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Spherical matrix functions on locally compact groups of a certain type

By

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Introduction

Let G be a locally compact unimodular group and K a compact subgroup of G. Let δ be an equivalence class of irreducible representations of K of degree d, and $k\to D(k)$ an irreducible unitary matrix representation of K belonging to δ. We put $\chi_\delta(k)=d\cdot\mathrm{trace}\, D(k)$.

A $p\times p$-matrix valued continuous function $U=U(x)$ on G is called a spherical matrix function of type δ if it satisfies the following four conditions;

(i) $U^\circ(x)=U(x)$, where $U^\circ(x)=\int_K U(kxk^{-1})dk$,

(ii) $U*\chi_\delta(x)=U(x)$, where $U*\chi_\delta(x)=\int_K U(xk^{-1})\chi_\delta(k)dk$,

(iii) $\{U(x);\, x\in G\}$ is an irreducible family of matrices,

(iv) $\int_K U(kxk^{-1}y)dk=U(x)U(y)$ for any $x, y\in G$,

where dk is the normalized Haar measure on K.

We assume that G has a continuous decomposition $G=SK$ $(S\cap K=\{e\})$, where S is a closed subgroup of G and e is the unit element in G. Let $s\to\Lambda(s)$ be a finite-dimensional irreducible matrix representation of S. We put

$$\tilde{W}(x)=\overline{D(k)}\otimes\Lambda(s^{-1}) \qquad (x=ks,\ k\in K,\ s\in S),$$

$$W(x)=\tilde{W}^\circ(x^{-1})\equiv\int_K \tilde{W}(kx^{-1}k^{-1})dk,$$

then $W(x)$ satisfies the above conditions (i), (ii), and (iv), and its each "irreducible component" is a spherical matrix function of type δ.

Conversely, are all spherical matrix functions of type δ given in this way? If G is a motion group or a connected semi-simple Lie group with finite center and if K is a maximal compact subgroup of G, then we have an affirmative answer [1]. But, in general, the author obtained a weaker result only for quasi-bounded spherical matrix functions. Namely, for a quasi-bounded spherical matrix function U of type δ, we can find an irreducible Banach representation $s\to\Lambda(s)$ of S such that U is equivalent to an "irreducible component" of $W(x)$. Here $W(x)=\tilde{W}^\circ(x^{-1})$ with $\tilde{W}(x)=\overline{D(k)}\otimes\Lambda(s^{-1})$ $(x=ks,\ k\in K,\ s\in S)$, and in this case, the author does not know whether the representation $s\to\Lambda(s)$ is finite-dimensional or not.

The quasi-boundedness of spherical matrix functions make it possible for us to utilize Banach algebras in our study. In a Banach algebra a maximal regular left ideal is closed, but in a more general algebra we don't know whether it is closed or not. This is just the reason why we need the qausi-boundedness of spherical matrix functions.

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§1. Quasi-bounded spherical matrix functions

Let G be a locally compact unimodular group, and K a compact subgroup of G. Let δ be an equivalence class of irreducible representations of K and $\chi_\delta(k)$ $(k\in K)$ be as in the introduction. A $p\times p$-matrix valued continuous function $U=U(x)$ on G is called a spherical matrix function of type δ if it satisfies the four conditions (i)~(iv) in the introcdution.

A function $\rho(x)$ on G is called a semi-norm on G if it satisfies the following conditions;

(i) $\rho(x)>0$ for all $x\in G$,

(ii) $\rho(xy)\leqq\rho(x)\rho(y)$ for any $x, y\in G$,

(iii) lower semi-continuous,

(iv) bounded on every compact subset.

If a spherical matrix function U satisfies the inequality

$$|u_{ij}(x)|\leqq\alpha\rho(x) \qquad (1\leqq i, j\leqq p)$$

for a semi-norm $\rho(x)$ and a positive constant α, where $u_{ij}(x)$ is the (i,j)-matrix element of $U(x)$, then U is called quasi-bounded.

