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Cauchy problem in Gevrey classes for non-strictly hyperbolic equations of second order

By

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§ 1. Introduction

In his remarkable article [1], Y. Ohya considered the Cauchy problem for linear partial differential equations of order m which has real characteristic roots of constant multiplicity and proved its well-posedness in the Gevrey classes $\gamma_{loc}^{(a)}(1<a<m/(m-1))$ and the existence of a finite domain of dependence. There no condition is assumed on the lower order terms, which differs very much from the well-posedness in $\mathcal{E}$, cf. [9]. These facts seem to imply that Gevrey classes are suitable spaces to treat hyperbolic differential equations.

Since then the Cauchy problem in Gevrey classes has been studied in detail from various viewpoints, e.g. Leray-Ohya [2], Steinberg [4], Beals [5], Ivriĭ [6], etc. However we should remark the followings. In [1], [2], [4], the smoothenss of the characteristic roots play an essential role. In [5], [6], the smoothness of the characteristic roots is not assumed, but it is assumed in [5] that the coefficients do not depend on time variable t and also that the characteristic roots do not vanish, and in [6] that the coefficients of the principal part of the differential operator are analytic.

Now we state our result. Consider the partial differential equation of second order

$$L[u]=\delta^2 u-\partial_i(a^{ij}\partial_j u)-b^i\partial_i u-cu=f(x, t), \tag{1.1}$$

$(x, t)\in\Omega=\boldsymbol{R}^n\times[0, h]$, $h>0$, where $\delta=\partial_t+a^i\partial_i+b^0$, $a^{ij}(x, t)=a^{ji}(x, t)$, it is supposed that repeated indices are summed from 1 to n, e.g. $\partial_i(a^{ij}\partial_j u)=\sum_{i,j=1}^n \partial_i(a^{ij}\partial_j u)$[1].

Definition 1.1. $(\gamma_{loc}^{(a)}, \gamma^{(a)}, \gamma_0^{(a)})$.

We say that $\phi(x)\in\mathcal{E}$ *belongs to* $\gamma_{loc}^{(a)}$ *if for any compact set* K, *there exist*

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1) Throughout this paper, we use the following abbreviations and function spaces: $x=(x_1, x_2, \cdots, x_n)$, $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$, $p=(p_1, p_2, \cdots, p_n)$; p_i are non-negative integers, $|p|=p_1+p_2+\cdots+p_n$, $e_i=(0, \cdots, 1, \cdots, 0)$, $\partial_t=\partial/\partial t$, $\partial_i=\partial/\partial x_i$, $\partial^p=\partial_1{}^{p_1}\partial_2{}^{p_2}\cdots\partial_n{}^{p_n}$, $\partial^p\phi(x)=\phi_{(p)}(x)$, $\partial_i\phi(x)=\{\phi(x)\}'_{x_i}$, $(u, v)=\int_{\boldsymbol{R}^n}u(x)v(x)dx$, $\|u\|^2=\int_{\boldsymbol{R}^n}|u(x)|^2dx$.

$\phi\in\mathcal{E}$ means that ϕ is an infinitely differentiable function, $\phi(x)\in\mathcal{D}_{L^2}{}^\infty$ means that $\phi(x)$ and all of its derivatives (in the distribution sense) are square integrable. $\phi(x, t)\in\mathcal{D}_{L^2}{}^\infty[0, h]$ means that $t\to\phi(x, t)\in\mathcal{D}_{L^2}{}^\infty$, $0<t<h$, is infinitely differentiable, cf. [8].

two constants ρ and C such that

$$|\partial^p\phi(x)|\leqq\frac{|p|!^a}{\rho^{|p|}}C,\quad x\in K,\ \text{for any } p. \tag{1.2}$$

If (1.2) holds for any x, we say that $\phi(x)$ belongs to $\gamma^{(a)}$. $\phi(x)\in\gamma_0{}^{(a)}$ means that $\phi(x)\in\gamma^{(a)}$ has a compact support.

We assume

$$\begin{cases} \text{i) the coefficients } \in\gamma^{(a)}(\Omega), \\ \text{ii) } a^i(x,t),\ a^{ij}(x,t) \text{ are real-valued}, \\ \text{iii) } a^{ij}(x,t)\xi_i\xi_j\geqq 0 \quad \text{for any } (x,t,\xi)\in\Omega\times\boldsymbol{R}^n. \end{cases} \tag{1.3}$$

Let $\partial_i b^{ij}\partial_j$, $b^{ij}=b^{ji}$, be the principal part of the commutator $[\delta, \partial_i a^{ij}\partial_j]$ and assume also

$$\begin{cases} \text{either iv): there exists a constant } A \text{ such that} \\ \qquad b^{ij}(x,t)\xi_i\xi_j\geqq -Aa^{ij}(x,t)\xi_i\xi_j,\quad \text{for any } (x,t,\xi)\in\Omega\times\boldsymbol{R}^n, \\ \text{or iv}'\text{): there exists a constant } A \text{ such that} \\ \qquad b^{ij}(x,t)\xi_i\xi_j\leqq Aa^{ij}(x,t)\xi_i\xi_j,\quad \text{for any } (x,t,\xi)\in\Omega\times\boldsymbol{R}^n. \end{cases} \tag{1.4}$$

Then our main result is

Theorem 1.1. *Assume* (1.3) *and* (1.4), *then if* $1<a<2$, *for any given* $f(x,t)\in\gamma_{loc}^{(a)}(\Omega)$ *and any given initial data* $(u(x,0), \partial_t u(x,0))\in\gamma_{loc}^{(a)}(\boldsymbol{R}^n)$, *there exists a solution* $u(x,t)$ *of the equation* (1.1) *in* Ω, *which belongs to* $\gamma_{loc}^{(a)}(\Omega)$. *Moreover the solution is unique in* $\mathcal{E}^2(\Omega)$.

Remark 1.1. To put it in the concrete,

$$b^{ij}=(a^{ij})_t'+a^k(a^{ij})'_{x_k}-(a^i)'_{x_k}a^{kj}-a^{ik}(a^j)'_{x_k}. \tag{1.5}$$

Example 1. Consider the differential equation

$$\partial_t{}^2u-\partial_i(a^{ij}\partial_j u)-b^0\partial_t u-b^i\partial_i u-cu=f(x,t), \tag{1.6}$$

$(x,t)\in\Omega$, assuming that a^{ij} do not depend on t, i.e. $a^{ij}=a^{ij}(x)$. In this case, $b^{ij}(x,t)\equiv 0$. Therefore by Theorem 1.1, we can see that if we assume only (1.3), the Cauchy problem for the equation (1.6) is well-posed in $\gamma_{loc}^{(a)}$, $1<a<2$.

Example 2. Consider the differential equation

$$\partial_t{}^2u-\partial(a\partial u)-b^0\partial_t u-b\partial u-cu=f(x,t), \tag{1.7}$$

$(x,t)\in\boldsymbol{R}^1\times[0,h]$. Consider the following two simple but typical cases: 1) $a(x,t)=\phi(x)t^k$; $\phi(x)\geqq 0$, $k\geqq 0$ is an integer, 2) $a(x,t)=\phi(x)(h-t)^k$; $\phi(x)\geqq 0$, $k\geqq 0$ is an integer. In case of 1), if we take $A=0$, then iv) in (1.4) is satisfied. In case of 2), if we take $A=0$, then iv′) in (1.4) is satisfied. Therefore in both cases, by Theorem 1.1, the Cauchy problem for (1.7) whose coefficients $\in\gamma^{(a)}(\Omega)$ is well-posed in $\gamma_{loc}^{(a)}$, $1<a<2$.

Now we explain the outline of the proof. At first, we prove Theorem 1.1

in a restricted form. Namely we prove the existence of a solution $u(x,t)\in\gamma^{(\alpha)}(\Omega)$ of the equation (1.1) for the restricted right-hand term and initial data: $f(x,t)\in\gamma_0^{(\alpha)}(\Omega)$, $(u(x,0),\ \partial_t u(x,0))\in\gamma_0^{(\alpha)}(\boldsymbol{R}^n)$, by the method of successive approximation, where the theorem of Oleinik and the lemma of Sobolev are used. Next we show the existence of a finite domain of dependence. Finally we obtain Theorem 1.1 by the procedure of partition of unity.

Remark 1.2. (*Lemma of Sobolev*). There exists a constant $c(n)$, which depends only on the space dimension n, such that

$$\sup|u(x)|\leqq c(n)\sum_{|p|\leqq[n/2]+1}\|\partial^p u(x)\|.$$

Remark 1.3. *Let* $a^i(x,t)$, $a^{ij}(x,t)$, $b^0(x,t)\in\mathscr{B}(\Omega)$. *Assume* ii), iii) *in* (1.3) *and* (1.4), *then the Cauchy problem for*

$$L_0[u]=\delta^2u-\partial_i(a^{ij}\partial_j u)=f(x,t),\qquad (x,t)\in\Omega, \tag{1.8}$$

is well-posed in $\mathscr{D}^\infty_{L^2}$ *and also in* $\mathcal{E}$. *Moreover there exists a finite domain of dependence.*[2] In (7), O. A. Oleinik considered in case of $\delta=\partial_t$, and proved the well-posedness in $\mathscr{D}^\infty_{L^2}$. We shall give a rough sketch of the proof of this theorem in Appendices.

