J. Math. Kyoto Univ. (JMKYAZ)
18-1 (1978) 95-120

Uniqueness of the factorization under composition of certain entire functions

By

Hironobu URABE

(Communicated by Prof. Y. Kusunoki, Dec. 4, '76, Revised March 19, '77)

Introduction

After the classical works of G. Julia [19] and P. Fatou [9], [10] on the iteration and composition theory for polynomials or rational functions, I. N. Baker has investigated the theory in the case of transcendental entire functions since 1955 and obtained many results. In particular, he generalized the minimum modulus theorem concerning entire functions of order less than 1/2 ([2] Theorem 3) and further proved, using Fatou's theory of iteration, interesting theorems concerning the permutability of transcendental entire functions ([2], [3], [4]). In 1968, F. Gross [13] and M. Ozawa [24] proved independently that certain entire functions do not have any factorization (by composition) into trascendental entire factors. Since then, there have appeared many results in factorization theory, by applying Nevanlinna theory etc. However, most of these recent results (except [21], [26]) concern the impossibility of factorization, that is, the primeness, the pseudoprimeness and so on.

In this paper, we shall treat certain composite functions of two or three prime functions, which belong to certain special classes. For the functions of these classes one can show the forms of their factors (Theorems 1 and 2, proved first by S. Koont [21], except one of the conclusions in Theorem 1). We shall give a simpler proof of these two theorems in § 2. Using these facts as key lemmas, we shall proceed to prove our main Theorems 3, 4, 5, 6 and 7, which assert that the factorization by composition of certain entire functions is unique up to linear polynomial factors. In the following, we shall give more detailed contents to these theorems.

The function $z+e^z$ is a prime function which is fundamental in factorization theory of transcendental entire functions, so it seems important to ask whether or not the composite function $F(z)=(z+e^z)\circ(z+e^z)$ is uniquely factorizable (see § 1). One of the purposes of this paper was to solve this question. At a glance, it seems difficult to know into what factors (other than those of the above factorization itself) $F(z)$ are factorized, because the order of $F(z)$ is not finite and the function $z+e^z$ (considered as the left factor of $F(z)$) has infinitely many zeros. However, $F(z)$ may be written as $z+H(z)$, where $H(z)=e^z(1+\exp(e^z))$ is periodic with period $2\pi i$, and this was the clue to solve the above problem. Indeed, letting any non-trivial factorization of F be $F(z)=f\circ g(z)=f(g(z))$, we can conclude that f and g have the same form as F (Theorem 2). Using this fact, we can prove Theorem 3, which includes the affirmative answer to our question above. In Theorems 4, 5 and 6, we consider the same problem for certain entire functions which are reduced to $(ze^z)\circ(z+e^z)$ in the simplest case. In Theorem 7, we consider the factorization of the functions of the form; $F(z)=(z+H_1(z))\cdot\exp[H_2(z)]$, where entire functions H_1 and $\exp[H_2]$ have period $2\pi i$. Under the condition that the order of $H_1(z)$ is finite, we shall prove that this function $F(z)$ is uniquely factorizable. Further if there exists an entire function $H_3(z)$ satisfying the identical relation $H_2(z)=H_3(z+H_1(z))$, then $F(z)=(z\cdot\exp[H_3(z)])\circ(z+H_1(z))$ is the only factorization up to equivalent factorizations. While, if there is no such an identical relation, then $F(z)$ is prime.

In addition to the above mentioned theorems, we shall give several results concerning the primeness of certain entire functions as well as generalizations of certain known results.

Here the author wishes to express his deepest gratitudes to Professor Y. Kusunoki for his encouragement and guidance. The author also wishes to express his hearty thanks to Professors M. Ozawa, I. N. Baker, F. Gross and C.-C. Yang for their valuable suggestions.

§ 1. Definitions and Preliminaries

A meromorphic function $F(z)=f\circ g(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors, respectively, provided that f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be *prime* (*pseudo-prime*, *left-prime*, *right-*

prime) if every factorization of the above form into factors implies that either f is linear or g is linear (either f is rational or g is a polynomial, f is linear whenever g is transcendental, g is linear whenever f is transcendental, resp.). When factors are restricted to entire functions, the factorization is called to be *in entire sense* (prime in entire sense, etc.). If F is a non-periodic entire function, then it is known that F is prime if F is prime in entire sense (cf. [15]).

Now it is well-known that $F(z)=z+e^z$ is prime. The primeness of this function was stated by P. Rosenbloom [29] without proof and proved for the first time by Gross [13]. The function $z+e^z$ has special properties such that it has no fixed points and no multiple zeros, it is periodic mod a non-constant polynomial and it is of smaller growth in some angular sector. Therefore the primeness of this $F(z)$ can be proved by several ways, and the proofs have suggested the extensions of factorization theory into several directions (cf. [6], [12], [13], [25]). By these facts the function $z+e^z$ has occupied the significant position in factorization theory.

Assume that a non-constant entire function $F(z)$ has two factorizations $f_1 \circ f_2 \circ \dots \circ f_n(z)$ and $g_1 \circ g_2 \circ \dots \circ g_m(z)$ into non-linear entire factors. If $m=n$ and if with suitable linear polynomials $T_j (j=1, \dots, n-1)$ the relations $f_1(z)=g_1 \circ T_1^{-1}(z)$, $f_2(z)=T_1 \circ g_2 \circ T_2^{-1}(z)$, $\dots$, $f_n(z)=T_{n-1} \circ g_n(z)$ hold, then the two factorizations are called *equivalent* (in entire sense). If every factorization of $F(z)$ into non-linear, prime, entire factors is equivalent, then we say that $F(z)$ is *uniquely factorizable.* Of course, prime functions are considered to be uniquely factorizable.

So far, the following two classes of entire functions have yielded numerous types of prime functions (cf. [6], [16], [25] etc.). For a non-zero constant b, following Koont [21], we define

$$\boldsymbol{J}(b)=\{F(z)=H(z)+cz\,;\ H \text{ is entire, periodic with period } b \ (H(z+b)=H(z)) \text{ and } c \text{ is a non-zero constant.}\}$$

$$\boldsymbol{L}(b)=\{F(z)=H_1(z)+z\cdot e^{H_2(z)}\,;\ H_1 \text{ and } e^{H_2} \text{ are entire, periodic with period } b.\}$$

Evidently, $\boldsymbol{J}(b)\subset \boldsymbol{L}(b)$. In fact, the function in $\boldsymbol{L}(b)$ such that $H_2(z)$ is constant belongs to $\boldsymbol{J}(b)$. Note that, for example, $z+e^{e^z}\in \boldsymbol{J}(2\pi i)$ and the primeness of this function was proved by Gross [16] (cf. [25]).

Using results of C.-C. Yang [35], S. Koont [21] recently proved two fundamental theorems concerning factors of functions in $\boldsymbol{J}(b)$ and $\boldsymbol{L}(b)$. We shall prove these theorems (Theorem 1 has become sharper) in § 2 by the simpler arguments. (The first proof of a special case of Theorem 3 essentially gave this. The author then used Baker-Gross' theorem ([6] Theorem 1, cf. also the argument in [34]).) Next, applying these theorems, we want to prove in § 3 that certain functions in $\boldsymbol{J}(2\pi i)$ or $\boldsymbol{L}(2\pi i)$ are uniquely factorizable. Note that $(z+e^z)\circ(z+e^z)\in\boldsymbol{J}(2\pi i)$ and $(ze^z)\circ(z+e^z)\in\boldsymbol{L}(2\pi i)$.

We denote by $M(r, f)$ the maximum modulus of (an entire function) $f(z)$ for $|z|=r$. And we shall use Nevanlinna's notations such as $T(r, f)$, $m(r, f)$, $N(r, a, f)$ and $S(r, f)$ without recalling the definitions ([23]). If $f(z)$ is entire, it is clear $T(r, f)=m(r, f)$. For a meromorphic function $f(z)$, we denote by $\rho(f)$ the order of f, by $\underline{\rho}(f)$ the lower order of f and by $\rho^*(f)$ the exponent of convergence of the zeros of $f(z)$. (About these notions, see for example [17] p. 16-25.). In the following, we shall use these notions only for entire functions.

For the factorization theory of entire functions, the following Pólya's lemma will be crucial.

Lemma 1 (*Pólya* [27]) *Suppose $f(z)$, $g(z)$ and $h(z)$ are non-constant entire functions such that $f(z)=g(h(z))$. If $h(0)=0$, then there exists a constant c with $0<c<1$ such that*

$$M(cM(r/2, h), g)\leqq M(r, f) \quad (r\geqq r_0).^{*)}$$

(Here the condition $h(0)=0$ is not essential which means that if we add the condition $r\geqq r_0$, then the condition $h(0)=0$ can be removed. Note that the inequality $M(r, f)\leqq M(M(r, h), g)$ is clearly valid.)

Let $F(z)$ be a transcendental entire function of finite order (finite lower order) and assume that $F(z)$ can be written as $F(z)=f(g(z))$ with transcendental entire functions f and g, then from Lemma 1, we can conclude (as Pólya showed) that the order $\rho(f)=0$ and $\rho(g)\leqq\rho(F)$ (the lower order $\underline{\rho}(f)=0$ and $\underline{\rho}(g)\leqq\underline{\rho}(F)$, resp.).

