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Notes on induced maps of Moore families

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Let $\tilde{S}$ and $\tilde{T}$ be Moore families on S and T respectively. Then a map $F: S\longrightarrow T$ induces maps $F_*: \tilde{S}\longrightarrow\tilde{T}$ and $F^*: \tilde{T}\longrightarrow\tilde{S}$. We study the lattice theoretic properties of these maps. In the latter half we treat inductive limits which will be applied in [3] to study ideals of germs of functions and their 'zero filters'.

Lattices of Moore families

A lattice L is called complete if it contains the least upper bound $\bigvee_{\lambda\in\Lambda} a_\lambda$ and the greatest lower bound $\bigwedge_{\lambda\in\Lambda} a_\lambda$ for any subset $\{a_\lambda\}_{\lambda\in\Lambda}\subset L$. Let $\mathfrak{m}$ be a cardinal number greater than 1 and let $\Phi: L\longrightarrow L'$ be a map of complete lattices. We define Φ to be an $(\mathfrak{m}\vee)$-*morphism* (or to be $(\mathfrak{m}\vee)$-*continuous*) if $\Phi(\bigvee_{\lambda\in\Lambda} a_\lambda)=\bigvee_{\lambda\in\Lambda}\Phi(a_\lambda)$ holds for any subset $\{a_\lambda\}_{\lambda\in\Lambda}\subset L$ such that $\sharp\Lambda\leqq\mathfrak{m}$. If L is $(\mathfrak{m}\vee)$-continuous for any $\mathfrak{m}\geqq 2$ we define it an $(\forall\vee)$-*morphism* (or $(\forall\vee)$-*continuous*). For the sake of convenience, we call an order preserving map a $(1\vee)$-*morphism* (or $(1\vee)$-*continuous*). Dually we can define $(\mathfrak{m}\wedge)$-*morphisms.* If Φ is both $(\mathfrak{m}\vee)$-continuous and $(\mathfrak{n}\wedge)$-continuous we call it an $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-*morphism* (or $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-*continuous*).

Let S be a set and $\tilde{S}=\{X_\lambda\}_{\lambda\in\Lambda}$ be a subfamily of the family $P(S)$ of all subsets of S. $\tilde{S}$ is called a *Moore family* on S if it contains S and $\bigcap_{\lambda\in K} X_\lambda$ for any $K\subset\Lambda$ (cf. [1]). A Moore family forms a complete lattice with respect to the order of inclusion. Let $c: P(S)\longrightarrow\tilde{S}$ be the associated *closure operation* defined by $c(X)=\bigcap_{\substack{Y\supset X\\ Y\in\tilde{S}}} Y$. If $Y=c(X)$ we say that X (or its elements) *generates* Y. If an element of $\tilde{S}$ is generated by one point of S, it is called *principal.* Let $\tilde{S}$ and $\tilde{T}$ be Moore families on S and T respectively and $\varphi : S \longrightarrow T$ be a map. We define the *direct induced map* (or ideal map) $\varphi_* : \tilde{S} \longrightarrow \tilde{T}$ and the *inverse one* $\varphi^* : \tilde{T} \longrightarrow \tilde{S}$ by $\varphi^*(X) = c\{\varphi(X)\}$, $\varphi^*(Y) = c\{\varphi^{-1}(Y)\}$. Of course these are order-preserving $((1\vee)$-continuous). Let us consider the following conditions about φ :

(a) $\varphi^{-1}(Y) \in \tilde{S}$ *(i.e.* $\varphi^*(Y) = \varphi^{-1}(Y)$ *for any* $Y \in \tilde{T}$).

(a′) $\varphi(X) \in \tilde{T}$ *(i.e.* $\varphi_*(X) = \varphi(X)$ *for any* $X \in \tilde{S}$).

(b) $\varphi^* \circ \varphi_*(X) = X \vee \varphi^*(\mathrm{O})$ *for any* $X \in S$, *where* O *denotes the minimal element of* $\tilde{T}$.

(a) is a fairly natural condition: it is satisfied in many practical cases.

