J. Math. Kyoto Univ. (JAKYAZ)
18-1 (1978) 71-85

On the generalized local cohomology and its duality

By

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(Communicated by Prof. M. Nagata, April 2, 1976)

§ 0. Introduction.

Let $(R, \mathfrak{M}, K)$ be a commutative, Noetherian local ring with the non-zero multiplicative identity and all the modules considered be unitary throughout.

The main purpose of this paper is to study the generalized local cohomology $H^i_{\mathfrak{M}}(*, *)$ introduced by J. Herzog:

$$H^i_{\mathfrak{M}}(M, N) = \varinjlim_m \mathrm{Ext}^i_R(M/\mathfrak{M}^m M, N)$$

for R-modules M and N, [5] (1. 1. 1). This is in fact a generalized one of the usual local cohomology $H^i_{\mathfrak{M}}(*)$: for any R-module N, $H^i_{\mathfrak{m}}(R, N) = H^i_{\mathfrak{M}}(N)$.

As is well known, the vanishing (or non-vanishing) of the local cohomology module $H^i_{\mathfrak{M}}(N)$ of a finitely generated (abbreviated to f.-g. from now on) R-module N reflects some important character of N, say dimension and depth of N. It is quite reasonable to ask when the generalized local cohomology module $H^i_{\mathfrak{M}}(M, N)$ of R-modules M and N vanishes (or never.)

Our first result, Theorem (2. 3), states that the lower bound of i's for which $H^i_{\mathfrak{M}}(M, N) \neq 0$ (for f.-g. non-zero modules M and N over R) coincides with the $\mathrm{depth}_R N$. As to the upper bound, we must require some restrictions on either M or N: for all sufficiently large i's $H^i_{\mathfrak{M}}(M, N) = 0$ if and only if either $Pd_R(M) < \infty$ or $Id_R(N) < \infty$, (2. 4). Where Pd_R (resp. Id_R) denotes the *projective* (resp. *injective*) *dimension* over R. We shall mainly treat the case when $Pd_R(M) < \infty$ in this paper.

In the next step, we want to express the functor $H^i_{\mathfrak{M}}(*, *)$ by means of the well-known functors. For that purpose, the construction of the so called "*Duality theorem*" will be the objective of our work. One already has the duality theorem with respect to the (usual) local cohomology: *in case R is a Cohen-Macaulay (abbreviated to C.-M.) ring with* $\dim(R)=r$

$$H^i_{\mathfrak{M}}(N) \cong \mathrm{Hom}_{\hat{R}}(\mathrm{Ext}_{\hat{R}}^{r-i}(N, \Omega), I_R(k))$$

for all i and for any f.-g. $\hat{R}$-module N with Ω the module of dualizing differentials, i. e., $\Omega=(H^r_{\mathfrak{M}}(R))'$. Where $I_R(k)$ denotes the *injective envelope* of the residue field k and $(*)'$ is the functor taking the $I_R(k)$ *-dual* $\mathrm{Hom}_R(*, I_R(k))$ and $(\hat{*})$ is the functor taking the (maximal ideal adic) *completion* of modules over the local ring.

What we must attach importance is the fact that *the duality above holds for all i's only if R is* C.-M.. We then try to construct the duality theorem with respect to the functor $H^i_{\mathfrak{M}}(*, *)$ and introduce a new category which is a subcategory of the category of f.-g. R-modules in § 3.

The following is the key.

Theorem (3.5). *Let* $d=\mathrm{depth}(R)$ *and M be a f.-g. R-module then the conditions below are equivalent.*

(i) *There exists an integer $e \geqslant 0$ such that*

$$(H^i_{\mathfrak{M}}(M, N))' \cong \mathrm{Ext}_{\hat{R}}^{e-i}(\hat{N}, \Omega(M))$$

for all i and for any f.-g. R-module N where $\Omega(M)$ denotes $(H^e_{\mathfrak{M}}(M, R))'$.

(ii) $H^i_{\mathfrak{M}}(M, N)=0$ *for all $i>d$ and for any R-module N.*

If M satisfies the equivalent conditions above, $e=d$.

We shall denote by $\mathscr{D}(R)$ *the category of f.-g. R-modules M satisfying the conditions in the theorem* and make an investigation into the category. To begin with, $\mathscr{D}(R)$ *is the subcategory of the category* $\mathscr{P}(R)$ *of f.-g. R-modules of finite projective dimension*, (2. 4), and what is more important is the fact that *for any* $M \in \mathscr{D}(R)$

$$\Omega(M) \cong \hat{M} \otimes_{\hat{R}} \mathrm{D}^d_R(R) \in \mathscr{I}(\hat{R})$$

where $D^d_R(R)=(H^d_{\mathfrak{M}}(R))'$ with $d=\mathrm{depth}(R)$ and $\mathscr{I}(R)$ denotes *the category of f.-g. R-modules of finite injective dimension*, (3. 7) and (3. 11).

We shall show in the latter half of § **3** that $\mathscr{D}(R)$ has a quite simple characterization as below.

Theorem (**3.16**). *Let M be a f.-g. R-module and* depth$(R)=d$, *then the following conditions are equivalent.*

(i) $M\in\mathscr{D}(R)$.

(ii) $M\in\mathscr{P}(R)$ *and* $\mathrm{Supp}_{\hat{R}}(\hat{M})\subseteq\mathrm{Supp}_{\hat{R}}(D_R^d(R))$.

