J. Math. Kyoto Univ. (JMKYAZ)
18-1 (1978) 121-130

Cardinals, isols, and the growth of functions

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(Communicated by Prof. H. Yoshizawa, March 30, 1977)

1. Introduction

Let ω be the non-negative integers and let Λ_Z be the cosimple isols. To each $X\in\Lambda_Z$ we can associate a unique degree of unsolvability Δ_X (c. f. [3]). This is the degree of any co-r. e. $\xi\in X$. Throughout this paper $\boldsymbol{d}$ is a non-recursive r. e. degree and Λ_d is the set $\{X\in\Lambda_Z\,|\,\Delta_X\leq\boldsymbol{d}\}$. In this paper we are concerned with the first order theory of $(\Lambda_d, +)$ where $+$ is isolic addition. We study this structure by means of a first order language L containing individual variables $u_0, u_1, \ldots, v_0, v_1, \ldots$, a binary functor $+$ denoting addition, and a binary predicate $=$ denoting equality. L is interpreted in ω or Λ_d in the usual way. Because ω and Λ_d are commutative semigroups we take the liberty of putting of L in the normal form $\sum_{i<n} a_iu_i$ where $\sum$ denotes summation, $a_i\in\omega$, and a_iu_i is the term consisting of u_i summed with itself a_i times. An *AE special Horn sentence* is a sentence of L having the form

$$(1)\qquad (\forall u_0, \ldots, u_{m-1})(\alpha\rightarrow(\exists v_0, \ldots, v_{n-1})\beta)$$

where $\alpha(u_0, \ldots, u_{m-1})$ has the form

$$(2)\qquad \bigwedge_{j<q}\left(\sum_{k<m}a_{jk}u_k=\sum_{k<m}a'_{jk}u_k\right)$$

and $\beta(u_0, \ldots, u_{m-1}, v_0, \ldots, v_{n-1})$ has the form

$$(3)\qquad \bigwedge_{j<r}\left(\sum_{k<m}b_{jk}u_k+\sum_{k<n}c_{jk}v_k = \sum_{k<m}b'_{jk}u_k+\sum_{k<n}c'_{jk}v_k\right).$$

In [6] it is shown that

Proposition A: *If φ is an AE special Horn sentence and $\omega \models \varphi$ then $\Lambda_d \models \varphi$.*

In order to get a converse to this result we define an *AE special sentence* to be a sentence of L having the form

$$(4) \qquad (\forall u_0, \ldots, u_{m-1})(\alpha \rightarrow (\exists v_0, \ldots, v_{n-1}) \bigvee_{i<p} \beta_i)$$

where α has the form (2) and each β_i has form (3). A *Horn reduct* of this sentence is any one of the *AE* special Horn sentences

$$(5) \qquad (\forall u_0, \ldots, u_{m-1})(\alpha \rightarrow (\exists v_0, \ldots, v_{n-1}) \beta_i).$$

From Proposition *A* it readily follows that if φ is an *AE* special sentence having a Horn reduct Ψ such that $\omega \models \Psi$ then $\Lambda_d \models \varphi$. A degree $\boldsymbol{d}$ is called *high* if $\boldsymbol{d}' = \boldsymbol{0}''$ where prime denotes jump. In [6] it is shown that if $\boldsymbol{d}$ is a high degree and φ is an *AE* special sentence such that $\Lambda \models \varphi$ then φ has a Horn reduct Ψ such that $\omega \models \Psi$. The proof of this result uses infinite indecomposable isols, and it is known from [6] that Λ contains such isols if and only if $\boldsymbol{d}$ is high. This accounts for the high requirement on $\boldsymbol{d}$. We relax this with

Theorem 1. *If φ is an AE special sentence then $\Lambda_d \models \varphi$ if and only if φ has a Horn reduct Ψ such that $\omega \models \Psi$.*

Our proof of Theorem 1 involves a lemma which is interesting in its own right. It asserts that a regressive isol is 3-meager if and only if it is multiple-free (c. f. Section 3 for definitions).