1 Lemma. *Let* $\varphi: S \longrightarrow T$ *and* $\psi: R \longrightarrow S$ *be maps of sets with Moore families. Suppose that* φ *satisfies* (a). *Then we have the following:*

(i) $c\{\varphi(A)\} = c\{\varphi(c(A))\}$ *for any* $A \subset S$.

(ii) $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$, $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.

(iii) $\varphi_* \circ \varphi^*(\mathrm{O}) = \mathrm{O}$, $\varphi_*(\mathrm{O}) = \mathrm{O}$, $\psi^* \circ \psi_*(R) = R$, $\psi^*(S) = R$.

(iv) *If* φ *is surjective,* φ_* *is also so.*

(v) *If* φ_* *is surjective,* $\varphi_* \circ \varphi^*$ *is the identity and hence* $\varphi_*(S) = T$.

Proof. (i) $\varphi^{-1}(c\{\varphi(A)\}) = c\{\varphi^{-1}(c\{\varphi(A)\})\} \supset c(A)$. Hence $c\{\varphi(A)\} = c \circ c\{\varphi(A)\} \supset c\{\varphi\{c(A)\}\} \supset c\{\varphi(A)\}$ and $c\{\varphi(A)\} = c\{\varphi(c(A))\}$.

(ii), (iii), (iv) We omit the proofs.

(v) If $Y \in \tilde{T}$ there exists $X \in \tilde{S}$ such that $\varphi_*(X) = Y$. Then we have

$$Y \supset c\{\varphi(\varphi^{-1}(Y))\} = \varphi_* \circ \varphi^*(Y) = \varphi_* \circ \varphi^* \circ \varphi_*(X) \supset \varphi_*(X) = \mathrm{Y},$$

proving that $\varphi_* \circ \varphi^*$ is the identity, q. e. d.

2 Theorem. *Let* $\varphi : S \longrightarrow T$ *be a map of sets with Moore families.*

(i) *If* φ *satisfies (a),* φ_* *is* $(\forall \vee)$*-continuous and* φ^* *is* $(\forall \wedge)$*-continuous.*

(ii) *If* φ *satisfies* (a), (b) *and if* φ_* *is surjective then* φ^* *is* $(\forall \vee, \forall \wedge)$*-continuous.* φ_* *and* φ^* *induces mutually inverse* $(\forall \vee, \forall \wedge)$*-morphisms between* $\tilde{T}$ *and* $\tilde{S}/\varphi^*(\mathrm{O}) = \{X \vee \varphi^*(\mathrm{O}) : X \in \tilde{S}\}$.

(iii) *Suppose that* φ *satisfies* (a) *and* (b), φ_* *is surjective and that* $\varphi^*(\mathrm{O}) \vee (\bigwedge_\lambda X_\lambda) = \bigwedge_\lambda \{\varphi^*(\mathrm{O}) \vee X_\lambda\}$ *holds for any* $\{X_\lambda\} \subset \tilde{S}$ *such that* $\# \{X_\lambda\} \leqq \mathfrak{m}$. *Then* φ_* *is* $(\forall \vee, \mathfrak{m} \wedge)$*-continuous.*

Remark. If L is a complete lattice and if $a \in L$, $L/a = \{x \vee a : x \in L\}$ is a complete lattice with respect to the induced order. Its $\vee a_\lambda$ and $\wedge a_\lambda$ coincide with those in L.

Proof. (i) If $\{X_\lambda\}\subset\tilde{S}$ and $\{Y_\lambda\}\subset\tilde{T}$,

$$\varphi_*(\bigvee X_\lambda)\supset\bigvee\varphi_*(X_\lambda)\supset\varphi_*\circ\varphi^*(\bigvee\varphi_*(X_\lambda))\supset\varphi_*(\bigvee\varphi^*\circ\varphi_*(X_\lambda))\supset \varphi_*(\bigvee X_\lambda),$$

$$\varphi^*(\bigwedge Y_\lambda)\subset\bigwedge\varphi^*(Y_\lambda)\subset\varphi^*\circ\varphi_*(\bigwedge\varphi^*(Y_\lambda))\subset\varphi^*\circ(\bigwedge\varphi_*\circ\varphi^*(Y_\lambda))\subset \varphi^*(\bigwedge Y_\lambda).$$