Furthermore, with the aid of the lemma (3.13) (ii) below

$$\mathrm{Supp}_{\hat{R}}(D_R^d(R))=\{\mathfrak{p}\in\mathrm{Spec}(\hat{R})\ ;\ \mathrm{depth}(\hat{R}_{\mathfrak{p}})+\dim(\hat{R}/\mathfrak{p})=d\},$$

we get

Corollary (**3.21**). *R is C.-M. if and only if* $\mathscr{D}(R)=\mathscr{P}(R)$.

In § 4, one of the main by-products of our work will be given. The following theorem, which is obtained by combining our results and *R.* Y. Sharp's in [11], provides an important characterization of the category $\mathscr{D}(R)$.

Theorem (**4.6**). *Assume that R is a Noetherian local ring of* depth $(R)=d$ *which is a quotient ring of a local Gorenstein ring S of dimension s. Let* $\omega=\mathrm{Ext}_S^{s-d}(R,S)$, *then the following two (exact) functors set up the equivalence of the categories $\mathscr{D}(R)$ and $\mathscr{I}(R)$:*

$$\omega\otimes_R(*):\mathscr{D}(R)\longrightarrow\mathscr{I}(R)$$

and $\mathrm{Hom}_R(\omega,*):\mathscr{I}(R)\longrightarrow\mathscr{D}(R)$.

Moreover the functors respect the supports of the modules and take C.-M. *modules to* C.-M. *modules in each categories.*

For those who are familiar to the notion of the canonical module, it is easy to see that the f.-g. R-module ω behaves like it. In fact, *in case R is* C.-M., *ω coincides with the canonical module of R* and **Theorem** (**4.6**) reduces to the results stated in [12]. Some results in [3] also follow from our duality theorem.

In § **5**, some important results in [**5**] are extended to the case when the canonical module exists.

As to the definitions and properties of C.-M. *rings* (or *modules*), *Gorenstein rings* (or *modules*), depth, grade and *the canonical module of* C.-M. *ring*, refer to [1], [4] and [12]. Spectral sequence arguements used in this paper are quite standard and can be seen in [2].

§ 1. Preliminaries.

(I. 1) Notations. The following notations are used throughout this paper without further statements.
$\mathcal{M}(R)$ = the category of R-modules (and R-homomorphisms.)
$\mathcal{F}(R)$ = the category of f.-g. R-modules.
$\mathcal{P}(R)$ = the category of f.-g. R-modules of *finite projective dimension.*
$\mathcal{I}(R)$ = the category of f.-g. R-modules of *finite injective dimension.*
$\hat{M}$ = the maximal-ideal-adic-completion of an R-module M.
$H^i_{\mathfrak{M}}(*)$ = the i-th *local cohomology functor*, for $i \geqslant 0$, ([6] or [9].)
$I_R(k)$ = the *injective envelope* of the residue field k of the local ring R.
$M' = \mathrm{Hom}_R(M, I_R(k))$ for an R-module M.

(1. 2) Definition. ([5] (**1. 1. 1**)). *Define the bi-functor*

$$H^i_{\mathfrak{M}}(*, *) : \mathcal{M}(R) \times \mathcal{M}(R) \longrightarrow \mathcal{M}(R)$$

by $H^i_{\mathfrak{M}}(M, N) := \varinjlim_m \mathrm{Ext}^i_R(M/\mathfrak{M}^m M, N)$ *for* M *and* N *in* $\mathcal{M}(R)$.

$\mathrm{E}^i_R(*, *) := (H^i_{\mathfrak{M}}(*, *))'$. Note that $H^i_{\mathfrak{M}}(R, *) = H^i_{\mathfrak{M}}(*)$ the usual loca cohomology functor. $\mathrm{D}^i_R(*) := (H^i_{\mathfrak{M}}(*))'$.

In [5], J. Herzog gave the basic observations of the functors $H^i_{\mathfrak{M}}(*, *)$ and $\mathrm{E}^i_R(*, *)$. We quote several of them for the convenience of the readers.

(1. 3) Remarks. ([5] (**1. 1. 3**)). (**i**) *For any* M *and* $N \in \mathcal{F}(R)$ $\mathrm{E}^i_R(M, N) = \mathrm{E}^i_{\hat{R}}(\hat{M}, \hat{N}) \in \mathcal{F}(\hat{R})$.
(**ii**) $H^i_{\mathfrak{M}}(M, *)$ *is the i-th right derived functor of* $H^0_{\mathfrak{M}}(M, *)$.

(1. 4) Theorem. (Satz (1. 1. 6) [5]). *Let* M *and* $N \in \mathcal{F}(R)$, $x = (x_1, \ldots, x_n)$ *be elements in* $\mathfrak{M}$ *which generate an* $\mathfrak{M}$*-primery ideal,* $K.(x^m; R)$ *denote the Koszul complex oj* R *with respect to* $x^m = (x_1^m, \ldots, x_n^m)$ *and* $F.$ *be a finite free resolution of* M. *If* C^m *denotes the simple complex associated to the double complex* $K.(x^m; R) \otimes F.$, $H^i_{\mathfrak{M}}(M, N) \cong \varinjlim_m H^i(\mathrm{Hom}_R(C^m, N))$.

