u_0(x),\dots, u_{m-1}(x))\in \Pi\mathscr{D}$ and the second term $f(x, t)\in\mathscr{D}_{x,t}$, and define (see O. A. Oleinik [8]) $u_{\varepsilon N}$, u, f_ε by

$$(3.7)\qquad u_{\varepsilon N}=u_0+tu_1+\cdots+\frac{t^{m-1}}{(m-1)!}u_{m-1}+\frac{t^m}{m!}\left(\frac{\partial}{\partial t}\right)^m u_\varepsilon\Big|_{t=0} +\cdots+\frac{t^{N+m}}{(N+m)!}\left(\frac{\partial}{\partial t}\right)^{N+m} u_\varepsilon\Big|_{t=0}$$

$$(3.8)\qquad u=u_\varepsilon-u_{\varepsilon N},\quad P_\varepsilon u=f-P_\varepsilon u_{\varepsilon N}=f_\varepsilon$$

then we have

$$(3.9)\qquad \left(\frac{\partial}{\partial t}\right)^j f_\varepsilon|_{t=0}=0 \qquad j=0, 1,\dots, N.$$

Using the same method as the reduction in §§ 1 and 2, the equation (3.8) can be represented in the form

$$(3.10)\quad \{\Pi_{\varepsilon,m}(x, t; D, D_t)-i\tilde{C}_{\varepsilon,m-1}(x, t; D, D_t)\}u+R_{\varepsilon,m-2}(x, t; D, D_t)u=i^{-m}f.$$

Let $a_{\varepsilon,i}(x, t; D)$ $(0\leq i\leq m-2)$ be a pseudo-differential operator of order i, then $R_{\varepsilon,m-2}$ can be expressed as a linear combination of the operators $a_{\varepsilon,i}$ $(0\leq i\leq m-2)$ (by lemma 4.1 in Mizohata and Ohya [5]) as follows:

$$(3.11)\quad R_{\varepsilon,m-2}(x, t; D, D_t)=a_{\varepsilon,0}(x, t; D)\partial^{(\varepsilon)}_{m-s-2}\cdots\partial^{(\varepsilon)}_1\Lambda^{(\varepsilon)}_s\cdots\Lambda^{(\varepsilon)}_1 +a_{\varepsilon,1}(x, t; D)\partial^{(\varepsilon)}_{m-s-3}\cdots\partial^{(\varepsilon)}_1\Lambda^{(\varepsilon)}_s\cdots\Lambda^{(\varepsilon)}_1+\cdots+a_{\varepsilon,m-2}(x, t; D).$$

We now rewrite the equation (3.10) as a system by substituting (3.11)

$$(3.12)\qquad D_tV_\varepsilon(x, t)=H_0^{(\varepsilon)}(x, t; D)\Lambda V_\varepsilon(x, t)+B_0^{(\varepsilon)}(x, t; D)V_\varepsilon(x, t) +(t+\varepsilon)^{-1}B_1V_\varepsilon(x, t)+(t+\varepsilon)F_\varepsilon(x, t),$$

where B_1 is a constant matrix.

Uniform boundedness of the operator norm of $H_0^{(\varepsilon)}(x, t; D)$, $B_0^{(\varepsilon)}(x, t; D)$ and also that of the diagonalizer $N^{(\varepsilon)}(x, t; D)$ of $H_0^{(\varepsilon)}$ imply

$$(3.13)\quad \|V_\varepsilon(t)\|_k \leq C(k, T')\left[\int_0^t \frac{1}{s}\|V_\varepsilon(s)\|_k ds + \int_0^t \|f_\varepsilon(s)\|_k ds\right] \qquad (k=1, 2,\ldots).$$

If we define $\tilde{f}_\varepsilon$, W_ε, by

$$f_\varepsilon(x, t) = t^{N+1}\tilde{f}_\varepsilon(x, t), \quad W_\varepsilon(t) = \int_0^t \frac{1}{s}\|V(s)\|_k ds$$

and choose the integer N such that $N+1 \geq C(k, T')$, then (3.13) yields

$$(3.14)\qquad tW'_\varepsilon(t) \leq (N+1)W_\varepsilon(t) + (N+1)t^{N+2}\max_{0\leq t\leq T'}\|\tilde{f}_\varepsilon(t)\|_k\,.$$