About the relation between $M(r, f)$ and $T(r, f)$ for an entire function $f(z)$, the following lemma is fundamental.

Lemma 2 (*cf.* [17] p. 18) *Let $f(z)$ be entire, then we have*

$$T(r, f)\leqq\log M(r, f)\leqq 3\cdot T(2r, f) \quad (r\geqq r_0).$$

Lemma 3 (*cf.* [21]) *If $f(z)\in\boldsymbol{L}(b)$ $(b\neq 0)$ and f is non-linear, then we have* $\log M(r, f)\geqq T(r, f)\geqq k_0 r (r\geqq r_0)$ *for some constant $k_0>0$, hence* $\underline{\rho}(f)\geqq 1$.

*) In the case where some assertion (*) is valid for sufficiently large values of r, we write simply that (*) is valid for $r\geqq r_0$ (r_0 is not same in all cases).

For completeness, we prove Lemma 3. Let $f(z)=H_1(z)+z\cdot\exp[H_2(z)]$. Assume that $H_2(z)$ is not constant. Then the function $f(z+b)-f(z)=b\cdot\exp[H_2(z)]$ satisfies $\log M(r, f(z+b)-f(z))\geqq k_0 r\ (r\geqq r_0)$, as is easily seen by Lemma 1. Hence we can conclude $\log M(r, f)\geqq k_0' r\ (r\geqq r_0)$. Next assume that H_2 is constant and write $f(z)=H(z)+cz$ with $H(z+b)=H(z)$. It will be sufficient to prove that $\log M(r, H)\geqq k_0 r\ (r\geqq r_0)$ for some $k_0>0$. As we can assume without loss of generality that the value 0 is taken by $H(z)$, we have $n(r, 0, H)\geqq(2k_0/\log 2)\cdot r\ (r\geqq r_0)$ for some k_0 whence we obtain

$$N(r, 0, H)\geqq\int_{r/2}^{r}\frac{n(t, 0, H)}{t}dt\geqq(\log 2)\cdot n(r/2, 0, H)\geqq k_0 r\ (r\geqq r_0).$$

Since $\log M(r, H)\geqq T(r, H)\geqq N(r, 0, H)$, we have the conclusion.

We note also the following well-known fact.

Lemma 4 *Let $H(z)=h(e^z)$ be a non-constant entire function which is periodic with period $2\pi i$ ($h(z)$ is holomorphic in $0<|z|<\infty$ and has the Laurent expansion $\sum_{-\infty}^{\infty} a_k z^k$). If $H(z)$ is of exponential type (order 1 and mean type $M(r, H)=O(\exp[Kr])$, as $r\to\infty$, for some constant K), then the number of coefficients a_k which are not zero is finite.*

The proof of this lemma can be done, using Lemma 1, as follows. Since $H(z)=h(e^z)=h_1(e^z)+h_2(e^{-z})$, where $h_1(z)=\sum_0^\infty a_k z^k$ and $h_2(z)=\sum_1^\infty a_{-k}z^k$, it is enough to prove that h_1 and h_2 are both polynomials, under the hypothesis that $H(z)=h(e^z)$ is of exponential type. But if $h(e^z)$ is of exponential type, then $h_1(e^z)$ and $h_2(e^{-z})$ are both so. In fact, noting $M(r, h_1(e^z))=\max\{|h_1(e^z)|;\ |z|=r,\ -\pi/2\leqq\arg z\leqq\pi/2\}+O(1)\ (r\geqq r_0)$, We have $M(r, h_1(e^z))\leqq M(r, H)+O(1)$. Hence $h_1(e^z)$ must be exponential type. Then by Lemma 1,

$$M(ce^{r/2}, h_1)=M[cM(r/2, e^z), h_1]\leqq M(r, h_1(e^z))\leqq e^{Kr}\ (r\geqq r_0)$$

for some positive constants c $(0<c<1)$ and K. Hence we have $M(r, h_1)\leqq r^N\ (r\geqq r_0)$ for some positive integer N, which means that h_1 is a polynomial. Similarly h_2 is a polynomial. Thus we have done.

§ 2. Two Fundamental Theorems

We shall prove here the following two theorems concerning the factors for functions in $\boldsymbol{J}(b)$ and $\boldsymbol{L}(b)$. These are used subsequently as key lemmas.

Theorem 1 *Let $F(z)\in\boldsymbol{L}(b)$ $(b\neq 0)$ and $F(z)=f(g(z))$ with non-*

linear entire functions f *and* g, *then we have* $f(z)\in \boldsymbol{L}(b')$ *for some* $b'\neq 0$ *and* $g(z)\in \boldsymbol{J}(b)$.

Theorem 2 *Let* $F(z)\in \boldsymbol{J}(b)$ $(b\neq 0)$ *and* $F(z)=f(g(z))$ *with non-linear entire functions* f *and* g, *then we have* $f(z)\in \boldsymbol{J}(b')$ *for some* $b'\neq 0$ *and* $g(z)\in \boldsymbol{J}(b)$.

These striking theorems were proved for the first time by Koont [21], except the conclusion $f(z)\in \boldsymbol{L}(b')$ in Theorem 1. But his proof seems complicated. Here we wish to prove these theorems by the simpler argument, which is due to Gross [14]. (About this argument, also cf. Baker-Gross [6].) Formerly in [33], the author used this argument and obtained a result concerning a problem of Gross on the periodicity of entire functions (cf. §5).

In the following, we shall use symbols H, H_j (j: a natural number) for periodic entire functions with some periods.

Proof of Theorem 1. Let $F(z)=H_1(z)+z\cdot\exp[H_2(z)]=f(g(z))$, where entire functions H_1 and $\exp(H_2)$ are periodic with period $b\neq 0$, f and g are non-linear entire functions. Then we have

(1) $\quad f(g(z+nb))-f(g(z))=nb\cdot e^{H_2(z)}$ (n: any integer).

From this relation, one sees that the functions $[g(z+nb)-g(z)]$ cannot vanish if n is a non-zero integer. Hence we obtain

(2) $\quad g(z+b)-g(z)=e^{p(z)}$ and $g(z+2b)-g(z)=e^{q(z)}$

for some entire functions $p(z)$ and $q(z)$. From (2), we deduce

(3) $\quad e^{p(z+b)}+e^{p(z)}=e^{q(z)}$, for all z.

Hence we have that $p(z+b)-p(z)=\text{const.}=c$, by Picard's theorem. (In fact, on accout of the relation (3) $\exp[p(z+b)-p(z)]$ cannot assume three values 0, -1 and ∞.) Thus we obtain

(4) $\quad p(z)=H_3(z)+\frac{c}{b}\cdot z=c_1z+H_3(z)$,

where $H_3(z+b)=H_3(z)$ and $c_1=c/b$. By (2) and (4), we have

(5) $\quad g(z+b)-g(z)=e^{c_1z+H_3(z)}$.

Taking an entire function $h(z)$ satisfying $[(\exp(c_1b))h(z+b)-h(z)]=1$, we have that g has the form; $g(z)=H_4(z)+h(z)\exp[c_1z+H_3(z)]$, with $H_4(z+b)=H_4(z)$. If $\exp[c_1b]\neq 1$, we may take $h(z)=\text{const.}=c_2$, with $c_2=1/(\exp[c_1b]-1)$. If $\exp[c_1b]=1$, we may take $h(z)=az$, with

$a=1/b$. Hence we may write

(6) $\quad g(z)=H_4(z)+e^{c_1z+H_3(z)}$, if $e^{c_1b}\neq 1$, or

(7) $\quad g(z)=H_4(z)+z\cdot e^{c_1z+H_3(z)}$, if $e^{c_1b}=1$.

We can rule out the possibility (6) as follows. We may assume that the n-th power of $\exp[c_1b]$ is not equal to 1 for any non-zero integer n, since otherwise $g(z)$ becomes periodic (with period nb), which means that $F(z)=f(g(z))$ is periodic, contrary to the non-periodicity of $F(z)$. Since $g(z+nb)=g(z)+(\exp[nc_1b]-1)\exp[c_1z+H_3(z)]$, we have from (1)

(8) $\quad f(g(z)+(\exp[nc_1b]-1)e^{c_1z+H_3(z)})=f(g(z))+nb\cdot e^{H_2(z)}$

for any integer n. If $|\exp(c_1b)|=1$, then the left hand side of (8) is bounded for all n. If $|\exp(c_1b)|>1\,(<1)$, it is bounded when n moves negative (positive resp.) integers, while the right hand side of (8) is unbounded with respect to n. This contradiction shows that the case (6) does not occur.

Consider the case (7). We prove that $c_1z+H_3(z)$ must be constant. Since $g(z+nb)=g(z)+nb\cdot\exp[c_1z+H_3(z)]$, from (1) we have

(9) $\quad f(g(z)+nb\cdot\exp[c_1z+H_3(z)])=f(g(z))+nb\cdot e.^{H_2(z)}$

When z moves some compact set $\boldsymbol{K}_1$, the value taken by the function $H_5(z)=\exp[c_1z+H_3(z)]$ moves some compact set $\boldsymbol{K}_2$ whose interior covers the full unit circle (say). This is evident. In fact, for open disks $\boldsymbol{D}_m=\{|z|<m\}$ $(m=1, 2, \ldots)$, then the union of the sets $H_5(\boldsymbol{D}_m)$ $(m\geqq 1)$ covers the unit circle (a compact set) by Picard's theorem, and $H_5(\boldsymbol{D}_m)$ is open by the fact that the holomorphic function is an open mapping, hence there exists a positive integer m such that $H_5(\boldsymbol{D}_m)$ covers the unit circle. Then $\boldsymbol{K}_1$ may be taken as the closure of $\boldsymbol{D}_m$. Note that the compact set $\boldsymbol{K}_2=H_5(\boldsymbol{K}_1)$ has a positive distance from the origin, since $H_5(z)$ does not vanish.