Let $(\Lambda_d^*, +)$ be the difference group formed from Λ. As an application of Theorem 1 we show

Theorem 2. *The first order theory of $(\Lambda_d^*, +)$ is independent of $\boldsymbol{d}$ (and in fact is the same as that of Λ^*).*

The significance of this result follows from [6] where we show that there is an *EAE* sentence which holds in Λ_d if and only if $\boldsymbol{d}$ is a high degree. Thus we can describe properties of $\boldsymbol{d}$ by means of Λ_d, but we cannot do so by means of Λ_d^*.

The methods developed so far apply equally well to the Dedekind cardinals. Let *ZF* be Zermelo-Fraenkel set theory (without the axiom of choice) and let Δ be the Dedekind cardinals, i. e., those cardinals x such that $x \neq x+1$.* In [2] it is mentioned in passing that

* From now on we tacitly assume that *ZF* is consistent (particularly as a hypothesis in Theorem 3).

Proposition B. *If φ is an AE special Horn sentence and $\omega \models \varphi$ then $ZF \vdash [\Delta \models \varphi]$.*

It follows as a corollary that if φ is an AE special sentence having a Horn reduct Ψ such that $\omega \models \Psi$ then $ZF \vdash [\Delta \models \varphi]$. Using infinite indecomposable cardinals (necessarily Dedekind) we got the following converse in [5]. If φ is an AE special sentence and $ZF \vdash [\Delta \models \varphi]$ then φ has a Horn reduct Ψ such that $\omega \models \Psi$. We obtain interesting modifications of this result by getting rid of the infinite indecomposables. This can be done in two ways. First, we can work in ZFO ($=ZF+$an axiom which asserts that every set can be linearly ordered). We can then prove in ZFO that every indecomposable cardinal is finite. Second, we can still work in ZF, but restrict ourselves to Δ_R (=cardinals of Dedeind sets of reals). Since the reals are linearly ordered we can prove in ZF that every indecomposable in Δ_R is finite. By using a growth rate analysis of combinatorial functions we obtained the following result in [5]. If φ is an AE special sentence then $ZFO \vdash [\Delta \models \varphi]$ if and only if φ has a Horn reduct Ψ such that $\omega \models \Psi$. The methods used to obtain this result fail for Δ_R; however we can modify the proof of Theorem 1 so as to get

Theorem 3. *If φ is an AE special sentence then $ZF \vdash [\Delta_R \models \varphi]$ if and only if φ has Horn reduct Ψ such that $\omega \vdash \Psi$.*

At no added expense we also obtain the main result of [5] (but with a different proof).

Theorem 3 (second part). *If φ is an AE special sentence then $ZFO \vdash [\Delta \models \Psi]$ if and only if φ has a Horn reduct Ψ such that $\varphi \models \Psi$.*

The key notion used in the proofs of Theorems 1 and 3 is that of an infinite multiple-free object. Although this notion is much less restrictive than that of an infinite indecomposable object, it is still sufficiently pathological to provide interesting counter-examples.

2. Main reduction

We start with a result due to Bradford.

Lemma 1 ([1]). *If $x, y \in \omega$ then $\sim(x=1 \wedge y=0)$ if and only if $(\exists z \in \omega)(2z \leq x+y \wedge x \leq 3z)$.*

Proof. Assume $\sim(x=1 \wedge y=0)$. If x is even take $z=x/2$. Clearly $2z=x \leq x+y$ and $x \leq 3x/2$. If $x \neq 1$ and x is odd take $z=(x-1)/2$. Then $2z=x-1 \leq x+y$. If $3z < x$ then $(3x/2)-(1/2) < x$ so $x < 1$ and x is even.

If $x=1$ and $y\neq 0$ take $z=1$. Then $2z=2\leq x+y$ and $x=1\leq 3=3z$. This proves the left to right implication. Conversely, assume $x=1 \wedge y=0$. Then $2z\leq x+y$ implies $z=0$ so that $x\nleq 3z$.