These prove the assertions.
(ii) By (1 ; v), φ_* and φ^* induce mutually inverse order isomorphisms between T and $S/\varphi^*(O)$. It is easy to see that order isomorphisms are $(\forall\bigvee,\forall\bigwedge)$-continuous.
(iii) If $\#\{X_\lambda\}\leqq\mathfrak{m}$,

$$\varphi^*(\bigwedge\varphi_*(X_\lambda))=\bigwedge\varphi^*\circ\varphi_*(X_\lambda)=\bigwedge(X_\lambda\vee\varphi^*(O))$$
$$=(\bigwedge X_\lambda)\vee\varphi^*(O)=\varphi^*\circ\varphi_*(\bigwedge X_\lambda).$$

Hence $\bigwedge\varphi_*(X_\lambda)=\varphi_*(\bigwedge X_\lambda)$ by (1 ; v) *i.e.* φ is $(\mathfrak{m}\bigwedge)$-continuous,

q. e. d.

Let us call a Moore family $\tilde{S}$ *finitary* if its associated closure operation c is finitary *i.e.* $X\subset S$ belongs to $\tilde{S}$ if $c(Y)\subset X$ for any finite subset Y of X. The following is known (cf. [1, VIII, § 4]) :

3 Lemma. (i) *If $\tilde{S}$ is finitary and if $\{X_\lambda\}\subset\tilde{S}$ is a directed subset, $\bigcup_\lambda X_\lambda\in\tilde{S}$.*
(ii) *If $\tilde{S}$ is finitary and if $x\in c(A)$ there exists a finite subset F of A such that $x\in c(F)$.*

4. Proposition. *Let $\varphi: S\longrightarrow T$ satisfy (a) and (a').*
(i) *If φ is surjective and if $\tilde{S}$ is finitary, $\tilde{T}$ is finitary.*
(ii) *If φ is injective and if $\tilde{T}$ is finitary, $\tilde{S}$ is finitary.*

The proof is easy.

Example 1. Let $\tilde{E}$ be the set of ideals of a commutative ring E with unity 1. $\tilde{E}$ is a finitary Moore family. Suppose that $\varphi: E\longrightarrow F$ is a unitary $(\varphi(1)=1)$ ring homomorphism. Then we have the following.
(i) φ satisfies *(a)*.
(ii) If φ is surjective it satisfies *(a)*, *(a')* and *(b)*.
(iii) If F is flat over E, φ_* is $(\forall\bigvee, 2\bigwedge)$-continuous (cf. [2]).

Example 2. The set $\tilde{L}$ of ideals of a lattice L is a finitary Moore family. A $(2\bigvee)$-morphism of lattices $\varphi: L\longrightarrow K$ satisfies *(a)*. A surjective $(2\bigwedge)$-morphism satisfies *(a')*.

Example 3. Let $\tilde{P}(A)$ be the family of nonvoid dual ideals of the complete Boolean lattice $P(A)$ of A (cf. [1]). Of course this is a finitary Moore family on $P(A)$. It is just the family of filters on A except the maximal element $P(A)\in\tilde{P}(A)$. If $f: B\longrightarrow A$ is a map we can define a map $\varphi=\varphi_f: P(A)\longrightarrow P(B)$ by $\varphi(\cdot)=f^{-1}(\cdot)$. φ satisfies (a) and (b). If $c\{x\}$ is a principal dual ideal of $P(A)$ we have $\bigwedge_\lambda(c\{x\}\vee X_\lambda)=c\{x\}\vee(\bigwedge_\lambda X_\lambda)$ for any $\{X_\lambda\}\subset\tilde{P}(A)$. Since $O_{\tilde{P}(B)}=\{B\}$ and $\varphi^*(O_{\tilde{P}(B)})=c\{f(B)\}$ is principal, $\bigwedge_\lambda\{\varphi^*(O_{\tilde{P}(B)})\vee X_\lambda\}=\varphi^*(O_{\tilde{P}(B)})\vee(\bigwedge_\lambda X_\lambda)$.† Now suppose that f is injective. Then φ is surjective (and satisfies (a') also). Hence φ_* and φ^* are $(\forall\vee, \forall\wedge)$-continuous in this case.