From this fact and (9), we obtain for some positive constants A and B,

(10) $\quad M(An, f)\leqq Bn+O(1),\ (n\geqq n_0).$

To prove this inequality, we must verify that, letting $\boldsymbol{K}_3(n)=(g(z)+nbH_5(z))(\boldsymbol{K}_1)$ the image-set of $\boldsymbol{K}_1$ by the function $g(z)+nb\cdot H_5(z)$, the complement of $\boldsymbol{K}_3(n)$ has a relatively compact connected component including the origin for each $n\geqq n_0$. For this purpose, we may assume (choosing a suitable circle near the unit circle if necessary) that for any z with $|z|=1$, there exists an open disk U_z with center at z such that some connected component of $H_5^{-1}(\partial U_z)$ is a simple closed curve in $\boldsymbol{K}_1$. Take a finite open covering of the unit circle by such disks U, then noting that $g(z)$ is bounded (w, r. t. n) on $\boldsymbol{K}_1$ and that $nb\cdot H_5(z)$ becomes large for large n, we can conclude that the complement of $\boldsymbol{K}_3(n)$ has a relatively compact component incluning the origin. Then the inequality (10) follows from (9) by a simple estimate, noting the maximum modulus principle.

From (10), we have $\lim\inf_{r\to\infty}(\log M(r, f)/\log r)\leqq 1$. By Liouville's theorem, we conclude that $f(z)$ is a linear polynomial, which is contrary to hypothesis. Thus we have proved that $c_1z+H_3(z)$ is constant. Hence by (7) we obtain

(11) $\quad g(z)=H_4(z)+c_2z$, for some $c_2\neq 0$,

that is, $g(z)\in\boldsymbol{J}(b)$. Further we shall prove $f(z)\in\boldsymbol{L}(b')$ for some $b'\neq 0$.

Since $g(z+nb)=g(z)+nbc_2$, the relation (1) becomes $f(g(z)+nbc_2)-f(g(z))=nb\cdot\exp(H_2(z))$. Because $g(z)$ takes every values*), we can conclude from the above relation that the functions $(f(z+bc_2)-f(z))$ and $(f(z+2bc_2)-f(z))$ have no zeros. Repeating the argument at the beginning of this proof (cf. (2), (3), (4), (5)), we obtain that $f(z)$ can be written as

(6′) $\quad f(z)=H_6(z)+e^{c_3z+H_7(z)}$, if $e^{c_3bc_2}\neq 1$ or

(7′) $\quad f(z)=H_6(z)+z\cdot e^{c_3z+H_7(z)}$, if $e^{c_3bc_2}=1$,

where H_j is entire with $H_j(z+bc_2)=H_j(z)$ $(j=6, 7)$. In the case (6′), the relation $F(z+nb)-F(z)=nb\cdot\exp(H_2(z))$ reduces to

$$(e^{nc_3bc_2}-1)\cdot e^{c_3g(z)+H_7(g(z))}=nb\cdot e^{H_2(z)}\ (n:\text{ any integer})$$

which is clearly impossible. Hence only the case (7′) is possible. Thus we have proved $f(z)\in\boldsymbol{L}(b')$ with $b'=bc_2\neq 0$, which completes the proof of Theorem 1.

Proof of Theorem 2. Let $F(z)=H(z)+cz=f(g(z))$ with non-linear

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*) In fact, if $g(z)=c'$ has no roots for some c', then by (11), we have $c_2z+H_4(z)-c'=\exp[a(z)]$ for some entire function $a(z)$. By the upper conclusion (note $\boldsymbol{J}(b)\subset\boldsymbol{L}(b)$), $a(z)\in\boldsymbol{J}(b)$ so that $a'(z)$ is periodic with period b. Since also the derivative of $\exp[a(z)]$ has period b, we deduce that $\exp[a(z)]$ itself has period b, which is impossible. Thus $g(z)$ takes every values. Further we can prove that there exists no identical relation such as $z+H(z)=h(z)\exp[a(z)]$, where H, h and a are entire with $H(z+b)=H(z)$ and $\rho(h)<1$ (cf. proof of Theorem 7),

entire functions f and g. Since $F\in\boldsymbol{J}(b)$ and $\boldsymbol{J}(b)\subset\boldsymbol{L}(b)$, by Theorem 1 we have $g\in\boldsymbol{J}(b)$. Now $g(z)$ can be written as in (11). In this case, $g'(z)=H_4'(z)+c_2$ is periodic with period b. As $F'(z)=f'(g(z))g'(z)=H'(z)+c$ is also periodic with period b, $f'(g(z))$ must be so, that is, $f'(g(z+b))=f'(g(z))$. Since $g(z+b)=g(z)+c_2b$, we obtain $f'(z+c_2b)=f'(z)$, whence we have that $f(z+c_2b)-f(z)=\text{const.}=c'$. Here $c'\neq 0$, otherwise $F(z)$ becomes periodic, which is impossible. Therefore we obtain

$$(12)\qquad f(z)=\frac{c'}{c_2b}\,z+H_8(z),$$

where $H_8(z+c_2b)=H_8(z)$. Thus $f(z)\in\boldsymbol{J}(b')$ with $b'=c_2b\neq 0$.

Corollary 1 *Let $F(z)=H(z)+cz$ be a non-linear entire function in $\boldsymbol{J}(b)$ $(b\neq 0)$ which is of finite lower order, then $F(z)$ is prime. (cf. Lemma* 11'.)

This follows directly from Theorem 2, Lemmas 1 and 3, since the non-linear left and right factors of $F(z)$ are both necessarily transcendental and further the lower order of the left-factor is positive so that, if F is not prime, then $F(z)$ cannot be of finite lower order.

Further we have

Corollary 2 *If $F(z)=H(z)+ze^z\in\boldsymbol{L}(b)$ $(b\neq 0)$, then $F(z)$ is prime.*

In fact, using Theorem 1, if $F=f(g)$ and f is non-linear, then we have that $g(z)$ must be linear (cf. proof of Theorem 4, (20)).

Corollary 3 *Let $F(z)=ze^{H(z)}\in\boldsymbol{L}(b)$ $(b\neq 0)$, then $F(z)$ is prime.*

In fact, let $F=f(g)$ with non-linear entire functions f and g. Then we may write $f(z)=ze^{p(z)}$ and $g(z)=ze^{q(z)}=cz+H_1(z)$, where $p(z)$, $q(z)$ and $H_1(z)$ are non-constant entire functions such that $H_1(z+b)=H_1(z)$. The relation $g(z+b)-g(z)=cb$ becomes $(z+b)e^{q(z+b)}-ze^{q(z)}=cb$. This will be clearly impossible. Because, if $\exp[q(z+b)-q(z)]\not\equiv 1$, the left hand side of this relation has zeros, and if $\exp[q(z+b)-q(z)]\equiv 1$, then the left hand side becomes $be^{q(z)}$, which is non-constant. (also cf. Lemma 9).

Corollary 4 *Let $F(z)$ be a non-linear function in $\boldsymbol{L}(b)$ $(b\neq 0)$ which does not belong to $\boldsymbol{J}(b)$. Then no entire function $f(z)$ can satisfy the identical relation $f(f(z))=F(z)$.*

Proof. Assume that an entire function $f(z)$ does satisfy the identity

$f(f(z))=F(z)$. By Theorem 1, $f(z)$ must belong to both classes $\boldsymbol{L}(b')$ and $\boldsymbol{J}(b)$ for some $b'(\neq 0)$. Hence $f(z)=cz+H_1(z)=H_2(z)+z\cdot\exp[H_3(z)]$ for all z, where H_1 has period b, H_2 and $\exp(H_3)$ have period b'. Considering the function $f'(z+b')-f'(z)=b'H_3'(z)\cdot\exp[H_3(z)]$, this entire function has periods b and b'. Using this fact and noting that non-constant entire functions cannot be doubly periodic, anyway, we can conclude that $H_3(z)=\text{const.}=c_1$ (say). Then we have $f(f(z))=e^{c_1}cz+e^{c_1}H_1(z)+H_2(cz+H_1(z))$. As $b'=cb$ (cf. proof of Theorem 1), $f(f(z))$ now belongs to $\boldsymbol{J}(b)$, contrary to hypothesis.