Corollary. *If $p>1$ and $x_j\in\omega$ for $j<p$ then* $\sim(x_i=1\wedge\bigwedge_{i\neq j<p}x_j=0)$ *if and only if* $(\exists z\in\omega)\ (2z\leq\sum_{j<p}x_j\wedge x_i\leq 3z)$.

Let φ be an *AE* special sentence of the form (4) (having p disjuncts) and let θ_p be

$$(6)\qquad (\forall u_0, \ldots, u_{p-1})\bigvee_{i<p}(\exists v)\,(2v\leq\textstyle\sum_{j<p}u_j\wedge u_i\leq 3v).$$

Lemma 2. *If φ is an AE special sentence such that $p>1$ and $\Lambda_d\models\varphi$, but no Horn reduct of φ is true in ω, then $\Lambda_d\models\theta_p$.*

Proof. Since no Horn reduct of φ is true in ω, for each $i<p$ there are $x_{0i}, \ldots, x_{(m-1)i}\in\omega$ such that

$$(7)\qquad \omega\models\alpha(x_{0i}, \ldots, x_{(m-1)i}),$$

$$(8)\qquad \omega\models(\forall v_0, \ldots, v_{n-1})\sim\beta_i(x_{0i}, \ldots, x_{(m-1)i}, v_0, \ldots, v_{n-1}).$$

Let $\alpha'(u_0, \ldots, u_{p-1})$ be

$$\alpha(\textstyle\sum_{j<p}x_{0j}u_j, \ldots, \sum_{j<p}x_{(m-1)j}u_j)$$

and let $\beta_i'(u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})$ be

$$\beta_i(\textstyle\sum_{j<p}x_{0j}u_j, \ldots, \sum_{j<p}x_{(m-1)j}u_j, v_0, \ldots, v_{n-1}).$$

Since $\alpha(u_0, \ldots, u_{m-1})$ is a conjunction of linear homogeneous equations, and any linear combination of solutions of such a system is itself a solution, (7) implies that

$$(9)\qquad \omega\models(\forall u_0, \ldots, u_{p-1})\alpha'.$$

It also follows from (8) that

$$(10)\qquad \omega\models(\forall u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})\,(\beta_i'\rightarrow \sim(u_i=1\wedge\textstyle\bigwedge_{i\neq j<p}u_j=0))$$

for each $i<p$.

Applying Lemma 1 we get

$$(11)\qquad \omega\models(\forall u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})\,(\beta_i'\rightarrow (\exists v)\,(2v\leq\textstyle\sum_{j<p}u_j\wedge u_i\leq 3v))$$

for each $i<p$. Proposition A applies directly to (9) and by eliminating $u_0\leq u_1$ with $(\exists v)\,(u_0+v=u_1)$, we may also apply it to (11). Thus

(12) $\Lambda_d \models (\forall u_0, \ldots, u_{p-1})\alpha'$,

(13) $\Lambda_d \models (\forall u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})(\beta'_i \rightarrow (\exists v)(2v \leq \sum_{j<p} u_j \wedge u_i \leq 3v))$

for each $i<p$. Now by hypothesis $\Lambda_d \models \varphi$ so be recalling the definitions of α', β'_i we obtain

(14) $\Lambda_d \models (\forall u_0, \ldots, u_{p-1})(\alpha' \rightarrow (\exists v_0, \ldots, v_{n-1})\bigvee_{i<p}\beta'_i)$.

Combining (12), (13), and (14) gives $\Lambda_d \models \theta_p$. q. e. d.

3. Applications to isols

Denote the domain, range of a function f by δf, ρf respectively and let $\mathrm{Req}(\xi)$ be the recursive equivalence type of ξ. Let t be a retraceable function with $\rho t = \tau$ and $T = \mathrm{Req}(\tau)$. Then t has a *special retracing function* p, i. e., a partial recursive function p such that

(i) $\rho t \subseteq \delta p$,

(ii) $p(t_0) = t_0$ and $(\forall n) p(t_{n+1}) = t_n$,

(iii) $\rho p \subseteq \delta p$, and

(iv) $(\forall x \in \delta p) p(x) \leq x$.