Inductive limits of sets with Moore families.

In the first section we have studied the importance of the condition (a). Here we treat inductive systems in the *category* $\mathcal{M}$ whose objects are sets with Moore families and whose morphisms are maps satisfying (a). It is easy to see that a morphism is an epimorphism (resp. a monomorphism) if it is set-theoretically so. By (1 ; ii) the correspondences $(S, \varphi)\longrightarrow(\tilde{S}, \varphi_*)$ and $(S, \varphi)\longrightarrow(\tilde{S}, \varphi^*)$ are respectively a covariant and a contravariant functor from $\mathcal{M}$ into the category of ordered sets (suitably defined). We always assume that the index set Λ of an inductive system is a *directed set.*

5. Theorem. ‡ *An inductive system* $\{S_\lambda, \varphi_{\mu\lambda}\}$ *in* $\mathcal{M}$ *has an inductive limit* $\lim S_\mu$ *unique up to isomorphism. That is, if* $\varphi_\lambda: S_\lambda\longrightarrow\lim S_\mu$ *are the set-theoretical inductive maps, there exists a Moore family* $(\lim S_\mu)^\sim$ *on* $\lim S_\mu$ *such that*:

(i) φ_λ *are morphisms.*

(ii) *There exists a unique morphism* $\lim\psi_\mu: \lim S_\mu\longrightarrow T$ *with* $\lim\psi_\mu\circ\varphi_\lambda=\psi_\lambda$ *for any given system* $\{\psi_\mu: S_\mu\longrightarrow T\}$ *of morphisms satisfying* $\psi_\mu\circ\varphi_{\mu\lambda}=\psi_\lambda$.

Proof. We have only to put

$$(\lim S_\mu)^\sim=\{X\in P(\lim S_\mu): \varphi_\lambda^{-1}(X)\in\tilde{S}_\lambda \text{ for any } \lambda\in\Lambda\}.$$

Remark. If $M\subset\Lambda$ is a cofinal set,

$$(\lim S_\mu)^\sim=\{x\in P(\lim S_\mu): \varphi_\lambda^{-1}(X)\in\tilde{S}_\lambda \text{ for any } \lambda\in M\}.$$

6 Proposition. *Suppose that all* $\tilde{S}_\lambda$ *are finitary. Then we have the following*:

(i) $\varphi_{\lambda *}(X)=\bigcup_{\mu\geqq\lambda}\varphi_{\mu}(\varphi_{\mu\lambda *}(X))$.

(ii) $\varphi_{\lambda'}^{*}\circ\varphi_{\lambda *}(X)=\bigcup_{\mu\geqq\lambda,\lambda'}\varphi_{\mu\lambda'}^{-1}(\varphi_{\mu\lambda *}(X))$.

(iii) *If all* $\varphi_{\mu\lambda}$ *satisfy* (*b*), φ_{λ} *do also so.*

Proof. (i) Let us put $\bigcup_{\mu\geqq\lambda}\varphi_{\mu}(\varphi_{\mu\lambda *}(X))=A$. Then

$$\varphi_{\lambda'}^{-1}(A)=\bigcup_{\mu\geqq\lambda}\varphi_{\lambda'}^{-1}\circ\varphi_{\mu}(\varphi_{\mu\lambda *}(X))=\bigcup_{\mu\geqq\lambda}\ \bigcup_{\nu\geqq\mu,\lambda'}\varphi_{\nu\lambda'}^{-1}\circ\varphi_{\nu\mu}(\varphi_{\mu\lambda *}(X))$$
$$\subset\bigcup_{\nu\geqq\lambda,\lambda'}\varphi_{\nu\lambda'}^{-1}(\varphi_{\nu\lambda *}(X))\subset\bigcup_{\nu\geqq\lambda,\lambda'}\varphi_{\lambda'}^{-1}\circ\varphi_{\nu}\circ\varphi_{\nu\lambda'}\circ\varphi_{\nu\lambda'}^{-1}(\varphi_{\nu\lambda *}(X))$$
$$\subset\bigcup_{\nu\geqq\lambda,\lambda'}\varphi_{\lambda'}^{-1}\circ\varphi_{\nu}(\varphi_{\nu\lambda *}(X))=\varphi_{\lambda'}^{-1}(\bigcup_{\nu\geqq\lambda,\lambda'}\varphi_{\nu}(\varphi_{\nu\lambda *}(X)))\subset\varphi_{\lambda'}^{-1}(A).$$

