J. Math. Kyoto Univ. (JMKYAZ)
17-2 (1977) 225-243

On the cohomology of irreducible symmetric spaces of exceptional type

By

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(Received May 14, 1976)

§1. Introduction

The 1-connected irreducible symmetric spaces are classified by E. Cartan [10], among them the symmetric spaces ***FII***, ***EIII***, ***EIV***, and ***EVII*** are torsion free and the ring structure of their integral cohomology is known [1], [13], [14]. On the other hand the remaining symmetric spaces of exceptional type have 2-torsions, in fact, these spaces are homogeneous spaces G/U of compact simply connected exceptional Lie groups G over subgroups U having the fundamental group of order 2. Except the case $G/U=\boldsymbol{EI}$, U is a maximal rank maximal subgroup of G and it is the identity component of the centralizer of an element x [6]. Let C be the identity component of the centralizer of a suitable one dimensional torus containing x. By [7], both G/C and U/C have torsion free cohomology of vanishing odd dimensional parts.

So, we have the following program to determine the cohomology ring of the symmetric spaces G/U: (I) To determine $H^*(G/C)$. (II) To compute $H^*(G/U)$ by the spectral sequence associated with the fibering $U/C\to G/C\to G/U$. In the cases $G/U=\boldsymbol{G}$, ***FI***, ***EII***, ***EV*** the group C is torsion free and (I) may be done by Theorem 2.1 of [12]. In the cases $G/U=\boldsymbol{G}$, ***FI***, ***EII***, ***EVI***, ***EIX*** we have that U/C is a 2-sphere S^2 and (II) may be done by use of the Gysin exact sequence.

In §2, we shall fix the subgroups U, C and the homogeneous spaces U/C. In §3, we shall summarize general properties on the cohomology of homogeneous spaces G/H of G over a maximal rank subgroup H. In §4, we shall apply our program to the symmetric space $\boldsymbol{FI}=F_4/U$, where $U=S^3\cdot Sp(3)$, $C=T^1\cdot Sp(3)$ with $S^3\cap Sp(3)=T^1\cap Sp(3)\cong \boldsymbol{Z}_2$, and our results are stated as follows:

$$H^*(F_4/T^1\cdot Sp(3))=\boldsymbol{Z}[t, u, v, w]/(t^3-2u,\ u^2-3t^2v+2w,\ 3v^2-t^2w,\ v^3-w^2)$$

where $\deg t=2$, $\deg u=6$, $\deg v=8$ *and* $\deg w=12$ (Theorem 4.4);

$$H^*(\boldsymbol{FI})=\boldsymbol{Z}[f_4,f_8,f_{12}]/(f_4^3-12f_4f_8+8f_{12},f_4f_{12}-3f_8^2,f_8^3-f_{12}^2) \quad \text{(free part)}$$

$$+\boldsymbol{Z}_2<\chi,\chi^2>\otimes\Delta(f_8,f_{12}) \qquad \text{(torsion part)},$$

where $\deg\chi=3$ *and* $\deg f_i=i$ $(i=4, 8, 12)$ (Theorem 4.11).

Here, $A<x_1,\dots,x_n>$ indicates a free A-module with an A-base $\{x_1,\dots,x_n\}$.

In §5, we shall prove that the cohomology groups of the 1-connected irreducible symmetric spaces of exceptional type are odd torsion free (Theorem 5.1).

§2. Symmetric spaces of exceptional type

We discuss the symmetric spaces $G/U=$ ***G*****,** ***FI*****,** ***EII*****,** ***EV*****,** ***EVI EVIII*** and ***EIX***, where G is an appropriate compact 1-connected exceptional Lie group and U is a maximal rank subgroup of G. Denote by $Z_0(A)$ the identity component of the centralizer of a subset A of G. Let T be a maximal torus of U, and let $\pi\colon V\to T$ be the universal covering.

By [6; Remark 1], $U=Z_0(x)$ for an element $x\in T$ which is determined as follows. We use the root system $\{\alpha_i\}$ of [9]. Then $x=\pi(\bar{x})$ is determined by the equalities $\alpha_k(\bar{x})=\frac{1}{2}$ and $\alpha_i(\bar{x})=0$ for $i\neq k$, where k takes the values in the following table. Let L be a line in V given by the equalities $\alpha_i(\bar{t})=0$ for $i\neq k$ and let $T^1=\pi(L)$ and $C=Z_0(T^1)$. Then we have the following table [6]:

$G/U=$	***G***	***FI***	***EII***	***EV***	***EVI***	***EVIII***	***EIX***
G =	G_2	F_4	E_6	E_7	E_7	E_8	E_8
k =	2	1	2	2	1	1	8
type of U:	$A_1\times A_1$	$A_1\times C_3$	$A_1\times A_5$	A_7	$A_1\times D_6$	D_8	$A_1\times E_7$
type of C:	$T^1\times A_1$	$T^1\times C_3$	$T^1\times A_5$	$T^1\times A_6$	$T^1\times D_6$	$T^1\times D_7$	$T^1\times E_7$

These groups U and C are described as follows.

Theorem 2.1. *U, C and U/C are given as follows:*

$G/U=$	***G***	***FI***	***EII***	***EV***	***EVI***	***EVIII***	***EIX***
U =	$S^3\cdot S^3$	$S^3\cdot Sp(3)$	$S^3\cdot SU(6)$	$SU(8)/\boldsymbol{Z}_2$	$S^3\cdot Spin(12)$	$Ss(16)$	$S^3\cdot E_7$
C =	$T^1\cdot S^3$	$T^1\cdot Sp(3)$	$T^1\cdot SU(6)$	$T^1\cdot SU(7)$	$T^1\cdot Spin(12)$	$T^1\cdot Spin(14)$	$T^1\cdot E_7$
$U/C=$	S^2	S^2	S^2	$P_7(\boldsymbol{C})$	S^2	$\frac{SO(16)}{SO(2)\times SO(14)}$	S^2

in which $S^3 \cap H = T^1 \cap H \cong \boldsymbol{Z}_2$ *if* $U = S^3 \cdot H$, $T^1 \cap SU(7) \cong \boldsymbol{Z}_7$, $T^1 \cap Spin(14) \cong \boldsymbol{Z}_4$ *and* $Spin(12)/(T^1 \cap Spin(12)) = Ss(12)$, *where* $Ss(4m) = Spin(4m)/\boldsymbol{Z}_2$ *denotes the semi-spinor group.*

Proof. According to [5], each weight w of G is a linear map $w: V \to \boldsymbol{R}^1$ such that $w(\mathrm{Ker}\,\pi) \subset \boldsymbol{Z}$, it is identified with an element of $H^1(T) = \mathrm{Hom}(H_1(T), \boldsymbol{Z})$ by the isomorphism $\mathrm{Ker}\,\pi \cong \pi_1(T) \cong H_1(T)$, and $H^1(T)$ is a free abelian group generated by the fundamental weights $w_1, w_2, \dots, w_l$ ($l = \mathrm{rank}\, G$) which are given by $2<w_i, \alpha_j>/<\alpha_j, \alpha_j> = \delta^i_j$.

Since U is compact, connected and semi-simple, the universal covering $p: \tilde{U} \to U$ is a finite covering. Then $\tilde{T} = p^{-1}(T)$ is a maximal torus of $\tilde{U}$ since every maximal torus contains $\mathrm{Ker}\, p \subset$ the center of $\tilde{U}$. The covering map π is factored through $p\tilde{\pi}: V \to \tilde{T} \to T$. Thus $\mathrm{Ker}\,\tilde{\pi} \subset \mathrm{Ker}\,\pi$ and every weight of G is also a weight of $\tilde{U}$, and this gives $p^*: H^1(T) \to H^1(\tilde{T})$.

Let $\tilde{\alpha} = m_1\alpha_1 + \cdots + m_l\alpha_l$ be the highest root. By [6], $\tilde{U}(U)$ has a system of the simple roots $\{\alpha_i (i \neq k), -\tilde{\alpha}\}$. Let $\{u_i (i \neq k), u\}$ be the fundamental weights with respect to this simple root system. It is directly verified that

$$(*) \qquad w_k = -n_k \cdot u \quad \text{and} \quad w_i = u_i - n_i \cdot u \ (i \neq k) \quad \text{for} \quad n_i = m_i|\alpha_i|^2/|\tilde{\alpha}|^2 .$$

Then (*) gives the induced homomorphism $p^*: H^1(T) \to H^1(\tilde{T}) = \boldsymbol{Z}<u_i, u>$.

In our cases we see that

$$(**) \qquad n_k = 2 \quad \text{and} \quad n_i \text{ is odd for some } i \neq k .$$

It follows from (*) and (**) that the index of $\mathrm{Im}\, p^*$ is 2 and $\mathrm{Ker}\, p \cong \boldsymbol{Z}_2$, that is, p is a double covering.