(1. 5) Theorem. (*Satz* (1. 1. 8) [5]). *Let* $\phi : R \longrightarrow S$ *be a local homomorphism of local rings such that* S *is a finite R-module by* ϕ. *Then there are natural homomorphisms for* $i \geqslant 0$

$$\phi^i : \mathrm{E}^i_R(M, N) \longrightarrow \mathrm{E}^i_S(M \otimes_R S, N)$$

for any $M \in \mathcal{F}(R)$ *and for any* $N \in \mathcal{F}(S)$.
If, further, ϕ *is flat or* M *is free,* ϕ^i *are isomorphisms.*

(1. 6) Corollary. (*Kor.* 1. 1. 9) [5]). *Under the same situation as above,* $\mathrm{D}^i_R(*)$ *and* $\mathrm{D}^i_S(*)$ *are equivalent functors on* $\mathscr{F}(R)$ *to* $\mathscr{F}(\hat{R})$ *for all* $i \geqslant 0$.

(1. 7) Proposition. ((1. 2. 1) and (1. 2. 3) [5]). *For M and N in* $\mathscr{F}(R)$, *there exist convergent spectral sequences*

$$\mathrm{Tor}_p^{\hat{R}}(\hat{M}, \mathrm{D}^q_R(N)) \Longrightarrow \mathrm{E}^{p+q}_R(M, N)$$

and $$\mathrm{E}^p_R(M, \mathrm{D}^{r-q}_R(N)) \Longrightarrow \mathrm{Tor}^{\hat{R}}_{p+q-r}(\hat{M}, \hat{N}) \text{ with } r = \dim(R).$$

We add another spectral sequence, which plays an important role in § 3.

(1. 8) Proposition. *For M and* $N \in \mathscr{F}(R)$, *there is a convergent spectral sequence*

$$\mathrm{D}^p_R(\mathrm{Ext}^q_R(M, N)) \Longrightarrow \mathrm{E}^{p+q}_R(M, N).$$

Proof. Consider the double complex $\mathrm{D}_m^{\bullet\bullet} = \mathrm{Hom}_R(\mathrm{K}.(x^m; R) \otimes_R F., N) = \mathrm{Hom}_R(\mathrm{K}.(x^m; R), \mathrm{Hom}_R(\mathrm{F}., N))$, here follow the notations in (1. 4). It is easy to see that $\mathrm{Hom}_R(\mathrm{C}^m, N)$ (in (1. 4)) is isomorphic to the simple complex associated to $\mathrm{D}_m^{\bullet\bullet}$. We have the spectral sequence

$$H^p(\mathrm{Hom}_R(\mathrm{K}.(x^m; R), \mathrm{Ext}^q_R(M, N)) \Longrightarrow H^{p+q}(\mathrm{Hom}_R(\mathrm{C}^m, N))$$

and take $\varinjlim_m$ of the both ends.

(1. 9) Lemma. (Theorem (6. 1) [9]). *If M is an R-module with finite (Krull-)dimension m, then* $H^i_{\mathfrak{M}}(M) = 0$ *for* $i > m$.

§ 2. Some results on the non-vanishing of $H^i_{\mathfrak{M}}(*,*)$.

We begin this section with the introduction of another expression of the functor $H^i_{\mathfrak{M}}(*, *)$.

(2. 1) Proposition. *Let* $M \in \mathscr{F}(R)$, $N \in \mathscr{M}(R)$ and $J^{\cdot}$ *be the minimal infective resolution of N. Then for all* $i \geqslant 0$,

$$H^i_{\mathfrak{M}}(M, N) \cong H^i(H^0_{\mathfrak{M}}(\mathrm{Hom}_R(M, J^{\cdot}))) \cong H^i(\mathrm{Hom}_R(M, H^0_{\mathfrak{M}}(J^{\cdot}))).$$

Proof is preceded by the lemma below.

(2. 2) Lemma. *For M and N as above,*

$$H^0_{\mathfrak{M}}(M, N) \cong \mathrm{Hom}_R(M, H^0_{\mathfrak{M}}(N)) \cong H^0_{\mathfrak{M}}(\mathrm{Hom}_R(M, N)).$$

Proof. Since $\text{length}_R(M/\mathfrak{M}^n M)<\infty$ for all $n\geqslant 0$, $\text{Hom}_R(M/\mathfrak{M}^n M, N)=\text{Hom}_R(M/\mathfrak{M}^n M, H^0_{\mathfrak{M}}(N))$. The natural surjection $M\longrightarrow M/\mathfrak{M}^n M$ induces the injection

$$\text{Hom}_R(M/\mathfrak{M}^n M, H^0_{\mathfrak{M}}(N))\longrightarrow\text{Hom}_R(M, H^0_{\mathfrak{M}}(N)).$$

On the other hand, for any $f\in\text{Hom}_R(M, H^0_{\mathfrak{M}}(N))$, there exists an integer $n>0$ such that $\mathfrak{M}^n f(M)=0$ and therefore

$$\text{Hom}_R(M, H^0_{\mathfrak{M}}(N))=\bigcup_{n>0}\text{Hom}_R(M/\mathfrak{M}^n M, H^0_{\mathfrak{M}}(N))$$
$$\cong\varinjlim_n \text{Hom}_R(M/\mathfrak{M}^n M, N)=H^0_{\mathfrak{M}}(M, N).$$

Since $\text{Hom}_R(M/\mathfrak{M}^n M, N)\cong\text{Hom}_R(R/\mathfrak{M}^n, \text{Hom}_R(M, N))$, the second is rather easy.

