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Plessner points, Julia points, and ρ^*-points[1)]

By

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1. Introduction.

Let $f(z)$ be a function defined in the unit disk $D(|z|<1)$. As in [14], a point $e^{i\theta}$ on the unit circle $C(|z|=1)$ is called a Plessner point of f provided each angular cluster set $C(f, e^{i\theta})$ of f at $e^{i\theta}$ coincides with the extended plane. Following [6], we call a point $e^{i\theta}$ a Julia point of f provided in each Stolz angle Δ having one vertex at $e^{i\theta}$ the function f assumes all values on the Riemann sphere except possibly two. For $z, z' \in D$, we denote by $\rho(z, z')$ the non-Euclidean metric $\rho(z, z') = \frac{1}{2}\log[(1+a)/(1-a)]$, where $a = |(z'-z)/(1-\bar{z}z')|$. We call $\rho(z, z')$ the ρ-distance between z and z'. As in [9], a sequence $\Delta(n)$ of disks in D is called a sequence of cercles de remplissage for f provided that the ρ-diameters of $\Delta(n)$ tend to zero, and the images $f(\Delta(n))$ cover all of the Riemann sphere, with the possible exception of two sets $E(n)$ and $G(n)$ whose spherical diameters tend to zero as $n\to\infty$. The sequence $\{z_n\}$ of centres of the disks $\{\Delta(n)\}$ is called a sequence of ρ-points for f. A point $e^{i\theta}$ is called a ρ^*-point of f provided each Stolz angle Δ with one vertex at $e^{i\theta}$ possesses a sequence of ρ-points of f.

The content of this article has six more sections. In section 2, we discuss the inclusion property among Plessner points, Julia points, and ρ^*-points. Then we present a sufficient condition of normal functions in section 3. In section 4, we construct some holomorphic

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functions with all points of ρ^*-points while in section 5 we deal with dense sets of ρ^*-points. After that we discuss some measurable and topological properties among these three different sets. Finally, on the last section, we study the relation between the asymptotic behaviour and the distribution of ρ-points.

Recently, in this Journal (Kyoto University), we co-worked with Gauthier on the distribution of cercles de remplissage for functions having spiral asymptotic values. We solved a previous conjecture of Gauthier about a function f which is meromorphic in D and approaches some value quickly on a spiral, then f has many sequences of ρ-points. More precisely, we have shown the following two results [9, Theorems 1 and 2].

Theorem A. *Let f be a function meromorphic in D. Suppose that for some spiral σ, f satisfies*

$$(*) \qquad |f(z)-w| = \exp(-1/(1-|z|))^{\alpha+\varepsilon} \text{ for } z \epsilon \sigma,$$

where $\alpha \geq 1$, $\varepsilon > 0$, and w is a complex number. Then either $f \equiv w$ or in each Stolz angle of opening π/α, f has a sequence of ρ-points.

The same conclusion holds if we replace the condition (*) by a restricted spiral and a function omitting some value $v \neq w$. To state the result precisely, we call σ a bounded hyperbolic spiral, if for any point $e^{i\theta}$ on C and any segment L in D with one vertex at $e^{i\theta}$, the sequence of points $\{z_n\}$, where $\bigcup_{n=1}^{\infty}\{z_n\} = \sigma \cap L$, $|z_n| \nearrow 1$, satisfies $\limsup_{n\to\infty} \rho_n < \infty$, where $\rho_n = \rho(z_n, z_{n+1})$.

Theorem B. *Let f be meromorphic in D and omit some value v. Suppose that for some bounded hyperbolic spiral σ, f tends to a value w different from v along σ. Then either $f \equiv w$, or in each Stolz angle, f has a sequence of ρ-points.*

The above Theorem B will be used to prove some holomorphic functions having all points of C to be ρ^*-points in Section 4. And the farther extension of Theorem A concerning the relation between the asymptotic behaviour and the distribution of ρ-points will be studied on the last section.

2. Inclusion property.

For a function f defined in D, we denote the sets of all Plessner points, Julia points, and ρ^*-points by $P(f)$, $J(f)$, and $\rho^*(f)$ respectively.

We have the following simple but interesting result.

Theorem 1. $\rho^*(f) \subset J(f) \subset P(f)$.

Proof. Let $e^{i\theta} \in \rho^*(f)$, Δ be a Stolz angle with one vertex at $e^{i\theta}$, and Δ' be a proper subangle of Δ having one vertex at the same point $e^{i\theta}$. Then by the definition of ρ^*-point, Δ' possesses a sequence of ρ-points $\{z_n\}$ of f. Consider the corresponding sequence of cercles de remplissage $\{\Delta(n)\}$. Since the ρ-diameters of $\Delta(n)$ tend to zero and Δ' is a proper subangle of Δ, there is a positive integer N such that $\bigcup_{n=N}^{\infty} \Delta(n) \subset \Delta$. It follows from a well-known result (see [9, Lemma 3]) that f assumes every value on the Riemann sphere except possibly two infinitely often in the union $\bigcup_{n=N}^{\infty} \Delta(n)$. This shows that $e^{i\theta} \in J(f)$ and therefore $\rho^*(f) \subset J(f)$. The proof of $J(f) \subset P(f)$ follows immediately from the definition of Julia point.