From the known type of U, we have the existence of a compact 1-connected simple Lie group H such that $\tilde{U} = H$ or $\tilde{U} = S^3 \times H$ ($S^3 = Sp(1)$).

Consider the case that $\tilde{U} = S^3 \times H$. We have that $\tilde{T} = T_1 \times T_2$ for $T_1 = S^3 \cap \tilde{T}$ and $T_2 = H \cap \tilde{T}$. Obviously $p(T_1) = T^1$ and $\tilde{C} = p^{-1}(C) = T^1 \times H$. The inclusions $i_j: T_j \to \tilde{T}$ ($j = 1, 2$) induce projections i_j^* of $H^1(\tilde{T})$ onto $H^1(T_j)$ given by $i_1^*(u_i) = 0$ ($i \neq k$) and $i_2^*(u) = 0$.

Now, (*) and (**) show that $i_j^* \circ p^*$ ($j = 1, 2$) are onto, thus that $\mathrm{Ker}\, p$ cannot be contained in T_1 nor in T_2. Thus the restrictions $p|S^3$ and $p|H$ are injective, and by putting $p(S^3) = S^3$ and $p(H) = H$ we have

$$U = S^3 \cdot H, \quad C = T^1 \cdot H, \quad S^3 \cap H = T^1 \cap H \cong \boldsymbol{Z}_2 \quad \text{and} \quad U/C = S^3/T^1 = S^2 .$$

Next consider the case $G/U = \boldsymbol{EV}$, then $G = E_7$ and $\tilde{U} = SU(8)$. We may assume that $\tilde{T} = T^8 \cap SU(8)$ for the canonical maximal torus $T^8 = U(1) \times \cdots \times U(1)$ of $U(8)$ and that

$$\alpha_1 = t_2 - t_3, \quad \alpha_3 = t_3 - t_4, \dots, \quad \alpha_7 = t_7 - t_8 \quad \text{and} \quad -\tilde{\alpha} = t_1 - t_2$$

for the canonical basis t_i of $H^1(\tilde{T})$ with the relation $t_1+\cdots+t_8=0$. Then $T_1=p^{-1}(T^1)$ consists of $(z^{-7}, z,\dots, z)$, $z\in U(1)$, and $\mathrm{Ker}\, p\cong \boldsymbol{Z}_2$ is generated by $(-1, -1,\dots, -1)$. By taking $U(1)\times U(7)\subset U(8)$, $SU(7)=SU(8)\cap U(7)$ and $p(SU(7))=SU(7)$, we have easily

$$\tilde{C}=p^{-1}(C)=T_1\cdot SU(7),\quad T_1\cap SU(7)\cong \boldsymbol{Z}_7,$$

$$U=SU(8)/\boldsymbol{Z}_2,\quad C=T^1\cdot SU(7),\quad T^1\cap SU(7)\cong \boldsymbol{Z}_7$$

and

$$U/C=\tilde{U}/\tilde{C}=SU(8)/T_1\cdot SU(7)=U(8)/(U(1)\times U(7))=P_7(\boldsymbol{C}).$$

Consider the case $G/U=\boldsymbol{EVIII}$, then $G=E_8$ and $\tilde{U}=Spin(16)$. Let $p'\colon Spin(16)\to SO(16)$ be the double covering. We may assume that $T'=p'(\tilde{T})$ is a maximal torus $SO(2)\times\cdots\times SO(2)$ of $SO(16)$ and that

$$\alpha_2=t_7+t_8,\quad \alpha_3=t_7-t_8,\dots,\quad \alpha_8=t_2-t_3\quad\text{and}\quad -\tilde{\alpha}=t_1-t_2$$

for the canonical basis t_i of $H^1(T')$. We have directly

$$u=t_1,\quad u_2=\frac{1}{2}\sum_{j=1}^{8} t_j,\quad u_3=u_2-t_8\quad\text{and}\quad u_i=\sum_{j=1}^{10-i} t_j\ (i=4, 5, 6, 7, 8)$$

which gives the injection $p'^*\colon H^1(T')\to H^1(\tilde{T})$. Thus $\mathrm{Im}\, p'^*$ is spanned by the elements

$$u,\quad u_8,\quad u_7,\quad u_6,\quad u_5,\quad u_4,\quad u_3-u_2\quad\text{and}\quad 2u_2\,.$$

On the other hand $(n_1,\dots, n_8)=(m_1,\dots, m_8)=(2, 3, 4, 6, 5, 4, 3, 2)$ and $k=1$ in $(*)$. Thus $\mathrm{Im}\, p^*$ is spanned by the elements

$$2u,\quad u_2+u,\quad u_3,\quad u_4,\quad u_5+u,\quad u_6,\quad u_7+u\quad\text{and}\quad u_8\,,$$

and $\mathrm{Im}\, p^*\neq \mathrm{Im}\, p'^*$. This shows that $U\neq SO(16)$ hence that U must be $Ss(16)$ and $p'(\mathrm{Ker}\, p)\cong \boldsymbol{Z}_2$ coincides with the center of $SO(16)$.

Next put $T_1=$identity component of $p^{-1}(T^1)$, $\tilde{C}=p^{-1}(C)$ and consider the canonical inclusion $SO(2)\times SO(14)\to SO(16)$, then we see that $T_1=p'^{-1}(SO(2))$ and $p'(\tilde{C})=SO(2)\times SO(14)$. Put $Spin(14)=p'^{-1}(SO(14))$, then $p'|T_1$ and $p'|Spin(14)$ are double coverings and $\mathrm{Ker}\, p'=T_1\cap Spin(14)\cong \boldsymbol{Z}_2$. Let $z_1\in SO(2)$ and $z_2\in SO(14)$ be the diagonal matrices of the diagonal elements -1. Then z_2 and z_1z_2 $(=(z_1, z_2))$ generate the center of $SO(14)$ and $SO(16)$ respectively. Choose elements $x_1\in T_1$ and $x_2\in Spin(14)$ such that $p(x_i)=z_i$ $(i=1, 2)$, then they are of order 4 and $x_1^2=x_2^2$ generates $\mathrm{Ker}\, p'$. Since $p'(x_1x_2)=z_1z_2$ generates the center $p'(\mathrm{Ker}\, p)$ of $SO(16)$, $\mathrm{Ker}\, p\cong \boldsymbol{Z}_2$ is generated by x_1x_2 or $x_1x_2^{-1}$. It follows that $p|T_1$ and $p|Spin(14)$ are isomorphisms and, by putting $p(Spin(14))=Spin(14)\subset Ss(16)\subset E_8$, we have

$$C=p(\tilde{C})=T^1\cdot Spin(14),\quad T^1\cap Spin(14)\cong \boldsymbol{Z}_4$$

and $$U/C=\tilde{U}/\tilde{C}=p'(\tilde{U})/p'(\tilde{C})=SO(16)/(SO(2)\times SO(14)).$$

Finally we consider the subgroup $U=S^3\cdot Spin(12)$ of E_7, the double covering $p\colon \tilde{U}=S^3\times Spin(12)\to U$, the inclusion $j_2\colon Spin(12)\to\tilde{U}$ and the projection $\pi\colon U\to U/S^3$. Let $p'\colon Spin(12)\to SO(12)$ be the double covering and choose maximal tori $T'=SO(2)\times\cdots\times SO(2)$ of $SO(12)$ and $T_2=p'^{-1}(T')$ of $Spin(12)$ and put $\tilde{T}=T_1\times T_2$, $T=p(\tilde{T})$. T/T^1 is a maximal torus of $U/S^3=C/T^1$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
T_2 & \xrightarrow{i_2} & \tilde{T}=T_1\times T_2 & \xrightarrow{p} & & T & \\
\downarrow & & \downarrow & & \swarrow & & \searrow^{\pi} \\
Spin(12) & \xrightarrow{j_2} & \tilde{U}=S^3\times Spin(12) & \xrightarrow{p} & U & & T/T^1 \\
 & \searrow^{id} & \downarrow^{\pi_2} & & & \searrow^{\pi}\ \swarrow & \\
 & & Spin(12) & \xrightarrow{\bar{p}} & & U/S^3, &
\end{array}$$

where i_2 and j_2 are injections to the second factors, p and $\bar{p}$ are double coverings, π and π_2 are the projections and the other maps are the inclusions.