If a topologically irreducible representation of G on a Banach space contains δ p-times, then it gives us a quasi-bounded $p\times p$-spherical matrix function $U=U(x)$ of type δ [1]. Conversely every quasi-bounded spherical matrix function is given by a topologically irreducible representation of G on a Banach space.

§2. Banach algebras A_ρ, A_ρ°, $L_\rho(G)*\bar{\chi}_\delta$, and $L_\rho^\circ(\delta)$

Let G and K be as in §1. We assume that there exists a closed subgroup S of G such that

$$G=SK, \qquad S\cap K=\{e\},$$

where e is the unit element in G, and that the decomposition $x=sk$ $(s\in S, k\in K)$ is continuous. Fix a left Haar measure $d\mu(s)$ on S and denote by dk the normalized Haar measure on K, then $dx=d\mu(s)dk$ is a Haar measure on G.

Let $\rho(x)$ be a semi-norm on G. We shall denote by $L_\rho(G)$ the Banach algebra of measurable functions f on G satisfying

$$\|f\|_\rho=\int_G|f(x)|\rho(x)dx<+\infty.$$

For an equivalence class δ of irreducible representations of K of degree d, we choose an irreducible representation $k \to D(k)$ of K belonging to δ such that all $D(k)$ are unitary matrices. Put

$$L_\rho(G)*\bar{\chi}_\delta = \{f*\bar{\chi}_\delta ; f \in L_\rho(G)\}$$

where $\bar{\chi}_\delta$ is the complex conjugate of χ_δ, then this is clearly a closed subalgebra of $L_\rho(G)$.

For a $d \times d$-matrix valued measurable function $F(s)$ on S, we write $f_{ij}(s)$ for its (i, j)-matrix element. Then we shall denote by A_ρ the Banach space of all $F(s)$ which satisfy

$$\|F\|_\rho = d \cdot \operatorname*{Max}_{1 \leq i, j \leq d} \int_S |f_{ij}(s)| \rho(s) d\mu(s) < +\infty .$$

For F, $G \in A_\rho$, we define a product $F*G$ as

$$F*G(s) = \int_S F(t) G(t^{-1}s) d\mu(t).$$

With this product A is a Banach algebra, namely we have the inequality $\|F*G\|_\rho \leqq \|F\|_\rho \|G\|_\rho$.

Now we have two Banach algebras $L_\rho(G)*\bar{\chi}_\delta$ and A_ρ. Define a linear mapping Φ of $L_\rho(G)*\bar{\chi}_\delta$ to A_ρ as

$$\Phi(f)(s) = \int_K \overline{D(k)} f(sk^{-1}) dk.$$

If we choose a positive constant C such that $\rho(k) \leqq C$ for all $k \in K$, then we have

$$\rho(x) \leqq \rho(xk)\rho(k^{-1}) \leqq C\rho(xk) \qquad (x \in G,\ k \in K).$$

From this, we easily obtain an inequality

$$\|\Phi(f)\|_\rho \leqq dC \|f\|_\rho .$$

This implies that Φ is continuous. Moreover we can easily show that Φ is a bijection and that

$$\Phi^{-1}(F)(x) = d \cdot \operatorname{trace}[F(s)\overline{D(k)}] \qquad (x = sk,\ s \in S,\ k \in K),$$
$$\|\Phi^{-1}(F)\|_\rho \leqq d^2 C \|F\|_\rho .$$

Therefore Φ is an isomorphism between two Banach spaces $L_\rho(G)*\bar{\chi}_\delta$ and A_ρ, but this is not an isomorphism of Banach algebras.