Naturally, we may ask whether the inclusion is proper. The following two theorems answer this question.

Theorem 2. *There exists a holomorphic function f in D such that $J(f) = \phi$ and* $\text{mea}.P(f) = 2\pi$.

Proof. Let f be the elliptic modular function which omits three values 0, 1, and ∞ in D. Clearly $J(f) = \phi$. Moreover, by a theorem of F. Bagemihl [1, Theorem 1], we know that $\text{mea}.P(f) = 2\pi$.

Theorem 3. *There exists a holomorphic function f in D such that $\rho^*(f) = \phi$ and $\text{mea}.J(f) = 2\pi$.*

Proof. Let $F(z) = \sum z^{2^n}$. Then we have

$$\begin{aligned} |F'(z)| &= |1 + 2z + 4z^3 + 8z^7 + \cdots| \\ &\leq 1 + 2r + 2(r^3 + r^3) + 2(r^7 + r^7 + r^7 + r^7) + \cdots \\ &< 2[1 + r + (r^2 + r^3) + (r^4 + r^5 + r^6 + r^7) + \cdots] \\ &= 2\sum r^n = 2/(1-r) = 2/(1-|z|). \end{aligned}$$

It follows that $|F'(z)|/(1+|F(z)|^2) \leq |F'(z)| \leq 2/(1-|z|) \leq 4/(1-|z|^2)$. By a theorem of Noshiro, Lehto, and Virtanen [21, p. 87, Theorem 7], we know that F is normal in D.

Now, applying a theorem of Littlewood, Paley, and Zygmund [7, p. 228], there is a choice of signs $\{\varepsilon_n\}$ such that the function $f(z) = \sum \varepsilon_n z^{2^n}$ has a finite radial limit almost nowhere. By using the same method as above, we find that

$$|f'(z)|/(1+|f(z)|^2)\leq 4/(1-|z|^2).$$

Therefore $f(z)$ is also normal in D. According to [9, Lemma 1] $f(z)$ has no sequence of ρ-points. We thus have $\rho^*(f)=\phi$.

It remains to prove mea. $J(f)=2\pi$. By a theorem of Privalov [5, Theorem 8.1 and Corollary 1], we know that the set of Fatou points of f with Fatou value equal to infinite must be of measure zero, and therefore the set of all Fatou points of f with finite or infinite Fatou value must also be of measure zero. It follows from a theorem of Plessner [5, Theorem 8.2] that the measure of $P(f)$ equals to 2π. It is sufficient to prove that $J(f)=P(f)$. Let $e^{i\theta}\in P(f)$, if $e^{i\theta}\notin J(f)$, then there would be a Stolz angle Δ having one vertex at $e^{i\theta}$ such that f omits at least one value v in Δ. By virtue of the Gross-Iversen's Theorem [5, Theorem 5.8], we can see that v is an asymptotic value of f at $e^{i\theta}$. It follows from Lehto-Virtanen's Theorem [15, Theorem 2], f has the angular limit v at $e^{i\theta}$ which is absurd. We thus establish that $P(f)\subset J(f)$ and therefore by Theorem 1, we have $J(f)=P(f)$. This completes the proof.

3. Gap series and normal functions.

The function $F(z)=\sum z^{2^n}$ constructed in the previous theorem present a method to find a sufficient condition of normal functions. For this, we have the following general theorem.

Theorem 4. *Let $f(z)=\sum a_n z^{k_n}$ be a holomorphic function in D. If $|a_n|\leq M$ and $\liminf_{n\to\infty} k_n/k_{n-1}=k>1$, then $f(z)$ is normal in D.*

Proof. From the given condition, there is a positive integer N such that

$$k_n\geq ck_{n-1} \text{ provided } n\geq N, \text{ where } 1<c<k.$$

This in turns implies that

$$[k_n/c]\geq k_{n-1},$$

where $n\geq N$ and $[x]$ denotes the greatest integer $\leq x$.