Remark 1.4. We give here the definition of $\Gamma_x^{(\alpha)}$, $\Gamma_x^{(\alpha)}[0,h]$ (Gevrey classes in the L^2-sense), which will be used in §§3 and 4. $\phi(x)\in\mathscr{D}^\infty_{L^2}$ is said to belong to $\Gamma_x^{(\alpha)}$ if there exist two constants ρ and C such that

$$\|\partial^p\phi(x)\|\leqq\frac{|p|!^\alpha}{\rho^{|p|}}C,\qquad\text{for any } p.$$

$\phi(x,t)\in\mathscr{D}^\infty_{L^2}[0,h]$ is said to belong to $\Gamma^{(\alpha)}[0,h]$ if there exist two constants ρ and C such that

$$\sup_{0\leqq t\leqq h}\|\partial_t^k\partial^p\phi(x,t)\|\leqq\frac{(|p|+k)!^\alpha}{\rho^{|p|+k}}C,\qquad\text{for any } p \text{ and } k.$$

Taking the lemma of Sobolev into account, we can see the following relations:

$$\gamma_0^{(\alpha)}\subset\Gamma_x^{(\alpha)}\subset\gamma^{(\alpha)},\quad \gamma_0^{(\alpha)}(\Omega)\subset\Gamma^{(\alpha)}[0,h]\subset\gamma^{(\alpha)}(\Omega).$$

§2. Estimate of a solution of $L_0[u]=f$, under the assumption iv)

In this and the following sections, we assume (1.3), iv) in (1.4). Our aim in this section is to estimate the solution $u(x,t)\in\mathscr{D}^\infty_{L^2}[0,h]$ of

$$L_0[u]=f(x,t) \tag{2.1}$$

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2) Let C_{x_0,t_0}, $(x_0,t_0)\in\Omega$, be a backward cone defined by

$$C_{x_0,t_0}=\{(x,t)\in\Omega;\ \mu|x-x_0|<t_0-t\},$$

where $\mu^{-1}=\sup_{\substack{(x,t)\in\Omega\\|\xi|=1}}|a^i(x,t)\xi_i+\sqrt{a^{ij}(x,t)\xi_i\xi_j}|$. Then the latter part of the theorem means that if $u(x,t)\in\mathcal{E}^2$ be a solution of (1.8) where $f(x,t)\equiv0$ in C_{x_0,t_0}, and if $(u(x,0),\ \partial_t u(x,0))\equiv0$ on $C_{x_0,t_0}\cap\{t=0\}$, then $u(x,t)$ vanishes identically in C_{x_0,t_0}.

with null initial data.

Our method mentioned below is based on the idea of O. A. Oleinik in [7]. However, we should remark that we need to obtain an energy inequality in so refined form as to be useful for the argument in the following section.

Let τ; $0\leqq\tau\leqq h$, be a parameter, and $v_p=v_p(x, t; \tau)$ be the solution of the (hyperbolic) Cauchy problem

$$\delta^*[v]=\partial^p u,\quad v|_{t=\tau}=0, \tag{2.2}$$

where $\delta^*[v]=-\partial_t v-\partial_i[a^i v]+\overline{b^0}v$. Let us start from the following identity:

$$\begin{aligned}(-1)^p 2\mathrm{Re}(u, L_0^*[\partial^p v_p]) &= 2\mathrm{Re}(\partial^p u, L_0^*[v_p])+(-1)^p 2\mathrm{Re}(u, [L_0^*, \partial^p]v_p)\\ &= 2\,\mathrm{Re}(\partial^p u, \delta^*[\partial^p u])-2\,\mathrm{Re}(\delta^*[v_p], \partial_i a^{ij}\partial_j v_p)\\ &\quad +4\,\mathrm{Re}([\partial^p, \delta]u, \partial^p u)+2\,\mathrm{Re}([\partial^p, a^{ij}]\partial_i u, \partial_j v_p)\\ &\quad +2\,\mathrm{Re}([[\partial^p, \delta], \delta]u, v_p),\end{aligned} \tag{2.3}$$

where $L_0^*=\delta^{*2}-\partial_i a^{ij}\partial_j$, $[L_0^*, \partial^p]=L_0^*\partial^p-\partial^p L_0^*$, and so on.

From now on, we consider each term of the right side of (2.3).

$$\text{The 1-st term}\geqq -d/dt\|u\|_p^2-C_1\|u\|_p^2, \tag{2.4}$$

where

$$\|u\|_p=\|\partial^p u\|,\quad C_1=\sum_{i=1}^{n}\sup_{(x,t)\in\Omega}|(a^i)'_{x_i}(x, t)|+2\sup_{(x,t)\in\Omega}|b^0(x, t)|.$$

Hereafter we use C_i to denote a constant which does not depend on p.

$$\begin{aligned}\text{The 2-nd term}=&-d/dt(a^{ij}\partial_i v_p, \partial_j v_p)+(b^{ij}\partial_i v_p, \partial_j v_p)\\ &-((a^k)'_{x_k}a^{ij}\partial_i v_p, \partial_j v_p)+2\,\mathrm{Re}(\overline{b^0}a^{ij}\partial_i v_p, \partial_j v_p)\\ &-2\,\mathrm{Re}(\{(a^k)''_{x_i x_k}-(\overline{b^0})'_{x_i}\}v_p, a^{ij}\partial_j v_p),\end{aligned}$$

and so, if we use the assumption iv) in (1.4),

$$\begin{aligned}\text{the 2-nd term}\geqq &-d/dt(a^{ij}\partial_i v_p, \partial_j v_p)-(A+C_1)(a^{ij}\partial_i v_p, \partial_j v_p)\\ &-2C_2\|v_p\|(a^{ij}\partial_i v_p, \partial_j v_p)^{1/2}.\end{aligned} \tag{2.5}$$

Next is the third term. Because that $[\partial^p, \delta]=\sum_{|q|\geq 1}C_q{}^p a^i_{(q)}\partial_i\partial^{p-q}+\sum_{|q|\geq 1}C_q{}^p b^0_{(q)}\partial^{p-q}$,

$$\begin{aligned}\text{the 3-rd term}\geqq &-4n\sum_{s\geqq 1}C_s{}^l\langle s-1\rangle\|u\|_{l+1-s}\|u\|_l\\ &-4\sum_{s\geqq 1}C_s{}^l\langle s\rangle\|u\|_{l-s}\|u\|_l,\end{aligned} \tag{2.6}$$

where $|p|=l$, $\|u\|_k=\max_{|q|=k}\|u\|_q$, $\langle s\rangle=\{s!^a/(2\rho)^s\}A_1$; ρ and A_1 be some constants, we assumed that $|a^i_{(q)}|\leqq\langle|q|-1\rangle$, $|b^0_{(q)}|\leqq\langle|q|\rangle$, and we used the relation: $\sum_{|q|=s}C_q{}^p=C_s{}^l$.

$$\begin{aligned}\text{The 4-th term}=&\,2\,\mathrm{Re}\sum_{|q|=1}C_q{}^p(a^{ij}_{(q)}\partial_i\partial^{p-q}u, \partial_j v_p)\\ &-2\,\mathrm{Re}\sum_{|q|\geqq 2}C_q{}^p(a^{ij}_{(q)}\partial_i\partial_j\partial^{p-q}u, v_p)\\ &-2\,\mathrm{Re}\sum_{|q|\geqq 2}C_q{}^p(a^{ij}_{(q+e_j)}\partial_i\partial^{p-q}u, v_p).\end{aligned}$$

To estimate the first term of the right side of this identity, we use the following lemma, whose proof will be given in Appendix.

Lemma 2.1. *(Cf. Oleinik,* [7]) *Let $a^{ij}(x)\in\mathcal{B}_x$ be real-valued functions, $a^{ij}(x)=a^{ji}(x)$, and we assume the condition* iii) *in* (1.3). *Then, if $|q|=1$, it holds that*

$$(*)\qquad |(a^{ij}_{(q)}\partial_i u,\ \partial_j v)|\leqq \text{const.}\|u\|_1\{(a^{ij}\partial_i v,\ \partial_j v)^{1/2}+\|v\|\},\quad \text{for } u,\ v\in C_0^\infty(\boldsymbol{R}^n).$$

Taking this lemma into account,

$$(2.7)\qquad \begin{aligned}\text{the 4-th term} \geqq &-2C_3 l\|u\|_l(a^{ij}\partial_i v_p,\ \partial_j v_p)^{1/2}-2C_4 l\|u\|_l\|v_p\|\\ &-2n^2\|v_p\|\sum_{s\geqq 2}C_s{}^l\langle s-2\rangle\|u\|_{l+2-s}\\ &-2n^2\|v_p\|\sum_{s\geqq 2}C_s{}^l\langle s-1\rangle\|u\|_{l+1-s},\end{aligned}$$

where we assumed that $|a^{ij}_{(q)}|\leqq\langle|q|-2\rangle$.