For every $f \in L_\rho(G)$, we put

$$f^\circ(x) = \int_K f(kxk^{-1}) dk,$$

then the subspace

$$L_\rho^\circ(\delta) = \{f^\circ ; f \in L_\rho(G)*\bar{\chi}_\delta\}$$

is a closed subalgebra of $L_\rho(G)*\bar{\chi}_\delta$, and $f \to f^\circ$ is a continuous projection of $L_\rho(G)*\bar{\chi}_\delta$ onto $L_\rho^\circ(\delta)$. Therefore this projection induces a continuous one $F \to F^\circ$ of A_ρ onto a closed subspace denoted by A_ρ°. Namely,

$$F^{\circ}=\Phi(f^{\circ}) \qquad (f=\Phi^{-1}(F)).$$

For any $f,\ g\in L_{\rho}(G)*\bar{\chi}_{\delta}$, it is easy to show that

$$\Phi(f*g^{\circ})=\Phi(f)*\Phi(g^{\circ})=\Phi(f)*(\Phi(g))^{\circ},$$

hence we have the following

Lemma 1. *$A_{\rho}^{\circ}=\Phi(L_{\rho}^{\circ}(\delta))$ is a closed subalgebra of A_{ρ} and Φ maps isomorphically the Banach algebra $L_{\rho}^{\circ}(\delta)$ onto A_{ρ}°.*

Since $(f*g^{\circ})^{\circ}=f^{\circ}*g^{\circ}$, we obtain the equality

$$(F*G^{\circ})^{\circ}=F^{\circ}*G^{\circ} \qquad (F,\ G\in A_{\rho}).$$

§3. Main theorem

Denote by $\boldsymbol{C}^{d}$ the vector space of all column vectors with d complex numbers, and by e_i $(1\leqq i\leqq d)$ the vector whose i-th component is 1 and all the others are 0. For a Banach space H with a norm $\|\cdot\|_{H}$, the tensor product space $\boldsymbol{C}^{d}\otimes H$ is also a Banach space with the norm

$$\|\sum_{i=1}^{p} e_i\otimes v_i\|=\sum_{i=1}^{d}\|v_i\|_{H}.$$

Then our aim in this section is to prove the following

Theorem. *Let G be a locally compact unimodular group with a continuous decomposition $G=SK$, where S is a closed subgroup and K is a compact subgroup of G such that $S\cap K=\{e\}$. Let δ be an equivalence class of irreducible representations of K with degree d. If $U=U(x)$ be a quasi-bounded $p\times p$-spherical matrix function on G of type δ, then there exists a topologically irreducible representation $\{\Lambda(s), H\}$ of S on a Banach space H with the following property. Fix an irreducible unitary matrix representation $k\rightarrow D(k)$ of K belonging to δ and put*

$$\tilde{W}(x)=\overline{D(k)}\otimes\Lambda(s^{-1}) \qquad (x=ks,\ k\in K,\ s\in S),$$

$$W(x)=\tilde{W}^{\circ}(x^{-1})\equiv\int_{K}\tilde{W}(kx^{-1}k^{-1})dk.$$

Then there exists a $W(x)$-invariant p-dimensional subspace L of the Banach space $\boldsymbol{C}^{d}\otimes H$ such that $W(x)|_{L}$ is equivalent to $U(x)$, namely, with respect to a suitable base of L, the matrix corresponding to the operator $W(x)|_{L}$ is equal to $U(x)$ for all $x\in G$.

Since U is quasi-bounded, there exist a positive constant α and a semi-norm $\rho(x)$ such that

$$|u_{ij}(x)|\leqq\alpha\rho(x) \qquad (1\leqq i, j\leqq d),$$

where $u_{ij}(x)$ is the (i,j)-matrix element of $U(x)$. Then

$$f\longrightarrow U(f)=\int_{G}U(x)f(x)dx$$

is a p-dimensional irreducible matrix representation of the algebra $L_\rho^\circ(\delta)$. Therefore, by Lemma 1,

$$F \longrightarrow U(F) = U(\Phi^{-1}(F))$$

is also a p-dimensional irreducible matrix representation of the algebra A_ρ°. Let $\mathfrak{E} \in A_\rho^\circ$ be an element for which $U(\mathfrak{E})$ is the unit matrix. Then there exists a maximal left ideal $\mathfrak{A}$ in A_ρ° of codimension p such that $\mathfrak{E}$ is a right identity modulo $\mathfrak{A}$ and that the natural representation of A_ρ° on $A_\rho^\circ/\mathfrak{A}$ is equivalent to $F \rightarrow U(F)$. In general, a left ideal $\mathfrak{a}$ in an algebra is called regular if there exists a right identity modulo $\mathfrak{a}$.