Let $p^0(x) = x$ and $(\forall n) p^{n+1}(x) = p(p^n(x))$. Then $p^*(x) = (\text{least } n)\ [p^{n+1}(x) = p^n(x)]$ and $\bar{p}(x) = \{p^n(x) \mid n \leq p^*(x)\}$. It follows from (i)—(iv) that $\delta p^* = \delta \bar{p} = \delta p$ and p^*, $\bar{p}$ are partial recursive, the latter in an extended sense. We say that t is *3-meager* if for every partial recursive function h, if $\rho t \subseteq \delta h$ and $h(\rho t) \subseteq \rho t$ then $h(t_n) \leq t_n$ for all but finitely many n. If this occurs then τ and T are also called *3-meager* and the latter is necessarily in Λ. A fundamental result of [6] is that each Λ_d contains at least one 3-meager T. Considerably stronger results along these lines have been obtained by McLaughlin [9]. If α, $\beta \subseteq \omega$ then α is *separated* from β if there exist disjoint r. e. sets α_0, β_0 such that $\alpha \subseteq \alpha_0$ and $\beta \subseteq \beta_0$. An isol Y is *multiple-free* if $2X \leq Y$ implies that $X \in \omega$, i. e., X is finite. These notions are connected by

Lemma 3. *A regressive isol is 3-meager if and only if it is multiple-free.*

Proof. ($\rightarrow$) Let t be a 3-meager function with special retracing function p, $\rho t = \tau$ and $T = Req(\tau)$. If $2X \leq T$ then we can find pairwise disjoint r. e. sets η_i, $i<3$, such that if $\xi_i = \eta_i \cap \tau$ then $\tau = \xi_0 \cup \xi_1 \cup \xi_2$ and $\mathrm{Req}(\xi_0) = X = \mathrm{Req}(\xi_1)$. Thus there is a one-one partial recursive function f such that $\xi_0 \subseteq \delta f$ and $f(\xi_0) = \xi_1$. By standard methods we may also assume that $\delta f = \eta_0$ and $\rho f = \eta_1$. We are going to describe a certain function g with $\delta g=\eta_0\cup\eta_1\cup\eta_2$. If $x\in\eta_0$ then $g(x)=f(x)$, if $x\in\eta_1$ then $g(x)=f^{-1}(x)$, and if $x\in\eta_2$ then $g(x)=x$. Clearly g is a one-one partial recursive function, $\tau\subseteq\delta g$, $g(\tau)=\tau$, and g is equal to its own inverse. By 3-meagerness there is an integer n_0 such that $g(t_n)\leq t_n$ for all $n>n_0$. By one-oneness there is an integer $n_1>n_0$ such that $p^*(g(t_n))>n_0$ for all $n>n_1$. Consequently $g(g(t_n))\leq g(t_n)$ for all $n>n_1$. But $g(g(t_n))=t_n$ since g is selfinverse and thus $g(t_n)=t_n$ for all $n>n_1$. This can only happen if ξ_0 is finite.

($\leftarrow$) Now suppose that t is retraceable but not 3-meager. Then there is a partial recursive function h such that $\tau\subseteq\delta h$, $h(\tau)\subseteq\tau$, and $h(t_n)>t_n$ for infinitely many n. By using the special retracing function p (of t) we may also assume that for all n either $h(t_n)=t_n$ or $h(t_n)=t_{n+1}$. Let $\zeta=\{x\in\delta h\mid h(x)>x \text{ and } p(h(x))=x\}$. Then $\zeta\cap\tau$ is separated from $\tau-\zeta$. It follows from [4] that $\zeta\cap\tau$ is retraceable, enumerated say, by the retraceable function s. Let $\sigma_0=\{s_{2n}\mid n<\omega\}$ and let $\sigma_1=\{x\in\tau\mid p(x)\in\sigma_0\}$. To complete our proof we show that

(a) $\sigma_0\cap\sigma_1=\phi$,

(b) there exist pairwise disjoint r. e. sets θ_i, $i<3$, such that $\sigma_0\subseteq\theta_0$, $\sigma_1\subseteq\theta_1$, and $\tau-(\sigma_0\cup\sigma_1)\subseteq\theta_2$,

(c) σ_0 is recursively equivalent to σ_1.