Hence $\varphi_{\lambda'}^{-1}(A)=\bigcup_{\nu\geqq\lambda,\lambda'}\varphi_{\nu\lambda'}^{-1}(\varphi_{\nu\lambda *}(X))\in\tilde{S}_{\lambda'}$ by (3). Then $A\in(\lim S_{\mu})^{\sim}$.
Since $\varphi_{\lambda}(X)\subset A\subset\varphi_{\lambda *}(X)$, we have $A=\varphi_{\lambda *}(X)$.

(ii) is obvious from the above calculation.

(iii) Since $\varphi_{\mu\lambda *}(\mathrm{O}_{\lambda})=\mathrm{O}_{\mu}$ and $\varphi_{\lambda *}(\mathrm{O}_{\lambda})=\mathrm{O}$, we have

$$\varphi_{\lambda}^{*}\circ\varphi_{\lambda *}(X)=\bigvee_{\mu\geqq\lambda}\varphi_{\mu\lambda}^{*}\circ\varphi_{\mu\lambda *}(X)=\bigvee_{\mu\geqq\lambda}(X\vee\varphi_{\mu\lambda}^{*}(\mathrm{O}_{\mu}))$$
$$=X\vee(\bigvee_{\mu\geqq\lambda}\varphi_{\mu\lambda}^{*}(\mathrm{O}_{\mu}))=X\vee(\bigvee_{\mu\geqq\lambda}\varphi_{\mu\lambda}^{*}\circ\varphi_{\mu\lambda *}(\mathrm{O}_{\lambda}))$$
$$=X\vee\varphi_{\lambda}^{*}(\mathrm{O}),\qquad\text{q. e. d.}$$

For the application in [3], we consider the following condition:

(C) (i) $\varphi_{\mu\lambda *}(S_{\lambda})=S_{\mu}$ *for any* $\mu\geqq\lambda$.

(ii) $\varphi_{\lambda}(X)=\varphi_{\mu}(\varphi_{\mu\lambda *}(X))$ *for any* $\mu\geqq\lambda$ *and* $X\in\tilde{S}_{\lambda}$.

(ii) is equivalent to the following:

(ii)′ *If* $X\in\tilde{S}_{\lambda}$ *and* $b\in\varphi_{\mu\lambda *}(X)$, *there exist* $\nu\geqq\mu$ *and* $a\in X$ *such that* $\varphi_{\nu\lambda}(a)=\varphi_{\nu\mu}(b)$.

7 **Proposition.** *If* $\{S_{\lambda},\varphi_{\mu\lambda}\}$ *satisfies* (*C*) *and if all* $\tilde{S}_{\lambda}$ *are finitary, then* φ_{λ} *are epimorphisms,* φ_{λ} *satisfy* (a′) *and* $(\lim S_{\mu})^{\sim}$ *is also finitary.*

Proof. Obvious from (6; i) and (4; i).

Let $\{S_{\lambda},\varphi_{\mu\lambda}\}$ and $\{T_{\lambda},\psi_{\mu\lambda}\}$ be inductive systems in $\mathscr{M}$ and $\zeta_{\lambda}:S_{\lambda}\longrightarrow T_{\lambda}$ be morphisms satisfying $\zeta_{\mu}\circ\varphi_{\mu}=\psi_{\lambda\mu}\circ\zeta_{\lambda}$ for any $\mu\geqq\lambda$.

8 **Theorem.** (i) $(\lim\zeta_{\mu})_{*}\circ\varphi_{\lambda *}=\psi_{\lambda *}\circ\zeta_{\lambda *}$, $\varphi_{\lambda}^{*}\circ(\lim\zeta_{\mu})^{*}=\zeta_{\lambda}^{*}\circ\psi_{\lambda}^{*}$.