In the fifth term, several commutators are contained. For simplicity, let us see only about a typical one:

$$\begin{aligned}&[[\partial^p,\ a^i\partial_i],\ a^j\partial_j]u\\ &=\sum_{|q|\geqq 2}C_q{}^{p+e_i}\sum_{|r|\geqq 1}^{|q-r|\geqq 1}C_r{}^q\frac{p_i+1-r_i}{p_i+1}a^i_{(r)}a^j_{(q-r)}\partial_i\partial_j\partial^{p-q}u-\sum_{|q|\geqq 1}C_q{}^p a^j a^i_{(q+e_j)}\partial_i\partial^{p-q}u.\end{aligned}$$

Therefore, if we assume that $\sum_r C_r{}^q|a^i_{(r)}||a^j_{(q-r)}|\leqq\langle|q|-2\rangle$,

$$\begin{aligned}2\,\mathrm{Re}([[\partial^p,\ a^i\partial_i],\ a^j\partial_j]u,\ v_p)\geqq &-2n^2\|v_p\|\sum_{s\geqq 2}C_s{}^{l+1}\langle s-2\rangle\|u\|_{l+2-s}\\ &-2n^2\|v_p\|\sum_{s\geqq 1}C_s{}^l\langle s-1\rangle\|u\|_{l+1-s}.\end{aligned}$$

One can estimate the others in the same way, and can see that, as a whole,

$$(2.8)\qquad \begin{aligned}\text{the 5-th term}\geqq &-(C_5 l+C_6 l^2)\|u\|_l\|v_p\|\\ &-C_7\sum_{s\geqq 3}C_s{}^{l+2}\langle s-2\rangle\|u\|_{l+2-s}\|v_p\|.\end{aligned}$$

Thus by (2.4)~(2.8),

$$(2.9)\qquad \begin{aligned}(-1)^p 2\,\mathrm{Re}(u,\ L_0{}^*[\partial^p v_p])\geqq &-d/dt\{\|u\|_p{}^2+(a^{ij}\partial_i v_p,\ \partial_j v_p)\}\\ &-2\gamma(l+1)\{\|u\|_p{}^2+(a^{ij}\partial_i v_p,\ \partial_j v_p)\}-C_2\|v_p\|^2\\ &-C_8 l\|u\|_l{}^2-(C_9 l+C_{10}l^2)\|u\|_l\|v_p\|\\ &-C_{11}\|v_p\|\sum_{s\geqq 3}C_s{}^{l+2}\langle s-2\rangle\|u\|_{l+2-s}\\ &-C_{12}\|u\|_l\sum_{s\geqq 2}C_s{}^{l+1}\langle s-1\rangle\|u\|_{l+1-s},\end{aligned}$$

where $2\gamma=\max\{A+C_1+C_2,\ C_3\}$.

Next, multiply the both sides by $e^{2\gamma(l+1)t}$, and integrate them from 0 to τ. If we define $[u]_l(\tau)$ and $[v_p](\tau)$ by

$$(2.10)\qquad \begin{aligned}&[u]_l(\tau)^2=\int_0^\tau\|u\|_l(t)^2e^{2\gamma(l+1)t}dt,\\ &[v_p](\tau)^2=\int_0^\tau\|v_p\|(t)^2e^{2\gamma(l+1)t}dt.\end{aligned}$$

respectively and remark that $u|_{t=0}=0$, $\partial_t u|_{t=0}=0$, $v_p|_{t=\tau}=0$, then

$$
\begin{aligned}
&(-1)^p 2\,\mathrm{Re}\int_0^\tau (u, L_0^*[\partial^p v_p])e^{2\gamma(l+1)t}dt \\
(2.11)\qquad &\geqq -\|u\|_p(\tau)^2 e^{2\gamma(l+1)\tau} - C_2[v_p](\tau)^2 - C_8 l[u]_l(\tau)^2 \\
&\quad -(C_9 l + C_{10} l^2)[u]_l[v_p] \\
&\quad -\{C_{12}[u]_l + C_{13}(l+2)[v_p]\} \sum_{s\geqq 2} C_s{}^{l+1}\langle s-1\rangle [u]_{l+1-s}(\tau)e^{\gamma(s-1)\tau},
\end{aligned}
$$

where we used that $C_{s+1}^{l+2} \leqq \{(l+2)/3\}\, C_s{}^{l+1}$.

On the other hand, integrating by parts,

$$
\begin{aligned}
\text{the left side of (2.11)} &= 2\,\mathrm{Re}\int_0^\tau (\partial^p f, v_p)e^{2\gamma(l+1)t}dt \\
&\quad -(-1)^p 2\,\mathrm{Re}(u, \delta^*[\partial^p v_p])_{t=\tau} e^{2\gamma(l+1)\tau} \\
&\quad +8(-1)^p \gamma(l+1)\,\mathrm{Re}\int_0^\tau (u, \delta^*[\partial^p v_p])e^{2\gamma(l+1)t}dt \\
&\quad -8\gamma^2(l+1)^2\,\mathrm{Re}\int_0^\tau (\partial^p u, v_p)e^{2\gamma(l+1)t}dt.
\end{aligned}
$$

Therefore, remarking that $\delta^*[\partial^p v_p]|_{t=\tau} = \partial^{2p} u|_{t=\tau}$,

$$
\begin{aligned}
\text{the left side of (2.11)} &\leqq 2\,\mathrm{Re}\int_0^\tau (\partial^p f, v_p)e^{2\gamma(l+1)t}dt \\
(2.12)\qquad &\quad -2\|u\|_p(\tau)^2 e^{2\gamma(l+1)\tau} + 8\gamma(l+1)[u]_l(\tau)^2 \\
&\quad +\{8\gamma^2(l+1)^2 + 8nA_1\gamma(l+1)l\}\,[u]_l[v_p] \\
&\quad +8n\gamma(l+1)[v_p] \sum_{s\geqq 2} C_s{}^{l+1}\langle s-1\rangle [u]_{l+1-s}(\tau)e^{\gamma(s-1)\tau}.
\end{aligned}
$$

Now we prepare a lemma:

Lemma 2.2. It holds the following inequality:

$$
(2.13)\qquad (l+1)[v_p](\tau) \leqq C_{14}[u]_l(\tau).
$$

Proof. Since v_p is the solution of (2.2), it is easily seen that

$$
d/dt\|v_p\| \geqq -(C_1/2)\|v_p\| - \|u\|_p,
$$

where C_1 is the same constant as in (2.4). Therefore

$$
\|v_p\|(t) \leqq \int_t^\tau \|u\|_p(s) e^{1/2\, C_1(s-t)} ds.
$$

Denote the right side by $\phi(t)$, then

$$
[v_p](\tau)^2 \leqq \int_0^\tau \phi(t)^2 e^{2\gamma(l+1)t}dt.
$$

Denote the right side by I^2, then

$$
\begin{aligned}
I^2 &= \frac{1}{2\gamma(l+1)}\int_0^\tau \phi(t)^2 \{e^{2\gamma(l+1)t}\}_t{}' dt \\
&= \frac{-\phi(0)^2}{2\gamma(l+1)} - \frac{1}{\gamma(l+1)}\int_0^\tau \phi(t)\phi'(t) e^{2\gamma(l+1)t}dt.
\end{aligned}
$$

Because $\phi'(t)=-\|u\|_p(t)-(C_1/2)\phi(t)$,

$$I^2\leqq\frac{1}{\gamma(l+1)}[u]_l I+\frac{C_1}{2\gamma(l+1)}I^2.$$

Since $2\gamma\geqq A+C_1+C_2$, $1-C_1/\{2\gamma(l+1)\}\geqq(A+C_1)/(2\gamma)$. So

$$\frac{A+C_1}{2\gamma}I^2\leqq\frac{1}{\gamma(l+1)}[u]_l I.$$

Thus (2.13) has been proved. q.e.d.

If we use this lemma, by (2.11) and (2.12) we have that

$$\begin{aligned}\|u\|_p(\tau)^2e^{2\gamma(l+1)\tau}\leqq 2\,\mathrm{Re}\int_0^\tau(\partial^p f,\ v_p)e^{2\gamma(l+1)t}dt\\ +2k\gamma(l+1)[u]_l(\tau)^2+C_{15}[u]_l(\tau)R_l(\tau),\end{aligned}\tag{2.14}$$

where k ($\geqq 8$) is a constant independent of $|p|=l$, and

$$R_l(t)=\sum_{s\geqq 2}C_s{}^{l+1}\langle s-1\rangle[u]_{l+1-s}(t)e^{\gamma(s-1)t}.\tag{2.15}$$

Finally we consider the first term of (2.14). Let g_p be the solution of the (hyperbolic) Cauchy problem

$$\delta[g]=\partial^p f,\quad g|_{t=0}=0.\tag{2.16}$$

Then, integrating by parts,

$$2\,\mathrm{Re}\int_0^\tau(\partial^p f,\ v_p)e^{2\gamma(l+1)t}dt\leqq C_{16}[g_p][u]_l,\tag{2.17}$$

where Lemma 2.2 was used again. Now that g_p is a solution of (2.16), we can easily see that

$$\|g_p\|(t)\leqq C_{17}\int_0^t\|f\|_l(s)ds,\qquad 0\leqq t\leqq h.$$

Therefore, if we define $(f)_l(t)$, $F_l(t)$ by

$$(f)_l(t)=\int_0^t\|f\|_l(s)ds,\quad F_l(t)^2=\int_0^t(f)_l(s)^2e^{2\gamma(l+1)s}ds,\tag{2.18}$$

respectively, then

$$[g_p]\leqq C_{17}F_l(t).\tag{2.19}$$

Thus we have obtained the following proposition.