Then $S=Req(\sigma_0)$ is infinite and $2S\leq T$ which implies that T is not multiple-free.

Re. (a). If $x\in\sigma_1$ then $p(x)\in\sigma_0\subseteq\zeta$ and hence there are integers m, n such that $s_{2m}=t_n=p(x)$. Then $s_{2m}\in\zeta$ and so $s_{2m}<h(s_{2m})=x=h(t_n)=t_{n+1}\leq s_{2m+1}<s_{2(m+1)}$. This implies that $x\notin\sigma_0$.

Re. (b). Suppose that $y\in\tau$. We describe an effective process by which we can decide whether $y\in\sigma_0$ or $y\in\sigma_1$ or neither. First, it is clear that we can decide whether $y\in\sigma_0$. If $y\notin\sigma_0$ see if there is an $x\in\bar{p}(y)\cap\sigma_0$ such that $y=h(x)$. If there is such an x then y goes into σ_1, otherwise into $\tau-(\sigma_0\cup\sigma_1)$.

Re. (c). $\sigma_0\subseteq\zeta$ and the restriction of h to ζ is a one-one partial recursive function mapping σ_0 onto σ_1. q. e. d.

A regressive isol X is called *strongly universal* if for every $R\subseteq\omega\times\omega$, the graph of a function r, $(\exists Y\in\Lambda)R_\Lambda(X, Y)$ implies that r is almost recursive increasing.

Corollary (*Barback*). *Every multiple-free regressive isol is strongly universal.*

Proof. Use Lemma 3 and the fact (c. f. [6]) that every 3-meager isol is *strongly universal* (with thanks to Barback for having provided us with his own unpublished proof of this corollary). q. e. d.

An isol Y is *highly decomposable* if Y is infinite and for every infinite $X \leq Y$ there are U, V, both infinite, such that $X = U + V$. In [4] it is shown that every infinite regressive isol is highly decomposable.

Proof of Theorem 1. Let φ be AE special sentence of the form (4) having Horn reducts Ψ of the form (5). It follows from Proposition A that if $\omega \models \Psi$ for some Ψ then $\Lambda_d \models \varphi$. For the converse assume that $\Lambda_d \models \varphi$ but no Ψ is true in ω. If $p=1$ then this amounts to $\omega \models \sim\varphi$. Thus there are integers $x_0, \ldots, x_{m-1} \in \omega$ such that both

$$(\forall v_0, \ldots, v_{n-1}) \sim\beta(x_0, \ldots, x_{m-1}, v_0, \ldots, v_{n-1})$$

and

$$\alpha(x_0, \ldots, x_{m-1})$$

are true in ω. By Proposition A they are both true in Λ_d and hence $(x_0, \ldots, x_{m-1})$ is a counterexample to φ in Λ_d i. e., $\Lambda_d \models \sim\varphi$, which is a contradiction. Since the same argument will work for Dedekind cardinals (via Proposition B and later Proposition C) we shall assume throughout the rest of this paper that $p>1$. By Lemma 2 we have $\Lambda_d \models \theta_p$ where θ_p is given by (6). Let T be a 3-meager isol in Λ_d. Then T is highly decomposable and by Lemma 3 it is multiple-free. Let $X_0, \ldots, X_{p-1}$ be infinite isols which sum to T. Each $X_i \in \Lambda_d$ and for any $V \in \Lambda_d$, if $2V \leq \sum_{i<p} X_i = T$ then V is finite and we cannot have $X_i \leq 3V$. Thus each disjunct of θ_p fails and $\Lambda_d \models \sim\theta_p$. Contradiction. q. e. d.