Put $f=\pi\, p\, i_2\colon T_2\to T/T^1$, then it is easy to see that $f=\bar{p}|T_2$ and this is a double covering. As in the previous case, we compare $\bar{p}$ with p', then the image of $p'^*\colon H^1(T')\to H^1(T_2)$ is spanned by ($t_1=u=0$ in $H^1(T_2)$)

$$t_2=u_7,\quad t_3=u_6-u_7,\dots,\quad t_6=u_3-u_4\quad\text{and}\quad t_7=2u_2-u_3\,.$$

Since $\pi\circ p=f\circ\pi_2\colon T_1\times T_2\to T/T^1$, $\operatorname{Im} f^*=i_2^*(\operatorname{Im}(p^*\circ\pi^*))\subset i_2^*(\operatorname{Im} p^*\cap \operatorname{Im}\pi_2^*)$. We have $\operatorname{Im}\pi_2^*=<u_2, u_3,\dots, u_7>$ and $\operatorname{Im} p^*=<-2u,\ u_i-n_iu\ (i=2, 3, 4, 5, 6, 7)>$ for $(n_1,\dots, n_7)=(m_1,\dots, m_7)=(2, 2, 3, 4, 3, 2, 1)$. Thus $\operatorname{Im} p^*\cap\operatorname{Im}\pi_2^*$ is spanned by

$$u_2,\quad u_3+u_7,\quad u_4,\quad u_5+u_7,\quad u_6\quad\text{and}\quad 2u_7.$$

The same holds for its i_2^*-image. Since f is a double covering, $\operatorname{Im} f^*$ has index 2. Thus $\operatorname{Im} f^*=i_2^*(\operatorname{Im} p^*\cap\operatorname{Im}\pi_2^*)\neq\operatorname{Im} p'^*$, which implies that $U/S^3=C/T^1=Spin(12)/(T^1\cap Spin(12))=Ss(12)$. q.e.d.

Corollary 2.2. *The symmetric spaces in* Theorem 2.1 *are* 1*-connected and have the* 2*-dimensional homotopy group isomorphic to* $\boldsymbol{Z}_2$.

The maximal subgroup of maximal rank in the exceptional Lie groups are classified in [6], among them the followings are the cases that the quotient spaces are not symmetric spaces:

$G=$	F_4	E_6	E_7	E_8	E_8	E_8
$k=$	2	4	3 or 5	7	5	2
type of U:	$A_2\times A_2$	$A_2\times A_2\times A_2$	$A_2\times A_5$	$A_2\times E_6$	$A_4\times A_4$	A_8

Proposition 2.3. *Corresponding to each case we put*

$H=$	$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	E_6	$SU(5)$	$\{e\}$
$n=$	2	2	2	2	4	8

For the first five cases, $U=SU(n+1)\cdot H$, $SU(n+1)\cap H=T^1\cap H\cong \boldsymbol{Z}_{n+1}$, $C=(SU(n)\cdot T^1)\cdot H$, $U/C=P_n(\boldsymbol{C})$ *and* $SU(n)\cap T^1\cong \boldsymbol{Z}_n$, *in which* $SU(n+1)\cap H$ *is the intersection of the centers of* $SU(n+1)$ *and* H. *For the second case,* $SU(3)\cap H$ *is not a subgroup of any factors of* $H=SU(3)\times SU(3)$.

For the last case, $U=SU(9)/\boldsymbol{Z}_3$, $C=SU(8)\cdot T^1$, $U/C=P_8(\boldsymbol{C})$ *and* $SU(8)\cap T^1\cong \boldsymbol{Z}_8$.

Here, T^1 consists of the diagonal matrices of $SU(n+1)$ of the diagonal element $(z,\dots, z, z^{-n})$. The proof of this proposition is similar to that of Theorem 2.1, and omitted.

Corollary 2.4. *The homogeneous space* G/U *is* 1*-connected and its* 2*-dimensional homotopy group is isomorphic to* $\boldsymbol{Z}_3$, *except the fifth case where it is isomorphic to* $\boldsymbol{Z}_5$.

§3. Cohomology of several homogeneous spaces

In this section we summarize some general properties of several homogeneous spaces. Throughout the section, G denotes a compact connected Lie group and T a maximal torus of G. $Z_0(A)$ denotes the identity component of the centralizer of a subset A in G.

At first we have the following proposition which is a slight generalization of Theorem A of Bott [7].

Proposition 3.1. *Let* $C=Z_0(S)$ *for a torus* $S\subset T$. *Then* $H^*(G/C)$ *is torsion free and* $H^{odd}(G/C)=0$.

Proof. In the case that G is simply connected, the assertion is true by Bott [7]. In the general case we have a finite covering $p: \tilde{G}=T_1\times H\to G$, where T_1 is a torus and H is a simply connected Lie group. Denote the inverse images of T, S and C respectively by $\tilde{T}$, $\tilde{S}$ and $\tilde{C}$. Then $\tilde{T}=T_1\times T_2$, where T_2 is a maximal torus of H. Let $\tilde{S}_0$ be the identity component of $\tilde{S}$

and $S_2 \subset H$ be the image of $\tilde{S}_0$ by the projection $T \to T_2$. We shall show that $\tilde{C} = Z_0(\tilde{S}_0) = T_1 \times Z_0(S_2)$, from which the assertion will follow since H is simply connected, S_2 is a torus and $G/C = \tilde{G}/\tilde{C} = H/Z_0(S_2)$. $\tilde{C}$ is connected since $\mathrm{Ker}\, p \subset \tilde{T} \subset \tilde{C}$. Then $\tilde{C} = Z_0(\tilde{S}_0)$ since $p: Z_0(\tilde{S}_0) \to C = Z_0(S)$ is a local isomorphism. Let $x = (x_1, x_2)$ be a generating element of $\tilde{S}_0$, then x_2 generates S_2 and we have $Z_0(\tilde{S}_0) = Z_0(x) = Z_0(x_1) \times Z_0(x_2) = T_1 \times Z_0(S_2)$. q.e.d.

Note that $C = T$ if $S = T$.

Let H be a subgroups of G containing T. Denote by $\Phi(H) = N_H(T)/T$ the Weyl group of H. $\Phi(H)$ is a subgroup of $\Phi(G)$ and it operates on G/T. The projection $p: G/T \to G/H$ commutes with the operation of $\Phi(H)$ which operates trivially on G/H. Thus we have that

(3.1) *the image of* $p^*: H^*(G/H; A) \to H^*(G/T; A)$ *is contained in the invariant subalgebra* $H^*(G/T; A)^{\Phi(H)}$, *where* $\boldsymbol{Z} \subset A \subset \boldsymbol{Q}$ *or* $A = \boldsymbol{Z}_p$.

By Borel [4]

(3.2) $p^*: H^*(G/H; \boldsymbol{Q}) \longrightarrow H^*(G/T; \boldsymbol{Q})^{\Phi(H)}$ *is an isomorphism.*

$H^{odd}(G/H; A) = 0$ if p^* of (3.1) is injective since $H^{odd}(G/T) = 0$. Conversely if $H^{odd}(G/H; A) = 0$, then the spectral sequence with coefficient A associated with the fibering $H/T \to G/T \to G/H$ collapses since $H^{odd}(H/T; A) = 0$. Then p^* of (3.1) is a split monomorphism. $H^*(G/T; A)^{\Phi(H)}$ is a direct factor of $H^*(G/T; A)$. Comparing the ranks by (3.2) we have

Proposition 3.2. $p^*: H^*(G/H; A) \to H^*(G/T; A)^{\Phi(H)}$ *is an isomorphism if and only if* $H^{odd}(G/H; A) = 0$. *In particular, it is an isomorphism if*

(α) $H = Z_0(S)$ *for a torus* S *and* A *is arbitrary,*

or (β) $\frac{1}{p} \in A$ *for each prime* p *such that the* p*-torsion of* $H^*(H)$ *is non-trivial.*

The assertion for (α) follows from Proposition 3.1. By Borel [2] the assertion for (β) holds for $A = \boldsymbol{Z}_q$ (q: prime to p), and then for general A.

Let U and C be the identity components of the centralizers which are discussed in the previous section, that is, U is a maximal subgroup of maximal rank in G.