Lemma 2. *Put*

$$\mathfrak{M} = \{F \in A_\rho ; (G * F)^\circ \in \mathfrak{A} \quad \textit{for all} \quad G \in A_\rho\},$$

then $\mathfrak{M}$ is a regular left ideal in A_ρ, and we have $\mathfrak{M} \cap A_\rho^\circ = \mathfrak{A}$. Moreover $\mathfrak{E}$ is a right identity modulo $\mathfrak{M}$.

Proof. It is clear that $\mathfrak{M}$ is a left ideal in A_ρ. For any F, $G \in A_\rho$, we have

$$\begin{aligned}\{G*(F*\mathfrak{E}-F)\}^\circ &= \{(G*F)*\mathfrak{E}\}^\circ - (G*F)^\circ \\ &= (G*F)^\circ * \mathfrak{E} - (G*F)^\circ \in \mathfrak{A}.\end{aligned}$$

Therefore $\mathfrak{E}$ is a right identity modulo $\mathfrak{M}$ in A_ρ.

The inclusion $\mathfrak{A} \subset \mathfrak{M} \cap A_\rho^\circ$ is clear. If $\mathfrak{E} \in \mathfrak{M}$, it follows that $A_\rho^\circ * \mathfrak{E} \subset \mathfrak{A}$ but this is impossible because the natural representation of A_ρ° on $A_\rho^\circ/\mathfrak{A}$ is irreducible. This implies $\mathfrak{M} \cap A_\rho^\circ \subsetneqq A_\rho^\circ$. Since $\mathfrak{M} \cap A_\rho^\circ$ is a proper left ideal which contains $\mathfrak{A}$, we obtain $\mathfrak{M} \cap A_\rho^\circ = \mathfrak{A}$. q.e.d.

Let $\mathfrak{M}_0$ be a maximal left ideal in A_ρ containing $\mathfrak{M}$. Then $\mathfrak{M}_0$ is regular ($\mathfrak{E}$ is a right identity modulo $\mathfrak{M}_0$). It is well known that a regular maximal left ideal in a Banach algebra is closed, and hence $\mathfrak{M}_0$ is closed. Since $\mathfrak{E} \notin \mathfrak{M}_0$, it follows that $\mathfrak{M}_0 \cap A_\rho^\circ = \mathfrak{A}$. From this, the space $A_\rho^\circ/\mathfrak{A}$ can be considered as a p-dimensional subspace of $A_\rho/\mathfrak{M}_0$. As usual, we can introduce a norm $\|\cdot\|$ in $A_\rho/\mathfrak{M}_0$ with which $A_\rho/\mathfrak{M}_0$ is a Banach space. Denote by $\prod(F)$ the natural representation of the Banach algebra A_ρ on $A_\rho/\mathfrak{M}_0$. Then it is algebraically irreducible and we have

$$\|\prod(F)X\| \leqq \|F\|_\rho \|X\|$$

for $F \in A_\rho$ and $X \in A_\rho/\mathfrak{M}_0$. The subspace $A_\rho^\circ/\mathfrak{A}$ of $A_\rho/\mathfrak{M}_0$ is invariant under $\prod(A_\rho^\circ)$ and $F \rightarrow \prod(F)|_{A_\rho^\circ/\mathfrak{A}}$ is an irreducible representation of A_ρ° equivalent to $F \rightarrow U(F)$.

We shall denote by $L_\rho(S)$ the Banach algebra of all functions f on S satisfying

$$\|f\|_\rho = \int_S |f(s)| \rho(s) d\mu(s) < +\infty.$$

Let E_{ij} be the $d \times d$-matrix whose (i, j)-matrix element is 1 and all the others are 0. Define $(fE_{ij})(s)=f(s)E_{ij}$, then $fE_{ij}\in A_\rho$ for all $f\in L_\rho(S)$. Now we put

$$\pi_{ij}(f)=\prod(fF_{ij}) \qquad (1\leqq i, j\leqq d).$$

Then clearly we have a relation

$$\pi_{ij}(f)\pi_{kl}(g)=\delta_{jk}\pi_{il}(f*g)$$

where δ_{jk} is the Kronecker's delta.