Proof of (2. 1). $H^i_{\mathfrak{M}}(M, N)\cong H^i(\varinjlim_n \text{Hom}_R(M/\mathfrak{M}^n M, J^{\cdot}))$

$\cong H^i(H^0_{\mathfrak{M}}(M, J^{\cdot}))\cong H^i(H^0_{\mathfrak{M}}(\text{Hom}_R(M, J^{\cdot})))\cong H^i(\text{Hom}_R(M, H^0_{\mathfrak{M}}(J^{\cdot})))$.

(2. 3) Theorem. *Let M and N be non-zero f.-g. R-modules and* $t=\text{depth}_R(N)$. *Then* $H^t_{\mathfrak{M}}(M, N)\neq 0$ *and* $H^i_{\mathfrak{M}}(M, N)=0$ *for* $i<t$.

Proof. Since $\text{length}_R(M/\mathfrak{M}^n M)<\infty$, $\text{Ext}^i_R(M/\mathfrak{M}^n M, N)=0$ for $i<t$ and $H^i_{\mathfrak{M}}(M, N)=0$. Let $J^{\cdot}$ and $\mathscr{A}^i$ be the minimal injective resolution of N and the boundary map $J^i\longrightarrow J^{i+1}$. $H^0_{\mathfrak{M}}(J^i)=0$ for $i<t$ and $\text{Hom}_R(M, H^0_{\mathfrak{M}}(J^t))\neq 0$. By **(2. 1)**, $H^t_{\mathfrak{M}}(M, N)\cong H^t(\text{Hom}_R(M, H^0_{\mathfrak{M}}(J^{\cdot})))=\ker(\text{Hom}_R(M, H^0_{\mathfrak{M}}(\mathscr{A}^t)))\cong\ker(H^0_{\mathfrak{M}}(\text{Hom}_R(M, \mathscr{A}^t)))=H^0_{\mathfrak{M}}(\text{Hom}_R(M, J^t))\cap\ker(\text{Hom}_R(M, \mathscr{A}^t))$. Since $\ker(\text{Hom}_R(M, \mathscr{A}^t))$ is an essential submodule of $\text{Hom}_R(M, J^t)$, the last intersection is non-zero. Q. E. D.

(2. 5) Remarks. (**i**) Let M and $N\in\mathscr{F}(R)$ such that $\text{Supp}(M)\cap\text{Supp}(N)=\{\mathfrak{M}\}$, then $\text{Hom}_R(M, H^0_{\mathfrak{M}}(J^{\cdot}))=\text{Hom}_R(M, J^{\cdot})$ for *the minimal injective resolution* $J^{\cdot}$ of N. By (2. 1), $H^i_{\mathfrak{M}}(M, N)\cong\text{Ext}^i_R(M, N)$ for all $i\geqslant 0$.

(**ii**) In (2. 3), we found the lower bound of i for which $H^i_{\mathfrak{M}}(M, N)\neq 0$. As to the upper bound, we must require some restriction on either of M or N. In fact, if $H^i_{\mathfrak{M}}(M, *)=0$ for $i\gg 0$, then $\text{Pd}_R M<\infty$: $\text{E}^i_R(M, k)\cong\text{Tor}^{\hat{R}}_{i}(\hat{M}, k)$. If $H^i_{\mathfrak{M}}(*, N)=0$ for $i\gg 0$, then $\text{Id}_R N<\infty$: $H^i_{\mathfrak{M}}(k. N)\cong\text{Ext}^i_R(k, N)$.

§ 3. Duality.

(3. 1) Lemma. *For* $M\in\mathscr{P}(R)$ *and for an* $N\in\mathscr{M}(R)$, $H^i_{\mathfrak{M}}(M, N)=0$ for $i>\text{Pd}_R M+\dim(N)$.

Consequently, for $M\neq 0$, *in* $\mathscr{F}(R)$, M *is of finite projective dimension if and only if* $H^i_{\mathfrak{M}}(M, *)=0$ *for* $i\gg 0$.

Proof. By (1. 8), there exists a spectral sequence

$$E_2^{p,q}=D_R^p(\mathrm{Ext}_R^q(M, N))\Longrightarrow E_R^{p+q}(M, N).$$

$E_2^{p,q}=0$, if $p>\dim(N)$ or if $q>\mathrm{Pd}_R M$, since $\dim(\mathrm{Ext}_R^q(M, N))\leqslant\dim(N)$. Consequently, $E_R^j(M, N)=0$, if $j>\dim(N)+\mathrm{Pd}_R M$.

(3. 2) **Remark.** For $M\neq 0$, $\in\mathscr{P}(R)$, *there exists an integer* $e\geqq 0$ *such that* $H^i_{\mathfrak{M}}(M, *)=0$ *for* $i>e$ *and* $H^e_{\mathfrak{M}}(M, R)\neq 0$: let e be the greatest i such that $H^i_{\mathfrak{M}}(M, R)\neq 0$, then by (3. 1) it follows that $H^i_{\mathfrak{M}}(M, N)=0$ for any $N\in\mathscr{F}(R)$ and for all $i>e$.