Proposition 3.3. *In the case* $\pi_1(U) \cong \boldsymbol{Z}_p$ (p: *prime*), *the canonical projection* $G/T \to G/U$ *induces an isomorphism*

$$H^*(G/U; \boldsymbol{Z}[1/p]) \cong H^*(G/T; \boldsymbol{Z}[1/p])^{\Phi(U)} = H^*(G/T; \boldsymbol{Z}[1/p]) \cap H^*(G/T; \boldsymbol{Q})^{\Phi(U)}.$$

Proof. First consider the case that the type of U is classical. Let $\tilde{U}$ be the universal covering group of U. Then either $H^*(\tilde{U})$ is torsion free or $H^*(\tilde{U})$ is odd torsion free and $p=2$. Since $H^*(U; \boldsymbol{Z}[1/p]) \cong H^*(\tilde{U}; \boldsymbol{Z}[1/p])$, ($\beta$) of Proposition 3.2 is satisfied for $H=U$ and $A=\boldsymbol{Z}[1/p]$. Thus we have

$$H^*(G/U; \boldsymbol{Z}[1/p]) \cong H^*(G/T; \boldsymbol{Z}([1/p])^{\Phi(U)}$$

It remains the cases $(G, U)=(E_8, S^3\cdot E_7)$ $(p=2)$ and $(G, U)=(E_8, SU(3)\cdot E_6)$ $(p=3)$. We see that $U/C=P_{p-1}(\boldsymbol{C})$ in these cases. Consider the fibering

$$P_{p-1}(\boldsymbol{C}) \xrightarrow{i} G/C \xrightarrow{q} G/U\,.$$

By the homotopy exact sequence, $\pi_2(G/U) \cong \pi_1(U) \cong \boldsymbol{Z}_p$ and by $\pi_2(G/C) \cong \boldsymbol{Z}$, we have that $i_*: \pi_2(P_{p-1}(\boldsymbol{C})) \to \pi_2(G/C)$ is of degree p, and the same is true for H_2 and H^2. Thus $i^*: H^*(G/C; \boldsymbol{Z}[1/p]) \to H^*(P_{p-1}(\boldsymbol{C}); \boldsymbol{Z}[1/p])$ is surjective since $H^*(P_{p-1}(\boldsymbol{C}))$ is multiplicatively generated by $H^2(P_{p-1}(\boldsymbol{C}))$. This shows that the spectral sequence associated with the above fibering collapses, q^*: $H^*(G/U; \boldsymbol{Z}[1/p]) \to H^*(G/C; \boldsymbol{Z}[1/p])$ is injective and $H^{odd}(G/U; \boldsymbol{Z}[1/p])=0$. Then the proposition follows from Proposition 3.2. q.e.d.

Corollary 3.4. *$H^{odd}(G/U) \subset \mathrm{Tor.}\, H^*(G/U)$ and* Tor. $H^*(G/U)$ *consists of only the p-torsion part. The symmetric spaces of* Theorem 2.1 *have vanishing odd torsion part.*

A general method to determine $H^*(G/H)$ for a torsion free maximal rank subgroup H of G has been given in Theorem 2.1 of [12]. The followings are the cases that this theorem can be applied.

Proposition 3.5. *$H^*(C)$ is torsion free for C of the first four cases of* Theorem 2.1 *and the cases except the fourth one of* Proposition 2.3.

Proof. In the first four cases C contains the subgroup H such that $C/H = T^1/\boldsymbol{Z}_2$. In the remaining cases C contains the subgroup $SU(n)\times H$ such that $C/(SU(n)\times H)=T^1/\boldsymbol{Z}_{n(n+1)}$ or $=T^1/\boldsymbol{Z}_n$. In all cases, C is a total space of a principal bundle over a circle with a connected structure group. Thus C is the product of the circle and the structure group which has torsion free cohomology group, and the proposition follows. q.e.d.

Finally we consider the case that $U/C=S^2$ in Theorem 2.1. Then we have the Gysin exact sequence which reduces to exact sequences

$$(3.3)\qquad 0 \longrightarrow H^{2i-3}(G/U; A) \xrightarrow{\cdot\chi} H^{2i}(G/U; A) \xrightarrow{q^*} H^{2i}(G/C; A) \xrightarrow{\theta} H^{2i-2}(G/U; A) \xrightarrow{\cdot\chi} H^{2i+1}(G/U; A) \longrightarrow 0\,,$$

where $\chi \in H^3(G/U; A)$, $2\chi = 0$ *and* A *is a principal ring with unit.*

Proposition 3.6. *If* $U/C = S^2$ *then* Tor. $H^*(G/U) = \chi \cdot H^*(G/U)$ *is an elementary 2-group.*

§4. Cohomology of the symmetric space *FI*

4.1. In the sequel to the last sentence in Proposition 4.1, the suffix of each cohomology class indicates the degree of the class. The mod p cohomology of F_4 is given as follows:

(4.1)
$$H^*(F_4; \boldsymbol{Z}_2) = \Lambda(x_3, x_5, x_{15}, x_{23}) \otimes \boldsymbol{Z}_2[x_6]/(x_6^2),$$
$$x_5 = \mathrm{Sq}^2 x_3, \quad x_6 = \beta x_5 = x_3^2;$$
$$H^*(F_4; \boldsymbol{Z}_3) = \Lambda(x_3, x_7, x_{11}, x_{15}) \otimes \boldsymbol{Z}_3[x_8]/(x_8^3),$$
$$x_7 = \mathscr{P}^1 x_3, \quad x_8 = \beta x_7$$

and
$$H^*(F_4; \boldsymbol{Z}_p) = \Lambda(x_3, x_{11}, x_{15}, x_{23}) \quad \textit{for} \quad p \geq 5.$$

Since x_3 is universally transgressive, $x_5 \in H^*(F_4; \boldsymbol{Z}_2)$ and $x_7 \in H^*(F_4; \boldsymbol{Z}_3)$ are transgressive with respect to the fibering

(4.2)
$$F_4 \xrightarrow{\pi} F_4/C \xrightarrow{i} BC.$$

Let $\delta_6 \in H^6(BC)$ and $\delta_8 \in H^8(BC)$ be classes such that their mod p ($p=2$ for δ_6 and $p=3$ for δ_8) reductions are the transgression images of $x_5 = \mathrm{Sq}^2 x_3$ and $x_7 = \mathscr{P}^1 x_3$ respectively. As is seen in the proof of Proposition 3.5, $C = T^1 \cdot Sp(3)$ is homeomorphic to the product $S^1 \times Sp(3)$ of a circle S^1 and $Sp(3)$. Thus $H^*(C) = \Lambda(s_1, s_3, s_7, s_{11})$. According to Borel [2], we have

$$H^*(BC) = \boldsymbol{Z}[t_2, t_4, t_8, t_{12}]$$

and by putting $t_j = i^*(t_j) \in H^j(F_4/C)$

$$H^*(F_4/T^1 \cdot Sp(3); \boldsymbol{Q}) = \boldsymbol{Q}[t_2, t_4, t_8, t_{12}]/(\sigma_4, \sigma_{12}, \sigma_{16}, \sigma_{24})$$

where $\sigma_i \in \boldsymbol{Z}[t_2, t_4, t_8, t_{12}]$ is an element of degree i and it is a transgression image, in rational coefficient, of the generator x_{i-1} of $H^*(F_4; \boldsymbol{Q}) = \Lambda(x_3, x_{11}, x_{15}, x_{23})$. Now apply Theorem 2.1 of [12], then we have

Proposition 4.1. *There exist generators* $\gamma_6, \gamma_8 \in H^*(F_4/T^1 \cdot Sp(3))$ *and relations* $\rho_j, \rho'_k \in \boldsymbol{Z}[t_i, \gamma_6, \gamma_8; i = 2, 4, 8, 12]$ $(j = 4, 12, 16, 24; k = 6, 8)$ *such that*

$$H^*(F_4/T^1\cdot Sp(3))=\boldsymbol{Z}[t_2, t_4, t_8, t_{12}, \gamma_6, \gamma_8]/(\rho_4, \rho_{12}, \rho_{16}, \rho_{24}, \rho'_6, \rho'_8),$$

(4.3) $$\pi^*(\gamma_6)\equiv x_6 \pmod 2, \quad \pi^*(\gamma_8)\equiv x_8 \pmod 3,$$

$$\rho'_6=2\cdot\gamma_6+\delta_6 \quad and \quad \rho'_8=3\cdot\gamma_8+\delta_8,$$

where ρ_j *is determined by the maximality of the integer* n *in*

(4.4) $$n\cdot\rho_j\equiv\sigma_j \mod(\rho_i, \rho'_6, \rho'_8;\ i<j).$$

Remark. *The situation is similar for* $(G, C)=(E_6, T^1\cdot SU(6))$, *and* Proposition 4.1 *holds for* $H^*(E_6/T^1\cdot SU(6))$ *by adding generators* t_6, t_{10} *and relations* ρ_{10}, ρ_{18}.

