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Formal degree and Clebsch-Gordan coefficient

By

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1. Let G be a unimodular locally compact group, and $\omega=\{\mathfrak{H}, U_g\}$ be a unitary representation of G. Here $\mathfrak{H}$ is the space of representation ω and U_g's are its representation operators.

We call ω L^2-representation if and only if ω is irreducible and there exists a non-zero vector v in $\mathfrak{H}$ such that $<U_g v, v>$ is a square integrable function of g in G with respect to the right Haar measure dg on G.

For an L^2-representation ω, the following properties are known (cf. [1]).

1) For any vectors u, w in $\mathfrak{H}$, $<U_g u, w>$ is square integrable.

2) For a fixed non-zero vector v in $\mathfrak{H}$, the map

$$\mathfrak{H}\ni u \longrightarrow <U_g u, v> \in L^2(G)$$

is an intertwining operator from ω to the right regular representation $\mathfrak{R}=\{L^2(G), R_g\}$ of G.

3) For any representation $\tau=\{\mathfrak{K}, V_g\}$ which is disjoint to ω, and any vectors u, v in $\mathfrak{H}$, any vectors x, y in $\mathfrak{K}$ for which $<V_g x, y>$ is square integrable,

$$\int_G <U_g u, v>\overline{<V_g x, y>}dg=0.$$

4) There exists a positive number $d(\omega)$, depending only on ω, such that

$$\int_G <U_g u, v>\overline{<U_g w, z>}dg=d(\omega)^{-1}<u, w><z, v>$$

for any u, v, w, z in $\mathfrak{H}$.

We call the number $d(\omega)$, the *formal degree* of ω.

On the other hand, consider two unitary representations $\omega=\{\mathfrak{H}, U_g\}$, $\tau=\{\mathfrak{K}, V_g\}$ and an irreducible one $\sigma=\{\mathfrak{L}, W_g\}$. Take normalized vectors u, v, w in their representation spaces $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{L}$ respectively.

Assume that $\omega\otimes\tau$ contains σ as a discrete component. Denote $\tilde{\mathfrak{L}}$ the maximal subspace of $\mathfrak{H}\otimes\mathfrak{K}$ on which the restriction of $\omega\otimes\tau$ operates as a multiple of σ. It is evident that $\tilde{\mathfrak{L}}$ is uniquely determined invariant subspace, and the space of vectors

$\mathfrak{H}(w)=\{Aw;\ A$ is any intertwining operator from σ to $\omega\otimes\tau\}$

is a closed subspace of $\tilde{\mathfrak{L}}$. Put $P_{\mathfrak{L}}$ and P_w the projection of $\mathfrak{H}\otimes\mathfrak{K}$ to $\tilde{\mathfrak{L}}$ and $\mathfrak{H}(w)$ respectively.

We call the following non-negative real number the *Clebsch-Gordan coefficient* of $u\otimes v$ with respect to w:

$$\alpha(u, v; w)=\|P_w(u\otimes v)\|.$$

The purpose of this paper is to show a close relation between formal degrees and Clebsch-Gordan coefficients, and using this relation, to calculate the formal degree of the discrete series of $SL(2, R)$.

Hereafter we denote the conjugate representation of ω in the sense of G. W. Mackey by ω^*, and the image of v in $\mathfrak{H}$ by the natural conjugation map into the representation space $\mathfrak{H}^*$ of ω^* by v^*. For instance, $<U_g{}^*v^*, u^*>=\overline{<U_gv, u>}$. Obviously, if ω is an L^2-representation, ω^* is too and $d(w)=d(w^*)$ (cf. [2]).

2. Lemma 1. *For any normalized vectors u in $\mathfrak{H}$, v in $\mathfrak{K}$, w in $\mathfrak{L}$,*

$$<(U_g\otimes V_g)P_w(u\otimes v), P_w(u\otimes v)>=\alpha(u, v; w)^2<W_gw, w>.$$

Proof. From the definition of $\mathfrak{H}(w)$, there exists an intertwining isometric operator A from $\mathfrak{L}$ into $\mathfrak{H}\otimes\mathfrak{K}$ such that

$$P_w(u\otimes v)=\alpha(u, v; w)Aw.$$

This leads us to the above equality directly.

Lemma 2. *The component of $\omega\otimes\tau$ restricted to the space $(\mathfrak{H}\otimes\mathfrak{K})\ominus\tilde{\mathfrak{L}}$ is disjoint to σ.*

Proof. Obvious from the assumption of maximality on $\tilde{\mathfrak{L}}$.