Corollary 5 *Let $F(z)\in\boldsymbol{L}(b)$ $(b\neq 0)$ and $F=f_1\circ\dots\circ f_n(n\geqq 2)$ be a non-trivial factorization of $F(f_j$: non-linear) into entire factors. Then we have $f_1\in\boldsymbol{L}(b_1)$ and $f_j\in\boldsymbol{J}(b_j)$ for appropriate values b_1 and $b_j(2\leqq j\leqq n)$ with $b_n=b$. If $F(z)\in\boldsymbol{J}(b)$, then for $1\leqq j\leqq n$, we have $f_j\in\boldsymbol{J}(b_j)$ with $b_n=b$. (The b's are not zero.)*

Corollary 6 *Let $F(z)\in\boldsymbol{L}(b)$ $(b\neq 0)$ and $F=f_1\circ\dots\circ f_n$ be a non-trivial factorization of F into entire factors. If for every $\varepsilon>0$, $M(r, F)\leqq e_m(\varepsilon r)$ holds for a sequence of r's going to infinity, then we have $n\leqq m-1$. Here $e_m(z)=\exp[e_{m-1}(z)]$ and $e_1(z)=e^z(m\geqq 2)$. (Note that $\boldsymbol{J}(b)\subset\boldsymbol{L}(b)$.)*

Corollary 5 follows from Theorems 1 and 2. Corollary 6 follows from Corollary 5, combined with Lemmas 1 and 3.

Note that e^z and $\cos z$ (which are both of pseudo-prime) have an infinite number of non-equivalent factorizations. (Also cf. [21], [26].)

§ 3. Uniqueness of Factorization

3. 1. J. F. Ritt [28] settled the factorization problem by composition for polynomials, in which case, roughly speaking, the factorization is unique unless it includes factors such as the following three cases; $f_1(g_1(z))=f_2(g_2(z))$, where

(A) $f_1(z)=z^m$, $g_1(z)=z^n$ and $f_2(z)=z^n$, $g_2(z)=z^m$,

(B) $f_1(z)=z^m[h(z)]^n$, $g_1(z)=z^n$ and $f_2(z)=z^n$, $g_2(z)=z^m h(z^n)$,

(C) $f_1(z)=P_m(z)$, $g_1(z)=P_n(z)$ and $f_2(z)=P_n(z)$, $g_2(z)=P_m(z)$,

where m and n are positive integers, $h(z)$ is a polynomial and $P_n(z)$ is the n-th cosine polynomial (degree n) defined by $\cos nz=P_n(\cos z)$. Here it will be noteworthy that in cases (A) and (C), f_1 and g_1 are permutable: $f_1(g_1(z))=g_1(f_1(z))$. (For the permutability, we shall refer to Jacobsthal [18] in case of polynomials, Julia [20] for rational functions, and Baker [2], [3], [4] and Yang-Urabe [36] in the case of transcendental entire functions.) The case (B) offers examples of transcendental entire functions whose factorizations are not unique. For instance, if we take $F(z)=z^p\cdot\exp(z^p)$ with a prime number $p(\geqq 2)$, then $F(z)$ has two non-equivalent factorizations (as noted by Ozawa);

$$F(z)=z^p\circ(ze^{z^p/p})=(ze^z)\circ z^p,$$

where z^p, $z\cdot\exp[z^p/p]$ and ze^z are all prime, as is known and easily proved.

However, we can prove (easily) that $F(z)=z^pe^{pz}=z^p\circ(ze^z)$ is uniquely factorizable if p is a prime number and further $F(z)=P(z)e^{P(z)}=(ze^z)\circ P(z)$ is so if $P(z)$ is a non-linear polynomial which is prime and has at least one simple zero or two zeros with coprime multiplicities.

When both factors are transcendental, the entire function $F(z)=(ze^z)\circ(ze^z)$ may be the simplest example which is uniquely factorizable. We can generalize this example, for instance, to the function $(ze^{P(z)})\circ(ze^{Q(z)})$, where P and Q are some non-constant polynomials. Also, let $F(z)=(ze^z)\circ(h(z)e^z)$, where $h(z)$ is a non-constant entire function of order less than 1 ($\rho(h)<1$) with at least one simple zero, then $F(z)$ is uniquely factorizable. We wish to put here an outline of the proof of this last assertion. Let

$$F(z)=(ze^z)\circ(h(z)e^z)=f(g(z))$$

with non-linear entire functions f and g. Then by Borel-Nevanlinna's theorem (cf. [23] p. 72) and the fact that $h(z)$ has at least one simple zero, $f(z)$ must be transcendental. Further, noting Edrei-Fuch's theorem (Lemma 8), we have only to consider the following three cases: (i) $f(z)=h_1(z)e^{p(z)}$, where h_1(non-linear) and $p(z)$ ($\neq$ const.) are entire functions with $\rho(h_1)=0$, and $g(z)$ is a transcendental entire function with $\rho(g)<1$. (ii) $f(z)=ze^{p(z)}$ and $g(z)=h(z)e^{q(z)}$ with non-constant entire functions p and q. (iii) $f(z)=h_1(z)e^{p(z)}$ and $g(z)$ is a polynomial with deg $g\geqq 2$, where h_1 and p are non-constant entire functions with $\rho(h_1)<1/(\deg g)$ and hence $\rho(h_1(g))<1$ (cf. [34] Lemma 6).

In case (i), from $F=f(g)$, we obtain equations $h_1(g(z))=h(z)e^{d(z)}$ and $p(g(z))=z-d(z)+h(z)e^z$, where $d(z)$ is an entire function with $\rho(d)<1$ (Lemma 1). Then applying Goldstein's theorem (Lemma 10), it follows that $p(z)$ must be a polynomial. But then we have $\rho(g)=1$, which is a contradiction. In case (ii), we have a functional equation $q(z)+p(h(z)e^{q(z)})=z+h(z)e^z$. From this relation, noting $q(z)$ must be linear, we obtain equivalent factorizations. In case (iii), we have equations $h_1(g(z))=h(z)$ and $p(g(z))=z+h(z)e^z$, whence we can derive a contradiction as above. Hence it is seen that $F(z)=(ze^z)\circ(h(z)e^z)$ is uniquely factorizable. Thus the impossibility of certain functional equations will be useful in the subsequent studies.

3. 2. Main Theorems

Theorem 3 *Let $F(z)=(z+h(e^z))\circ(z+Q(e^z))$, where $h(z)$ is a non-constant entire function with the order $\rho(h(e^z))<\infty$ and $Q(z)$ is a non-constant polynomial. Then $F(z)$ is uniquely factorizable.*

Remark. If h is entire, then $\rho(\mathrm{h}(e^z))<\infty$ if $h(z)\in\mathscr{H}$ (cf. [36] Lemmas 1, 2).

Theorem 4 *Let $F(z)=(H(z)+z\cdot\exp[z+h(e^z)])\circ(z+P(e^z))$, where $H(z)$ and $h(z)$ are entire functions with $H(z+2\pi i)=H(z)$ and $\rho(h(e^z))<\infty$, and $P(z)$ is a non-constant polynomial. Assume that the function $H(z)+z\cdot\exp[z+h(e^z)]$ is prime, then $F(z)$ is uniquely factorizable.*

Corollary 7 *Let $F(z)=(H(z)+ze^z)\circ(z+h(e^z))\circ(z+P(e^z))$, where H, h and P are as in Theorem 4. Then $F(z)$ is uniquely factorizable.*

Theorem 5 *Let $F(z)=(H_1(z)+ze^z)\circ(z+H_2(z))$, where entire functions $H_j(j=1, 2)$ have period $2\pi i$. Assume that $z+H_2(z)$ is prime, then $F(z)$ is uniquely factorizable.*

Corollary 8 *Let $F(z)=(H(z)+ze^z)\circ(z+e_m(z))$, where H is entire with $H(z+2\pi i)=H(z)$ and $e_m(z)=\exp[e_{m-1}(z)]$, $e_0(z)=z(m\geqq 1)$. Then $F(z)$ is uniquely factorizable.*

Theorem 6 *Let $F(z)=(H_1(z)+ze_m(z))\circ(z+H_2(z))$, where entire functions $H_j(j=1, 2)$ have period $2\pi i$ with $\rho(H_2)<\infty$, and $e_m(z)$ is as in Corollary 8$(m\geqq 1)$. Then $F(z)$ is uniquely factorizable.*

Theorem 7 *Let $F(z)=(z+H_1(z))e^{H_2(z)}$, where entire functions H_1 and e^{H_2} have period $2\pi i$ with $\rho(H_1)<\infty$. Then $F(z)$ is uniquely factorizable.*

Remark. We shall show that, if $H_2(z)=H_3(z+H_1(z))$ has an entire solution $H_3(z)$, then $F(z)$ is uniquely factorized as $F(z)=(z\cdot\exp[H_3(z)])\circ(z+H_1(z))$, otherwise $F(z)$ is prime. (Note Corollaries 1 and 3.)

3. 3. For the proof of Theorem 3, we use the following Lemmas 5 and 6.

Lemma 5 *Let $F(z)=(z+H_1(z))\circ(z+H_2(z))$, where H_1 and $H_2(\neq$ const.) are entire, periodic with period $2\pi i$ such that the order of H_1 is finite and $H_2(z)$ is of exponential type. If $F(z)=f(g(z))$ with non-linear entire functions, then $g(z)$ must be of exponential type.*

Lemma 6 *Let $p(z)$ and $q(z)$ be holomorphic in $0<|z|<\infty$, and let $G(z)$ be entire. Assume that the relation*

(13) $$p(z)+q(ze^{p(z)})=G(z)$$

holds for all $z\neq 0$. Then both $p(z)$ and $q(z)$ must be entire functions.