Proposition 2.1. *The solution* $u(x,t)\in\mathcal{D}^\infty_{L^2}[0,h]$ *of the equation* (2.1) *with null initial data satisfies that*

$$\|u\|_l(t)^2e^{2\gamma(l+1)t}\leqq 2k\gamma(l+1)[u]_l(t)^2+2K[u]_l(t)\{F_l(t)+R_l(t)\},\tag{2.20}$$

where γ, k *and* K *are constants independent of* $l=|p|$.

Immediately we can get the

Proposition 2.2. *The solution* $u(x,t)\in\mathcal{D}_{L^2}^{\infty}[0,h]$ *of the equation* (2.1) *with null initial data satisfies that*

$$[u]_l(t)\leqq K\int_0^t\{F_l(s)+R_l(s)\}e^{k\gamma(l+1)(t-s)}ds. \tag{2.21}$$

§3. Existence of a solution, under the assumption iv)

Let us prove the existence of a solution of the equation

$$L[u]=f \tag{1.1}$$

with given initial data at $t=0$. We assume (1.3), iv) in (1.4). At first we consider the case where initial data are null. We construct a solution by the method of successive approximation. Namely we define $u_i(x,t)$ by

$$\begin{aligned}&L_0[u_1]=f,\ \text{with null initial data}\\&L_0[u_i]=M[u_{i-1}],\ \text{with null initial data},\ i\geqq 2,\end{aligned} \tag{3.1}$$

where $L_0=\delta^2-\partial_i a^{ij}\partial_j$, $M=b^i\partial_i+c$. Then the formal sum $\sum_{i=1}^{\infty}u_i(x,t)$ gives a formal solution of (1.1) with null initial data. So let us examine its convergence.

Successive estimate Suppose that

$$\|f\|_l(t)e^{\gamma(l+1)t}\leqq\frac{t^i}{i!}\frac{(l+r)!^a}{\rho^{l+r}}Ce^{k\gamma(l+r+1)t}(1+\beta t)^{l+r+1}, \tag{3.2}$$

where ρ, k and γ are the same constants as in the preceding section, C is a constant, β is a constant which will be determined later, l, i and r are non-negative integers. Under this assumption, let us estimate the solution $u(x,t)\in\mathcal{D}_{L^2}^{\infty}[0,h]$ of the equation

$$L_0[u]=f \tag{2.1}$$

with null initial data. For simplicity, we denote the right hand term of (3.2) by $\kappa_{i,l+r}(t)$.

From the definition (2.18), it follows that

$$\begin{aligned}&(f)_l(t)\leqq\kappa_{i+1,l+r}(t)e^{-\gamma(l+1)t},\\&F_l(t)\leqq\frac{1}{\sqrt{2k\gamma(l+r+1)}}\kappa_{i+1,l+r}(t).\end{aligned} \tag{3.3}$$

Therefore

$$K\int_0^t F_l(s)e^{k\gamma(l+1)(t-s)}ds\leqq\frac{K}{\sqrt{2k\gamma(l+r+1)}}\kappa_{i+2,l+r}(t). \tag{3.4}$$

If we use the proposition 2.2, we can prove the

Lemma 3.1. *Assume* (3.2) *and take* $\beta=2A_1K$, *then the solution* $u(x,t)\in\mathcal{D}_{L^2}^{\infty}[0,h]$ *of* (2.1) *with null initial data satisfies that*

$$[u]_l(t)\leqq\frac{2K}{\sqrt{2k\gamma(l+r+1)}}\kappa_{i+2,l+r}(t). \tag{3.5}$$

Proof. We show this by induction. For $l=0$, taking the proposition 2.2 into account, it is evident from (3.4). Next suppose that (3.5) is valid for all $l'\leqq l-1$. Then

$$R_l(t)\leqq\sum_{s\geqq 2}C_s{}^{l+1}\langle s-1\rangle\frac{2K}{\sqrt{2k\gamma(l+1-s+r+1)}}\kappa_{i+2,l+1-s+r}(t)e^{\gamma(s-1)t}$$
$$\leqq 2A_1K\kappa_{i+2,l+r}(t)(1+\beta t)^{-1}\sum_{s\geqq 2}\frac{1}{2^{s-1}\sqrt{2k\gamma(l+1-s+r+1)}}\frac{C_s{}^{l+1}}{C_{s-1}^{l+r}}.$$

$\sum_{s\geqq 2}\cdots\leqq\sqrt{l+1}/\sqrt{2k\gamma}$, because $C_s{}^{l+1}/C_{s-1}^{l+r}\leqq(l+1)/s$, $s(l+1-s+r+1)\geqq(l+1)$. Therefore

$$R_l(t)\leqq\frac{\sqrt{l+1}}{\sqrt{2k\gamma}}2A_1K\kappa_{i+2,l+r}(t)(1+\beta t)^{-1}. \tag{3.6}$$

Therefore

$$K\int_0^t R_l(s)e^{k\gamma(l+1)(t-s)}ds\leqq\frac{2A_1K^2\sqrt{l+1}}{\sqrt{2k\gamma}\beta(l+r+1)}\kappa_{i+2,l+r}(t)$$
$$\leqq\frac{K}{\sqrt{2k\gamma(l+r+1)}}\kappa_{i+2,l+r}(t).$$

By the proposition 2.2, the inequality (3.5) follows from the above one and (3.4). *q.e.d.*

Hereafter we fix the constant $\beta=2A_1K$. By the way, $F_l(t)$ has an another type of estimate as follows:

$$F_l(t)\leqq\frac{t^{1/2}}{\sqrt{2i+3}}\kappa_{i+1,l+r}(t). \tag{3.7}$$

Therefore

$$K\int_0^t F_l(s)e^{k\gamma(l+1)(t-s)}ds\leqq\frac{Kt^{3/2}}{\sqrt{2i+3}(i+5/2)}\kappa_{i+1,l+r}(t).$$

In the same way as Lemma 3.1, by the proposition 2.2, we can prove the

Lemma 3.2. *We assume* (3.2). *Then the solution* $u(x, t)$ *of* (2.1) *with null initial data satisfies that*

$$[u]_l(t)\leqq\frac{2Kt^{3/2}}{\sqrt{2i+3}(i+5/2)}\kappa_{i+1,l+r}(t). \tag{3.8}$$

Now let us apply the obtained estimates to the inequality (2.20) in Proposition 2.1. By (3.5),

$$2k\gamma(l+1)[u]_l(t)^2\leqq 4K^2\kappa_{i+2,l+r}(t)^2.$$

By (3.5) and (3.6),

$$2K[u]_l(t)R_l(t)\leqq\frac{A_1(2K)^3}{2k\gamma}\kappa_{i+2,l+r}(t)^2.$$

By (3.7) and (3.8),

$$2K[u]_l(t)F_l(t)\leqq 4K^2\kappa_{i+2,l+r}(t)^2.$$

Thus, by Proposition 2.1, we can get the

Proposition 3.1. *Assume* (3.2), *then the solution* $u(x, t)\in\mathcal{D}^{\infty}_{L^2}[0, h]$ *of* (**2.1**) *with null initial data satisfies that*

(3.9) $$\|u\|_l(t)e^{\gamma(l+1)t}\leqq K_1\kappa_{i+2,l+r}(t),$$

where K_1 *is a constant which does not depend on* l, i *and* r.

We need to estimate $\|\partial_t u\|_l$ too.