For every element $F\in A_\rho$, we put

$$(E_{ij}F)(s)=E_{ij}\cdot F(s) \qquad (1\leqq i, j\leqq d),$$

where the right hand side is the product of matrices E_{ij} and $F(s)$. The linear mapping $F\rightarrow E_{ij}F$ is clearly continuous.

Lemma 3. $E_{ij}\mathfrak{M}_0\subset\mathfrak{M}_0 \qquad (1\leqq i, j\leqq d)$.

Proof. For every open neighbourhood V of the unit e, we take a non negative continuous function e_V which vanishes outside of V and satisfies $\int_S e_V(s)d\mu(s)=1$. Then $e_V E_{ij}\in A_\rho$, and for any $F\in A_\rho$, we have

$$\|(e_V E_{ij})*F-E_{ij}F\|_\rho\longrightarrow 0 \qquad (V\rightarrow e).$$

Hence the lemma is proved. q.e.d.

Therefore we may consider that E_{ij} acts continuously on the Banach space $A_\rho/\mathfrak{M}_0$. Put

$$H_i=E_{ii}(A_\rho/\mathfrak{M}_0) \qquad (1\leqq i\leqq d),$$

then H_i is a closed subspace of $A_\rho/\mathfrak{M}_0$ and E_{ii} is a continuous projection onto H_i. Moreover it is clear that

$$A_\rho/\mathfrak{M}_0=H_1+\cdots+H_d \qquad \text{(direct sum)}$$

and that

$$E_{ij}H_j=H_i \qquad (1\leqq i, j\leqq d).$$

For any function $f\in L_\rho(S)$, we easily have the equality

$$\pi_{ii}(f)\circ E_{ij}=E_{ij}\circ\pi_{jj}(f) \qquad (1\leqq i, j\leqq d).$$

Lemma 4. *All* $\{\pi_{ii}(f), H_i\}$ *are algebraically irreducible representations of the algebra* $L_\rho(S)$, *and they are equivalent with one another.*

Proof. We have only to show that each H_i is invariant and algebraically irreducible under π_{ii}, but the former is clear. Let's prove the latter. Take a non-trivial invariant subspace H_1' of H_1 under π_{11}. We put $H_i'=E_{i1}H_1'$ $(1\leqq i\leqq d)$, then H_i' is invariant under π_{ii}. Let F be an arbitrary element in A_ρ with f_{ij} for its (i, j)-matrix element. For any vector $\sum_{i=1}^d Y_i\in H_1'+\cdots+H_d'$ where $Y_i=E_{i1}X_i$ $(X_i\in H_1')$,

$$\prod(F)(\sum_{i=1}^{d} Y_i)=\sum_{i,j=1}^{d}\pi_{ij}(f_{ij})(\sum_{k=1}^{d}E_{k1}X_k)$$
$$=\sum_{i=1}^{d}\sum_{k=1}^{d}\pi_{ik}(f_{ik})(E_{k1}X_k)$$
$$=\sum_{i=1}^{d}\sum_{k=1}^{d}\pi_{ii}(f_{ik})(E_{i1}X_k)\in H_1'+\cdots+H_d',$$

since $E_{i1}X_k\in H_i'$ for all i. Therefore the subspace $H_1'+\cdots+H_d'$ is invariant under $\prod(F)$ for all $F\in A_\rho$. This implies $H_1'+\cdots+H_d'=A_\rho/\mathfrak{M}_0$, hence $H_1'=H_1$. q.e.d.

Let $\pi(s)$ ($s\in S$) be the left translation on A_ρ, namely,

$$(\pi(s)F)(t)=F(s^{-1}t).$$

Then $\pi(s)$ is a continuous linear operator on A_ρ since we have

$$\|\pi(s)F\|_\rho\leqq\rho(s)\|F\|_\rho.$$

Moreover, we can prove that the function $s\to\pi(s)F$ on S is continuous for every $F\in A_\rho$. Therefore $\{\pi(s), A_\rho\}$ is a representation of S.