For $|z|=r>0$, we have

$$|f'(z)|\leq M(\textstyle\sum_1+\sum_N^\infty k_n r^{k_n-1}), \text{ where } \sum_1=\sum_1^{N-1} k_n r^{k_n-1},$$

$$\leq M_1\textstyle\sum_N^\infty k_n r^{k_n}, \text{ where } M_1 \text{ is some positive number,}$$

$$\leq M_2\{(1-1/c)k_N r^{k_N}+(1-1/c)k_{N+1}r^{k_{N+1}}+\cdots\},$$

where $M^2=\dfrac{cM_1}{c-1}$,

$$\leq M^2\{(1+r+\cdots+r^{k_N})+(r^{[k_{N+1}/c]}+\cdots+r^{k_{N+1}})+\cdots\}$$

$$\leq M_3/(1-r)=M_3/(1-|z|),\text{ where } M_3>M_2.$$

Thus $f(z)$ is normal in D on account of the same reasoning as in Theorem 3.

With the help of Theorem 4, we are able to give an alternative proof of Theorem 3 without using the theorem of Littlewood, Palay, and Zygmund. Our method was suggested by Professor Piranian.

Theorem 5. *There exists a sequence of positive integers $\{k_n\}$ such that the function $f(z)=\sum z^{k_n}$ satisfies $\rho^*(f)=\phi$ and* mea. $J(f)=2\pi$, *and further $f(z)$ has no finite asymptotic values.*

Proof. Since $\lim_{k\to\infty}(1-1/k)^k=e^{-1}$, inductively we can choose a sequence of positive integers $\{k_n\}$ such that

(1) $\qquad r_m^{k_{n+1}}\leq 1/2^{n+1}$,

(2) $\qquad r_m^{k_m-1}+\cdots+s_m^{k_m-1}\geq k_m/2$, and

(3) $\qquad \sum_{n=1}^{m-1}k_n\leq k_m/4$,

where $r_m=1-1/2k_m$, $s_m=1-1/k_m$, and $n=m$, $m+1,\cdots$.

Let $f(z)=\sum z^{k_n}$, then we have

$$|f(r_m e^{i\theta})-f(s_m e^{i\theta})|=|\sum_{n=1}^{\infty}(r_m^{k_n}-s_m^{k_n})e^{ik_n\theta}|$$

$$\geq(r_m^{k_m}-s_m^{k_m})-[\sum_{n=1}^{m-1}(r_m^{k_n}-s_m^{k_n})+\sum_{n=m+1}^{\infty}(r_m^{k_n}-s_m^{k_n})]$$

$$\geq(r_m-s_m)(r_m^{k_m-1}+\cdots+s_m^{k_m-1})-\sum_{n=1}^{m-1}(r_m^{k_n}-s_m^{k_n})-1/2^{m+1}, \qquad \text{(by (1))}$$

$$\geq(r_m-s_m)[k_m/2-\sum_{n=1}^{m-1}(r_m^{k_n-1}+\cdots+s_m^{k_n-1})]-o(1), \qquad \text{(by (2))}$$

$$\geq\frac{1}{2k_m}(k_m/2-\sum_{n=1}^{m-1}k_n)-o(1)\geq 1/8-o(1), \qquad \text{(by (3))}.$$

It follows that $|f(r_m e^{i\theta})-f(s_m e^{i\theta})|>1/10$, uniformly on $[0, 2\pi]$, as $m\to\infty$. Thus the function $f(z)$ has no finite asympotic values.

Now, by virtue of Theorem 4, $f(z)$ is normal in D and therefore we have $\rho^*(f)=\phi$. The proof of mea.$J(f)=2\pi$ lies on the same pattern as in Theorem 3.

Theorem 5 has an important application to the following problem asked by Lohwater and Piranian [16, p. 16]:

Whether there exists a meromorphic (or holomorphic) function $f(z)$ that has no radial limits and whose Nevanlinna characteristic function $T(r, f)$ ([20]), as a function of r, has arbitrarily slow growth.

This question has been answered by G. R. MacLane [17] and [18] for meromorphic and holomorphic functions respectively. From a well-known theorem of Nevanlinna [20, p. 220], we know that $T(r, f) \leq \log M(r, f)$, where $M(r, f) = \mathrm{Max}\,|f(z)|$, for $|z| = r$. Therefore, for holomorphic function, the above problem becomes to find a function f that has no radial limits and whose maximum modulus has arbitrarily slow growth. Naturally, we may ask whether we can require the function f to be normal in D. The following theorem answers this question.

Theorem 6. *There exists a normal holomorphic function $f(z)$ that has no finite radial limits and whose maximum modulus has arbitrarily slow growth.*

Proof. Let $1 < \mu(r) \uparrow \infty \, (r \uparrow 1)$ be given. Inductively we can choose a sequence of positive integers $\{k_n\}$ satisfying the equations (1), (2), and (3) in Theorem 5, and also the inequality

$$|z|^{k_n} \leq 2^{-n} \mu(|z|), \quad (0 \leq |z| < 1).$$

Then the function $f(z) = \sum z^{k_n}$ satisfies $M(r, f) \leq \sum 2^{-n} \mu(r) = \mu(r)$. The conclusion now follows from Theorem 5.