3. Main Theorem. *Let $\omega=\{\mathfrak{H}, U_g\}$ be a unitary representation of G, and $\tau=\{\mathfrak{K}, V_g\}$, $\sigma=\{\mathfrak{L}, W_g\}$ be two L^2-representations of G.*

Assume $\omega\otimes\tau$ contains σ^* as a discrete component. Then for any normalized vectors u_j in $\mathfrak{H}$, v_j in $\mathfrak{K}$, w_j in $\mathfrak{L}(j=1, 2)$,

1) *$\omega\otimes\sigma$ contains τ^* as a discrete component,*

2) $\alpha(u_1, v_1; w_1^*)=\alpha(u_1^*, v_1^*; w_1)$,

3) $d(\tau^*)\alpha(u_1, v_1; w_1^*)^2=d(\sigma^*)\alpha(u_1, w_1; v_1^*)^2$,

4) $\alpha(u_1, v_1; w_1^*)\alpha(u_2, w_2; v_2^*)=\alpha(u_1, w_1; v_1^*)\alpha(u_2, v_2; w_2^*)$.

Proof. At first the equality 2) is a direct conclusion of the definitions of α and the conjugation map.

For normalized vectors u in $\mathfrak{H}$, v in $\mathfrak{K}$, w in $\mathfrak{L}$, put

$$I_0=\int_G \langle U_g u, u\rangle\langle V_g v, v\rangle\langle W_g w, w\rangle dg$$
$$=\int_G \langle (U_g\otimes V_g)(u\otimes v), u\otimes v\rangle\overline{\langle W_g^* w^*, w^*\rangle} dg.$$

Now we put $z_1=P_{w*}(u\otimes v)$, $z_2=(P_{\mathfrak{L}*}-P_{w^*})(u\otimes v)$ and $z_3=(I-P_{\mathfrak{L}*})(u\otimes v)$, then z_j's are mutually orthogonal and $u\otimes v=z_1+z_2+z_3$. By lemma 1,

$$\langle (U_g\otimes V_g)z_1, z_1\rangle=\alpha(u, v; w^*)^2\langle W_g^* w^*, w^*\rangle.$$

The vectors z_1 and z_2 are of the σ^*-component in $\omega\otimes\tau$, therefore the functions $\langle (U_g\otimes V_g)z_j, z_k\rangle$ $(j, k=1, 2)$ are square integrable. And since $\tilde{\mathfrak{L}}^*$ is an invariant subspace,

$$\langle (U_g\otimes V_g)z_3, z_j\rangle=\langle (U_g\otimes V_g)z_j, z_3\rangle=0 \quad (j=1, 2).$$

Combining these results, we obtain the following,

$$\langle U_g u, u\rangle\langle V_g v, v\rangle=\langle (U_g\otimes V_g)(u\otimes v), u\otimes v\rangle=$$
$$=\sum_{j,k=1,2}\langle (U_g\otimes V_g)z_j, z_k\rangle+\langle (U_g\otimes V_g)z_3, z_3\rangle.$$

Because the left hand side and the first sum part of the right hand side are square integrable, the last term is too.

Thus by the orthogonality relations 3) and 4) in **1.**,

$$\int_G \langle (U_g\otimes V_g)z_j,z_k\rangle\overline{\langle W_g^* w^*, w^*\rangle} dg=0 \quad (\text{if } j \text{ or } k=2).$$
$$\int_G \langle (U_g\otimes V_g)z_3, z_3\rangle\overline{\langle W_g^* w^*, w^*\rangle} dg=0.$$

Consequently we get

$$I_0=\int_G \langle (U_g\otimes V_g)z_1,z_1\rangle\overline{\langle W_g^* w^*,w^*\rangle} dg=\alpha(u, v; w^*)^2 d(\sigma^*)^{-1}$$

Changing the roll of (σ, w) to (τ, v), we get analogously $I_0=\alpha(u, w; v^*)^2 d(\tau^*)^{-1}$.

The rest of Main theorem are deduced from this equality immediately.

4. Now we shall consider the case of $G=SL(2, R)$. Its L^2-representations D_n^+, D_n^- $(n=1, 3/2, 2, 5/2, \dots)$ are so-called of the discrete series. In the space of the representation D_n^+ (resp. D_n^-), there exists a complete orthonormal system $\{u_n^j;\ j=n,\ n+1, \dots$ (resp. $u_{-n}^j;\ j=-n, -n-1, \dots)\}$ consisting of K-finite vectors.