Multiplying (3.14) by t^{-N-2} and integrating from δ to t, and observing that $\delta^{-(N+1)}W_\varepsilon(\delta)$ tends to zero when $\delta \to +0$, we have

$$(3.15)\qquad W_\varepsilon(t) \leq (N+1)t^{N+2}\max_{0\leq t\leq T'}\|\tilde{f}_\varepsilon(t)\|_k\,.$$

From (2.11), (3.13) and (3.15), it follows

$$(3.16)\qquad |||u(x, t)|||_{k-1} \leq \tilde{c}(k, N, T')t^{N+1}\max_{0\leq t\leq T'}\|\tilde{f}_\varepsilon(t)\|_k\,.$$

By definition, the inequality

$$|||u_{\varepsilon N}(x, t)|||_k \leq C\Big[\,|||u(x, 0)|||_{N+m+k} + \max_{0\leq t\leq T'}\sum_{\substack{|\alpha|\leq N\\ j\leq N}}\Big\|\Big(\frac{\partial}{\partial x}\Big)^\alpha\Big(\frac{\partial}{\partial t}\Big)^j f(x, t)\Big\|\Big]$$

holds, and thus we get

$$(3.17)\quad |||u_\varepsilon(x, t)|||_k \leq C_{k,N,T'}\Big[\,|||u(x, 0)|||_{N+m+k} + \max_{0\leq t\leq T'}\sum_{\substack{|\alpha|\leq N\\ j\leq N+1}}\Big\|\Big(\frac{\partial}{\partial x}\Big)^\alpha\Big(\frac{\partial}{\partial t}\Big)^j f(x, t)\Big\|\Big]$$

where $C_{k,N,T'}$ is a constant independent of ε.

Consider the case $k=2$ for instance, then from (3.17) we have

$$(3.18)\qquad \|u_\varepsilon(x, t)\|_{m+1, L^2(\Omega')} \leq M \qquad (0<\varepsilon<\varepsilon_0),$$

and then there exists a subsequence $(u_{\varepsilon j})_{j=1,2,\ldots}$ which converges weakly in $\mathscr{E}^{m+1}_{L^2(\Omega')}$.

The limit function u is a unique solution of the Cauchy problem (1.1)–(1.2), which satisfies

$$\left(u, \frac{\partial}{\partial t}u, \dots, \left(\frac{\partial}{\partial t}\right)^{m-1}u\right) \in \mathscr{E}_t^0(\mathscr{D}_{L^2}^m \times \cdots \times \mathscr{D}_{L^2}^1) \qquad \text{(cf. Mizohata [4])}.$$

4. Reformulation of the conditions

We shall now express the condition A and B in a more explicit form. Let $\sigma_{m-1}(C_{m-1})$ denote the principal symbol of C_{m-1}, and set

$$\Pi_m^{(i)} = \partial_{m-s} \cdots \partial_1 \Delta_s \cdots \Delta_{i+1} \Delta_{i-1} \cdots \Delta_1 \Delta_i \qquad i = 1, 2, \dots, s. \tag{4.1}$$

$$\mathring{\Pi}_m^{(i)} = \partial_{m-s} \circ \cdots \circ \partial_1 \circ \Delta_s \circ \cdots \circ \Delta_{i+1} \circ \Delta_{i-1} \circ \cdots \circ \Delta_1 \qquad i = 1, 2, \dots, s. \tag{4.2}$$

Then

$$\Pi_m - \Pi_m^{(i)} = (\partial_{m-s} \cdots \partial_{i+1})(\partial_i)\{(\partial_{i-1} \cdots \partial_1 \Delta_s \cdots \Delta_1) - (\partial_{i-1} \cdots \partial_1 \Delta_s \cdots \cdots \Delta_{i+1} \Delta_{i-1} \cdots \Delta_1 \Delta_i)\}$$

implies

$$\sigma_{m-1}(\Pi_m - \Pi_m^{(i)}) \equiv 0 \ (\mathrm{mod}\,(\tau - \lambda_i)) \qquad i = 1, 2, \dots, s, \tag{4.3}$$

and by the same reason we get

$$\sigma_{m-1}(\Pi_m^{(i)} - \mathring{\Pi}_{m-1}^{(i)} \Delta_i) \equiv 0 \ (\mathrm{mod}\,(\tau - \mu_i)) \qquad i = 1, 2, \dots, s. \tag{4.4}$$