4.2. We shall determine the integral cohomology of $F_4/T^1\cdot Sp(3)$ and $F_4/Sp(3)$.

At first $H^*(BT)=\boldsymbol{Z}[w_1, w_2, w_3, w_4]$ for the fundamental weights $\{w_i\}$. Take new generators:

$$t=w_1, \quad y_1=w_2-w_3, \quad y_2=w_3-w_4 \quad \text{and} \quad y_3=w_4.$$

Let R_i (resp. $\tilde{R}$) be the reflection to the plane $\alpha_i=0$ (resp. $\tilde{\alpha}=0$) in the universal covering V of $T(i=1, 2, 3, 4)$. Then we have the following system of the generators of Weyl groups:

$$\Phi(F_4)=<R_1, R_2, R_3, R_4>, \quad \Phi(U)=<R_2, R_3, R_4, \tilde{R}>$$

$$\text{and} \quad \Phi(C)=<R_2, R_3, R_4>.$$

The reflections satisfy

$$R_i(w_i)=w_i-\sum_j(2<\alpha_i, \alpha_j>/<\alpha_j, \alpha_j>)w_j, \quad R_i(w_k)=w_k\ (k\neq i)$$

and $$\tilde{R}(w_i)=w_i-n_iw_1 \quad (n_i=m_i|\alpha_i|^2/|\tilde{\alpha}|^2).$$

Then we have the following table of the action:

	R_1	R_2	R_3	R_4	$\tilde{R}$
t	$-t+y_1+y_2+y_3$				$-t$
y_1		$t-y_1$	y_2		$-t+y_1$
y_2			y_1	y_3	$-t+y_2$
y_3				y_2	$-t+y_3$

where the blanks indicate the trivial action. It is easily seen that t is $\Phi(C)$-invariant, t^2 and the set $\{y_i(t-y_i);\ i=1, 2, 3\}$ are $\Phi(U)$-invariant and the set $S=\{\pm y_i, \pm(t-y_i), \pm(t-y_j-y_k), \pm(y_j-y_k)\}$ is $\Phi(F_4)$-invariant.

Put $z_i=y_i(t-y_i)$ $(i=1, 2, 3)$ and define $q_j\in H^{4j}(BT)$ and $s_n\in H^n(BT)$ respectively by

$$\sum_j q_j=\prod_i(1+z_i) \quad\text{and}\quad \sum_n s_n=\prod_{x\in S}(1+x).$$

Lemma 4.2. *$H^*(BT)^{\Phi(C)}=\boldsymbol{Z}[t, q_1, q_2, q_3]$, $H^*(BT)^{\Phi(U)}=\boldsymbol{Z}[t^2, q_1, q_2, q_3]$ and*

$$H^*(BT;\boldsymbol{Q})^{\Phi(F_4)}=\boldsymbol{Q}[s_4, s_{12}, s_{16}, s_{24}].$$

Proof. By the above definition the elements in the lemma are invariant for the corresponding Weyl group. In general $H^*(BG;\boldsymbol{Q})\cong H^*(BT;\boldsymbol{Q})^{\Phi(G)}$. For $G=C$, $H^*(BT;\boldsymbol{Q})^{\Phi(C)}=\boldsymbol{Q}[x_2, x_4, x_8, x_{12}]$. Let p_i be the i-th elementary symmetric function of y_1^2, y_2^2, y_3^2, then $q_i=(-1)^i p_i+tf_i$ for some f_i. As is well known, $\boldsymbol{Z}[p_1, p_2, p_3]$ is a direct factor of $\boldsymbol{Z}[y_1, y_2, y_3]$. From these facts it follows that $\boldsymbol{Z}[t, q_1, q_2, q_3]$ is a direct factor of $H^*(BT)^{\Phi(C)}$ with the same ranks for each dimension. Thus the first assertion is proved. The second assertion is proved similarly. The last assertion is essentially proved in Lemma 5.1 of [13], or it follows also from the following lemma. q.e.d.

Lemma 4.3. *$s_4/6=-t^2+q_1$, $s_{12}/3\equiv -t^6+4t^2q_2-8q_3 \mod (s_4)$, $s_{16}/10\equiv 3t^2q_3-q_2^2 \mod(s_4, s_{12})$ and $s_{24}/10\equiv -q_2^3+27q_3^2 \mod(s_4, s_{12}, s_{16})$.*

Proof. In the following computations, (i, j, k) runs the cyclic permutation of $(1, 2, 3)$. From the definitions

$$\sum_n s_n=\prod(1-y_i^2)(1-(t-y_i)^2)(1-(y_j-y_k)^2)(1-(t-y_j-y_k)^2)$$

$$=\prod(1-t^2+2z_i+z_i^2)(1-t^2+2(z_j+z_k)+(z_j-z_k)^2).$$

Thus $s_4=6(-t^2+q_1)$, and by putting $\sum_i z_i=q_1=t^2$ we have

$$\sum_n s_n\equiv\prod(1+2(q_2-3z_jz_k)+(-t^6+4t^2q_2-8q_3)+z_i^2(z_j-z_k)^2)$$

and $s_{12}\equiv 3(-t^6+4t^2q_2-8q_3) \mod(s_4)$. And, modulo (s_4, s_{12})

$$\sum_n s_n\equiv 1+10(-q_2^2+3t^2q_3)+10(2q_2^3-9t^2q_2q_3+27q_3^2)+\text{higher terms},$$

from which the last two formulas of the lemma follows. q.e.d.

The canonical map $BT\to BC$ induces an isomorphism

$$H^*(BC) \cong H^*(BT)^{\Phi(C)} = \boldsymbol{Z}[t, q_1, q_2, q_3].$$

So, in Proposition 4.1, we may use the following identification:

$$t = t_2, \quad q_i = t_{4i} \ (i = 1, 2, 3) \quad \text{and} \quad \sigma_j = s_j \ (j = 4, 12, 16, 24).$$

Then we have the following description of $H^*(F_4/C)$.

Theorem 4.4. *There exist elements $u \in H^6$ and $v \in H^8$ such that $2u = t^3$ and $3v = q_2$. Rewriting q_3 with $w \in H^{12}$, we have*

$$H^*(F_4/T^1 \cdot Sp(3)) = \boldsymbol{Z}[t, u, v, w]/(t^3 - 2u, u^2 - 3t^2v + 2w, 3v^2 - t^2w, v^3 - w^2).$$

Proof. Obviously we can take $\rho_4 = s_4/6 = -t^2 + q_1$ which must be the transgression image $\tau(x_3) = \rho_4$ of a generator x_3 of $H^3(F_4) \cong \boldsymbol{Z}$. We have

$$\tau(x_5) = \mathrm{Sq}^2\tau(x_3) = \mathrm{Sq}^2(t^2 + q_1) = \sum \mathrm{Sq}^2(y_it + y_i^2) = \sum y_it(y_i + t) = tq_1$$

and
$$\tau(x_7) = \mathscr{P}^1\tau(x_3) = \mathscr{P}^1(-t^2 + q_1) = -\mathscr{P}^1t^2 + \sum \mathscr{P}^2(y_it - y_i^2)$$
$$= t^4 + \sum(y_i^3t + y_it^3 + y_i^4) = t^4 + t^2q_1 + q_1^2 + q_2.$$

So, we can choose $\delta_6 = tq_1 \equiv t^3$ and $\delta_8 = t^4 + t^2q_1 + q_1^2 + q_2 \equiv 3t^4 + q_2 \mod (\rho_4)$. Then by putting $u = -\gamma_6$ and $v = -\gamma_8 - t^4$ and by using the relations $\rho_4 = \rho'_6 = \rho'_8 = 0$, it follows from Proposition 4.1 that

$$H^*(F_4/T^1 \cdot Sp(3)) = \boldsymbol{Z}[t, u, v, w]/(t^3 - 2u, \rho_{12}, \rho_{16}, \rho_{24}),$$

where $3v = q_2$, $w = q_3$ and the relations ρ_j $(j = 12, 16, 24)$ are determined by the maximality of the integer n in $n \cdot \rho_j \equiv s_j \mod (t^3 - 2u, \rho_i; i < j)$. It is easily computed that $\rho_{12} = -u^2 + 3t^2v - 2w$ $(n = 12)$, $\rho_{16} = t^2w - 3v^2$ $(n = 30)$ and $\rho_{24} = w^2 - v^3$ $(n = 270)$. q.e.d.

Corollary 4.5. *The following elements form an additive base of $H^*(F_4/T^1 \cdot Sp(3))$.*

deg=	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
	1	t	t^2	u	tu v	t^2u tv	t^2v w	uv tw	tuv v^2	t^2uv x	vw tx	tvw t^2x	y	z	tz	t^2z

where $x = uw - tv^2$, $y = 5w^2 - tuv^2$ and $z = uvw - 4ty$.

Proof. Note that $\dim. F_4/C = 30$. By simple computations it is seen that these elements span $H^*(F_4/C)$. Then their independence will follow from the

Poincaré polynomial $P(F_4/C, x)=(1-x^4)(1-x^{12})(1-x^{16})(1-x^{24})(1-x^2)^{-1}\cdot(1-x^4)^{-1}(1-x^8)^{-1}(1-x^{12})^{-1}=(1+x^8)(1+x^2+x^4+x^6+\cdots+x^{22})$. q.e.d.