This theorem is sharp because any unbounded normal holomorphic function must have the Fatou value ∞ on a dense subset of C [4, Corollary 1].

The function $f(z)$ of Theorem 6 is only a special case of this class of functions $f_m(z) = \sum k_n^m z^{k_n}$. In general, these functions $f_m(z)$ need not be normal in D. We shall now prove a necessary and sufficient conditions of $f_m(z)$ to be normal in D.

Theorem 7. *Let $f_m(z) = \sum k_n^m z^{k_n}$, where $k_n / k_{n-1} \to \infty$, as $n \to \infty$, then we have $f_m(z)$ is normal if $m \leq 0$ and conversely $f_m(z)$ is not normal if $m \geq 1$.*

Proof. The sufficient condition is an easy consequence of Theorem 4. We need only to prove that if $m \geq 1$ then $f_m(z)$ is not normal in D.

Our proof comes from that of Hayman [11, p. 22], we sketch it here. Let $|z| = e^{-1/k_n}$, then by [11, p. 22], we have

$$|f_m(z)| = [(1 + \mathrm{o}(1))/e] k_n^m \to \infty, \text{ and}$$

$$|f'_m(z)|/(1+|f_m(z)|^2)=(e+o(1))/k_n^{m-1}\to 0.$$

It follows from [11, Theorem 6] that $f_m(z)$ possesses asymptotic values at every point of C, and by the above estimate, these asymptotic values must all be infinite. Now, if $f_m(z)$ were normal in D, then by the aforementioned theorem of Lehto-Virtanen, $f_m(z)$ would have the angular limit ∞ at all points of C. This contradicts to the aforementioned theorem of Privalov.

Theorem 7 has an important application to the existence of a non-normal function such that its intergral is normal. As in [12], Hayman and Storvick answered a question of Drasin by showing that neither the derivative nor the integral of a normal function is necessarily normal.

Now, let $f(z)=\sum k_n z^{k_n}$, $k_n/k_{n-1}\to\infty$, then by Theorem 7, we know that $f(z)$ is not normal in D. We set

$$F(z)=\int_0^z f(w)\,dw=\sum (k_n/(k_n+1))z^{k_n+1}.$$

Again, by Theorem 7, we can see that the integral $F(z)$ is normal in D.

We now consider the function $f(z)$ as the derivative of $F(z)$. We thus obtain

Theorem 8. *There exists a normal function $f(z)$ for which the derivative $f'(z)$ is not normal and $f'(z)$ possesses the asymptotic value ∞ at every point of C.*

Unfortunately, the above method cannot produce a normal function such that its integral is not normal.

Another application of Theorem 4 is to determine whether a gap series has a point-tract or arc-tract [19, p. 5]. In this connection, MacLane [19, Example 5] proved that the function $f(z)=\sum a_n z^{k_n}$, where $a_n=(-1)^n/n$ and $k_n=4^n$, has no arc-tract. This result follows immediately from Theorem 4 and a theorem of MacLane [19, Theorem 17].

4. Holomorphic functions with all points of ρ^*-points.

The functions constructed in Theorems 2 and 3 both give the result $\rho^*(f)=\phi$. Therefore we may ask whether there is some function f for which $\rho^*(f)\neq\phi$. This can be seen from the following function constructed by Collingwood, and Piranian [6, Theorem 1], which is a Tsuji function and possesses the property $\rho^*(f)=C$,

$$f(z) = \sum a_n/(z - z_n),$$

where $z_n = (1 - n^{-1/2}) e^{i \log n}$, a_n are sufficiently small. However this function is only meromorphic in D. The following theorem gives us a holomorphic function satisfying $\rho^*(f) = C$, but f is not a Tsuji function.

Theorem 9. *There is a holomorphic function f in D for which we have $\rho^*(f) = C$.*

Proof. Let $x_n = 1 - 1/n^2$, $C_n = \{z : |z| = x_n\}$, Δ a Stolz angle having one vertex at the point $z = 1$, and let $r_n = C_n - \Delta$. By joining the lower end point of r_n to the upper point of r_{n+1}, we can construct a spiral α in D. Since

$$\rho_n = \rho(x_n, x_{n+1}) = \frac{1}{2} \log \frac{(1 - x_n x_{n+1}) + (x_{n+1} - x_n)}{(1 - x_n x_{n+1}) - (x_{n+1} - x_n)}$$
$$= \frac{1}{2} \log \frac{(n+1)^2 - 1}{n^2 - 1},$$

it follows that $\rho_n \to 0$ as $n \to \infty$.