From (4.3) and (4.4), we have

$$\sigma_{m-1}(C_{m-1}(x, t; D, D_t))\,|_{\tau=\mu_i} \equiv i\sigma_{m-1}(\mathring{\Pi}_{m-1}^{(i)} \circ \Delta_i - \mathring{\Pi}_{m-1}^{(i)} \Delta_i)\,|_{\tau=\mu_i} + P_{m-1}(x, t; \xi, \mu_i) \qquad (\mathrm{mod}\ (\lambda_i - \mu_i)). \tag{4.5}$$

Using the product formula for pseudo-differential operators we find

$$i\sigma_{m-1}(\mathring{\Pi}_{m-1}^{(i)} \circ \Delta_i - \mathring{\Pi}_{m-1}^{(i)} \Delta_i) = \left(\frac{\partial}{\partial \tau}\Pi_{m-1}^{(i)} \frac{\partial}{\partial t}\mu_i - \sum_{\alpha=1}^{l} \frac{\partial}{\partial \xi_\alpha}\mathring{\Pi}_{m-1}^{(i)} \frac{\partial}{\partial x_\alpha}\mu_i\right). \tag{4.6}$$

On the one hand, $P_m(x, t; \xi, \tau) = (\tau - \mu_i)\mathring{\Pi}_{m-1}^{(i)}(x, t; \xi, \tau)$ shows

$$\frac{\partial}{\partial \tau}\mathring{\Pi}_{m-1}^{(i)}\Big|_{\tau=\mu_i} = \frac{1}{2}\left(\frac{\partial}{\partial \tau}\right)^2 P_m\Big|_{\tau=\mu_i} \tag{4.7}$$

$$\frac{\partial}{\partial \xi_\alpha}\mathring{\Pi}_{m-1}^{(i)}\Big|_{\tau=\mu_i} = \frac{\partial^2}{\partial \tau \partial \xi_\alpha} P_m\Big|_{\tau=\mu_i} + \frac{1}{2}\left(\frac{\partial}{\partial \tau}\right)^2 P_m \frac{\partial}{\partial \xi_\alpha}\mu_i\Big|_{\tau=\mu_i} \tag{4.8}$$

therefore, if we define L_i by

(4.9) $$L_i(x, t; \xi)=\left\{P_{m-1}(x, t; \xi, \tau)+\frac{1}{2}\left(\frac{\partial}{\partial \tau}\right)^2 P_m \frac{\partial}{\partial t}\mu_i + \sum_{\alpha=1}^{l}\left(\frac{\partial^2}{\partial\tau\partial\xi_\alpha}P_m+\frac{1}{2}\left(\frac{\partial}{\partial\tau}\right)^2 P_m\frac{\partial}{\partial\xi_\alpha}\mu_i\right)\frac{\partial}{\partial x_\alpha}\mu_i\right\}\bigg|_{\tau=\mu_i}$$

then (4.5) means $L_i(x, t; \xi)\equiv C^0_{m-1}(x, t; \xi, \mu_i(x, t; \xi))$ (mod$(\lambda_i-\mu_i)$).

Exchange (λ_i) for (μ_i) in our considerations in §§1, 2, and 3, and define $\tilde{L}_i(x, t; \xi)$ by replacing λ_i with μ_i in (4.9) then we get the following:

Theorem 3. *If we replace $C^0_{m-1}(x, t; \xi, \mu_i)$ by L_i (or $\tilde{L}_i$) in the condition A and B, we obtain the equivalent conditions.*

Remark: From the expression $P_m(x, t; \xi, \tau)=(\tau-\mu_i)(\tau-\lambda_i)Q_i$, it follows $L_i(x, t; \xi)\equiv P_{m-1}(x, t; \xi, \mu_i)+Q_i(x, t; \xi, \mu_i)\left(\frac{\partial}{\partial t}\mu_i-\sum_{\alpha=1}^{l}\frac{\partial}{\partial x_\alpha}\mu_i\frac{\partial}{\partial\xi_\alpha}\lambda_i\right)$ (mod $(\lambda_i-\mu_i)$), and the right hand side coincides with (Σ) in A. Menikoff [7]. If we set $\lambda_i=\mu_i$, then L_i would be the Levi function of Mizohata and Ohya [6].