Lemma 3.3. *If we assume that*

$$\|u\|_l(t)e^{\gamma(l+1)t}\leqq\kappa_{i,l+r}(t),\quad |\partial^q a(x, t)|\leqq\langle|q|\rangle,$$

then it follows that

$$\|au\|_l(t)e^{\gamma(l+1)t}\leqq 2A_1\kappa_{i,l+r}(t).$$

Proof. Because $\partial^p[au]=\sum_q C_q{}^p(\partial^q a)\partial^{p-q}u$,

$$\begin{aligned}\|au\|_l e^{\gamma(l+1)t}&\leqq\sum_s C_s{}^l\langle s\rangle\|u\|_{l-s}e^{\gamma(l-s+1)t}e^{\gamma st}\\&\leqq\sum_s C_s{}^l\langle s\rangle\kappa_{i,l-s+r}(t)e^{\gamma st}\\&\leqq A_1\kappa_{i,l+r}(t)\sum_s 2^{-s}C_s{}^l/C_s{}^{l+r}\leqq 2A_1\kappa_{i,l+r}(t).\end{aligned}$$ *q.e.d.*

Since $\delta^2 u=\partial_i a^{ij}\partial_j u+f$, one can verify by this lemma that

(3.10) $$\|\delta^2 u\|_l e^{\gamma(l+1)t}\leqq 2A_1K_1\kappa_{i+2,l+r+2}(t)+\kappa_{i,l+r}(t),$$

where we assumed (3.2) and that $|a^{ij}_{(q)}|\leqq\langle|q|\rangle$and we used (3.9). Next let us derive the estimate of $\|\delta u\|_l(t)$ from the above inequality. For this purpose, consider the solution $v(x, t)$ of the (hyperbolic) Cauchy problem

(3.11) $$\delta[v]=g,\quad v|_{t=0}=0.$$

We want to give the estimate of $\|v\|_l(t)$, assuming that

(3.12) $$\|g\|_l(t)e^{\gamma(l+1)t}\leqq\kappa_{i,l+r}(t).$$

One can easily show that

$$\begin{aligned}d/dt\|v\|_p&\leqq\tfrac{1}{2}C_1\|v\|_l+\|g\|_l+\|[\partial^p, \delta]v\|\\&\leqq(\tfrac{1}{2}C_1+nA_1l)\|v\|_l+\|g\|_l+n\sum_{s\geqq 2}C_s{}^{l+1}\langle s-1\rangle\|v\|_{l+1-s},\end{aligned}$$

where C_1 is the same constant as in (2.4). Because $n\leqq K$ and because $\frac{1}{2}C_1+\gamma+nA_1l+\gamma l\leqq k\gamma(l+1)$,

$$d/dt\{\|v\|_l(t)e^{\gamma(l+1)t}\}\leqq k\gamma(l+1)\|v\|_l e^{\gamma(l+1)t}+\|g\|_l e^{\gamma(l+1)t}+KT_l(t),$$

where $T_l(t)=\sum_{s\geqq 2}C_s{}^{l+1}\langle s-1\rangle\|v\|_{l+1-s}e^{\gamma(l+1-s+1)t}e^{\gamma(s-1)t}$. Therefore

$$\|v\|_l(t)e^{\gamma(l+1)t}\leqq\int_0^t\{\|g\|_l(s)e^{\gamma(l+1)s}+KT_l(s)\}e^{k\gamma(l+1)(t-s)}ds.$$

If we use this inequality, we can prove the following lemma in the same way as Lemma 3.1.

Lemma 3.4. *Assume* (3.12), *then the solution* $v(x, t)$ *of* (3.11) *satisfies that*

(3.13) $$\|v\|_l(t)e^{\gamma(l+1)t}\leqq 2\kappa_{i+1,l+r}(t).$$

δu is a solution of (3.11) for $g=\delta^2 u$. Therefore by the above lemma one can get from (3.10) that

(3.14) $$\|\delta u\|_l e^{\gamma(l+1)t}\leqq 4A_1K_1\kappa_{i+3,l+2+r}(t)+2\kappa_{i+1,l+r}(t).$$

Since $\partial_t u=-a^i\partial_i u-b^0u+\delta u$, by Lemma 3.3, (3.9) and (3.14), we have the

Proposition 3.2. *Assume* (3.2), *then the solution* $u(x, t)\in\mathcal{D}^\infty_{L^2}[0, h]$ *of the equation* (**2.1**) *with null initial data satisfies that*

(3.15) $$\|\partial_t u\|_l(t)e^{\gamma(l+1)t}\leqq K_2\sum_{\nu=0}^{2}\kappa_{i+1+\nu,l+\nu+r}(t),$$

where K_2 *is a constant which does not depend on* l, i *and* r.

Finally we remark that by Lemma 3.3 one can easily show the

Proposition 3.3. *Assume that*

(3.16) $$|b^i_{(q)}|\leqq\langle|q|\rangle,\quad |c_{(q)}|\leqq\langle|q|\rangle,$$
$$\|u\|_l(t)e^{\gamma(l+1)t}\leqq\kappa_{i,l+r}(t),$$

then it follows that

(3.17) $$\|M[u]\|_l(t)e^{\gamma(l+1)t}\leqq K_3\kappa_{i,l+1+r}(t),$$

where $M=b^i\partial_i+c$ *and* K_3 *is a constant independent of* l, i *and* r.

We are ready now to prove the existence of a solution of (1.1).

Existence of a solution In (1.1) we assume that

(3.18) $$\|f\|_p\leqq\frac{|p|!^a}{\rho^{|p|}}C.$$

It is evident that

$$\|f\|_l(t)e^{\gamma(l+1)t}\leqq\kappa_{0,l}(t).$$

Apply Proposition 3.1, regarding $i=r=0$, then

$$\|u_1\|_l(t)e^{\gamma(l+1)t}\leqq K_1\kappa_{2,l}(t).$$

Apply Proposition 3.3, regarding $i=2$, $r=0$, then

$$\|M[u_1]\|_l(t)e^{\gamma(l+1)t}\leqq K_3K_1\kappa_{2,l+1}(t).$$

Apply Proposition 3.1 again, regarding $i=2$, $r=1$, then

$$\|u_2\|_l(t)e^{\gamma(l+1)t}\leqq K_3K_1{}^2\kappa_{4,l+1}(t).$$

If we repeat this argument, we can get the

Proposition 3.4. *Assume* (3.18), *then* $u_i(x, t)$ *defined by* (3.1) *satisfies that*

$$\| u_i \|_l(t) e^{\gamma(l+1)t} \leqq K_3^{i-1} K_1^i \kappa_{2i, l+i-1}(t), \tag{3.19}$$

$$\| M[u_i] \|_l(t) e^{\gamma(l+1)t} \leqq K_3^i K_1^i \kappa_{2i, l+i}(t). \tag{3.20}$$

By Proposition 3.2 and by (3.20), we have the

Proposition 3.5. *Assume* (3.18), *then* $u_i(x, t)$ *satisfies that*

$$\| \partial_t u_i \|_l(t) e^{\gamma(l+1)t} \leqq K_2 K_3^{i-1} K_1^{i-1} \sum_{\nu=0}^{2} \kappa_{2i-1+\nu, l+i-1+\nu}(t). \tag{3.21}$$

Now let us examine the convergence of $\sum_{i=1}^{\infty} u_i(x, t)$ by means of (3.19).

$$\sum_{i=1}^{\infty} \| u_i \|_l \leqq \sum_{i=1}^{\infty} K_3^{i-1} K_1^i \kappa_{2i, l+i-1}(t).$$

If we denote $\rho^{-1} e^{k\gamma h}(1+\beta h)$ by B_1, $K_3 K_1 \rho^{-1} e^{k\gamma h}(1+\beta h)$ by B_2, $K_1 C e^{k\gamma h}(1+\beta h)$ by B_3, then

$$\sum_{i=1}^{\infty} \| u_i \|_l \leqq t^2 B_3 B_1^l \sum_{i=1}^{\infty} (B_2 t^2)^{i-1} \frac{(l+i-1)!^a}{(2i)!}$$

$$\leqq t^2 B_3 (2^a B_1)^l l!^a \sum_{i=1}^{\infty} (2^a B_2 t^2)^{i-1} \frac{(i-1)!^a}{(2i)!},$$

where we used that $C_{i-1}^{l+i-1} \leqq 2^{l+i-1}$. Remark that $(i-1)!^2/(2i)! \leqq 4^{-(i-1)}$, then

$$\sum_{i=1}^{\infty} \| u_i \|_l(t) \leqq t^2 B_3 (2^a B_1)^l l!^a \sum_{i=1}^{\infty} (2^a/4 B_2 t^2)^{i-1} (i-1)!^{a-2}.$$

Therefore, if $1 \leqq a < 2$, the right hand term converges uniformly in $[0, h]$. If $a=2$, there exists h_0 ($\leqq h$) such that the right hand term converges uniformly in $[0, h_0]$.

Thus, if we put $u(x, t) = \sum_{i=1}^{\infty} u_i(x, t)$, we have

$$\| u \|_l(t) \leqq \text{const.} (2^a B_1)^l l!^a t^2, \quad \begin{cases} \text{for } 0 \leqq t \leqq h, & \text{if } 1 \leqq a < 2 \\ \text{for } 0 \leqq t \leqq h_0, & \text{if } a = 2. \end{cases} \tag{3.22}$$

The same consideration on $\sum_{i=1}^{\infty} \| \partial_t u_i \|_l$ gives

$$\| \partial_t u \|_l \leqq \text{const.} (2^a B_1)^{l+1} (l+1)!^a t, \quad \begin{cases} \text{for } 0 \leqq t \leqq h, & \text{if } 1 \leqq a < 2 \\ \text{for } 0 \leqq t \leqq h_0, & \text{if } a = 2, \end{cases} \tag{3.23}$$

where we used Proposition 3.5. Here, if neccessary, h_0 is supposed to be replaced with a smaller one.