Corollary 4.6. *For* $u\in H^6$, $v\in H^8$, $w\in H^{12}$ *and* $s\in H^{23}$ *we have*

$$H^*(F_4/Sp(3))=\boldsymbol{Z}[u, v, w, s]/(2u, u^2+2w, 3v^2, v^3-w^2, us, ws, s^2),$$

that is, $\mathrm{Tor.}\,H^*(F_4/Sp(3))=\boldsymbol{Z}_2<u, uv, uw, uvw>+\boldsymbol{Z}_4<w, vw>+\boldsymbol{Z}_3<v^2>$ *and* $H^*(F_4/Sp(3))/\mathrm{Tor.}=\Lambda(v, s)$.

Proof. The Gysin exact sequence associated with the fibering $S^1\to F_4/Sp(3)\to F_4/C$ is reduced to

$$0\longrightarrow H^{2i-1}(F_4/Sp(3))\xrightarrow{\theta}H^{2i-2}(F_4/C)$$

$$\xrightarrow{h}H^{2i}(F_4/C)\xrightarrow{p^*}H^{2i}(F_4/Sp(3))\longrightarrow 0,$$

where h is the multiplication with some $c\in H^2(F_4/C)$. Since $F_4/Sp(3)$ is 2-connected, we have $h(1)=\pm t$ and $h(\alpha)=\pm t\alpha$. Then, by Corollary 4.5,

$$H^{\mathrm{even}}(F_4/Sp(3))\cong\mathrm{Coker}\,h=\boldsymbol{Z}[u, v, w]/(2u, u^2+2w, 3v^2, v^3-w^2)$$

and $$H^{\mathrm{odd}}(F_4/Sp(3))\cong\mathrm{Ker}\,h=\boldsymbol{Z}<14tvw-9uv^2, t^2z>.$$

Putting $s=\theta^{-1}(14tvw-9uv^2)$ we have $\theta(p^*(v)s)=v\theta(s)=t^2z$. Then the corollary follows immediately. q.e.d.

4.3. Next we shall determine the cohomology of BU, $U=S^3\cdot Sp(3)$. Consider the fibering

$$U/C=S^2\longrightarrow BC\xrightarrow{\rho}BU$$

and the associated Gysin sequence

$$(4.5)_n\qquad 0\longrightarrow H^{2n-3}(BU)\xrightarrow{\cdot\chi}H^{2n}(BU)\xrightarrow{\rho^*}H^{2n}(BC)$$

$$\xrightarrow{\theta}H^{2n-2}(BU)\xrightarrow{\cdot\chi}H^{2n+1}(BU)\longrightarrow 0,$$

where $2\chi=0$ *and* $\theta(\rho^*(\alpha)\beta)=\alpha\cdot\theta(\beta)$.

Proposition 4.7. $H^*(BU)=\boldsymbol{Z}[\chi, t_4, u_4, u_8, u_{12}]/(2\chi)$ *where* $\rho^*(t_4)=t^2$ *and* $\rho^*(u_{4i})=q_i$ $(i=1, 2, 3)$.

Proof. $H^0(BU)=\boldsymbol{Z}<1>$ and $H^1(BU)=0$ since BU is 1-connected. Since

$\pi_2(BU)\cong\pi_1(U)\cong \boldsymbol{Z}_2$, $H^3(BU)\neq 0$, and by $(4.5)_1$, $H^3(BU)=\boldsymbol{Z}_2<\chi>$, $\theta(t)=\pm 2$ and $H^2(BU)=0$. Then by $(4.5)_2$, $H^5(BU)=0$ and $\rho^*\colon H^4(BU)\cong H^4(BC)=\boldsymbol{Z}<t,^2 q_1>$. Thus $H^4(BU)=\boldsymbol{Z}<t_4, u_4>$ for $t_4=\rho^{*-1}(t^2)$ and $u_4=\rho^{*-1}(q_1)$. Here we remark that

$(4.6)_m$ *if* $\rho^*\colon H^{4m}(BU)\longrightarrow H^{4m}(BC)$ *is surjective, then* $\theta\colon H^{4m+2}(BC)\longrightarrow H^{4m}(BU)$ *is injective and* $\mathrm{Im}\,\theta=2H^{4m}(BU)$. *Thus* $\cdot\chi\colon H^{4m-2}(BU)\cong H^{4m+1}(BU)$, $\cdot\chi\colon H^{4m-1}(BU)\cong H^{4m+2}(BU)$, $\cdot\chi\colon H^{4m}(BU)\otimes \boldsymbol{Z}_2\cong H^{4m+3}(BU)$ *and the sequence* $0\longrightarrow H^{4m-3}(BU)\xrightarrow{\cdot\chi} H^{4m}(BU)\xrightarrow{\rho^*} H^{4m}(BC)\longrightarrow 0$ *is exact.*

This follows from the exactness of $(4.5)_{2m}$ and $(4.5)_{2m+1}$ and the fact that $\cdot t\colon H^{4m}(BC)\cong H^{4m+2}(BC)$ (as free modules) and $\theta(\rho^*(\alpha)t)=2\alpha$.

By $(4.6)_1$, $H^6(BU)=\boldsymbol{Z}_2<\chi^2>$ and $H^7(BU)=\boldsymbol{Z}_2<t_4\chi, u_4\chi>$. The following lemma (4.7) will be proved later.

(4.7) $\chi^3\neq 0$ *in* $H^9(BU)$ *and* $<t_4\chi^3, u_4\chi^3>\cong \boldsymbol{Z}_2+\boldsymbol{Z}_2$ *in* $H^{13}(BU)$.

Then $\cdot\chi\colon H^6(BU)\to H^9(BU)$ is injective. By the exactness of $(4.5)_4$, $\rho^*\colon H^8(BU)\to H^8(BC)$ is surjective, and $u_8=\rho^{*-1}(q_2)$ exists. By $(4.6)_2$, $H^{10}(BU)=\boldsymbol{Z}_2<t_4\chi^2, u_4\chi^2>$. Again using the second part of (4.7) and $(4.5)_6$, we have the existence of $u_{12}=\rho^{*-1}(q_3)$.

Since $\sum H^{4m}(BC)$ is multiplicatively generated by $t^2=\rho^*(t_4)$ and $q_i=\rho^*(u_{4i})$ $(i=1, 2, 3)$, $\rho^*\colon H^{4m}(BU)\to H^{4m}(BC)$ is surjective for each m. Therefore the proposition is proved by applying $(4.6)_m$ inductively. q.e.d.

Proof of (4.7). Let K be the kernel of the natural homomorphism $S^3\times Sp(3)\to U=S^3\cdot Sp(3)$. Imbed $Sp(1)$ into $Sp(3)$ by the diagonal map. Then $K\subset S^3\times Sp(1)$, and $(S^3\times Sp(1))/K$ is isomorphic to $SO(4)$. Thus we have natural maps $SO(4)\to U$ and $j\colon BSO(4)\to BU$. It is easily seen that the imbedding of $Sp(1)$ into $Sp(3)$ induces a homomorphism of degree 3 of π_3. It follows that $H^*(U/SO(4); \boldsymbol{Z}_2)=H^*(Sp(3)/Sp(1); \boldsymbol{Z}_2)=0$ for degree <7. Thus $j^*\colon H^*(BU; \boldsymbol{Z}_2)\cong H^*(BSO(4); \boldsymbol{Z}_2)$ for degree <7. As is well known $H^*(BSO(4); \boldsymbol{Z}_2)=\boldsymbol{Z}_2[w_2, w_3, w_4]$. From the results of $H^*(BU)$ in lower dimensions we see that $j^*<\chi, t_4, u_4\ (\mathrm{mod}\,2)>=<w_3, w_2^2, w_4>$. Then (4.7) follows from $w_3^3\neq 0$ and $<w_2^2w_3^3, w_3^3w_4>\cong \boldsymbol{Z}_2+\boldsymbol{Z}_2$.

Corollary 4.8. $H^*(BU; \boldsymbol{Z}_2)=\boldsymbol{Z}_2[u_2, u_3, u_4, u_8, u_{12}]$, $u_i=j^*w_i$ $(i=2, 3)$.