Proof of Lemma 5. Since $F(z)\in \boldsymbol{J}(2\pi i)$, by Theorem 2 we have $f(z)\in \boldsymbol{J}(b)$ for some $b\neq 0$ and $g(z)\in \boldsymbol{J}(2\pi i)$. Now we have

(14) $$\log M(r, F)\leqq \log M([M(r, z+H_2(z)), z+H_1(z)]) \leqq [M(r, z+H_2(z))]^k\leqq e^{mkr}\,(r\geqq r_0)$$

for some positive integers m and k, since $\rho(H_1)$ is finite and H_2 is of exponential type. Further, noting that $M(r, f)\geqq e^{\delta r}(r\geqq r_0)$ with some positive constant δ (Lemma 3), we have, using Lemma 1,

(15) $$\log M(r, f(g))\geqq \log M[cM(r/2, g), f]\geqq \delta\, cM(r/2, g)\quad (r\geqq r_0).$$

Since $F=f(g)$, we have from (14) and (15) that $M(r, g)\leqq(\delta c)^{-1}$ exp $(2mkr)$ $(r\geqq r_0)$, which, combined with Lemma 3, implies that g is exponential type.

Proof of Lemma 6. Assume that $p(z)=\sum_{-\infty}^{\infty} a_k z^k$ and $a_{-k}\neq 0$ for some k with $1\leqq k<\infty$, then can choose $\{z_j\}$, $z_j\to 0\,(j\to\infty)$ such that

(16) $$|z_j\cdot \exp(p(z_j))|=1,\ (j=1, 2, \ldots).$$

Indeed, write $p(z)=p_1(z)+p_2(1/z)$, where $p_1(z)$ consists of terms of non-negative powers and $p_2(1/z)$ consists of terms of negative powers. If (16) is not valid for any $\{z_j\}$, then $|z\cdot \exp(p(z))|>1$ (or <1) in some neibourhood of $z=0$. Since $p_1(z)$ is bounded near $z=0$, from above inequality we have $|\exp[-p_2(1/z)]|<A|z|$ (or $|\exp(p_2(1/z))|<A/|z|$) in some neibourhood of $z=0$ for some positive constant A. This means that $|\exp(-p_2(z))|<1/A|z|$ (or $|\exp(p_2(z))|<A|z|$) for sufficiently large values of $|z|$. From this we have that $\exp(-p_2(z))$ is constant, $=0$, or $\exp(p_2(z))$ is at most a linear polynomial. This contradicts to hypothesis. Hence (16) must be satisfied for some $\{z_j\}$, $z_j\to 0$ as $j\to\infty$.

Let's take $z=z_j$ satisfying (16) as z in (13). Since z_j tends to zero as $j\rightarrow\infty$ and $|z_j\cdot\exp(p(z_j))|=1$ $(j=1, 2, \ldots)$, we conclude that $|p(z_j)|\geqq\mathrm{Re}\,p(z_j)\rightarrow\infty(j\rightarrow\infty)$, whence we deduce that the left hand side of (13) tends to ∞ as $j\rightarrow\infty$. While the right hand side of (13) remains bounded. This is impossible. Hence we have proved that $a_{-k}=0$ for any positive integer k, which shows that $p(z)$ is entire.

From (13), it is clear that the origin cannot be a pole of $q(z)$. Further the origin cannot be an essential singular point of $q(z)$, which follows from a theorem of Weierstrass, since $ze^{p(z)}$ $(z\neq 0)$ cover some punctured neibourhood of origin by Rouche's theorem. Thus $q(z)$ must be also an entire function.

Proof of Theorem 3. Let $F=f(g)$ with non-linear entire functions f and g. Since $F(z)\in\boldsymbol{J}(2\pi i)$, by Theorem 2 (cf. (11) and (12)) we have $f(z)=cz/(2\pi ic_2)+H_1(z)$ and $g(z)=c_2z+H_2(z)$, where $H_1(z+2\pi ic_2)=H_1(z)$, $H_2(z+2\pi i)=H_2(z)$, c and c_2 are some non-zero constants. From $F=f(g)$, we have $c=2\pi i$. Hence we have $f(z)=z/c_2+H_1(z)=z/c_2+H_3(z/c_2)$ and $g(z)=c_2(z+H_4(z))$, where $H_3(z+2\pi i)=H_3(z)$ and $H_4(z)=1/c_2\cdot H_2(z)$. Thus we may assume that

$$f(z)=z+H_1(z) \text{ and } g(z)=z+H_2(z),$$

where $H_j(z+2\pi i)=H_j(z)$ $(j=1, 2)$. Now we can write that $f(z)=z+p(e^z)$ and $g(z)=z+q(e^z)$, where $p(z)$ and $q(z)$ are some holomorphic functions in $0<|z|<\infty$. By Lemma 5, $g(z)$ must be of exponential type, whence by Lemma 4 we may write $q(z)=\sum_{-m}^{m}a_kz^k$, for some constants $a_k(-m\leqq k\leqq m)$ and some positive integer m. In the relation $F=f(g)$, cancelling z and then putting $w=e^z$, we obtain the relation $q(w)+p(w\cdot\exp[q(w)])=Q(w)+h(w\cdot\exp[Q(w)])$, $(w\neq 0)$. By Lemma 6, $q(z)$ is a polynomial and $p(z)$ is an entire function. Here p and q are non-constant, since otherwise f or g becomes linear. The above identical relation now can be written as

$$(17)\qquad p(ze^{q(z)})=Q(z)-q(z)+h(ze^{Q(z)}),\ (z\neq 0).$$

One finds that $\deg q=\deg Q$ and the arguments of the leading coefficients of q and Q are equal. In fact, if $\deg q\neq\deg Q$, or $\deg q=\deg Q$ and the arguments of the leading coefficients of q and Q are not equal, then we can choose a suitable radial straight line L on which $\exp[q(z)]$ tends to zero while $\exp[Q(z)]$ tends to ∞ as $z\rightarrow\infty$. We shall show that this state of affairs leads us to a contradiction. Since $\rho(h)=0$, there exists a sequence $\{r_n\}$, $r_n>0$ and $r_n\rightarrow\infty$ as $n\rightarrow\infty$ such that $\tilde{m}(r_n, h)\geqq M(r_n, h)^{1-\epsilon}$, where $\tilde{m}(r, h)$ is the minimum modulus of $h(z)$ for $|z|$

*) Boas R. P. : Entire functions (Academic Press, 1954), p. 51.

$=r^{*)}$, and since $h(z)$ is not constant, $M(r, h)\geqq r^{1-\epsilon}(r\geqq r_0)$, hence we have

(18) $\tilde{m}(r_n, h)\geqq r_n^{1/2}(n\geqq n_0)$,

here we take ε as $0<\varepsilon<1/4$. Letting $L: z=te^{i\theta}$ $(t>0)$, the equation $r_n=|z\cdot\exp[Q(z)]|$ has a solution $z=z_n, =t_ne^{i\theta}$ say, $(n\geqq n_0)$. In this case we may assume that $r_n\geqq\exp(\delta t_n)$ $(n\geqq n_0)$ for some positive constant δ so that we obtain from (18) $|h(z_n\cdot\exp[Q(z_n)])|\geqq\exp(\delta t_n/2)=\exp(\delta|z_n|/2)$, while we have that $|p(z_n\cdot\exp[q(z_n)])|+|Q(z_n)|+|q(z_n)|\leqq|z_n|^K$ $(n\geqq n_0)$ for some constant $K>0$ (noting $ze^{q(z)}\to 0$ as $z\to\infty$ on L). These inequalities and (17) mean that $\exp(\delta|z_n|/2)\leqq|z_n|^K(n\geqq n_0)$, which is clearly impossible.

Then again by (17), we obtain that $Q(z)-q(z)$ (polynomial) is bounded on some radial straight line, hence $Q(z)-q(z)=\text{const.}=-d$ (say). Then we have that $q(z)=Q(z)+d$ and from (17), $p(z)=h(e^{-d}z)-d$. Thus $f(z)=z+p(e^z)=z-d+h(e^{z-d})$ and $g(z)=z+q(e^z)=z+Q(e^z)+d$. Taking $T(z)=z+d$, we obtain that

$$f(z)=(z+h(e^z))\circ T^{-1}(z) \text{ and } g(z)=T(z)\circ(z+Q(e^z)),$$

which shows that two factorizations $f(g(z))=(z+h(e^z))\circ(z+Q(e^z))$ are equivalent, which is to be proved.

3. 4. Proof of Theorem 4. Let $F=f(g)$ with non-linear entire functions f and g. Since $F\in\boldsymbol{L}(2\pi i)$, by Theorem 1 we can write

(19) $f(z)=H_1(z)+ze^{K(z)}$ and $g(z)=z+H_2(z)$,

where non-constant entire functions $H_j(j=1, 2)$ and $e^{K(z)}$ have period $2\pi i$. Then

$$\begin{aligned} f(g(z)) &= H_1(z+H_2(z))+H_2(z)\cdot e^{K(z+H_2(z))}+z\cdot e^{K(z+H_2(z))} \\ &= H(z+P(e^z))+P(e^z)e^{(z+h(e^z))\circ(z+P(e^z))}+z\cdot e^{(z+h(e^z))\circ(z+P(e^z))}. \end{aligned}$$

Considering the function $F(z+2\pi i)-F(z)$ and cancelling the periodic parts, we have $\exp[K(z+H_2(z))]=\exp[(z+h(e^z))\circ(z+P(e^z))]$, hence

(20) $K(z+H_2(z))=(z+h(e^z))\circ(z+P(e^z))$, (mod $2\pi i$).