If an integer e is chosen as above for some $M\neq 0$, $\in\mathscr{P}(R)$ *then* $H^e_{\mathfrak{M}}(M, *)$: $\mathscr{F}(R)\longrightarrow\mathscr{M}(R)$ *is an additive covariant right exact functor and there exists a natural isomorphism*

$$H^e_{\mathfrak{M}}(M, N)\cong H^e_{\mathfrak{M}}(M, R)\otimes_R N \text{ for any } N\in\mathscr{F}(R).$$

(3. 3) **Proposition.** *Assume that* $R=\hat{R}$, M *be non zero*$\in\mathscr{P}(R)$ *for which an integer* $e\geqslant 0$ *is chosen as* (3. 2). *Then there are pairings for all i and for any* $N\in\mathscr{F}(R)$

$$H^i_{\mathfrak{M}}(M, N)\times\mathrm{Ext}_R^{e-i}(N, \Omega(M))\longrightarrow I_R(k),$$

where $\Omega(M)$ *denotes the non-zero f.-g.* $\hat{R}$*-module* $E_R^e(M, R)$.

Proof. There defined the pairings ((6. 1) [6]),

$$H^i_{\mathfrak{M}}(M, N)\times\mathrm{Ext}_R^{e-i}(N, \Omega(M))\longrightarrow H^e_{\mathfrak{M}}(M, \Omega(M)).$$

By (3. 2), $H^e_{\mathfrak{M}}(M, \Omega(M))\cong H^e_{\mathfrak{M}}(M, R)\otimes_R(H^e_{\mathfrak{M}}(M, R))'$ and the natural map $H^e_{\mathfrak{M}}(M, \Omega(M))\longrightarrow I_R(k)$ is defined. Compositions of them give rise to the pairings.

(3. 4) **Proposition.** *Assume that* $R=\hat{R}$, M *be as* (3. 3) *and an integer e is chosen for M as* (3. 2). *For an integer s the following conditions are equivalent.*

(**i**) *The pairings in* (3. 3) *are perfect for* $e\text{-}s\leqq i\leqq e$.

(**ii**) $H^i_{\mathfrak{M}}(M, R)=0$ *for* $e\text{-}s\leqq i<e$.

Proof. First note that $E_R^e(M, R)\cong\mathrm{Hom}_R(R, \Omega(M))$. Both $E_R^e(M, *)$ and $\mathrm{Hom}_R(*, \Omega(M))$ are additive, left exact, contravariant functors. While $E_R^i(M, *)$ are the right derived functors of $E_R^e(M, *)$ for $e\text{-}s\leqq i<e$, if $E_R^i(M, R)=0$ for the same i's.

We have now come to one of our main results.

(3. 5) Theorem (Duality). *Let M be a f.-g. non zero R-module. The following conditions are equivalent.*
(i) $E_R^i(M, N) \cong \mathrm{Ext}_{\hat{R}}^{e-i}(\hat{N}, \Omega(M))$ *for some integer* $e \geqq 0$, *for all* i *and for any* $N \in \mathscr{F}(R)$. *Here* $\Omega(M) = E_R^e(M, R)$.
(ii) $E_R^i(M, N) = 0$ *for all* $i > d = \mathrm{depth}(R)$ *and for any* $N \in \mathscr{F}(R)$.
If M satisfies the conditions above, then $e = d = \mathrm{depth}(R)$.

Proof. Recall that $E_R^d(M, R) \neq 0$ and $E_R^i(M, R) = 0$ for $i < d$, (2. 3). Then the equivalence follows from (3. 4).

(3. 6) Definition. Let $\mathscr{D}(R)$ denote *the category of f.-g.* R-*modules M satisfying the equivalent conditions in* (**3. 5**). Note that $\mathscr{D}(R) \subseteq \mathscr{P}(R)$ by (**2. 4**)(**ii**).

In the rest of this section, some properties of modules in $\mathscr{D}$(R) are studied.

(3. 7) Theorem. *Let M be an R-module in* $\mathscr{D}$(R), *then*

$$\mathrm{Hom}_{\hat{R}}(D_R^d(R), \hat{M} \otimes_{\hat{R}} D_R^d(R)) \cong \hat{M}, \text{ for } d = \mathrm{depth}(R).$$

Consequently, $\mathrm{Supp}_{\hat{R}} \hat{M} = \mathrm{Supp}_{\hat{R}}(\hat{M} \otimes_{\hat{R}} D_R^d(R))$.

Proof. We may assume that $R = \hat{R}$. There exists a convergent spectral sequence

$$E_2^{p,q} = E_R^p(M, D_R^{r-q}(R)) \Longrightarrow \mathrm{Tor}_{p+q-r}^R(M, R)$$

where $r = \dim(R)$, see (1. 7). $D_R^{r-q}(R) = 0$ for $r - q < d = \mathrm{depth}(R)$. By (3. 5), $E_R^p(M, D_R^{r-q}(R)) = 0$ for $p > d$. By the standard spectral sequence argument, $E_R^d(M, D_R^d(R)) \cong M \otimes_R R \cong M$. On the other hand by the duality (3. 5), $E_R^d(M, D_R^d(R)) \cong \mathrm{Hom}_R(D_R^d(R), E_R^d(M, R))$ and we have the required isomorphism, if it is shown that $E_R^d(M, R) \cong M \otimes_R D_R^d(R)$. By (**1. 7**) again, there exists

$$E_2^{p,q} = \mathrm{Tor}_p^R(M, D_R^q(R)) \Longrightarrow E_R^{p+q}(M, R).$$

$E_2^{p,q} = 0$ if $p < 0$ or if $q < d = \mathrm{depth}(R)$, and the assertion is valid.