Lemma 5. $\pi(s)\mathfrak{M}_0\subset\mathfrak{M}_0$ *for all* $s\in S$.

Proof. For every open neighbourhood V of s, we take a function e_V as in the proof of Lemma 3. Then for any function $f\in L_\rho(S)$, we obtain $e_V*f\in L_\rho(S)$ and

$$\|e_V*f-\pi(s)f\|_\rho\longrightarrow 0 \qquad (V\to s),$$

where$(\pi(s)f)(t)=f(s^{-1}t)$. Let E be the unit matrix of degree d, then $e_VE\in A_\rho$ and

$$e_VE*F\longrightarrow\pi(s)F \qquad (V\to s)$$

in A_ρ. Since $\mathfrak{M}_0$ is closed, the lemma is now proved. q.e.d.

This lemma implies that the linear operator $\pi(s)$ naturally induces a linear operator on $A_\rho/\mathfrak{M}_0$ which is also denoted by $\pi(s)$. Since $\|\pi(s)X\|\leqq\rho(s)\|X\|$, $\{\pi(s), A_\rho/\mathfrak{M}_0\}$ is a representation of S.

Lemma 6. *Each subspace* H_i *is invariant under* $\pi(s)$ *for all* $s\in S$.

Proof. Since $\pi(s)\circ E_{ii}=E_{ii}\circ\pi(s)$, the lemma is clear. q.e.d.

Now we put

$$\pi_{ii}(s)=\pi(s)|_{H_i} \qquad (1\leqq i\leqq d)$$

for every $s\in S$. Then for any $f\in L_\rho(S)$, we have

$$\pi_{ii}(f)=\int_S\pi_{ii}(s)f(s)d\mu(s).$$

Therefore all representations $\{\pi_{ii}(s), H_i\}$ of S are topologically irreducible and equivalent with one another. Let $\{\Lambda(s), H\}$ be a topologically irreducible representation of S on a Banach space H such that there exists an isomorphism I_i of H_i onto H satisfying

$$I_i \circ \pi_{ii}(s) = \Lambda(s) \circ I_i \qquad (s \in S)$$

and

$$I_i = I_j \circ E_{ji} \qquad (1 \leqq i, j \leqq d).$$

As before, we denote by e_i $(1 \leqq i \leqq d)$ the column vector whose i-th component is 1 and all the others are 0. Let I be an isomorphism of $A_\rho/\mathfrak{M}_0$ onto $C^d \otimes H$ defined by

$$I(\sum_{i=1}^{d} X_i) = \sum_{i=1}^{d} e_i \otimes I_i X_i \qquad (X_i \in H_i).$$

Then, for every $F \in A_\rho$ whose (i, j)-matrix element is f_{ij},

$$\begin{aligned}
&[(\sum_{i,j=1}^{d} E_{ij} \otimes \Lambda(f_{ij})) \circ I](\sum_{k=1}^{d} X_k) \\
&= (\sum_{i,j=1}^{d} E_{ij} \otimes \Lambda(f_{ij}))(\sum_{k=1}^{d} e_k \otimes I_k X_k) \\
&= \sum_{i,j=1}^{d} e_i \otimes \Lambda(f_{ij}) I_j X_j = \sum_{i,j=1}^{d} e_i \otimes I_j \pi_{jj}(f_{ij}) X_j \\
&= \sum_{i,j=1}^{d} e_i \otimes I_i E_{ij} \pi_{jj}(f_{ij}) X_j \\
&= \sum_{i,j=1}^{d} e_i \otimes I_i \prod (f_{ij} E_{ij}) X_j = [I \circ \prod (F)](\sum_{k=1}^{d} X_k).
\end{aligned}$$

Therefore the representation

$$F \longrightarrow \sum_{i,j=1}^{d} E_{ij} \otimes \Lambda(f_{ij})$$

of A_ρ on the Banach space $C^d \otimes H$ is equivalent to $F \to \prod(F)$ on $A_\rho/\mathfrak{M}_0$.