Now the right hand side of (20) is uniquely factorizable by Theorem 3. Hence we may write

(21) $K(z)=z+h(e^z)$ and $z+H_2(z)=z+P(e^z)$, or

(22) $K(z)=z$ and $z+H_2(z)=(z+h(e^z))\circ(z+P(e^z))$.

In case (21), from (19) $g(z)=z+P(e^z)$, so that we have $H_1(z)=H$

(z) and $f(z)=H(z)+z\cdot\exp[z+h(e^z)]$. In case (22), we have from $F=f(g)$ and (19) that $H(z)+z\cdot\exp[z+h(e^z)]=f(z+h(e^z))$. By the assumption the left hand side of this relation is prime. Since f is non-linear, this is impossible. Hence we can rule out the case (22). Thus we have done.

Proof of Corollary 7. We repeat the argument in the proof of Theorem 4. Letting $F=f(g)$, we have (19). We shall study two cases (21) and (22). In case (21), since $g(z)=z+P(e^z)$, we have $f(z)=H_1(z)+z\cdot\exp[z+h(e^z)]=(H(z)+ze^z)\circ(z+h(e^z))$. Assume that $f(z)=f_1(g_1(z))$ with non-linear entire functions f_1 and g_1, then by Theorem 1 we may write

(23) $f_1(z)=H_2(z)+ze^{H_3(z)}$ and $g_1(z)=z+H_4(z)$.

From $f=f_1(g_1)$, we conclude as in the proof of Theorem 4 that $H_3(z+H_4(z))=z+h(e^z)$. The right hand side of this is of finite order, so that, noting Lemmas 1 and 3, $H_3(z)$ must be linear. We may assume that $H_3(z)=z$. Hence $g_1(z)=z+H_4(z)=z+h(e^z)$. Then from (23) and $f=f_1(g_1)$, we have $H_2(z)=H(z)$ so that we obtain

$$f_1(z)=H(z)+ze^z,\ g_1(z)=z+h(e^z) \text{ and } g(z)=z+P(e^z).$$

In case (22), noting $g(z)=z+H_2(z)=(z+h(e^z))\circ(z+P(e^z))$, we have from $F=f(g)$, $f(z)=H(z)+ze^z$. Thus in any cases, the factorization $F(z)=(H(z)+ze^z)\circ(z+h(e^z))\circ(z+P(e^z))$ is the only one into prime factors up to equivalent factorizations, which is to be proved.

3. 5. The proof of Theorem 5 will become clear from that of Corollary 8. Hence we prove only Corollary 8. For this purpose we shall need the following fact, which is a generalization of known results (cf. [16], [25]).

Theorem 8 *Let $F(z)=e_m(z+P(e^z))+Q(z)$, where P and $Q(\not\equiv$ const.) are polynomials and $m\geqq 1$. Then $F(z)$ is prime.*

Corollary 9 *Let $F(z)=z+e_m(z)$ $(m\geqq 1)$, then $F(z)$ is prime.*

Corollary 10 *Let $F(z)=P(e^z)e^{e^z}+Q(z)$ with polynomials $P(\not\equiv 0)$ and $Q(\not\equiv$const.). Then $F(z)$ is prime.*

Corollary 9 follows directly from Theorem 8 if we take $P(z)\equiv 0$ and $Q(z)\equiv z$. And the proof of Theorem 8 below essentially shows Corollary 10, which is noted in [34] (remark after Theorem 3) without proof. To prove Theorem 8, we use the following lemma due to Ozawa.

Lemma 7 ([25]) *Let $F(z)$ be an entire function satisfying the inequality $N(r, 0, F')\geqq k_0 m(r, F)$ $(=T(r, F))$ for all positive number $r\notin E$ with a set E of finite linear measure and for some $k_0>0$. Assume that the system of equations*

(24) $\quad F(z)=c$ *and* $F'(z)=0$

have only finitely many common roots for any constant c $(\neq\infty)$. Then $F(z)$ is left-prime in entire sense.

Proof of Theorem 8. Consider $N(r, 0, F')$. Letting

$$G(z)=\frac{(1+P'(e^z)e^z)e_1(z+P(e^z)).....e_m(z+P(e^z))}{(1+P'(e^z)e^z)e_1(z+P(e^z))...e_m(z+P(e^z))+Q'(z)}$$

we have, by the second main theorem of Nevanlinna,

$$T(r, G)\leqq N(r, \infty, G)+N(r, 0, G)+N(r, 1, G)$$
$$+0(\log[rT(r, G)]),$$

for $r\notin E_G$ with a set E_G of finite linear measure. Note that the denominator of $G(z)$ is equal to $F'(z)$. Then using Clunie's theorem (cf. [17] p. 54), we obtain $N(r, 0, F')\geqq(1-\varepsilon)m(r, F)=k_0 m(r, F)$ for $r\notin E_G$, with $k_0=1-\varepsilon$ $(0<\varepsilon<1)$ (cf. [25]). Thus the first condition in Lemma 7 is satisfied.

Next consider the equation (24). We may suppose, substituting $Q(z)$ by $Q(z)-c$, that $c=0$. Then (24) can be written as

$$(25)\qquad \begin{cases} e_m(z+P(e^z))+Q(z)=0 \\ e_m(z+P(e^z)).....\ e_1(z+P(e^z))(1+P'(e^z)e^z)+Q'(z)=0. \end{cases}$$

Assume that the equations (25) have an infinite number of common roots $\{z_n\}_1^\infty$. Then from (25) we have

$$(26)\qquad \frac{Q'(z_n)}{Q(z_n)}=e_{m-1}(z_n+P(e^{z_n}))....e_1(z_n+P(e^{z_n}))(1+e^{z_n}P'(e^{z_n}))$$

for $n=1, 2, \ldots$. By the first equation of (25), $e_m(z_n+P(e^{z_n}))\to\infty$ as $n\to\infty$. Hence we must have $e_k(z_n+P(e^{z_n}))\to\infty$ as $n\to\infty$ for $k=1, \ldots, m-1$, and $e^{z_n}\to\infty$ as $n\to\infty$. Then the right hand side of (26) tends to ∞ as $n\to\infty$, while the left hand side of (26) tends to zero as $n\to\infty$. This is a contradiction. Hence the equation (24) can have at most a finite number of common roots for any c. Thus $F(z)$ is left-prime in entire sense by Lemma 7.

By the result of Baker-Gross ([6] Theorem 3), the right factor of $F(z)$ (which is periodic mod a polynomial) cannot be a polynomial of degree greater than 2. Let $F(z)=f(R(z))$ with a quadratic polynomial $R(z)$. In this case, substituting the variable, we may assume that $e_m[(z+z_0)+P(\exp(z+z_0))]+Q(z+z_0)$ is an even function for some z_0. But this is impossible. Thus $F(z)$ is prime in entire sense. Since F is non-periodic, $F(z)$ is prime as is known and easily proved ([15]).

Proof of Corollary 8. Letting $F=f(g)$ with non-linear entire functions f and g, we can write as in (19) that $f(z)=H_1(z)+z\cdot\exp(H_2(z))$ and $g(z)=z+H_3(z)$, where H_1, $\exp(H_2)$ and H_3 are entire, periodic with period $2\pi i$. From $F=f(g)$, similarly as in the proof of Theorem 4, we have

(27) $$H_2(z+H_3(z))=z+e_m(z)+2k\pi i$$

for some integer k. Here the right hand side of (27) is prime by Corollary 9, hence $H_2(z)$ is linear and we (may) have $g(z)=z+H_3(z)=z+e_m(z)$. Therefore we obtain $f(z)=H(z)+ze^z$. Thus we have proved that two factorizations $F(z)=(H(z)+ze^z)\circ(z+e_m(z))=f(g(z))$ are equivalent, hence we have done.

3. 6. Proof of Theorm 6. Let $F=f(g)$, where f and g are non-linear entire functions. Since $F\in \boldsymbol{L}(2\pi i)$, we can write f and g as in (19).

(19′) $$f(z)=H_3(z)+z\cdot e^{K(z)} \text{ and } g(z)=z+H_4(z).$$

Then as before we obtain $\exp[K(z+H_4(z))]=e_m(z+H_2(z))$. Hence

(28) $$K(z+H_4(z))=e_{m-1}(z+H_2(z))+2k_1\pi i \quad (k_1\text{: an integer}),$$

so that $K(z)=2k_1\pi i+\exp[U_1(z)]$ for some entire function $U_1(z)$, since $z+H_4(z)=c$ has roots for any constant c (cf. footnote at p. 102). Therefore we have $U_1(z+H_4(z))=e_{m-2}(z+H_2(z))+2k_2\pi i$ for some integer k_2. This implies that $U_1(z)=2k_2\pi i+\exp[U_2(z)]$ for some entire function $U_2(z)$. Thus we have $U_2(z+H_4(z))=e_{m-3}(z+H_2(z))+2k_3\pi i$ for some integer k_3. Repeating this process, we arrive at $U_{m-2}(z)=2k_{m-1}\pi i+\exp[U_{m-1}(z)]$ and

(29) $$U_{m-1}(z+H_4(z))=z+H_2(z)+2k_m\pi i,$$

where U_{m-2} and U_{m-1} are some non-constant entire functions, k_{m-1} and k_m are some integers. Since the right hand side of (29) is prime by Corollary 1 (cf. [6]), $U_{m-1}(z)$ must be linear. Putting $U_{m-1}(z)=az+b$, we have from (29) that $a=1$ and $H_4(z)=H_2(z)+c$ with $c=-b+2k_m\pi i$. Then we have

(30) $$g(z)=z+H_2(z)+c.$$

From (28) we have $K(z)=e_{m-1}(z-c)+2k_1\pi i$ and further from $F=f(g)$ we have

(31) $f(z)=H_1(z-c)+(z-c)e_m(z-c).$

Taking $T(z)=z+c$, we obtain from (30) and (31) that $f(z)=(H_1(z)+ze_m(z))\circ T^{-1}(z)$ and $g(z)=T(z)\circ(z+H_2(z))$. Therefore two factorizations $F(z)=f(g(z))=(H_1(z)+ze_m(z))\circ(z+H_2(z))$ are equivalent, which is to be proved.