Clearly for any point $e^{i\theta}$ and any segment L with one endpoint at $e^{i\theta}$, the sequence of points $\{z_n\}$, where $\cup \{z_n\} = \alpha \cap L$, satisfies $\rho_n = \rho(z_n, z_{n+1}) \to 0$. This shows that α is a bounded hyperbolic spiral.

Now, it is easy to see that α belongs to the Arakélian's class [10, Definition 1. 8] and therefore by the Arakélian's tangential approximation [10, Theorem 1.11], we can find a nonconstant holomorphic function f which tends to zero along α. By virtue of [9, Theorem 2], we can see for any point $e^{i\theta}$ and any Stolz angle Δ having one vertex at $e^{i\theta}$, f possesses a sequence of ρ-points. We thus establish that $e^{i\theta}$ is a ρ^*-point of f and therefore $\rho^*(f) = C$.

5. Holomorphic functions with dense sets of ρ^*-points.

The conformal mapping f^* constructed by Lappan and Piranian [14] has a dense set of Plessner points on C. However f^* is normal in D and therefore by [9, Lemma 1], it has no sequence of ρ-points. Thus the set of ρ^*-points of f^* is empty. On the other hand, if E is a set of measure 0 on C, then by [6, Theorem 3], there exists a Tsuji function f of bounded characteristic for which every point of E is a ρ^*-point. But this function f is not holomorphic in D. For holomorphic function, we shall prove the following result which generalizes to that of [14].

Theorem 10. *If E is a countable dense set on C, then there is a* *holomorphic function f of bounded characteristic for which every point of E is a ρ^*-point and mea.$\rho^*(f)=0$.*

To prove this theorem, we shall need three lemmas.

Lemma 1. *If $\{R_m\}$, $m=1, 2,\ldots$, is a countable system of rays, there is a sequence of points $\{p_n\}$, where $|p_n|=1-1/n^2$, such that each ray R_m contains infinitely many of the points $p_{m,n}$, where $\{p_{m,n}\}$ is a rearrangement of $\{p_n\}$, for which $\rho_{m,n}=\rho(p_{m,n}, p_{m,n+1})\to 0$, as $n\to\infty$.*

Proof. With respect to this order R_1, $R_2, \ldots$, we choose

$$\begin{aligned}&(1-1/2^2)e^{i\theta_1}\in R_1, (1-1/3^2)e^{i\theta_2}\in R_1,\ (1-1/4^2)e^{i\theta_3}\in R_2, \ldots,\\ &(1-1/n^2)e^{i\theta_{k_n}}\in R_1,\ (1-1/(n+1)^2)e^{i\theta_{k_n+1}}\in R_2, \ldots,\\ &(1-1/(n+a)^2)e^{i\theta_{k_n+a}}\in R_{a+1}, \ldots\end{aligned}$$

Let $p_n=(1-1/n^2)e^{i\theta_{k_n}}$, then clearly each R_m contains infinitely many of the points p_n.

The distribution of the index set $\{n\}$ reads as follows:

$$\begin{array}{l}1,\ 2,\ 4, \ldots,\ 1+n(n+1)/2, \ldots\\ 3,\ 5,\\ 6,\\ \vdots\\ m(m+1)/2, \ldots, [m(m+1)+(n-1)(2m+n-2)]/2, \ldots\end{array}$$

where $n=0, 1, 2, \ldots$, and $m=1, 2, \ldots$ From this arrangement, we can see for each ray R_m, the sequence of points $p_{m,n}\in R_m$ can be written as

$$p_{m,n}=(1-a_{m,n})e^{i\theta_{m,n}}, \text{ where } a_{m,n}=4/[m(m+1)+(n-1)(2m+n-2)]^2.$$

It is obvious that $\rho(p_{m,n}, p_{m,n+1})\le M\rho(|p_{m,n}|, |p_{m,n+1}|)$, for some positive number M, and therefore it is sufficient to prove that

$$\rho'_{m,n}=\rho(|p_{m,n}|, |p_{m,n+1}|)\to 0, \text{ as } n\to\infty.$$

We have the following computation

$$\begin{aligned}\rho'_{m,n}&=\frac{1}{2}\log\frac{(1-(1-a_{m,n})(1-a_{m,n+1}))+(a_{m,n}-a_{m,n+1})}{(1-(1-a_{m,n})(1-a_{m,n+1}))-(a_{m,n}-a_{m,n+1})}\\ &=\frac{1}{2}\log\frac{a_{m,n}-a_{m,n}a_{m,n+1}}{a_{m,n+1}-a_{m,n}a_{m,n+1}}\\ &=\frac{1}{2}\log\frac{[m(m+1)+n(2m+n-1)]^2-4}{[m(m+1)+(n-1)(2m+n-2)]^2-4}\to 0, \text{ as } n\to\infty.\end{aligned}$$

This completes the proof of Lemma 1.