Thus the existence of a solution of (1.1) with null initial data has been proved, which is a function of Gevrey class of order a with respect to x. Moreover we can prove that if $f(x, t)$ belongs to $\Gamma^{(a)}[0, h]$, then the obtained solution $u(x, t)$ also belongs to $\Gamma^{(a)}[0, h]$ (or to $\Gamma^{(a)}[0, h_0]$ in case of $a=2$), cf. [1].

Up to now, our consideration has been restricted to the case where initial data are null. Now consider the Cauchy problem; $L[u] = f(x, t)$, $u|_{t=0} = \phi(x)$, $\partial_t u|_{t=0} = \psi(x)$. Assume that $f(x, t) \in \Gamma^{(a)}[0, h]$, $\phi(x)$ and $\psi(x) \in \Gamma_x^{(a)}$, then $f(x, t) - L[\phi + t\psi]$ belongs to $\Gamma^{(a)}[0, h]$. Therefore, as shown above, one can find a solution $v(x, t)$ of the equation

$$L[v] = f - L[\phi + t\psi],$$

with null initial data. Besides this solution $v(x, t)$ belongs to $\Gamma^{(\alpha)}[0, h]$, (or to $\Gamma^{(\alpha)}[0, h_0]$ in case of $\alpha=2$). Put $u=v+\phi+t\psi$, then $u(x, t)$ gives a desired solution.

Thus we have obtained the

Theorem 3.1. *Assume* (1.3), iv) *in* (1.4). *Then, if* $1\leqq\alpha<2$, *for any* $f(x, t)\in\Gamma^{(\alpha)}[0, h]$, *and any initial data* $\phi(x)$, $\psi(x)\in\Gamma_x^{(\alpha)}$, *there exists a solution* $u(x, t)$ *of the equation* (1.1) *in* Ω *which belongs to* $\Gamma^{(\alpha)}[0, h]$ *and satisfies that* $u|_{t=0}=\phi$, $\partial_t u|_{t=0}=\psi$. *If* $\alpha=2$, *there exists* h_0 $(\leqq h)$ *such that there exists a solution* $u(x, t)\in\Gamma^{(\alpha)}[0, h_0]$ *of* (1.1) *in* $\boldsymbol{R}^n\times[0, h_0]$.

If we remark the lemma of Sobolev, we have also the

Corollary 3.1. *Under the same assumptions as in Theorem* 3.1, *if* $1<\alpha<2$, *for any* $f(x, t)\in\gamma_0^{(\alpha)}(\Omega)$ *and any initial data* $\phi(x)$, $\psi(x)\in\gamma_0^{(\alpha)}(\boldsymbol{R}^n)$, *there exists a solution* $u(x, t)$ *of the equation* (1.1) *in* Ω *which belongs to* $\gamma^{(\alpha)}(\Omega)$ *and satisfies that* $u|_{t=0}=\phi(x)$, $\partial_t u|_{t=0}=\psi(x)$. *If* $\alpha=2$, *there exists* h_0 $(\leqq h)$ *such that there exists a solution* $u(x, t)\in\gamma^{(\alpha)}(\boldsymbol{R}^n\times[0, h_0])$ *of the equation* (1.1) *in* $\boldsymbol{R}^n\times[0, h_0]$.

§4. Existence of a solution, under the assumption iv′)

We assume (1.3), iv′) in (1.4). Also in this case, one can prove the existence of a solution in the same way as where iv) is assumed, except a few points. We use the method of successive approximations, as well. Below we only indicate the points different from where iv) is assumed.

At first, let us estimate the solution $u(x, t)\in\mathscr{D}_{L^2}^{\infty}[0, h]$ of

$$L_0[u]=f(x, t) \tag{4.1}$$

with null initial data. We start from the following identity:

$$\begin{aligned} 2\,\mathrm{Re}(\partial^p\delta u, \partial^p L_0[u])&=2\,\mathrm{Re}(\partial^p\delta u, \delta\partial^p\delta u)\\ &-2\,\mathrm{Re}(\delta\partial^p u, \partial_i a^{ij}\partial_j\partial^p u)+2\,\mathrm{Re}(\partial^p\delta u, [\partial^p, \delta]\delta u)\\ &-2\,\mathrm{Re}(\partial^p\delta u, \partial_i[\partial^p, a^{ij}]\partial_j u)-2\,\mathrm{Re}([\partial^p, \delta]u, \partial_i a^{ij}\partial_j\partial^p u). \end{aligned} \tag{4.2}$$

The 1-st term $\geqq d/dt\|\delta u\|_p^2-C_1\|\delta u\|_p^2$.

Here and hereafter we use C_i to denote a constant which does not depend on p.

$$\begin{aligned}\text{The 2-nd term}\geqq\ & d/dt(a^{ij}\partial_i\partial^p u, \partial_j\partial^p u)-(b^{ij}\partial_i\partial^p u, \partial_j\partial^p u)\\ &-C_2(a^{ij}\partial_i\partial^p u, \partial_j\partial^p u)-C_3\|u\|_p^2,\end{aligned}$$

where $b^{ij}=(a^{ij})'_t+a^k(a^{ij})'_{x_k}-(a^i)'_{x_k}a^{kj}-a^{ik}(a^j)'_{x_k}$.

$$\text{The 3-rd term}\geqq -2n\|\delta u\|_p\sum_{s\geqq1}C_s{}^{l+1}\langle s-1\rangle\|\delta u\|_{l+1-s},$$

where $\langle k\rangle=\{k!^{\alpha}/(2p)^k\}A_1$, $\|\delta u\|_k=\max_{|q|=k}\|\delta u\|_q$ and we assumed that $|a^k_{(q)}|\leqq\langle|q|-1\rangle$, $|b^0_{(q)}|\leqq\langle|q|\rangle$, n is a dimension of the space variable $x=(x_1, \cdots, x_n)$, $|p|=l$.

$$\text{The 4-th term}=-2\,\mathrm{Re}(\partial^p\delta u, \sum_{|q|\geqq1}C_q{}^p\{a^{ij}_{(q)}\partial_i+a^{ij}_{(q+e_i)}\}\partial_j\partial^{p-q}u).$$

When $|q|=1$, by Oleinik's lemma in [7],

$$\|a^{ij}_{(q)}\partial_i\partial_j\partial^{p-q}u\|^2\leqq C_4\sum_{s=1}^{n}(a^{ij}\partial_i\partial_s\partial^{p-q}u,\ \partial_j\partial_s\partial^{p-q}u).$$

Therefore

$$\begin{aligned}\text{the 4-th term}\geqq &-2C_5\|\delta u\|_p\sum_{s=1}^{n}\sum_{|q|=1}C_q{}^p(a^{ij}\partial_i\partial_s\partial^{p-q}u,\ \partial_j\partial_s\partial^{p-q}u)\\ &-2n^2\|\delta u\|_p\sum_{s\geqq 2}C_s{}^{l+1}\langle s-2\rangle\|u\|_{l+2-s},\end{aligned}$$

where we assumed that $|a^{ij}_{(q)}|\leqq\langle|q|-2\rangle$.

As
$$\begin{aligned}\partial_i[\partial^p,\delta]u=&\sum_{|q|\geqq 1}C_q{}^p a^k_{(q)}\partial_i\partial_k\partial^{p-q}u+\sum_{|q|\geqq 1}C_q{}^p b^0_{(q)}\partial_i\partial^{p-q}u\\ &+\sum_{|q|\geqq 1}C^p_q a^k_{(q+e_i)}\partial_k\partial^{p-q}u+\sum_{|q|\geqq 1}C_q{}^p b^0_{(q+e_i)}\partial^{p-q}u,\end{aligned}$$

$$\begin{aligned}\text{the 5-th term}\geqq -2(a^{ij}\partial_i\partial^p u,\ \partial_j\partial^p u)^{1/2}\{&n\sum_{s\geqq 1}C_s{}^{l+1}\langle s-1\rangle\|u\|_{l+1-s}\\ &+\sum_{|q|\geqq 1}C_q{}^p\langle|q|-1\rangle(a^{ij}\partial_i\partial_k\partial^{p-q}u,\ \partial_j\partial_k\partial^{p-q}u)^{1/2}\\ &+\sum_{|q|\geqq 1}C_q{}^p\langle|q|\rangle(a^{ij}\partial_i\partial^{p-q}u,\ \partial_j\partial^{p-q}u)^{1/2}\},\end{aligned}$$

where we used that $|a^k_{(q)}|\leqq\langle|q|-1\rangle$, $|b^0_{(q)}|\leqq\langle|q|\rangle$, $|a^{ij}a^k_{(q+e_i)}a^k_{(q+e_j)}|^{1/2}\leqq\langle|q|-1\rangle$, $|a^{ij}b^0_{(q+e_i)}\overline{b^0_{(q+e_j)}}|^{1/2}\leqq\langle|q|\rangle$.