We know that Λ_d is a commutative semigroup (with respect to addition and 0 as the identity). Moreover it also satisfies the universal closures of

(15) $X+Z=Y+Z \rightarrow X=Y$,

(16) $nX=nY \rightarrow X=Y$,

for $0<n<\omega$, where nX has its usual inductive definition. From (15) and (16) we can easily show that Λ_d can be extended to a torsion free Abelian group (*TFAG*) which we call Λ_d^* (=the isolic integers of degree $\leq \boldsymbol{d}$, a typical member having the form $X-Y$ where $X, Y \in \Lambda_d$). One possible way to show that the theory of a *TFAG* is decidable is to give a complete set of axioms for it which is recursive. Such a method was devised in [11]. Let $(G, +)$ be a *TFAG*, m, $n \in \omega$, and $x_0, \ldots, x_{m-1} \in G$. $x_0, \ldots, x_{m-1}$ are said to be *strongly linearly independent* (mod n) if for each sequence $a_0, \ldots, a_{m-1} \in \omega$ and $y \in G$, $\sum_{i>m} a_i x_i = ny$ implies that each a_i is congruent to 0 (mod n) in the ordinary arithmetical sense. Without loss of generality we may assume that $0\leq a_i<n$ for each $i<m$. Let S be the set of all sequences of integers $a=(a_0, \ldots, a_{m-1})$ such that $0\leq a_i<n$ for $i<m$, but not all $a_i=0$. Thus $x_0, \ldots, x_{m-1}$ are strongly linearly independent (mod n) if for each $a\in S$ they satisfy

$$(17) \qquad (\forall y)\textstyle\sum_{i<m} a_i x_i \neq ny$$

in $(G, +)$. Clearly this can be expressed in our language L. Let Ψ_{mn} be a sentence of L which asserts that there exist m elements strongly linearly independent mod n.

Important fact (c. f. [11]). An extension of the theory of *TFAGs* is complete if and only if it is consistent and contains for any two numbers $m>0$, $n>1$ either Ψ_{mn} or its negation.

In [10] it is shown that $TFAG\cup\{\Psi_{mn}\,|\,m>0, n>1\}$ is a complete set of axioms for the isolic integers $(\Lambda^*, +)$. We do the same for $(\Lambda_d^*, +)$; however our proof must necessarily be different because [10] uses infinite indecomposables (which are not available in our context). Theorem 1 does the trick. We start with

Lemma 4. *If* $X\in\Lambda_d^*$, $0<n<\omega$, *and* $nX\in\Lambda_d$ *then* $X\in\Lambda_d$.

Proof. (16) is a special case of the more general

$$(18) \qquad nX\leq ny\rightarrow X\leq Y$$

for $0<n<\omega$ which Λ_d also satisfies. Thus suppose that $X=Z-Y$ where $Y, Z\in\Lambda_d$ and $nX=nZ-nY\in\Lambda_d$. Then $nY\leq nZ$ and hence by (18) $Y\leq Z$, i. e., $X=Z-Y\in\Lambda_d$. q. e. d.

Proof of Theorem 2. Suppose that $\Lambda_d\models\sim\Psi_{mn}$. This can be expressed by the fact that

$$(19) \qquad (\forall u_0, \ldots, u_{m-1})\textstyle\bigvee_{a\in S}(\exists v)(\sum_{i<m} a_i u_i=nv)$$

holds in Λ_d^*. By restricting the u_i to Λ_d and using Lemma 4 we see that (19) also holds in Λ_d. But clearly no Horn reduct of (19) holds in ω. Thus $\Lambda_d^*\models\Psi_{mn}$ and $TFAG\cup\{\Psi_{mn}\,|\,m>0, n>1\}$ is a complete axiomatization of $(\Lambda_d^*, +)$ by the important fact. Since our argument is independent of $\boldsymbol{d}$, Theorem 2 follows. q. e. d.

We conclude this section with a final word about Lemma 3. 3-meagerness is clearly an assertion about the growth rate of retraceable functions. On the other hand, multiple-freeness appears to be an algebraic property. It is surprising that they are equivalent.

4. Applications to cardinals

Throughout this section we assume Proposition B as well as

Proposition C. *If* φ *is an AE special Horn sentence and* $\omega \models \varphi$ *then* $ZF \vdash [\Delta_R \models \varphi]$.