5. Dependence domain

In this section, sometimes μ_{j+s} $(1\le j\le m-s)$ may be used in place of λ_j for the simplicity of notations. Similarly $\tilde{\lambda}_j$ is for $\tilde{\mu}_{j+s}$.

Change the coordinates, the so-called "space-like transformation" such that

(5,1) $$t'=\psi(x, t), \quad x'_\alpha=x_\alpha \qquad (1\le\alpha\le l)$$

$$\left(\frac{\partial\psi}{\partial t}\right)^2-\mu^2_{\max}\sum_{\alpha=1}^{l}\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2>0 \qquad (x, t)\in U$$

where U is a some neighborhood of the origin and

$$\mu_{\max}=\max_{1\le i\le m}\ \sup_{(x,t)\in\Omega,|\xi|=1}|\mu_i(x, t; \xi)|.$$

Consider the equation in τ

(5.2) $$\psi_t\tau-\mu_i(x, t; \psi_x\tau+\xi)=0 \qquad (1\le i\le m)$$

subordinate to the change of coordinates, then there exists a unique solution τ of (5.2) for any fixed $(x, t)\in U$ and ξ $(\neq 0)$.

If we denote this root by $\tilde{\mu}_i(x, t; \xi)$, then $\tilde{\mu}_i(x, t; \xi)$ can be seen sufficiently smooth, and we have

Lemma 1. *In a some neighborhood* U_0 *of the origin, we have*

(5.3) $\lambda_i(x, t; \psi_x\tilde{\mu}_i+\xi)-\mu_i(x, t; \psi_x\tilde{\mu}_i+\xi)=C_i(x, t; \xi)(\tilde{\lambda}_i(x, t; \xi)-\tilde{\mu}_i(x, t; \xi))$

$(1\leq i\leq s)$ *where* $C_i(x, t; \xi)$, *homogeneous of degree* 0, *is sufficiently smooth.*

Proof. First we note that $\psi_x\tilde{\mu}_i+\xi$ is not zero for any $(x, t)\in U$ and $\xi\,(\neq 0)$. Therefore, for suitable U_0 and for sufficiently small $|\zeta-\tau|$, there exists a smooth function $v_i(x, t; \zeta, \tau, \xi)$ such that

(5.4) $$\lambda_i(x, t; \psi_x\zeta+\xi)-\lambda_i(x, t; \psi_x\tau+\xi)=(\zeta-\tau)v_i(x, t; \zeta, \tau, \xi).$$

Put $\zeta=\tilde{\lambda}_i$, $\tau=\tilde{\mu}_i$ in (5.4) and take account of $\psi_t\tilde{\lambda}_i=\lambda_i(x, t; \psi_x\tilde{\lambda}_i+\xi)$, $\psi_t\tilde{\mu}_i=\mu_i(x, t; \psi_x\tilde{\mu}_i+\xi)$, then we have the result by setting $C_i=\psi_t-v_i$.

Lemma 2. (Mizohata and Ohya [6] Lemma 4.1) *By the transformation* (5.1) $D_t-\mu_i(x, t; D)$ *is transformed to*

(5.5) $$\Psi_i(x, t; D, D_t)(D_t-\tilde{\mu}_i(x, t; D))+e_i(x, t; D, D_t)$$

where $\Psi_i(x, t; \xi, \tau)$ *is homogeneous of degree* 0 *in* (ξ, τ) *such that* $\prod_{i=1}^{m}\Psi_i(x, t; \xi, \tau)=P_m(x, t; \psi_x, \psi_t)$, *and* $e_i(x, t; D, D_t)$ *is a pseudo-differential operator of degree* 0.

Assume that P is transformed to $\tilde{P}$ by (5.1) and put

$$i^{m-1}\tilde{C}_{m-1}=i^m(\tilde{P}_m-\tilde{\Pi}_m)+i^{m-1}\,\tilde{P}_{m-1}$$

then the lemma 2, (1.8) and (1.10) imply

(5.6) $$\tilde{C}^0_{m-1}=\sum_{i=1}^{s}e_i(x, t; \xi, \tau)\frac{\tilde{P}_m}{\psi_i(\tau-\tilde{\mu}_i)}+\sum_{i=1}^{m-s}e_{s+i}(x, t; \xi, \tau)\frac{\tilde{P}_m}{\psi_{s+i}(\tau-\tilde{\lambda}_i)}$$
$$+c_{m-1}(x, t; \psi_x\tau+\xi)+\cdots+c_0(x, t; \psi_x\tau+\xi)(\psi_t\tau$$
$$-\lambda_{m-s-1}(x, t; \psi_x\tau+\xi))\cdots(\psi_t\tau-\mu_1(x, t; \psi_x\tau+\xi)).$$