Now we put $E_p(t)^2=\|\delta u\|_p{}^2+(a^{ij}\partial_i\partial^p u,\ \partial_j\partial^p u)$, and denote $\max_{|p|=l}E_p(t)$ by $E_l(t)$. Then from the above consideration, we can get the following inequality.

$$\begin{aligned}\frac{d}{dt}E_p(t)^2\leqq &2E_l(t)\|f\|_l+(C_6+C_7 l)E_l(t)^2+C_3\|u\|_l{}^2\\ &+C_8E_l(t)\sum_{s\geqq 1}C_s{}^{l+1}\langle s-1\rangle E_{l+1-s}(t)\\ &+C_9E_l(t)\sum_{s\geqq 2}C_s{}^{l+2}\langle s-2\rangle\|u\|_{l+2-s},\end{aligned}\tag{4.3}$$

where the condition iv') in (1.4) was used.

Next we put $F_l(t)^2=E_l(t)^2+(l+1)^2\|u\|_l(t)^2$. Because

$$\begin{aligned}\frac{d}{dt}(l+1)^2\|u\|_p{}^2=&2(l+1)^2\,\mathrm{Re}(\partial^p\delta u,\ \partial^p u)-2\,\mathrm{Re}(l+1)^2(\partial^p\{a^k\partial_k+b^0\}u,\ \partial^p u)\\ \leqq&2(l+1)^2\|\delta u\|_l\|u\|_l+C_1(l+1)^2\|u\|_l{}^2\\ &+2n(l+1)^2\|u\|_l\sum_{s\geqq 1}C_s{}^{l+1}\langle s-1\rangle\|u\|_{l+1-s},\end{aligned}$$

we have from (4.3) that

$$\frac{d}{dt}F_l(t)^2\leqq 2F_l(t)\|f\|_l+2(\gamma_0+\gamma l)F_l(t)^2+2KF_l(t)\sum_{s\geqq 2}C_s{}^{l+2}\langle s-1\rangle F_{l+1-s}(t),$$

where γ_0, γ and K are constants independent of $l=|p|$. Here we used the followings: $C_s{}^{l+1}(l+1)/(l+2-s)\leqq C_s{}^{l+2}$, $C^{l+2}_{s+1}/(l+2-s)\leqq C_s{}^{l+2}$.

Therefore we have

$$F_l(t)\leqq\int_0^t\{\|f\|_l(s)+KR_l(s)\}e^{(\gamma_0+\gamma l)(t-s)}ds,\tag{4.4}$$

where $R_l(t)=\sum_{s\geqq 2}C_s{}^{l+2}\langle s-1\rangle F_{l+1-s}(t)$.

Successive estimate Now in (4.1) we assume that

$$\|f\|_l(t) \leqq \frac{t^i(l+r)!^\alpha}{i!\rho^{l+r}} Ce^{\gamma_0 t+\gamma(l+r)t}(l+\beta t)^{l+r}, \tag{4.5}$$

where ρ is the same constant as in $\langle k \rangle$, C is a constant, β is a constant which will be determined later, i and r are parameters which run over non-negative integers. For simplicity, we abbreviate the right hand term to $\kappa_{i,l+r}(t)$.

Remark that

$$\int_0^t \|f\|_l(s) e^{(\gamma_0+\gamma l)(t-s)} ds \leqq \kappa_{i+1,l+r}(t),$$

then, using the inequality (4.4), one can prove the following lemma in the same way as Lemma 3.1.

Lemma 4.1. *Assume* (4.5) *and take* $\beta = 8A_1K$, *then the solution* $u(x, t)$ *of* (**4.1**) *with null initial data satisfies that*

$$F_l(t) \leqq 2\kappa_{i+1,l+r}(t). \tag{4.6}$$

Since $\|\delta u\|_l \leqq F_l(t)$, it follows that

$$\|\delta u\|_l(t) \leqq 2\kappa_{i+1,l+r}(t). \tag{4.7}$$

Next is the estimate of $\|u\|_l(t)$. Remark that $\partial_t u + a^k \partial_k u + b^0 u = \delta u$, $u|_{t=0} = 0$, then one can easily verify that

$$d/dt \|u\|_l \leqq (\gamma_0 + \gamma l)\|u\|_l + \|\delta u\|_l + n \sum_{s \geqq 2} C_s{}^{l+1} \langle s-1 \rangle \|u\|_{l+1-s}.$$

Therefore we have

$$\|u\|_l(t) \leqq \int_0^t \{\|\delta u\|_l(s) + nT_l(s)\} e^{(\gamma_0+\gamma l)(t-s)} ds,$$

where $T_l(t) = \sum_{s \geqq 2} C_s{}^{l+1} \langle s-1 \rangle \|u\|_{l+1-s}(t)$.

If we use this inequality and (4.7), by induction we can get the following inequality:

$$\|u\|_l(t) \leqq 4\kappa_{i+2,l+r}(t). \tag{4.8}$$

Moreover, if we remark that $\partial_t u = -a^k \partial_k u - b^0 u + \delta u$, we can get from (4.7) and (4.8) that

$$\|\partial_t u\|_l(t) \leqq C_{10} \{\kappa_{i+2,l+1+r}(t) + \kappa_{i+1,l+r}(t)\}, \tag{4.9}$$

where C_{10} is a constant independent of l, i, r.

Taking into account the above, one can prove the existence of a solution in the same way as in §3. Namely we can obtain the

Theorem 4.1. *Assume* (1.3), iv') *in* (1.4), *then for any given* $f(x, t) \in \Gamma^{(\alpha)}[0, h]$ *and any given initial data* $(u(x, 0), \partial_t u(x, 0)) \in \Gamma_x^{(\alpha)}$, *if* $1 \leqq \alpha < 2$, *there exists a solution* $u(x, t)$ *of the equation* (1.1) *in* Ω, *which belongs to* $\Gamma^{(\alpha)}[0, h]$. *If* $\alpha = 2$, *there exists* h_0 $(\leqq h)$ *such that in* $\boldsymbol{R}^n \times [0, h_0]$ *a solution* $u(x, t)$ *exists, which belongs to* $\Gamma^{(\alpha)}[0, h_0]$.

By the lemma of Sobolev, we can also get the corollary which corresponds to Corollary 3.1.

§5. Uniqueness and dependence domain

Let C_{x_0,t_0}, $(x_0, t_0)\in\Omega$, be a backward cone defined by

$$C_{x_0,t_0}=\{(x, t)\in\Omega;\ \mu|x-x_0|<t_0-t\},$$

where $\mu^{-1}=\sup_{\substack{(x,t)\in\Omega\\|\xi|=1}}|a^i(x, t)\xi_i+\sqrt{a^{ij}(x, t)\xi_i\xi_j}|$. Consider $u_i(x, t)$ defined already by (3.1), assuming that $f(x, t)\equiv 0$ in C_{x_0,t_0}. Taking into account the remark 1.3, it follows inductively that $u_i(x, t)$ and $M[u_i](x, t)$ vanish identically in C_{x_0,t_0}. Therefore $u(x, t)=\sum_{i=1}^{\infty}u_i(x, t)\equiv 0$ in C_{x_0,t_0}. Thus we have

Proposition 5.1. *Assume* (1.3) *and* (1.4), *then, if* $1<\alpha<2$, *the Cauchy problem*: $L[u]=f$, $f\in\gamma_0^{(\alpha)}(\Omega)$, $(u(x, 0), \partial_t u(x, 0))\in\gamma_0^{(\alpha)}(\boldsymbol{R}^n)$, *has a solution* $u(x, t)\in\gamma^{(\alpha)}(\Omega)$ *which satisfies the following property*:

$$\begin{cases}\text{If } f(x, t)\equiv 0 \text{ in } C_{x_0,t_0},\ (u(x, 0), \partial_t u(x, 0))\equiv 0 \text{ on } C_{x_0,t_0}\cap\{t=0\},\\ \text{then } u(x, t)\equiv 0 \text{ in } C_{x_0,t_0}.\end{cases}$$

By means of this proposition, we can obtain the

Theorem 5.1. (**Uniqueness**). *Assume* (1.3), (1.4) *and that* $1<\alpha<2$. *Let* $u(x, t)\in\mathcal{E}^2$ *be a solution of* (1.1) *which satisfies that*

$$L[u](x, t)=f(x, t)\equiv 0 \text{ in } C_{x_0,t_0},$$
$$(u(x, 0), \partial_t u(x, 0))\equiv 0 \text{ on } C_{x_0,t_0}\cap\{t=0\},$$

then $u(x, t)$ *must be identically null in* C_{x_0,t_0}.