For such vectors, the step-up (-down) operators are given by

$$F^+(D_n^\pm)u_{\pm n}^j=\sqrt{(j+n)(j-n+1)}\ u_{\pm n}^{j+1},$$
$$F^-(D_n^\pm)u_{\pm n}^j=-\sqrt{(j-n)(j+n-1)}\ u_{\pm n}^{j-1}.$$

The vector u_n^n (resp. u_{-n}^{-n}) in the space of D_n^+ (resp. D_n^-) is characterized as the normalized vector of weight n (resp. $-n$) such that $F^-(D_n^+)u_n^n=0$ (resp. $F^+(D_n^-)u_{-n}^{-n}=0$) up to constant factor (cf. [3]).

In the space of $D_{1/2}^+\otimes D_p^+$, the vector $u_{1/2}^{1/2}\otimes u_p^p$ is the only normalized vector of weight $p+(1/2)$ and $F^-(D_{1/2}^+\otimes D_p^+)(u_{1/2}^{1/2}\otimes u_p^p)=0$. This means that $D_{1/2}^+\otimes D_p^+$ contains $D_{p+(1/2)}^+$ with multiplicity one and $u_{1/2}^{1/2}\otimes u_p^p$ just corresponds to the vector $u_{p+(1/2)}^{p+(1/2)}$. That is,

$$\alpha(u_{1/2}^{1/2},\ u_p^p;\ u_{p+(1/2)}^{p+(1/2)})=\alpha(u_{1/2}^{1/2},\ u_p^p;\ (u_{-p-(1/2)}^{-p-(1/2)})^*)=1.$$

While in the space of $D_{1/2}^+\otimes D_{-p-(1/2)}^-=D_{1/2}^+\otimes(D_{p+(1/2)}^+)^*$, vectors v of weight $-p$ are given by

$$v=\sum_{l\geq 0} a_l(u_{1/2}^{l+(1/2)}\otimes u_{-p-(1/2)}^{-p-l-(1/2)}).$$

Thus
$$F^+v=\sum_{l\geq 0} a_l((l+1)u_{1/2}^{l+(3/2)}\otimes u_{-p-(1/2)}^{-p-l-(1/2)}+$$
$$+\sqrt{l(l+2p)}\ u^{l+1/2}\otimes u_{-p-(1/2)}^{-p-l+(1/2)})$$
$$=\sum_{l\geq 0}(a_l(l+1)+a_{l+1}\sqrt{(l+1)(l+2p+1)})(u_{1/2}^{l+(3/2)}\otimes u_{-p-(1/2)}^{-p-l-(1/2)}).$$

The equality $F^+v=0$ gives $a_{l+1}=-a_l\sqrt{(l+1)/(l+2p+1)}$. Therefore $a_l=(-1)^l\sqrt{l!(2p)!/(l+2p)!}\ a_0(l\geq 0)$. By the additional condition $\|v\|=1$, we get

$$a_0=\alpha(u_{1/2}^{1/2},\ u_{-p-(1/2)}^{-p-(1/2)};\ u_{-p}^{-p})=((2p-1)/2p)^{1/2}.$$

Substitute this to the formula 3) of the Main theorem, then we obtain

$$d(D_p^-)/d(D_{p+(1/2)}^+)=d(D_p^+)/d(D_{p+(1/2)}^-)=$$
$$=\alpha(u_{1/2}^{1/2}, u_{-p-(1/2)}^{-p-(1/2)};\ u_{-p}^{-p})^2/\alpha(u_{1/2}^{1/2},\ u_p^p;\ u_{p+(1/2)}^{p+(1/2)})^2=(2p-1)/2p$$
$$(p=1,\ 3/2,\ 2,\ 5/2,\ \dots).$$

Summarizing these results, if we put $b_0=d(D_1^+)$,

$$d(D_p^+)=d(D_p^-)=(2p-1)b_0 \quad (p=1,\ 3/2,\ 2,5/2,\ \ldots).$$

The determination of b_0 depends of the normalization of the Haar measure on G. For instance, for the normalization as

$$\int_G |<U_g u_1^1,\ u_1^1>|^2\, dg (=d(D_1^+))=1, \text{ we get}$$

$$d(D_p^+)=d(D_p^-)=(2p-1).$$

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