Q. E. D.

(3. 8) Corollary. *If* $M \in \mathscr{D}(R)$, *then* $\mathrm{Supp}_{\hat{R}}(\hat{M}) \subseteq \mathrm{Supp}\,(D_R^d(R))$ *with* d = depth(R).

(3. 9) Remark. By (1. 9) and (2. 3), a *f.-g.* R-module N is a C.-M. *module* if and only if *there exists an integer* $n \geqslant 0$ *such that* $H_{\mathfrak{m}}^i(N) = 0$ *if* $i \neq n$.

(3.10) Proposition. *If $M\in\mathcal{D}(R)$ and if N is a C.-M. R-module of dimension n, then for all i,*

$$\mathrm{Tor}_{i-n}^{\hat{R}}(\hat{M},\ \mathrm{D}_R^n(N))\cong\mathrm{Ext}_{\hat{R}}^{d-i}(\hat{N},\ \hat{M}\otimes_{\hat{R}}\mathrm{D}_R^d(R)).$$

Proof. We may assume that $R=\hat{R}$. By (3. 9) and (1. 7), $\mathrm{Tor}_{i-n}^R(\hat{M},\ \mathrm{D}_R^n(N))\cong\mathrm{E}_R^i(M,\ N)$. The assertion follows from (3. 5).

(3.11) Corollary. *If $M\in\mathcal{D}(R)$, then*

$$\beta_i^{\hat{R}}(\hat{M})=\mu_{\hat{R}}^{d-i}(\hat{M}\otimes_{\hat{R}}\mathrm{D}_R^d(R)),\ \textit{for all } i.$$

Where $\beta_i^R(N)=\dim_k(\mathrm{Tor}_i^R(N,\ k))$ and $\mu_R^i(N)=\dim_k(\mathrm{Ext}_R^i(k,\ N))$ for a *f.-g.* R-module N.

Consequently, we have $\mathrm{Id}_{\hat{R}}(\hat{M}\otimes_{\hat{R}}\mathrm{D}_R^d(R))<\infty$ *for any* $M\in\mathcal{D}(R)$.

Proof. Since k is a 0-dimensional C.-M. R-module, (3. 10) can be applied.

We now proceed to give the sufficient conditions for $M\in\mathcal{F}(R)$ to belong to the category $\mathcal{D}(R)$. For that purpose several lemmata are needed.

(3.12) Lemma. *For any $M\neq 0,\ \in\mathcal{F}(R)$*

$$\mathrm{depth}(R)\leqq\mathrm{grade}_R M+\dim(M)\leqq\dim(R).$$

Proof. See ([8] (4. 8) Chap. I).

(3.13) Lemma. *Assume that $R=\hat{R}$ and $d=\mathrm{depth}(R)$. Then*
(i) *for any $\mathfrak{p}\in\mathrm{Spec}(R)$ with $\dim(R/\mathfrak{p})=n$,*

$$\widehat{(\mathrm{D}_R^d(R))_{\mathfrak{p}}}\cong\mathrm{D}_{R_{\mathfrak{p}}}^{d-n}(R_{\mathfrak{p}}).$$

(ii) $\mathrm{Supp}_R(\mathrm{D}_R^d(R))=\{\mathfrak{p}\in\mathrm{Spec}(R)\ ;\ \mathrm{depth}(R_{\mathfrak{p}})+\dim(R/\mathfrak{p})=d\}$.

Proof. Since R is complete local, there exists a surjective ring homomorphism $\phi: S\longrightarrow R$ with a complete local Gorenstein ring of dimension s. Let $\mathfrak{q}=\phi^{-1}(\mathfrak{p})$, then $\dim(S_{\mathfrak{q}})=s-n$. By **(1. 6)** and by the standard duality theory (**[6]**), for all $i\geqq 0$, $\mathrm{D}_R^i(R)\cong\mathrm{D}_S^i(R)\cong\mathrm{Ext}_S^{s-i}(R,\ S)$. Since $S_{\mathfrak{q}}$ is also a local Gorenstein ring of dimension $s-n$,

$$\widehat{(\mathrm{D}_R^d(R))_{\mathfrak{p}}}\cong\widehat{\mathrm{Ext}_{S_{\mathfrak{q}}}^{s-d}(R_{\mathfrak{p}},\ S_{\mathfrak{q}})}\cong\mathrm{Ext}_{S_{\mathfrak{q}}}^{s-d}(\widehat{R_{\mathfrak{p}}},\ \widehat{S_{\mathfrak{q}}})=\mathrm{D}_{S_{\mathfrak{q}}}^{d-n}(R_{\mathfrak{p}}).$$

As to (ii), by (3. 12), $d-\dim(R/\mathfrak{p})\leqq\mathrm{depth}(R_{\mathfrak{p}})$. For $\mathfrak{p}\in\mathrm{Spec}(R)$ with $\dim(R/\mathfrak{p})=n$, $\mathfrak{p}\in\mathrm{Supp}\ \mathrm{D}_R^d(R)$ if and only if $\mathrm{depth}(R_{\mathfrak{p}})=d-n$ by **(2. 3)**.