The next lemma is a result of the Contraction Principle as introduced by Heins [13] which is based upon the following Harnack's inequality

$$\left|\frac{|w|-|z|}{1-|wz|}\right|\le\left|\frac{w-z}{1-\bar{w}z}\right|\le\frac{|w|+|z|}{1+|wz|},\qquad w,\ z\in D.$$

Lemma 2. *For any Blaschke product*

$$B(z)=\Pi\frac{\bar{p}_n}{p_n}\cdot\frac{p_n-z}{1-\bar{p}_n z}$$

there corresponds the Blaschke product

$$b(z)=\Pi\frac{|p_n|-z}{1-|p_n|z}$$

such that $|b(r)|=m(r;\ b)\le m(r;B)$ *and* $M(r;B)\le M(r;b)=b(-r)$, *where* $m(r;\ f)=\min|f(re^{i\theta})|$, $M(r;\ f)=\mathrm{Max}|f(re^{i\theta})|$, $0\le\theta<2\pi$.

The last lemma we need comes from that of Bagemihl and Seidel [3, Example 3].

Lemma 3. *The function* $f(z)=\exp\left(\frac{1+z}{1-z}\right)\Pi\frac{x_n-z}{1-x_n z}$, *where* $x_n=1-1/n^2$, *is of bounded characteristic in* D *for which* $\lim_{n\to\infty}|f(y_n)|=\infty$, *where* $y_n=(x_n+x_{n+1})/2$.

Proof of Theorem 10. Since $E=\cup\{z_n\}$ is countable we can clearly choose a countable system of rays R_m, $m=1, 2,\ldots$, such that every ray R_m has one endpoint at a point $z_n\in E$, and further such that every Stolz angle with one vertex at such a point z_n contains infinitely many of the rays R_m. By virtue of Lemma 1, there is a sequence of points $\{p_n\}$, where $|p_n|=1-1/n^2$, such that each ray R_m contains infinitely many of the points $p_{m,n}$ for which $\rho_{m,n}\to 0$, as $n\to\infty$. Since $\sum(1-|p_n|)=\sum 1/n^2<\infty$, the Blaschke product

$$B(z)=\Pi\frac{\bar{p}_n}{p_n}\frac{p_n-z}{1-\bar{p}_n z}$$

defines a nonconstant bounded holomorphic function in D with $B(p_n)=0$ for every n. We define the function $f(z)=F(z)B(z)$, where $F(z)=\exp(-\sum(z+z_n)/(z-z_n)n^2)$. Evidently $f(z)$ is holomorphic and bounded characteristic in D, and therefore by Nevanlinna's Theorem [5, Theorem 2. 18] $f(z)$ possesses radial limits at almost all points of C, so that mea.$p(f)=0$. Thus by Theorem 1, we have mea.$\rho^*_{\sqcup}(f)=0$.

It remains to prove that each point of E is a ρ^*-point. For any point $e^{i\theta_k}\in E$ and any ray R_m with one vertex at $e^{i\theta_k}$, by virtue of [9, Lemma 2], it is sufficient to find two sequence of points $\{P_n\}$, $\{Q_n\}$ such that P_n, Q_n, $\in R_m$, P_n, $Q_n\to e^{i\theta_k}$, $\rho(P_n, Q_n)\to 0$ as $n\to\infty$ and $\chi(f(P_n), f(Q_n))\geq M>0$, where $\chi(a, a')$ is the chordal distance between a and a'.

Now, let us consider the sequence $\{p_n\}$ and the subsequence $\{p_{m,n}\}$ of $\{p_n\}$ for which $p_{m,n}\in R_m$, $m=1, 2,\ldots$ We set $P_n=p_{m,n}$ and $Q_n=(p_{m,n}+p_{m,n+1})/2$, then by Lemma 1, we have $\rho_{m,n}\to 0$, as $n\to\infty$ which in turns implies

$$\rho(P_n, Q_n)\to 0, \text{ as } n\to\infty.$$

Since $f(P_n)=0$, it is sufficient to prove that $\lim_{n\to\infty}|f(Q_n)|>0$.

Let $F_k(z)=\exp((e^{i\theta_k}+z)/(e^{i\theta_k}-z))\, B(z)$, then by the property of the function $F(z)$, it is sufficient to prove that $\lim_{n\to\infty}|F_k(Q_n)|>0$.