Put

$$\tilde{W}(x) = \overline{D(k)} \otimes \Lambda(s^{-1}) \qquad (x = ks,\ k \in K,\ s \in S).$$

For any function $f \in L_\rho(G) * \chi_\delta$, we denote by f_{ij} the (i, j)-matrix element of $F = \Phi(f) \in A_\rho$, then

$$\begin{aligned}
\sum_{i,j=1}^{d} E_{ij} \otimes \Lambda(f_{ij}) &= \sum_{i,j=1}^{d} E_{ij} \otimes \int_S \Lambda(s) d\mu(s) \int_K \overline{d_{ij}(k)} f(sk^{-1}) dk \\
&= \int \sum_{i,j=1}^{d} (E_{ij} \otimes \Lambda(s)) \overline{d_{ij}(k)} f(sk^{-1}) d\mu(s) dk \\
&= \int \overline{D(k^{-1})} \otimes \Lambda(s) f(sk) d\mu(s) dk \\
&= \int_G \tilde{W}(x^{-1}) f(x) dx.
\end{aligned}$$

Denote by $W(x)$ the operator valued function $\tilde{W}^\circ(x^{-1})$, i.e.,

$$W(x) = \int_K \tilde{W}(kx^{-1}k^{-1}) dk,$$

then we have

$$\sum_{i,j=1}^{d} E_{ij}\otimes\Lambda(f_{ij})=\int_G W(x)f(x)dx$$

for all $f\in L_p^\circ(\delta)$. Since $f\to\Phi(f)$ is an isomorphism of the Banach algebra $L_p^\circ(\delta)$ onto A_p°, the mapping

$$f\longrightarrow\int_G W(x)f(x)dx\,|_L,$$

where $L=I(A_p^\circ/\mathfrak{A})$, is a p-dimensional irreducible representation of the algebra $L_p^\circ(\delta)$ equivalent to $f\to U(f)$.

Lemma 7. *The subspace L of $\boldsymbol{C}^d\otimes H$ is invariant under $W(x)$ for all $x\in G$.*

Proof. From the definition of $\tilde{W}(x)$ we can easily show that the $\boldsymbol{C}^d\otimes H$-valued function $x\to\tilde{W}(x^{-1})a$ on G is continuous for every $a\in\boldsymbol{C}^d\otimes H$. Therefore $x\to W(x)a$ is also continuous since K is compact.

Assume there exists a vector $a_0\in L$ such that $W(x_0)a_0\notin L$ for some $x_0\in G$. For every open neighbourhood V of x_0, we take a non negative continuous function e_V which vanishes outside of V and satisfies $\int_G e_V(x)dx=1$.
For an arbitrarily given $\varepsilon>0$, we have

$$\|W(x)a_0-W(x_0)a_0\|<\varepsilon\quad\text{for all }x\in V,$$

if V is small enough, where $\|\cdot\|$ is the norm in $\boldsymbol{C}^d\otimes H$ defined at the beginning of this section. Then

$$\begin{aligned}&\left\|\int_G W(x)a_0e_V(x)dx-W(x_0)a_0\right\|\\&=\left\|\int_G\{W(x)-W(x_0)\}a_0e_V(x)dx\right\|<\varepsilon.\end{aligned}$$

This implies $\int_G W(x)a_0e_V(x)dx\notin L$ if V is small enough. It is clear that $W*\chi_\delta=W$ and that $W^\circ=W$, hence we obtain

$$\int_G W(x)a_0e_V^\circ*\bar{\chi}_\delta(x)dx\notin L.$$

This contradicts that $e_V^\circ*\bar{\chi}_\delta\in L_p^\circ(\delta)$. q.e.d.

The proof of the theorem is now completed.

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Reference

[1] H. Shin'ya, Spherical functions and spherical matrix functions on locally compact groups, Lectures in Mathematics, Dep. Math. Kyoto Univ., Kinokuniya, Tokyo, (1974).