(3. 14) Lemma. *Assume that* $R=\hat{R}$, $d=\mathrm{depth}(R)$ *and* $M\neq 0\in\mathscr{P}(R)$. *If* $\dim(\mathrm{Ext}_R^q(M, R))+q\leqq d$, *for* $\mathrm{grade}_R M\leqq q\leqq \mathrm{Pd}_R M$, *then* $\mathrm{E}_R^i(M, N)=0$ *for all* $i>d$ *and for any* $N\in\mathscr{F}(R)$.

Prrof. By (1. 8), there exists a convergent spectral sequence

$$\mathrm{E}_2^{p,q}=\mathrm{D}_R^p(\mathrm{Ext}_R^q(M, R))\Longrightarrow \mathrm{E}_R^{p+q}(M, R).$$

If $p+q>d$, then $p>d-q\geqq\dim(\mathrm{Ext}_R^q(M, R))$ for $\mathrm{grade}_R M\leqq q\leqq\mathrm{Pd}_R M$. $\mathrm{E}_2^{p,q}=0$ if $p+q>d$ and $\mathrm{E}_R^i(M, R)=0$ for $i>d$. Since $\mathrm{Pd}_R M<\infty$, $\mathrm{E}_R^i(M, *)=0$ for $i\gg 0$ on $\mathscr{M}(R)$ and the assertion is sustained.

We are now ready to prove the main theorem below.

(3. 15) Theorem. *Let* M *be a f.-g.* R*-module of* $\mathrm{Pd}_R M<\infty$ *such that* $\mathrm{Supp}_{\hat{R}}\hat{M}\subseteq\mathrm{Supp}_{\hat{R}}(\mathrm{D}_R^d(R))$ *with* $d=\mathrm{depth}(R)$, *then* M *belongs to the category* $\mathscr{D}(R)$.

Proof. We may assume that $R=\hat{R}$. For any q fixed with $\mathrm{grade}_R M\leqq q\leqq\mathrm{Pd}_R M$, let $\mathfrak{p}\in\mathrm{Supp}_R\mathrm{Ext}_R^q(M, R)$ such that $\dim(R/\mathfrak{p})=\dim(\mathrm{Ext}_R^q(M, R)=n$. Since $\mathfrak{p}\in\mathrm{Supp}_R M$, $\mathfrak{p}\in\mathrm{Supp}_R(\mathrm{D}_R^p(R))$ and by (3. 13) $\mathrm{depth}(R_\mathfrak{p})=d-n$. On the other hand, $\mathrm{Ext}_{R_\mathfrak{p}}^q(M_\mathfrak{p}, R_\mathfrak{p})\neq 0$ implies the inequality $q\leqq\mathrm{Pd}_{R_\mathfrak{p}}M_\mathfrak{p}\leqq\mathrm{depth}(R_\mathfrak{p})=d-n$, *i.e.*, $\dim(\mathrm{Ext}_R^q(M, R))+q\leqq d$ and apply (3. 14). Q. E. D.

Preceding arguments are summarized as below.

(3. 16) Theorem. *Let* M *be a f.-g.* R*-module and* d *be the depth of* R. *Then the following conditions are equivalent.*

(i) $M\in\mathscr{D}(R)$.

(ii) $M\in\mathscr{P}(R)$ *and* $\mathrm{Supp}_{\hat{R}}(\hat{M})\subseteq\mathrm{Supp}_{\hat{R}}(\mathrm{D}_R^d(R))$.

An important class of modules in $\mathscr{D}(R)$ is given.

(3. 17) Proposition. *Let* M *be a f.-g.* R*-module of* $\mathrm{Pd}_R M<\infty$. *If* M *is* C.-M., *then* $M\in\mathscr{D}(R)$.

In particular, if R *is a* C.-M. *ring, then* $\mathscr{D}(R)=\mathscr{P}(R)$.

Proof. We may assume that $R=\hat{R}$. For any $\mathfrak{p}\in\mathrm{Supp}_R M$,

$$\begin{aligned}\mathrm{depth}(R_\mathfrak{p})&=\mathrm{Pd}_{R_\mathfrak{p}}M_\mathfrak{p}+\mathrm{depth}_{R_\mathfrak{p}}M_\mathfrak{p}\leqq\mathrm{Pd}_R M+\dim_{R_\mathfrak{p}}M_\mathfrak{p}\\&\leqq\mathrm{Pd}_R M+\dim(M)-\dim(R/\mathfrak{p})=\mathrm{depth}(R)-\dim(R/\mathfrak{p})\end{aligned}$$

and it follows that $\mathrm{depth}(R_\mathfrak{p})=\mathrm{depth}(R)-\dim(R/\mathfrak{p})$ and that $\mathfrak{p}\in\mathrm{Supp}_R(\mathrm{D}_R^d(R))$.

The second assertion is easy to see by (3. 16).

(3.18) Corollary. *Let* $M\in\mathscr{D}(R)$, $N\in\mathscr{F}(R)$ *such that* $n=\dim(N)$ *and* $\mathrm{Supp}(N)\subseteq\mathrm{Supp}(M)$. *Then* $H^n_{\mathfrak{M}}(M, N)\neq 0$.