To do this, we form the Blaschke product

$$b_k(z)=\Pi\frac{|\bar{p}_n|e^{i\theta_k}-z}{1-|p_n|e^{-i\theta_k}z}.$$

By virtue of Lemma 2, we know that $b_k(z)$ attains its minimum on the radius $[0, e^{i\theta_k}]$ and $|b_k(re^{i\theta_k})|=m(r;\, b_k)\leq m(r;B)$. Let $y_n=|Q_n|e^{i\theta_k}$ and

$$f_k(z)=\exp\,((e^{i\theta_k}+z)/(e^{i\theta_k}-z))b_k(z),$$

then by Lemma 3, we have $\lim_{n\to\infty}|f_k(y_n)|=\infty$. It follows that

$$\lim_{n\to\infty}|F_k(y_n)|\geq\lim_{n\to\infty}|f_k(y_n)|=\infty.$$

Applying the same idea as in [3, Example 3], we can switch from y_n to Q_n such that $\lim_{n\to\infty}|F_k(Q_n)|=\infty$. This completes the proof of Theorem 10.

6. Measurable and topological properties.

For a function f defined in D, let $F(f)$ denote the set of all Fatou points of f. The well-known Plessner Theorem [5, Theorem 8.2] claims that if $f(z)$ is meromorphic in D, then almost all points of C belong to $F(f)\cup P(f)$. We may ask whether the analogous theorem holds for either $J(f)$ or $\rho^*(f)$ instead of $P(f)$. The answer of this question is negative. Actually this can be seen from Theorems 1 and 2.

Let us make a precise theorem as follows.

Theorem 11. *There exists a holomorphic function f in D such that* mea. $F(f) \cup J(f) =$ mea. $F(f) \cup \rho^*(f) = 0$.

This theorem shows that there is no analogous result regard to the measurable property and therefore we may ask whether there is some similar result concerning about the topological property. More precisely, we may ask whether the Meier's analogue of Plessner's Theorem is true. For this purpose, let us recall to the definition of Meier points [5, p. 153]. A point $e^{i\theta}$ is called a Meier point if (1) the cluster set $C(f, e^{i\theta})$ is subtotal, and (2) the chordal cluster set $C_{\rho(\phi)}(f, e^{i\theta}) = C(f, e^{i\theta})$ for all values of ϕ in $[-\pi/2, \pi/2]$. We denote the set of all Meier points by $M(f)$. We shall prove

Theorem 12. *If $f(z)$ is meromorphic in D, then all points of C except a set of first category belong to $M(f) \cup \rho^*(f)$, and therefore to $M(f) \cup J(f)$.*

To prove this theorem, we first prove the following generalization of Collingwood and Lohwater [5, Theorem 8.9].

Theorem 13. *If $f(z)$ is normal in D, then all points of C except a set of first category belong to $M(f)$.*

Proof. Let $e^{i\theta} \in M(f)^c$, the complement of $M(f)$, then there is some chord $\rho(\phi)$ for which $C_{\rho(\phi)}(f, e^{i\theta}) \neq C(f, e^{i\theta})$. Since f is normal, by virtue of [9, Lemma 1], f has no sequence of ρ-points. It follows from [8, Theorem 1]

$$C_{\rho(\phi)}(f, e^{i\theta}) = \bigcap C_{\Delta}(f, e^{i\theta}),$$

where Δ varies over all Stolz angles which contain $\rho(\phi)$.

Evidently, there is some Stolz angle $\Delta(\phi)$ such that

$$C_{\Delta(\phi)}(f, e^{i\theta}) \neq C(f, e^{i\theta}).$$

According to the Maximality Theorem of Collingwood [5, Theorem 4. 10], the set $M(f)$ is of first category. This completes the proof.

Proof of Theorem 12. Suppose on the contrary that the complement $(M(f) \cup \rho^*(f))^c$ is of second category. Then the closure $\overline{(M(f) \cup \rho^*(f))^c}$ contains an arc A of C. For any point $e^{i\theta} \in A \cap (M(f) \cup \rho^*(f))^c$, there is a Stolz angle $\Delta(\theta)$ which contains no sequence of ρ-points. By using the same method as in [5, p. 155], there can be found an annular trapezoid T for which A is a part of the boundary of T and T contains no sequence of ρ-points.

Now, let $z=g(w)$ be a conformal mapping from $D_w(|w|<1)$ onto T and let $F(w)=f(g(w))$. Then by [9, Lemma 5], F is normal in D_w. Applying Theorem 13, we thus conclude that $M(F)^c$ is of first category in $g^{-1}(A)$ and so is $M(f)^c$ in A. This contradiction establishes our assertion.

The proof of the above two theorems depends heavily on Collingwood's Theorem. Naturally we may try to find some sort of analogous result of Collingwood's. By applying the same method as in the proof of Theorem 12, we obtain the generalization of Collingwood [21, p. 65, Theorem 8] concerning the dense set of Plessner points.