(ii) *If* $\varphi_{\lambda *}$ *is surjective,*

$$(\lim\zeta_{\mu})_{*}=\psi_{\lambda *}\circ\zeta_{\lambda *}\circ\varphi_{\lambda}^{*},\quad(\lim\zeta_{\mu})^{*}=\varphi_{\lambda *}\circ\zeta_{\lambda}^{*}\circ\psi_{\lambda}^{*}.$$

(iii) *Suppose that all* $\tilde{T}_{\mu}$ *are finitary,* ψ_{λ} *satisfies* (*b*), $\varphi_{\lambda *}$ *is surjective and that* $\zeta_{\nu}^{*}\circ\psi_{\nu\lambda *}(Y)\subset\varphi_{\nu\lambda *}\circ\zeta_{\lambda}^{*}(Y)$ *for all* $\nu\geqq\lambda$ *(or for all* $\nu\geqq\lambda$ *of a cofinal*

subset of Λ). *Then* $(\lim \zeta_\mu)^* \circ \psi_{\lambda *} = \varphi_{\lambda *} \circ \zeta_\lambda^*$.
(iv) *Suppose that all* $\tilde{S}_\mu$ *and* $\tilde{T}_\mu$ *are finitary,* φ_λ *and* ψ_λ *satisfy* (b) *and that* $\varphi_{\lambda *}$ *and* $\psi_{\lambda *}$ *are surjective. If* $\zeta_{\lambda *} \circ \varphi_{\nu\lambda}^*(\mathrm{O}_\nu) \supset \psi_{\nu\lambda}^*(\mathrm{O}_\nu)$ *holds for any* $\nu \geqq \lambda$ *we have* $\psi_\lambda^* \circ (\lim \zeta_\mu)_* = \zeta_{\lambda *} \circ \varphi_\lambda^*$. *Moreover if* $\zeta_{\lambda *}$ *is* $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-*continuous,* $(\lim \zeta_\mu)_*$ *is also so.*
(v) *Suppose the same as the first sentence of* (iv) *and that* $\zeta_\lambda^* \circ \psi_{\nu\lambda}^*(\mathrm{O}_\nu) \supset \varphi_{\nu\lambda}^*(\mathrm{O}_\nu)$ *holds for any* $\nu \geqq \lambda$. *If* ζ_λ^* *is* $(\mathfrak{m}\vee, \mathfrak{n}\wedge)$-*continuous,* $(\lim \zeta_\mu)^*$ *is also so.*

Proof. (i) is obvious from (1; ii).
(ii) $(\lim \zeta_\mu)_* = (\lim \zeta_\mu)_* \circ \varphi_{\lambda *} \circ \varphi_\lambda^* = \psi_{\lambda *} \circ \zeta_{\lambda *} \circ \varphi_\lambda^*$.
$(\lim \zeta_\mu)^* = \varphi_{\lambda *} \circ \varphi_\lambda^* \circ (\lim \zeta_\mu)^* = \varphi_{\lambda *} \circ \zeta_\lambda^* \circ \psi_\lambda^*$.
(iii) $(\lim \zeta_\mu)^* \circ \psi_{\lambda *}(Y) = \psi_{\lambda *} \circ \zeta_\lambda^*(Y \vee \psi_\lambda^*(\mathrm{O})) = c\{\varphi_\lambda \circ \zeta_\lambda^{-1}(c\{Y \cup \psi_\lambda^{-1}(\mathrm{O})\})\}$.
If $x \in \varphi_\lambda \circ \zeta_\lambda^{-1}(c\{Y \cup \psi_\lambda^{-1}(\mathrm{O})\})$, $x \in \varphi_\lambda \circ \zeta_\lambda^{-1}(c\{Y \cup F\})$ for some finite subset $F \subset \psi_\lambda^{-1}(\mathrm{O})$ by (3). Then

$$x \in \varphi_\nu \circ \varphi_{\nu\lambda} \circ \zeta_\lambda^{-1}(c\{Y \cup F\}) \subset \varphi_\nu \circ \zeta_\nu^{-1} \circ \psi_{\nu\lambda}(c\{Y \cup F\})$$
$$\subset \varphi_{\nu *} \circ \zeta_\nu^* \circ c\{\psi_{\nu\lambda}(Y \cup F)\}.$$