3. 7. For the proof of Theorem 7, we shall use the following lemmas.

Lemma 8 (*Edrei-Fuchs* [8]) *Let $f(z)$ and $g(z)$ are two transcendental entire functions. If $\rho^*(f)>0$, then we have necessarily $\rho^*(f(g))=\infty$.*

Lemma 9 (*Borel's unicity theorem cf.* [23]) *Let $a_j(z)$ $(j=0, 1, \ldots, n)$ be entire functions of order no greater that ρ, let $g_j(z)$ $(j=1, \ldots, n)$ be also entire and let $g_j(z)-g_k(z)$ $(j\neq k)$ be transcendental entire functions or polynomials of degree greater than ρ, then the identity*

$$\sum_{j=1}^{n} a_j\cdot e^{g_j(z)}=a_0(z)$$

holds only when $a_0(z)=a_1(z)=\ldots.=a_n(z)=0$ identicaly.

Proof of Theorem 7. Letting $F=f(g)$ with non-linear entire functions f and g, by the fact $F(z)\in \boldsymbol{L}(2\pi i)$ we can write as before

(32) $f(z)=H_3(z)+z\cdot e^{H_4(z)}$ and $g(z)=z+H_5(z)$,

where entire functions H_3, $\exp(H_4)$ and H_5 have period $2\pi i$. Since $\rho^*(F)$ is finite and $g(z)$ is transcendental, by Lemma 8 $\rho^*(f)$ cannot be positive. Hence

(33) $f(z)=H_3(z)+z\cdot e^{H_4(z)}=h(z)e^{q(z)}$

for some entire functions $h(z)$ and $q(z)$ with $\rho(h)=0$. From (33) we have

(34) $h(z+2\pi i)e^{q(z+2\pi i)}-h(z)e^{q(z)}=2\pi ie^{H_4(z)}.$

By Lemma 9, we obtain $q(z+2\pi i)-q(z)=\text{const.}=c_1$, say. Then (34) becomes $[e^{c_1}h(z+2\pi i)-h(z)]e^{q(z)}=2\pi i \exp(H_4(z))$. Since $\rho(h)=0$, we conclude that $e^{q(z)}=c_2\exp(H_4(z))$ for some $c_2\neq 0$. Going back to (33), we have $H_3(z)=(c_2h(z)-z)\exp(H_4(z))$. This means that $c_2h(z)-z(\not\equiv 0)$ is periodic, so $c_2h(z)=z+c_2'$ for some $c_2'\neq 0$. Hence from (33) we have $H_3(z)=c_3\exp(H_4(z))$ and $f(z)=(z+c_3)\exp(H_4(z))$ for some c_3. From $F=f(g)$ and (32), considering $F(z+2\pi i)-F(z)$, we obtain (as before)

(35) $\quad H_2(z)=H_4(z+H_5(z))+2k\pi i$ and $H_1(z)=H_5(z)+c_3$.

From (32) and (35) we have $g(z)=z+H_1(z)-c_3$ and hence

$$\begin{aligned} F(z) &= (z+H_1(z))e^{H_2(z)}=(z+H_1(z))e^{H_4(z+H_1(z)-c_3)} \\ &= ((z+c_3)e^{H_4(z)})\circ(z+H_1(z)-c_3)=f(g(z)), \end{aligned}$$

which is the only non-trivial factorization if $F(z)$ is not prime. Thus under the assumption that $F(z)$ is not prime, from (35) we have $H_2(z)=H_6(z+H_1(z))$ for some entire function $H_6(z)$ such that $\exp(H_6(z))$ has period $2\pi i$.

Remark. The functions $z+H_1(z)-c_3$ and $(z+c_3)\cdot\exp(H_4(z))$ are known to be prime by Corollary 1 or 3.

§4. Certain prime functions in $J(2\pi i)$ or $L(2\pi i)$

We with to note here the following several results.

Theorem 9 *Let $F(z)=z+h_1(e^z)+h_2(e^z)e^{az}$, where h_1 and $h_2(\not\equiv 0)$ are entire functions with $\rho(h_j)<1$ $(j=1, 2)$. Then $F(z)$ is prime, unless h_1 is a linear polynomial and $h_2(z)=cz^m$, for some non-zero constant c and some positive integer m.*

Theorem 10 *Let $F(z)=z+Q(e^z)+h(e^{az})$, where $h(\neq$ const.) is entire with $\rho(h(e^z))<\infty$ and Q is a polynomial, then $F(z)$ is prime.*

Theorem 11 *Let $F(z)=z+h(e^z)+Q(e^{az})$, where h is entire with $\rho(h)<1$ and Q is a (non-constant) polynomial, then $F(z)$ is prime.*

Theorem 12 *Let $F(z)=Q(e^z)+z\cdot\exp[z+P(e^z)]$ with polynomials P and Q. Then $F(z)$ is prime.*

Theorem 13 *Let $F(z)=(z+H_1(z))e^{H_2(z)}$, where $H_j(\neq$ const.) is entire, periodic with period $2\pi i$ with $\rho(H_j)<\infty$ $(j=1, 2)$, then $F(z)$ is prime.*

For the proof of Theorem 9, we shall need the following lemma.

Lemma 10 ([12]) *Let $F(z)=h_1(z)+h_2(z)e^z$, where h_1 and $h_2(\not\equiv 0)$ are entire functions with $\rho(h_j)<1$ $(j=1, 2)$, then $F(z)$ is right-prime. Further if h_1 is non-constant, then F is prime.*

Proof of Theorem 9. Let $F=f(g)$ with non-linear entire functions

f and g. Then by Theorem 2, Lemma 6 and the proof of Lemma 5, we may assume that $f(z)=z+q(e^z)$ and $g(z)=z+Q(e^z)$, where $q(z)$ ($\neq$ const.) is entire and $Q(z)$ ($\neq$ const.) is a polynomial. From the relation $F=f(g)$, we have (cf. proof of Theorem 3)

$$q(ze^{Q(z)})=h_1(z)-Q(z)+h_2(z)e^z. \tag{36}$$

Here the right hand side of (36) is right-prime by Lemma 10, hence $q(z)$ is a polynomial. Further if $h_1-Q\neq$ const., then the right hand side of (36) is prime by Lemma 10, so that q must be linear. But in this case, a special case of Lemma 9 leads to a contradiction. If $h_1-Q=$const. ($=c_1$), noting that Q is a linear polynomial (which will be clear from (36)), we have that h_1 is linear and, using Borel's theorem (cf. [31] p. 279), we obtain that $q(z)=z^m+c_1$, hence $h_2(z)=c_2z^m$, for some constants c_1 and $c_2\neq 0$, and for some positive integer m. Thus, if F is not prime, then h_1 is linear and $h_2(z)=c_2z^m$, which is to be proved.

The proof of Theorem 10 can be done, using Theorem 2 and the following Lemma.

Lemma 11 ([34]) *Let $F(z)=h(e^z)+Q(z)$, where h($\neq$ const.) is entire with $\rho(h(e^z))<\infty$ and Q is a non-constant polynomial, then $F(z)$ is prime.*

The proof of Lemma 11 (in [34]) shows essentially the following fact, which is a conjecture of Gross ([16] Conjecture 2).

Lemma 11′ *Let $F(z)=H(z)+Q(z)$, where H($\neq$const.) is an entire function of finite lower order which is periodic with period $2\pi i$ and $Q(z)$ is a non-constant polynomial. Then $F(z)$ is left-prime. Further if $Q(z)$ has no quadratic right factor, then $F(z)$ is prime.*

Indeed, if $F=f(g)$ with transcendental entire functions h and g, then by Lemma 1 we have $\underline{\rho}(h)=0$. Then for any $\varepsilon>0$, there exists a sequence $\{r_n\}_1^\infty$, $r_n>0$ and $r_n\to\infty\,(n\to\infty)$ such that $\check{m}(r_n, h)\geqq M(r_n/2, h)^{1/3-\varepsilon}\,(n\geqq 1)$ even in the case $\underline{\rho}(h)=0$ (noting Lemma 2)*). Therefore from the argument in [34] $F(z)$ must be pseudo-prime and further we can get the above conclusion.