Theorem 14. *If $f(z)$ is meromorphic in D and $\rho^*(f)$ (or $J(f)$) is dense on an arc A of C, then $\rho^*(f)$ (or $J(f)$) is residual on A.*

Now, let $W_\rho(f)$ be the set of all points $e^{i\theta}$ of C for which the radial cluster set $C_\rho(f, e^{i\theta})$ is total. Collingwood [21, p. 65 Theorem 9] proved that if $f(z)$ is meromorphic in D, then the set $W_\rho(f)$ and $P(f)$ differ by a set of first category on C. But the same thing is not true for $\rho^*(f)$ or $J(f)$ instead of $P(f)$ due to Theorem 2 and another theorem of Bagemihl and Seidel [2, Theorem 5].

7. Spiral asymptotic and distribution of ρ-points.

In this last section, we are going to prove some relations between the asymptotic behaviour and the location of ρ-points.

Theorem 15. *If $f(z)$ is a nonconstant meromorphic function in D and has an asymptotic value c along a spiral α, then there is a residual set of radii such that each one of them contains a sequence of ρ-points.*

To prove this theorem, we need a uniqueness theorem for normal functions which is characterized by a sequence of Jordan arcs $\{J_n\}$ such that J_n tend to an arc on C uniformly. Such a sequence of Jordan arcs is called a Koebe sequence. The following theorem was first stated and proved by Bagemihl and Seidel [4, Theorem 1], but our method is different from them.

Theorem 16. *Let $f(z)$ be a normal meromorphic function in D. If $f(z)\to c$ along a Koebe sequence of arcs $\{J_n\}$, then $f(z)\equiv c$.*

Proof. Let $J_n \to A=\{e^{i\theta}:\ \theta_1\leq\theta\leq\theta_2\}$ as $n\to\infty$, $\Delta=\{z:0<|z|<1,\ \theta_1<$

$\arg z < \theta_2\}$, and let $T_{1/2}(A)$ be the domain consisting of all points of Δ where the harmonic measure of A with respect to Δ is at least 1/2. Then by a result of Hayman [11, Lemma 6], the function $g(z) = (f(z) - c)/(1+\bar{c}f(z))$ is holomorphic and bounded by 1 in $T_{1/2}(A)$ and $g(z)$ tends to zero along J_n.

Now, let $z = h(w)$ be a conformal mapping from D_w onto the region bounded by A and the subtended chord. Then the function $F(w) = g(h(w))$ is bounded holomorphic in D_w and tends to zero along $h^{-1}(J_n)$ which is clearly a Koebe sequence in D_w. It follows from the uniqueness theorem of Koebe's Lemma [5, p. 42] that $F(w) \equiv 0$, so is $f(z) \equiv c$.

Proof of Theorem 15. Let S be a sector bounded by two radii and a portion of C. We want to prove that S contains a sequence of ρ-points. To do this, we denote $z = h(w)$ a conformal mapping from D_w onto S. If S did not contain a sequence of ρ-points, then by [9, Lemma 5], the function $F(w) = f(h(w))$ would be normal in D_w. Clearly the intersection $\alpha \cap S$ is a Koebe sequence in D and so $h^{-1}(\alpha \cap S)$ in D_w. By virtue of Theorem 16, we would have $f(z) \equiv c$ which is absurd and therefore we conclude that S possesses a sequence of ρ-points. The assertion now follows from [2, Theorem 5].

In view of the above proof, we find that

Corollary. *Under the hypothesis of Theorem 15, the set $\rho^*(f)$ is dense on C.*

Theorem 15 describes that how many of the radii possess a sequence of ρ-points. But it does not tell us how many of rays with one vertex at a point of C possess a sequence of ρ-points. The following two theorems will give some answers of this problem.

Theorem 17. *Under the hypothesis of Theorem 15, there exists a residual set of points $\{z_\beta\}$ on C such that for each z_β, there is a residual set of rays R_ν terminating at z_β for which every R_ν possesses a sequence of ρ-points.*

Proof. According to Theorem 14 and the above Corollary, we can see that $\rho^*(f)$ is a residual set on C. The assertion now follows from [2, Theorem 9].

Now let us recall to the definition of bounded hyperbolic spiral [9] as was mentioned in the proof of Theorem 9. With the help of this special spiral and an application of [9, Theorem 2] and [2, Theorem 9], we can improve Theorem 17 as follows.

Theorem 18. *Suppose that $f(z)$ is a nonconstant meromorphic function and omits one value v in D. If $f(z)$ tends to a limit $w \neq v$ along a bounded hyperbolic spiral α then for each point $e^{i\theta}$ of C, there is a residual set of rays R_ν terminating at $e^{i\theta}$ such that each R_ν possesses a sequence of ρ-points.*

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Added in proof: Open problem.

Let $f_m(z)$ be the function defined in Theorem 7, where $k_n/k_{n-1}\to K>1$. We conjecture that f_m is normal if and only if $m\leq 0$. We posed this problem for $K=\infty$ in the Detroit Meeting and recently, it has been solved by L. R. Sons [22].