4.4. We shall determine the cohomology of the symmetric space $\boldsymbol{FI}=F_4/U$, $U=S^3\cdot Sp(3)$. First consider the homomorphism

$$q^*\colon H^*(\boldsymbol{FI}; \boldsymbol{Z}[1/2])\longrightarrow H^*(F_4/C; \boldsymbol{Z}[1/2])$$

induced by the projection of the fibering ($C=T^1\cdot Sp(3)$)

$$U/C=S^2 \longrightarrow F_4/C \xrightarrow{q} \boldsymbol{FI}=F_4/U .$$

Theorem 4.4 implies $H^*(F_4/C;\ \boldsymbol{Z}[1/2])=\boldsymbol{Z}[1/2][t,\ v,\ w]/(t^6-12t^2v+8w,\ 3v^2-t^2w,\ v^3-w^2)$. By Lemmas 4.2 and 4.3, $H^*(F_4/T;\ \boldsymbol{Q})^{\Phi(U)}\cong H^*(BT;\ \boldsymbol{Q})^{\Phi(U)}/(H^+(BT;\ \boldsymbol{Q})^{\Phi(F_4)})=\boldsymbol{Q}[t^2,\ q_1,\ q_2,\ q_3]/(s_4,\ s_{12},\ s_{16},\ s_{24})=\boldsymbol{Q}[t,\ v,\ w]/(s_{12},\ s_{16},\ s_{24})$. Thus it follows from Proposition 3.3

(4.8) *q^* defines an isomorphism*

$$H^*(\boldsymbol{FI};\ \boldsymbol{Z}[1/2])\cong\boldsymbol{Z}[1/2][t^2,\ v,\ w]/(t^6-12t^2v+8w,\ 3v^2-t^2w,\ v^3-w^2) .$$

Recall the Gysin sequence (3.3) ($A=\boldsymbol{Z}$ or $\boldsymbol{Z}_2$)

$$(4.9)_n \qquad 0 \longrightarrow H^{2n-3}(\boldsymbol{FI};\ A) \xrightarrow{\cdot\chi} H^{2n}(\boldsymbol{FI};\ A) \xrightarrow{q^*} H^{2n}(F_4/C;\ A)$$

$$\xrightarrow{\theta} H^{2n-2}(\boldsymbol{FI};\ A) \xrightarrow{\cdot\chi} H^{2n+1}(\boldsymbol{FI};\ A) \longrightarrow 0 ,$$

where $2\chi=0$, θ satisfies $\theta(q^*(\alpha)\beta)=\alpha\cdot\theta(\beta)$ and the sequence commutes with the mod 2 *reduction $H^*(\ \ )\to H^*(\ \ ;\ \boldsymbol{Z}_2)$.*

Lemma 4.9. (i) *Changing θ to $-\theta$ if it is necessary, we have $\theta(t)=2$. Put $f_4=\theta(u)$. Then $q^*(f_4)=t^2$, $H^1(\boldsymbol{FI})=H^2(\boldsymbol{FI})=H^5(\boldsymbol{FI})=0$, $H^3(\boldsymbol{FI})=\boldsymbol{Z}_2<\chi>$ and $H^4(\boldsymbol{FI})=\boldsymbol{Z}<f_4>$.*

(ii) *There exist elements $f_8\in H^8(\boldsymbol{FI})$ and $f_{12}\in H^{12}(\boldsymbol{FI})$ such that $q^*(f_8)=v$ and $q^*(f_{12})=w$.*

Proof. (i) $H^1(\boldsymbol{FI})=0$ since $\boldsymbol{FI}$ is 1-connected. Since $\pi_2(\boldsymbol{FI})\cong\pi_1(U)\cong\boldsymbol{Z}_2$, $H^3(\boldsymbol{FI})\neq 0$. By the exactness of $(4.9)_1$, $\theta(t)=2$ (changing θ by $-\theta$ if $\theta(t)=-2$), and $H^2(\boldsymbol{FI})=0$, $H^3(\boldsymbol{FI})=\boldsymbol{Z}_2<\chi>$. By the exactness of $(4.9)_2$, $H^5(\boldsymbol{FI})=0$ and q^*: $H^4(\boldsymbol{FI})\cong H^4(F_4/C)=\boldsymbol{Z}<t^2>$. Put $f=q^{*-1}(t^2)$, then $2\theta(u)=\theta(2u)=\theta(t^3)=\theta(q^*(f)t)=f\cdot\theta(t)=2f$. Since $H^4(\boldsymbol{FI})$ is free, $f_4=\theta(u)=f$.

(ii) Consider the following commutative diagram of natural maps:

$$\begin{array}{ccc} F_4/C & \xrightarrow{q} & F_4/U=\boldsymbol{FI} \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} \\ BC & \xrightarrow{\rho} & BU . \end{array}$$

By Proposition 4.7, $q^*(i^*(u_8))=i_0^*(\rho^*(u_8))=q_2=3v$ and $q^*(i^*(u_{12}))=i_0^*(\rho^*(u_{12}))=q_3=w$. Thus $q^*(f_{12})=w$ for $f_{12}=i^*(u_{12})$. $\theta(2v)=2\theta(v)=0$ since $2H^6(\boldsymbol{FI})=0$. So, there exists $\alpha\in H^8(\boldsymbol{FI})$ such that $q^*(\alpha)=2v$. By putting $f_8=i^*(u_8)-\alpha$, we have $q^*(f_8)=v$. q.e.d.

Now consider $(4.9)_n$ for $A=\boldsymbol{Z}_2$. By Theorem 4.4 we have

$$(4.10)\quad H^*(F_4/T^1\cdot Sp(3);\boldsymbol{Z}_2)=\boldsymbol{Z}_2[t,u,v,w]/(t^3,u^2-t^2v,v^2-t^2w,w^2-v^3)$$

$$=\boldsymbol{Z}_2[t]/(t^3)\otimes\Lambda(u,v,w).$$

Lemma 4.10. (i) *Let* $y_3=\chi \bmod 2$. *There exists* $y_2\in H^2(\boldsymbol{FI};\boldsymbol{Z}_2)$ *such that* $q^*(y_2)=t$. *Then we have* $y_3=\mathrm{Sq}^1y_2$, $\mathrm{Sq}^2y_3=y_2y_3$, $y_2^2=f_4 \bmod 2$, $\theta(u)=y_2^2$, $y_2^3=y_3^2$, $y_2^2y_3=0$, $y_2y_3^2=y_2^4\neq 0$ *and* $y_3^3=0$.

(ii) *Let* $y_8=f_8 \bmod 2$ *and* $y_{12}=f_{12} \bmod 2$, *then*

$$H^*(\boldsymbol{FI};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_2,y_3]/(y_2^3+y_3^2,y_2^2y_3)\otimes\Lambda(y_8,y_{12}).$$

Proof. (i) From Lemma 4.9, it follows $H^i(\boldsymbol{FI};\boldsymbol{Z}_2)=\boldsymbol{Z}_2\langle y_i\rangle$ for $i=2,3,4$, $\mathrm{Sq}^1y_2=y_3$, $y_3=\chi \bmod 2$, $y_4=f_4 \bmod 2$ and $\theta(u)=y_4$. By $(4.9)_1$, $q^*(y_2)=t$, $q^*(y_2^2)=t^2\neq 0$, thus $y_2^2=y_4$. By the exactness of $(4.9)_n$ $(n=2,3,4)$, we have $H^5(\boldsymbol{FI};\boldsymbol{Z}_2)=\boldsymbol{Z}_2\langle y_2y_3\rangle$, $H^6(\boldsymbol{FI};\boldsymbol{Z}_2)=\boldsymbol{Z}_2\langle y_3^2\rangle$, $H^7(\boldsymbol{FI};\boldsymbol{Z}_2)=0$ and $y_2y_3^2\neq 0$. Then $y_2^2y_3=0$ and $y_2^3=ay_3^2$ for some $a\in\boldsymbol{Z}_2$. Since $\mathrm{Sq}^1\mathrm{Sq}^2y_3=\mathrm{Sq}^3y_3=y_3^2\neq 0$, $\mathrm{Sq}^2y_3=y_2y_3$. By use of Cartan formula, $0=a(\mathrm{Sq}^1y_3)^2=\mathrm{Sq}^2(ay_3^2)=\mathrm{Sq}^2y_2^3=y_2(\mathrm{Sq}^1y_2)^2+y_2^2\mathrm{Sq}^2y_2=(a+1)y_2y_3^2$. Thus $a=1$, $y_2^3=y_3^2$, $y_2^4=y_2y_3^2$ and $y_3^3=y_2^3y_3=0$.

(ii) Put $F^*=\boldsymbol{Z}_2[y_2,y_3]/(y_2^3+y_3^2,y_2^2y_3)\otimes\Lambda(y_8,y_{12})=\{1,y_2,y_3,y_2^2,y_2y_3,y_2^3=y_3^2,y_2^4=y_2y_3^2\}\otimes\Lambda(y_8,y_{12})$, then we have the exactness of a sequence

$$0\longrightarrow F^{2n-3}\xrightarrow{\cdot y_3}F^{2n}\xrightarrow{q^*}H^{2n}(F_4/C;\boldsymbol{Z}_2)\xrightarrow{\theta}F^{2n-2}\xrightarrow{\cdot y_3}F^{2n+1}\longrightarrow 0,$$

where the homomorphisms are given by the multiplication with y_3 $(y_3^3=0)$, by a multiplicative q^* which carries y_2,y_3,y_8,y_{12} to $t,0,v,w$ respectively and by $\theta(t^iv^jw^k)=0$, $\theta(t^iuv^jw^k)=y_2^{i+1}y_8^jy_{12}^k(i=0,1,2;\ j,k=0,1)$. Applying the five lemma to the natural map of this sequence to $(4.9)_n$ of $A=\boldsymbol{Z}_2$, we have the assertion of (ii) by induction on n. q.e.d.