Since $F \subset \psi_\lambda^* \circ \psi_{\lambda *}(\mathrm{O}_\lambda) = \bigcup_{\nu \geq \lambda} \psi_{\nu\lambda}^*(\mathrm{O}_\nu)$, $F \subset \psi_{\nu\lambda}^*(\mathrm{O})$ for some ν. Then $\psi_{\nu\lambda}(F) \subset \psi_{\nu\lambda *} \circ \psi_{\nu\lambda}^*(\mathrm{O}_\nu) = \mathrm{O}_\nu \subset \psi_{\nu\lambda *}(Y)$. Hence

$$x \in \varphi_{\nu *} \circ \zeta_\nu^* \circ \psi_{\nu\lambda *}(Y) \subset \varphi_{\nu *} \circ \varphi_{\nu\lambda *} \circ \zeta_\lambda^*(Y) = \varphi_{\lambda *} \circ \zeta_\lambda^*(Y).$$

This proves that $(\lim \zeta_\mu)^* \circ \psi_{\lambda *}(Y) \subset \varphi_{\lambda *} \circ \zeta_\lambda^*(Y)$. The converse inclusion is obvious.
(iv) Since

$$\zeta_{\lambda *} \circ \varphi_\lambda^*(X) \supset \zeta_{\lambda *} \circ \varphi_\lambda^*(\mathrm{O}) = \zeta_{\lambda *}(\bigcup_{\nu \geq \lambda} \varphi_{\nu\lambda}^{-1}(\mathrm{O}_\nu)) \supset \bigcup_{\nu \geq \lambda} \zeta_{\lambda *} \circ \varphi_{\lambda\nu}^*(\mathrm{O}_\nu)$$
$$\supset \bigcup_{\nu \geq \lambda} \psi_{\nu\lambda}^*(\mathrm{O}_\nu) = \psi_\lambda^*(\mathrm{O})$$

by (6), we have

$$\psi_\lambda^* \circ (\lim \zeta_\mu)_*(X) = \psi_\lambda^* \circ \psi_{\lambda *} \circ \zeta_{\lambda *} \circ \varphi_\lambda^*(X) = \zeta_{\lambda *} \circ \varphi_\lambda^*(X) \vee \psi_\lambda^*(\mathrm{O})$$
$$= \zeta_{\lambda *} \circ \varphi_\lambda^*(X).$$

The second assertion follows from (2; ii) and (8; ii).
(v) is quite similar to (iv), q. e. d.

9 Corollary. *Suppose that* φ_λ *is an epimorphism and* ζ_λ *and* ψ_λ *satisfy* (a′). *Then* $\lim \zeta_\mu$ *satisfies* (a′) *also.*

Notes

† If a finitary Moore family forms a distributive lattice, it is Brouwerian (a generalization of the theorem of M. H. Stone, cf. [1; (V, 10)]). Hence $Y\vee(X_1\wedge X_2)=(Y\vee X_1)\wedge(Y\vee X_2)$, $Y\wedge(\bigvee_\lambda X_\lambda)=\bigvee_\lambda(Y\wedge X_\lambda)$ for any Y, X_1, X_2, $X_\lambda\in\tilde{P}(A)$.

‡ Let $\mathfrak{m}$ be a cardinal and $C(\mathfrak{m}\vee)$ be the category of $(\mathfrak{m}\vee)$-semilattice: the objects are ordered sets having $\vee a_\lambda$ for any subset $\{a_\lambda\}$ satisfying $\sharp\{a_\lambda\}\leqq\mathfrak{m}$ and the morphisms are $(\mathfrak{m}\vee)$-continuous maps. Any inductive system in $C(2\vee)$ has an inductive limit in it (false for infinite $\mathfrak{m}$). In our case $\lim \tilde{S}_\mu$ has the canonical structure of $(2\vee)$-semilattice and the canonical map θ: $\lim \tilde{S}_\mu\longrightarrow(\lim S_\mu)^\sim$ is $(2\vee)$-continuous.

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