Proof of Theorem 10. Let $F=f(g)$ with non-linear entire functions

*) cf. Ostrovskii I.V. : On defects of meromorphic functions with lower order less than one, Soviet Math. Dokl. 4 (1963) 587-591.

f and g. Then from $F\in\boldsymbol{J}(2\pi i)$ we may write as before, $f(z)=z+q(e^z)$ and $g(z)=z+R(e^z)$, where $q(\neq\text{const.})$ is entire and $R(\neq\text{const.})$ is a polynomial.*) From the relation $F=f(g)$, we obtain the identical relation

$$(37)\qquad q(ze^{R(z)})=Q(z)-R(z)+h(e^z).$$

If $Q(z)-R(z)\neq\text{const.}$, by Lemma 11′ the right hand side of (37) is leftprime, so $q(z)$ must be linear. Then we have from (37) that the function $(z+2\pi i)e^{R(z+2\pi i)}-ze^{R(z)}$ is a polynomial, which will be clearly impossible (cf. Lemma 9).

Assume that $Q(z)-R(z)=\text{const.}$ Then, noting $\rho(q)=0$ by Lemma 1, we can deduce a contradiction similarly as in the proof of Theorem 3. Since F is non-periodic, $F(z)$ must be prime.

The proof of Theorem 11 can be done quite similarly as that of Theorem 10, if we use the following lemma (hence omitted).

Lemma 12 ([34]) *Let $F(z)=h(z)+Q(e^z)$, where $h(z)$ is a non-constant entire function with $\rho(h)<1$ and $Q(z)$ is a non-constant polynomial. Then $F(z)$ is prime.*

Proof of Theorem 12. Letting $F=f(g)$ with non-linear entire functions f and g, since $F\in\boldsymbol{L}(2\pi i)$, by Theorem 2 we can write $f(z)=H_3(z)+z\cdot\exp(H_4(z))$ and $g(z)=z+H_5(z)$, where entire functions H_3, $\exp(H_4)$ and H_5 have period $2\pi i$. From $F=f(g)$ we can conclude as before that $z+P(e^z)=H_4(z+H_5(z))$. Since the left hand side of this is prime (Corollary 1), we may have $H_4(z)=z$ and $H_5(z)=P(e^z)$. Hence we obtain again from $F=f(g)$ that $H_3(z+P(e^z))+P(e^z)\exp[z+P(e^z)]=Q(e^z)$. Writing $H_3(z)=h(e^z)$ for some holomorphic function $h(z)$ in $0<|z|<\infty$, then the above relation reduces to

$$(38)\qquad h(ze^{P(z)})=-zP(z)e^{P(z)}+Q(z)\quad(z\neq0).$$

By the proof of Lemma 6, we conclude for (38) that $h(z)$ is entire. By Lemma 1, $\rho(h)=0$. Further we conclude from (38) that $h(z)$ must be a polynomial, using the argument in the proof of Theorem 3. (Note also that the right hand side of (38) is known to be pseudo-prime by Goldstein's theorem, [12] Theorem 1.) Then by Lemma 9 we can deduce a contradiction.

Proof of Theorem 13. Let $F(z)=f(g(z))$, where f and g are non-linear entire functions. Since $F\in\boldsymbol{L}(2\pi i)$ and $\rho^*(F)<\infty$, $f(z)$ and $g(z)$ can be written as $f(z)=z\cdot\exp[H_3(z)]$ and $g(z)=z+H_4(z)$, where entire functions $\exp(H_3)$ and H_4 have period $2\pi i$. From $F=f(g)$, we

*) Note that Lemmas 4 and 5 hold under certain weaker conditions.

have $H_3(z+H_4(z))=H_2(z)$. Since $\rho(H_2)<\infty$ and $g(z)=z+H_4(z)$ is necessarily transcendental, by Lemmas 1 and 3 this identity is possible only when $H_3(z)$ is linear, but in this case $H_2(z)=H_3(z+H_4(z))$ cannot be periodic. This contradiction shows that $F(z)$ is prime in entire sense. As $F(z)$ is not periodic, $F(z)$ is prime.

§5. Certain Results on the Periodicity

Inspired by the question of Gross [13]: Whether or not a non-constant entire function $f(z)$ is periodic when $f(f(z))$ is so, the author studied the periodicity of entire functions and obtained several results in [33], using the method which is closely connected with the argument in the proof of Theorem 1. Then we had to put the assumption that $f(z)$ is of finite order for getting a result from some moderate condition ([33] Theorem 2). In this section, we shall show that we can prove the same conclusion as Theorem 2 in [33], even when the order of $f(z)$ is not necessarily finite. The result is stated as follows.

Theorem 14 *Let $f(z)$ be a transcendental entire function. If there exists a sequence of positive integers $\{k_n\}_1^N$ (N: finite or infinite) such that for each z, at least one of $f^{(k_n)}(z)$ (k_n-th derivative of $f(z)$) is not zero, and $f^{(k_n-1)}(f(z))$ has the common period ($1\leqq n\leqq N$), then $f(z)$ or $f'(z)$ is necessarily periodic. ($f^{(0)}(z)=f(z)$). In fact, $f(z)$ can be written as $f(z)=cz+H(z)$, where c is a constant and $H(z)$ is a periodic entire function.*

Remark 1. The assumption that for each, z, at least one of the k_n-th derivative of $f(z)$ ($n=1, 2, \ldots$) is not zero, is always valid when $\{k_n\}_1^N$ is taken as the set $\{m;\ \text{integer}\geqq m_0\}$ for some positive integer m_0.

Remark 2. Take $f(z)=z+e_m(z)$ ($m\geqq 1$, integer), where $e_m(z)=\exp[e_{m-1}(z)]$, $e_0(z)=z$. Then $f^{(k)}(f(z))$ is periodic with period $2\pi i$ for any integer $k\geqq 1$. In this case, $f(z)$ is not periodic while $f'(z)$ is periodic. Thus Theorem 14 is best possible in this formulation.

We shall give here an outline of the proof of Theorem 14, which will be sufficient. We can assume that the common period is equal to b. Further by the assumption that for each z, at least one of $f^{(k_n)}(z)$ is not zero and $f^{(k_n-1)}(f(z))$ is periodic with period b ($n=1, \ldots, N$), we may assume that the functions $[f(z+b)-f(z)]$ and $[f(z+2b)-f(z)]$ have no zeros, otherwise by the unicity theorem for holomorphic functions $f(z)$ becomes periodic with period b or $2b$. Hence by the argument in the proof of Theorem 1, we have only to consider the following two cases.

(6″) $f(z)=H_1(z)+e^{H_2(z)+cz}$, if $e^{bc}\neq 1$,

(7″) $f(z)=H_1(z)+z\cdot e^{H_2(z)+cz}$, if $e^{bc}=1$,

where entire functions $H_j(z)$ $(j=1, 2)$ have period b.

In case (6″), we can proceed to rule out this case quite similarly as in [33] (cf. the first step of the proof of Theorem 1). In fact, we have the identity

$$g(f(z)+(\exp[ncb]-1)\cdot\exp[cz+H_2(z)])=g(f(z)) \quad (n=0, \pm 1, \ldots),$$

for some transcendental entire function $g(z)$. From the sequence {exp (ncb) ; n is any integer} $(\exp(ncb)\neq 1)$, taking a subsequence which converges, we conclude by the unicity theorem that $g(z)$ is a constant, which is clearly impossible.

In case (7″), $f(z+nb)=f(z)+nb\cdot\exp[cz+H_2(z)]$. By the assumption we have the identity, for some transcendental entire function $g(z)$,

$$g(f(z)+nb\cdot e^{cz+H_2(z)})=g(f(z)) \quad (n\text{: any integer}).$$

In this case, if $cz+H_2(z)$ is not constant, applying the argument in the proof of Theorem 1 (cf. the second step), we can conclude that $M(r, g)=O(1)$, hence $g(z)$ must be constant, contrary to the transcendency of $g(z)$. This contradiction shows that $cz+H_2(z)$ is a constant. Hence $f(z)=c_1z+H_1(z)$ for some constant c_1, so that $f'(z)=c_1+H'_1(z)$ is periodic with period b.

Remark. Let $f(z)=\cos z$, then $f(f(z))$ has period π, but $f(z)$ has period 2π.

Finally, we note the following result of Gross [14] as a Gorollary.

Corollary 11 *Let $f(z)$ be a transcendental entire function such that $f'(z)$ does not vanish. Assume $f(f(z))$ is periodic, then $f(z)$ is necessarily so.*

Proof. Assume $f(f(z))$ has period b $(\neq 0)$. Since $f'(z)$ does not vanish, the conditions in Theorem 14 are satisfied when $N=1$ and $k_1=1$. Then from Theorem 14 it is enough to consider the case: $f(z)=cz+H(z)$, where c is a constant and $H(z)$ is entire with $H(z+b)=H(z)$. We have to show $c=0$. By the periodicity of $f(f(z))$ and f'

$(z) = c + H'(z)$, $f'(f(z))$ is also periodic with period b, hence $H'(cz + H(z))$ is so. Since non-constant entire functions cannot be doubly periodic, we conclude that c is a rational number. Now $f(f(z)) = c^2z + cH(z) + H(cz + H(z))$, and note here that $cH(z) + H(cz + H(z))$ is periodic (with period mb for some non-zero integer m, since c is a rational number and $H(z)$ has period b). This shows that, if $c \neq 0$, $f(f(z))$ cannot be periodic. Hence $c = 0$, which means that $f(z)$ is periodic. Thus we have done.

KYOTO UNIV. OF EDUCATION
FUSHIMI-KU, KYOTO, 612
JAPAN

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