Theorem 4.11. *Let* $r_{12}=f_4^3-12f_4f_8+8f_{12}$, $r_{16}=3f_8^2-f_4f_{12}$ *and* $r_{24}=f_8^3-f_{12}^2$. *Then we have*

(i) $H^*(\boldsymbol{FI};\boldsymbol{Z}[1/2])=\boldsymbol{Z}[1/2][f_4,f_8,f_{12}]/(r_{12},r_{16},r_{24})$,

(ii) $H^*(\boldsymbol{FI};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_2,y_3,y_8,y_{12}]/(y_2^3+y_3^2,y_2^2y_3,y_8^2+y_2^2y_{12},y_8^3+y_{12}^2)$

and (iii) $H^*(\boldsymbol{FI})=\boldsymbol{Z}[\chi,f_4,f_8,f_{12}]/(2\chi,\chi f_4,\chi^3,r_{12},r_{16},r_{24})$

$=\boldsymbol{Z}[f_4,f_8,f_{12}]/(r_{12},r_{16},r_{24})$ (*free part*)

$+\boldsymbol{Z}_2\langle\chi,\chi^2\rangle\otimes\Lambda(f_8,f_{12})$ (*torsion part*).

Proof. (i) follows from (4.8) and Lemma 4.10. Since y_3, y_8, y_{12} are integral classes, $\mathrm{Sq}^1(y_3)=\mathrm{Sq}^1(y_8)=\mathrm{Sq}^1(y_{12})=0$. Then it follows from Lemma 4.10 that $\mathrm{Sq}^1 H^*(\boldsymbol{FI}; \boldsymbol{Z}_2)=\boldsymbol{Z}_2<y_3, y_2^3>\otimes\Lambda(y_8, y_{12})$ and the derived group of $H^*(\boldsymbol{FI}; \boldsymbol{Z}_2)$ with respect to Sq^1 is $\boldsymbol{Z}_2[y_2]/(y_2^3)\otimes\Lambda(y_8, y_{12})$. By use of Proposition 3.6, we have Tor. $H^*(\boldsymbol{FI})=\boldsymbol{Z}_2<\chi, \chi^2>\otimes\Lambda(f_8, f_{12})$, $\sum H^{4i}(\boldsymbol{FI})$ is the free part, $H^{4i}(\boldsymbol{FI})\otimes\boldsymbol{Z}_2\cong H^{4i}(\boldsymbol{FI}; \boldsymbol{Z}_2)$ and that $H^{4i}(\boldsymbol{FI})\to H^{4i}(\boldsymbol{FI}; \boldsymbol{Z}[1/2])$ is injective. From the last statement follow the relations $r_{12}=r_{16}=r_{24}=0$, and then $r_{16}=y_8^2+y_2^2y_{12}=0$, $r_{24}=y_8^3+y_{12}^2=0$ in $H^*(\boldsymbol{FI}; \boldsymbol{Z}_2)$. Thus (ii) is proved by Lemma 4.10. Now, consider $Z[f_4, f_8, f_{12}]/(r_{12}, r_{16}, r_{24})$. By tensoring $\boldsymbol{Z}_2$ and $\boldsymbol{Z}[1/2]$, we obtain $\sum H^{4i}(\boldsymbol{FI}; \boldsymbol{Z}_2)$ and $\sum H^{4i}(\boldsymbol{FI}; \boldsymbol{Z}[1/2])$ which have the same rank over $\boldsymbol{Z}_2$ and $\boldsymbol{Z}[1/2]$ respectively. This shows $\sum H^{4i}(\boldsymbol{FI}) =\boldsymbol{Z}[f_4, f_8, f_{12}]/(r_{12}, r_{16}, r_{24})$. The relations $2\chi=\chi f_4=\chi^3=0$ are obvious. So, (iii) is proved. q.e.d.

§5. Torsion in the cohomology of the irreducicle symmetric spaces of exceptional type

The purpose of this section is to prove the following

Theorem 5.1. *The cohomology groups of the irreducible symmetric spaces of exceptional type are odd torsion free.*

The symmetric spaces ***FII***, ***EIII*** and ***EVII*** are hermitian and their cohomology groups are torsion free. Also $H^*(\boldsymbol{EIV})=\Lambda(x_9, x_{17})$ is torsion free [1]. By Corollary 3.4, the cohomology groups of ***G***, ***FI***, ***EII***, ***EV***, ***EVI***, ***EVIII*** and ***EIX*** are odd torsion free.

It remains the symmetric space $\boldsymbol{EI}=E_6/PSp(4)$. The spaces ***EI***, $\boldsymbol{EIV} =E_6/F_4$ and $\boldsymbol{FI}=F_4/S^3\cdot Sp(3)$ are related to each other by

$$PSp(4)\cap F_4=S^3\cdot Sp(3).$$

Let $C=T^1\cdot Sp(3)\subset S^3\cdot Sp(3)\subset F_4\subset E_6$ and consider the fibering

$$F_4/C \xrightarrow{i'} E_6/C \longrightarrow E_6/F_4=\boldsymbol{EIV}. \tag{5.1}$$

Proposition 5.2. *$H^*(E_6/C)\cong H^*(F_4/C)\otimes H^*(\boldsymbol{EIV})$ as algebras.*

Proof. Since ***EIV*** is 8-connected,

$$0 \longrightarrow H^n(E_6/C) \xrightarrow{i'^*} H^n(F_4/C) \xrightarrow{\tau} H^{n+1}(\boldsymbol{EIV})$$

is exact for $n\leq 8$. Thus t and u are i'^*-images. $q_2=3v$ and $w=q_3$ are i^*-images for the map $i: F_4/C\to BC$ of (4.2). The map i can be extended over E_6/C. Thus $3v$ and w are i'^*-images. Since $H^9(\boldsymbol{EIV})\cong\boldsymbol{Z}$, v is also an i'^*-

image. Since $H^*(F_4/C)$ is multiplicatively generated by t, u, v and w, it follows that F_4/C is totally non-homologous to zero in E_6/C, and the spectral sequence associated with (5.1) collapses. Then we obtain an additive isomorphism of the proposition. This also shows that $i'^*: H^{2n}(E_6/C) \cong H^{2n}(F_4/C)$ for even $2n < 26$, and that the relations in $H^*(E_6/C)$ which correspond to those in Theorem 4.4 hold. q.e.d.

Since $Sp(4)/Sp(3) = S^{15}$, $PSp(4)/C = Sp(4)/(S^1 \times Sp(3)) = P_7(\boldsymbol{C})$ and we have a fibering

$$P_7(\boldsymbol{C}) \xrightarrow{i_1} E_6/C \xrightarrow{q_1} \boldsymbol{EI} = E_6/PSp(4). \tag{5.2}$$

Proposition 5.3. *q_1^*: $H^*(\boldsymbol{EI}; \boldsymbol{Z}[1/2]) \to H^*(E_6/C; \boldsymbol{Z}[1/2])$ is injective and $H^*(E_6/C; \boldsymbol{Z}[1/2]) \cong H^*(P_7(\boldsymbol{C}); \boldsymbol{Z}[1/2]) \otimes H^*(\boldsymbol{EI}; \boldsymbol{Z}[1/2])$ (additively).*

Proof. By concerning low dimensional homotopy groups, we see that i_1^*: $H^*(E_6/C; \boldsymbol{Z}[1/2]) \to H^*(P_7(\boldsymbol{C}); \boldsymbol{Z}(1/2])$ is surjective for degree $=2$, and then for all degrees since $H^*(P_7(\boldsymbol{C}))$ is multiplicatively generated by $H^2(P_7(\boldsymbol{C}))$. Thus the spectral sequence associated to (5.2) with coefficient $\boldsymbol{Z}[1/2]$ collapses. Then the proposition follows. q.e.d.

This proposition shows that $H^*(\boldsymbol{EI})$ is odd torsion free, and the proof of Theorem 5.1 has been established.

Finally recall from [5], $\boldsymbol{G} = G_2/SO(4)$,

$$H^*(\boldsymbol{G}; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_2, y_3]/(y_2^3 + y_3^2, y_2^2 y_3) \tag{5.3}$$

and $$H^*(\boldsymbol{G}) = \boldsymbol{Z}[\chi, f_4]/(2\chi, \chi f_4, \chi^3, f_4^3) = \boldsymbol{Z}[f_4]/(f_4^3) + \boldsymbol{Z}_2 < \chi, \chi^2 >$$

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