We shall not prove this here but only mention that it can be proved using Bradford's method [1]; however because all the cardinals involved are linearly ordered, an even easier proof (along the lines of [7]) is available.

Now suppose that φ is an AE special sentence of the form (4) having Horn reducts Ψ of the form (5). It follows from Propositions B and C that if $\omega \models \Psi$ for some Ψ then $ZFO \vdash [\Delta \models \varphi]$ and $ZF \vdash [\Delta_R \models \varphi]$. Thus our remaining task is to obtain converses. Assume that p of (4) is >1.

Lemma 5. *If* φ *is an AE special sentence then*
(i) *If* $ZFO \vdash [\Delta \models \varphi]$ *but no Horn reduct of* φ *is true in* ω *then* $ZFO \vdash [\Delta \models \theta_p]$.
(ii) *If* $ZF \vdash [\Delta_R \models \varphi]$ *but no Horn reduct of* φ *is true in* ω *then* $ZF \vdash [\Delta_R \models \theta_p]$.

Proof. If we follow the proof of Lemma 2 we see that our hypotheses guarantee (9) and (11). If we apply Propositions B and C instead of A we obtain analogues of (12), (13), and (14), being different only in the fact that $\Lambda_d \models \gamma$ is replaced by either $ZFO \vdash [\Delta \models \gamma]$ or $ZF \vdash [\Delta_R \models \gamma]$. q. e. d.

Proof of Theorem 3. Just as in the proof of Theorem 1 it will suffice to show that

(20) $\Delta \models \theta_p$ is not a theorem of ZFO,

(21) $\Delta_R \models \theta_p$ is not a theorem of ZF.

Now we can prove in ZF that if $x, y \in \Delta_R$ then $x+y \in \Delta_R$ and if $x \leq y \in \Delta_R$ then $x \in \Delta_R$. Consequently $ZF \vdash [(\Delta \models \theta_p) \models (\Delta_R \models \theta_p)]$ so that both (20) and (21) will follow from

(22) $\Delta_R \models \theta_p$ is not a theorem of ZFO.

We can prove in ZF that every infinite $x \in \Delta_R$ is highly decomposable. Then just as in the proof of Theorem 1, (22) will follows if we can find a model M of ZFO such that

(23) $M \models (\exists x \in \Delta_R)$ (x is infinite and multiple-free).

To our language of set theory add an individual constant $\boldsymbol{K}$ and a functor $\boldsymbol{\sigma}$. Then it is stated in [8] that if ZF is consistent then ZFO has a model M which satisfies

(i) every set can linearly ordered,

(ii) $\boldsymbol{K}$ is a set of reals which is dense in the canonical ordering of the reals,

(iii) $(\forall x)$ $\boldsymbol{\sigma}(x)$ is a finite subset of $\boldsymbol{K}$ and if x is a finite subset of $\boldsymbol{K}$ then $\boldsymbol{\sigma}(x)=x$,

(iv) if $\varphi(x_0, \ldots, x_{n-1}, y)$ and $(\exists ! z)\varphi(x_0, \ldots, x_{n-1}, z)$ then $\boldsymbol{\sigma}(y) \subseteq \bigcup_{i<n}\boldsymbol{\sigma}(x_i)$.

Now working entirely in M, we claim that the cardinal of $\boldsymbol{K}$ is an infinite multiple-free element of Δ_R. By (ii) $\boldsymbol{K}$ is an infinite set of reals, and it is clear that $\boldsymbol{K}$ is Dedekind if it is multiple-free. To prove the latter assume that x, y are disjoint subsets of $\boldsymbol{K}$ and that f is a one-one function mapping x onto y. If x is infinite choose $u \in x$ such that $f(u) \notin \sigma(f)$. By (iii) we have $\{u\}=\boldsymbol{\sigma}\{u\}$, $\{f(u)\}=\boldsymbol{\sigma}\{f(u)\}$, and by (iv) we have $\boldsymbol{\sigma}\{f(u)\} \subseteq \boldsymbol{\sigma}\{u\}$. Thus $f(u)=u$ which contradicts $x \cap y=\phi$. Therefore x must be a finite set. q. e. d.

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