Now for instance, consider $\tilde{C}^0_{m-1}(x, t; \xi, \tilde{\mu}_1)$. If we express $\tilde{C}^0_{m-1}(x, t; \xi, \tilde{\mu}_1)$ as a sum of $c_{m-1}(x, t; \psi_x\tilde{\mu}_1+\xi)$ and the rest $r(x, t; \xi)$, then it follows from (5.6) that $r(x, t; \xi)$ has a factor $(\tilde{\lambda}_1-\tilde{\mu}_1)$.

On the other hand, by the condition A and lemma 1, $c_{m-1}(x, t; \psi_x\tilde{\mu}_1+\xi)$ may be factorized as $(\tilde{\lambda}_1-\tilde{\mu}_1)$ times a symbol of some pseudo-differential operator.

In general, we have

(5.7) $\tilde{C}^0_{m-1}(x, t; \xi, \tilde{\mu}_i)=\tilde{T}_i(x, t; \xi)(\tilde{\lambda}_i(x, t; \xi)-\tilde{\mu}_i(x, t; \xi))$ $(1\leq i\leq s)$.

In other words, condition A is invariant under the transformation (5.1), then we obtain

Theorem 4. *Under the condition A, the solution $u(x, t)$ of the Cauchy problem*; $Pu=0$, $\left(\frac{\partial}{\partial t}\right)^j u(x, 0)=u_j(x)$ $0\leq j\leq m-1$, *has its support in*

$$\{x; \bigcup_{\xi} |x-\xi|\leq \mu_{\max} t, \xi\in \bigcup_{j=0}^{m-1} \operatorname{supp}(u_j)\}.$$

Remark: It is evident from the construction of the solution that under the condition B, the theorem 4 is also valid.

Proof. It is sufficient to prove that the local uniqueness holds when the condition A is satisfied locally. This is done by use of the localization of the equation and the energy inequality.

6. Final remarks

The methods of the preceding sections seem to be applicable to the case where the multiplicity of the characteristic roots is at most 3. But, since the computation of the symbols becomes much more complicated, in this section we treat only the equation of order 3 as an example. Here we omit the proof.

Consider the Cauchy problem

(6.1) $$P\left(x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)u=\left(\frac{\partial}{\partial t}\right)^3 u+\sum_{\substack{|\nu|+j\leq 3\\ j<3}} a_{\nu,j}(x, t)\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial t}\right)^j u=f$$

$$(x, t)\in R^l\times[0, T]$$

(6.2) $$\left(\frac{\partial}{\partial t}\right)^j u(x, 0)=u_j(x), \qquad j=0, 1, 2.$$

Express the principal part P_3 of P in the form

$$P_3\left(x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)=i^3\{D_t^3+h_1(x, t; D)D_t^2+\cdots+h_3(x, t; D)\}$$

and assume

$$\tau^3+h_1(x, t; \xi)\tau^2+\cdots+h_3(x, t; \xi)=(\tau-\lambda(x, t; \xi))(\tau-\mu(x, t; \xi))(\tau-\nu(x, t; \xi)).$$

Moreover, to obtain the explicit condition, we suppose

$$(6.3)\qquad \begin{aligned}\lambda(x, t; \xi)-\nu(x, t; \xi)&=g(x, t; \xi)\{\lambda(x, t; \xi)-\mu(x, t; \xi)\}\\ \lambda(x, t; \xi)-\mu(x, t; \xi)&=h(x, t; \xi)\{\mu(x, t; \xi)-\nu(x, t; \xi)\}\end{aligned}$$

where g, h are symbols of some pseudo-differential operators.