Proof. We show this by contradiction. We suppose that for some (x_1, t_1) in C_{x_0,t_0}, $u(x_1, t_1)\neq 0$. Consider the (backward) Cauchy problem:

(5.1) $\quad {}^tL[v]=0$ in $\boldsymbol{R}^n\times[0, t_1]$, $v|_{t=t_1}=0$, $\partial_t v|_{t=t_1}=\theta(x)$,

where $\theta(x)\in\gamma_0^{(\alpha)}(\boldsymbol{R}^n)$, $\operatorname{supp}[\theta(x)]\subset C_{x_0,t_0}\cap\{t=t_1\}$. By the transform of variables $\boldsymbol{\Phi}$: $y=-x$, $s=-t+t_1$, this problem is reduced to the equivalent Cauchy problem:

(5.1′) $\quad {}^t\mathcal{L}[w]=0$ in $\boldsymbol{R}^n\times[0, t_1]$, $w|_{s=0}=0$, $\partial_s w|_{s=0}=-\theta(-y)$,

where ${}^t\mathcal{L}={}^tL(-y, -s+t_1; -\partial_y, -\partial_s)$. One can easily verify that if L satisfies (1.3) and iv) (or iv′)) in (1.4), then ${}^t\mathcal{L}$ satisfies (1.3) and iv′) (or iv) respectively) in (1.4). Therefore by the proposition 5.1, we can see that there exists a solution $v(x, t)\in\gamma^{(\alpha)}(\boldsymbol{R}^n\times[0, t_1])$ of (5.1), whose support is contained in $C_{x_0,t_0}\cap\{t\leqq t_1\}$.

Taking the above into account,

$$0=\int_0^{t_1}\int_{R^n}L[u]v\,dxdt=\int_0^{t_1}\int_{R^n}u\,{}^tL[v]dxdt-\int_{R^n}u(x, t_1)\theta(x)dx$$
$$=-\int_{R^n}u(x, t_1)\theta(x)dx.$$

On the other hand, since $u(x_1, t_1) \neq 0$, we can chose $\theta(x)$ such that

$$\int_{R^n} u(x, t_1)\theta(x)dx \neq 0.$$

This is a contradiction. *q.e.d.*

Finally we remark that by the procedure of the partition of unity, we can obtain the theorem 1.1.

Appendices

A.1. Remarks on the Oleinik's theorem

We explain only our plan of the proof of the theorem stated in the remark 1.3. We use the same method as in [7], namely the method of elliptic regularization. Consider the Cauchy problem

$$L_{0,\varepsilon}[u] = L_0[u] - \varepsilon \Delta u = f, \quad \text{in } \Omega,\ \varepsilon > 0, \tag{1}$$

$$u|_{t=0} = \phi(x),\ \partial_t u|_{t=0} = \psi(x). \tag{2}$$

Since the equation (1) is strictly hyperbolic, this Cauchy problem is well-posed in $\mathcal{D}^\infty_{L^2}$, also in $\mathcal{E}$ and there exists a finite domain of dependence.

Lemma. *Assume the same as in the remark* 1.3, *then the solution* $u_\varepsilon(x,t)$ *of the Cauchy problem* (1)–(2) *satisfies that*

$$\|u_\varepsilon(\cdot, t)\|_k^2 \leqq C\{\|\phi\|^2_{k+1} + \|\psi\|_k^2\} + C'\|f(\cdot, t)\|_{k-2} + C''\int_0^t \|f(\cdot, t)\|^2_{k,0} dt, \tag{3}$$

where the constants C, C', C'' *depend on* k *but does not depend on* ε*).

The inequality (3) implies that $\{u_\varepsilon(x, t)\}_{\varepsilon>0}$ is a bounded set in $\mathcal{E}^m_{L^2}(\Omega)$, $m=0,1,2,\cdots$. Therefore one can extract a subsequence $\{u_{\varepsilon_j}(x, t)\}_{j=1,2,\cdots}$, $\varepsilon_j \to 0$ as $j \to \infty$, which converges weakly in $\mathcal{E}^m_{L^2}(\Omega)$ for any $m=0, 1, 2, \cdots$. We can see that there exists $u(x, t) \in \mathcal{E}^\infty_{L^2}(\Omega)$ $(\subset \mathcal{D}^\infty_{L^2}[0, h])$ such that for any p and k and for any $v \in L^2(\Omega)$

$$(\partial^p \partial_t^k u_{\varepsilon_j}, v)_{L^2(\Omega)} \longrightarrow (\partial^p \partial_t^k u, v)_{L^2(\Omega)}, \quad \text{as } j \to \infty^{**)}.$$

This gives a solution of the Cauchy problem

$$\begin{cases} L_0[u] = f(x, t) \ \text{in } \Omega, \\ u|_{t=0} = \phi(x),\ \partial_t u|_{t=0} = \psi(x). \end{cases} \tag{4}$$

Let $C^\varepsilon_{x_0,t_0}$ be a backward cone defined by

$$C^\varepsilon_{x_0,t_0} = \{(x, t) \in \Omega;\ \mu_\varepsilon |x - x_0| < t_0 - t\},$$

---

*) We used the following notations:

$$\|u(\cdot, t)\|_k = \sum_{|p|+j \leqq k} \|\partial^p \partial_t{}^j u(\cdot, t)\|_{L_x^2},\ \|f(\cdot, t)\|_{k,0} = \sum_{|p| \leqq k} \|\partial^p f(\cdot, t)\|_{L_x^2}.$$

**) Cf. [8], Chapter 2.

where $\mu_\varepsilon^{-1}=\sup_{|\xi|=1,(x,t)\in\Omega}|a^i(x,t)\xi_i+\sqrt{a^{ij}(x,t)\xi_i\xi_j+\varepsilon}|$. If for $\delta>0$, $f(x,t)\equiv 0$ in $C^\delta_{x_0,t_0}$, $\phi(x)\equiv\psi(x)\equiv 0$ on $C^\delta_{x_0,t_0}\cap\{t=0\}$, then for any $\varepsilon<\delta$, $u_\varepsilon(x,t)\equiv 0$ in $C^\delta_{x_0,t_0}$ and therefore $u(x,t)\equiv 0$ in $C^\delta_{x_0,t_0}$. Since $\delta>0$ is an arbitrary number, we have the

Proposition. *Assume the same as in the remark* 1.3. *Then for any* $f(x,t)\in C_0^\infty(\bar{\Omega})$ *and any initial data* $\phi(x)$, $\psi(x)\in C_0^\infty(\boldsymbol{R}^n)$, *there exists a solution* $u(x,t)\in C^\infty(\bar{\Omega})$ *of the Cauchy problem* (4), *which satisfies the following property*:

$$\begin{cases}\textit{If } f(x,t)\equiv 0 \textit{ in } C_{x_0,t_0} \textit{ and if } \phi(x)\equiv\psi(x)\equiv 0 \textit{ on}\\ C_{x_0,t_0}\cap\{t=0\}, \textit{ then } u(x,t)\equiv 0 \textit{ in } C_{x_0,t_0}.\end{cases}$$

If we use this proposition, we can prove the well-posedness in $\mathcal{E}$ and the existence of a finite domain of dependence, in the same way as in §5.

A.2. Proof of Lemma 2.1.

We define the operators Λ and R_j, $j=1,2,\cdots,n$, by

$$\Lambda u=\bar{\mathscr{F}}_{\xi\to x}[|\xi|\hat{u}(\xi)],\quad R_ju=\bar{\mathscr{F}}_{\xi\to x}\left[\frac{1}{2\pi i}\frac{\xi_j}{|\xi|}\hat{u}(\xi)\right].$$

Then $\Lambda=R_j\partial_j$, $\partial_j=-(2\pi)^2R_j\Lambda$, $R_j^*=-R_j$, $R_j^*R_j=(2\pi)^{-2}$. Then

$$\begin{aligned}(a^{ij}_{(q)}\partial_iu,\partial_jv)&=(2\pi)^4(a^{ij}_{(q)}R_i\Lambda u, R_j\Lambda v)\\&=(2\pi)^4(\Lambda u, a^{ij}_{(q)}R_i^*R_j\Lambda v)+(2\pi)^4(\Lambda u,[R_i^*,a^{ij}_{(q)}]R_j\Lambda v)\\&\leqq\text{const.}\|u\|_1\{\|a^{ij}_{(q)}R_iR_j\Lambda v\|+\|v\|\}.\end{aligned}$$

Here we used that R_i and $[R_i, a^{ij}_{(q)}]\Lambda$ are bounded operators in L_x^2.

By Oleinik's lemma in [7], for $|q|=1$,

$$\|a^{ij}_{(q)}R_iR_j\Lambda v\|^2\leqq\text{const.}(a^{ij}R_iR_s\Lambda v, R_jR_s\Lambda v).$$

By the way,

$$\begin{aligned}(a^{ij}R_iR_s\Lambda v, R_jR_s\Lambda v)&=(a^{ij}R_i\Lambda v, R_jR_s^*R_s\Lambda v)\\&\quad+\text{Re}([a^{ij},R_s]R_i\Lambda v, R_jR_s\Lambda v)\\&\leqq(2\pi)^{-6}(a^{ij}\partial_iv,\partial_jv)+\text{const.}\|v\|^2.\end{aligned}$$

q.e.d.

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Added in Proof.

After submitting this paper, the author was noticed that V. Ja. Ivriĭ had succeeded, with the different method from ours, in removing the condition of analyticity of the coefficients of the operator when the multiplicity of the characteristic roots are at most double. This means that the condition (1.4) may be removed.