$$(6.4)\qquad \begin{aligned}C_1(x, t; \xi)=[&-\sum_{j=0}^{l}(\xi_0-\mu)^{(j)}(\xi_0-\lambda)_{(j)}-\sum_{j=0}^{l}(\xi_0-\nu)^{(j)}(\xi_0-\lambda)_{(j)}\\ &-\sum_{j=0}^{l}(\xi_0-\nu)^{(j)}(\xi_0-\mu)_{(j)}+P_2^{(1)}]_{\xi_0=\frac{\lambda+\mu}{2}}\end{aligned}$$

$$C_2(x, t; \xi)=\Big[-\sum_{j=0}^{l}\Big\{\frac{\partial}{\partial\xi_j}P_3^{(1)}-\frac{1}{2}P_3^{(2)}(\xi_0-\lambda)^{(j)}\Big\}(\xi_0-\lambda)_{(j)}+P_2\Big]_{\xi_0=\lambda}$$

$$\begin{aligned}C_3(x, t; \xi)=\Big[&-\frac{1}{2}\sum_{j,i=0}^{l}(\xi_0-\nu)^{(ij)}\{(\xi_0-\mu)(\xi_0-\lambda)\}_{(ij)}\\ &-\sum_{i=0}^{l}(\xi_0-\nu)^{(i)}\{\sum_{j=0}^{l}(\xi_0-\mu)^{(j)}(\xi_0-\lambda)_{(j)}\}_{(i)}\\ &+\sum_{i=1}^{l}C_1^{(i)}\lambda_{(i)}-\frac{1}{2}P_2^{(2)}\sum_{j=0}^{l}(\xi_0-\mu)^{(j)}(\xi_0-\lambda)_{(j)}+P_1\Big]_{\xi_0=\lambda}\end{aligned}$$

where, to simplify our notations, we have set $\tau=\xi_0$, $t=x_0$ and

$$\frac{\partial}{\partial\xi_i}f=f^{(i)},\ \frac{\partial}{\partial x_i}f=f_{(i)},\ \frac{\partial}{\partial\xi_0}P=P^{(1)},\ \Big(\frac{\partial}{\partial\xi_0}\Big)^2P=P^{(2)}.$$

Now we define $M(x, t; \xi)$ and $L(x, t; \xi)$ by

$$(6.5)\qquad M(x, t; \xi)=\sum_{j=1}^{l}(\xi_0-\lambda)^{(j)}(\xi_0-\mu)_{(j)}-\sum_{j=0}^{l}(\xi_0-\mu)^{(j)}(\xi_0-\lambda)_{(j)}$$

$$(6.6)\qquad L(x, t; \xi)=\frac{1}{2}\sum_{j=1}^{l}\frac{\partial^2}{\partial\xi_j\partial x_j}C_2(x, t; \xi)+C_3(x, t; \xi)$$

then we have the following:

Proposition 1. *Let* $C_1(x, t; \xi)\equiv 0\ (\mathrm{mod}\,(\mu-\nu))$,

$$C_2(x, t; \xi)\equiv 0\ (\mathrm{mod}\,(\lambda-\mu)(\lambda-\nu))$$

$$M(x, t; \xi) \equiv 0 \pmod{(\lambda-\mu)},$$

$$L(x, t; \xi) \equiv 0 \pmod{(\lambda-\mu)}$$

then, for any given initial data and the second term such that $(u_0(x), u_1(x), u_2(x)) \in \mathscr{D}_{L^2}^{m+2} \times \mathscr{D}_{L^2}^{m+1} \times \mathscr{D}_{L^2}^{m}$, $f(x, t) \in \mathscr{E}_t^0(\mathscr{D}_{L^2}^{m})$, *there exists a unique solution* $u(x, t)$ *of the Cauchy problem* (6.1)-(6.2) *belonging to* $\mathscr{E}_t^0(\mathscr{D}_{L^2}^{m}) \cap \mathscr{E}_t^1(\mathscr{D}_{L^2}^{m-1})$.

Proposition 2. *Let*

$$tC_1(x, t; \xi) \equiv 0 \pmod{(\mu-\nu)},$$

$$tC_2(x, t; \xi) \equiv 0 \pmod{(\lambda-\mu)(\lambda-\nu)}$$

$$tM(x, t; \xi) \equiv 0 \pmod{(\lambda-\mu)},$$

$$t^2L(x, t; \xi) \equiv 0 \pmod{(\lambda-\mu)}$$

then, for any given initial data $(u_0(x), u_1(x), u_2(x)) \in \Pi\mathscr{D}_x$ *and the second term* $f(x, t) \in \mathscr{D}_{x,t}$, *there exists a unique solution* $u(x, t) \in \mathscr{E}_{x,t}$ *of the Cauchy problem* (6.1)-(6.2).

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