Proof. We may assume that $R=\hat{R}$. By the duality (3. 5), $\mathrm{E}^i_R(M, N)\cong\mathrm{Ext}^{d-i}_R(N, M\otimes_R\mathrm{D}^d_R(R))$. Let $\mathfrak{p}\in\mathrm{Supp}_R N$ such that $n=\dim(R/\mathfrak{p})$. Then $\mathrm{Ext}^{d-i}_{R_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}\otimes(\mathrm{D}^d_R(R))_\mathfrak{p})\neq 0$ if $d-i=\mathrm{Id}_{R_\mathfrak{p}}(M_\mathfrak{p}\otimes(\mathrm{D}^d_R(R))_\mathfrak{p})=\mathrm{depth}(R_\mathfrak{p})=d-n$, *i. e.*, if $i=n$.
$\mathrm{E}^n_R(M, N)=\mathrm{Ext}^{d-n}_R(N, M\otimes\mathrm{D}^d_R(R))\neq 0$ and $H^n_{\mathfrak{M}}(M, N)\neq 0$.

(3.19) Corollary. *For* $N\in\mathscr{F}(R)$ *with* $\dim(N)=n$, $H^n_{\mathfrak{M}}(N)\neq 0$.

Proof. We may assume that $R=\hat{R}$, hence that R is a Gorenstein local ring by (1. 6) and $R\in\mathscr{D}(R)$. Apply (3. 18).

(3.20) Remark. If M is a *f.-g. h-perfect* R-module (*i. e.* $\mathrm{grade}_R M=h=\mathrm{Pd}_R M<\infty$), then *the integer e determined for M as in* (3. 2) is equal to $h+\dim(M)$: $\mathrm{Ext}^i_R(M, R)=0$ if $i\neq h$, hence by (**1. 8**), $\mathrm{D}^j_R(\mathrm{Ext}^h_R(M, R))=\mathrm{E}^{j+h}_R(M, R)$ and it vanishes if $j>m=\dim(M)=\dim(\mathrm{Ext}^h_R(\mathrm{M}, R)$, while $\mathrm{E}^{m+h}_R(M, R)\neq 0$.

It, therefore, follows that *a perfect module M is in* $\mathscr{D}(R)$ *if and only if it is* C.-M..

(3.21) Corallary. *R is* C.-M. *if and only if* $\mathscr{D}(R)=\mathscr{P}(R)$.

Proof. If $\mathscr{D}(R)=\mathscr{P}(R)$, then $R\in\mathscr{D}(R)$ and by (3. 20) R is C.-M.. Only if part is proved in (3. 17).

§4. Equivalence of the categories $\mathscr{D}(R)$ and $\mathscr{I}(R)$.

This section is devoted to give a characterization of the category $\mathscr{D}(R)$ (under an acceptable restriction on R.) Our characterization of $\mathscr{D}(R)$ has a close connection with the results in [11] by Sharp: it treats the functor on $\mathscr{I}(R)$ to $\mathscr{P}(R)$, while ours the functor on $\mathscr{D}(R)$ to $\mathscr{I}(R)$ which finally sets up the equivalence of the categories $\mathscr{D}(R)$ and $\mathscr{I}(R)$.

All through this section $(R, \mathfrak{M}, k)$ is a Noetherian local ring of $\mathrm{depth}(R)=d$ which is a *homomorphic image of a local Gorenstein ring* S *of dimension s* and $\phi: S\longrightarrow R$ is the natural surjective ring homomorphism.

(4.1) Notations. *Let* ω *denote the f.-g. R-module*

$$\mathrm{Ext}^{s-d}_S(R, S).$$

(4.2) Remark. $\omega\otimes_R\hat{R}=\omega\otimes_S\hat{S}\cong\mathrm{D}^d_S(R)\cong\mathrm{D}^d_R(R)$.

(4. 3) Theorem([11] (2. 9)). **(i)** $\mathrm{Hom}_R(\omega, *)$ *is a functor on* $\mathscr{I}(R)$ *to* $\mathscr{P}(R)$.

(ii) *For any* $T \in \mathscr{I}(R)$, *there exists a natural isomorphism*

$$\omega \otimes_R \mathrm{Hom}_R(\omega, T) \cong T.$$

(iii) $\mathrm{Ann}_R(T) = \mathrm{Ann}_R(\mathrm{Hom}_R(\omega, T))$ for $T \in \mathscr{I}(R)$.

Proof. See the article referred to.

(4. 4) Proposition. *For* $M \in \mathscr{F}(R)$, M *is in* $\mathscr{D}(R)$ *if and only if* $M \in \mathscr{P}(R)$ *and* $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(\omega)$.

Proof. Straightforward. See (3. 16) and (4. 2).

(4. 5) Proposition. **(i)** $\omega \otimes_R (*)$ *is an exact functor on* $\mathscr{D}(R)$ *to* $\mathscr{I}(R)$.

(ii) $\mathrm{Hom}_R(\omega, *)$ *is an exact functor on* $\mathscr{I}(R)$ *to* $\mathscr{D}(R)$.

Proof. We may assume that $R = \hat{R}$.

(i) By (3. 11) for any $M \in \mathscr{D}(R)$, $\omega \otimes_R M \in \mathscr{I}(R)$. By (1. 7) we have $E_2^{p,q} = \mathrm{Tor}_p^R(M, D_R^q(R)) \Longrightarrow E_R^{p+q}(M, R)$ and $E_2^{p,q} = 0$ if $p < 0$ or if $q < d = \mathrm{depth}(R)$. $\mathrm{Tor}_1^R(M, D_R^d(R)) = E_2^{1,d} \cong E_\infty^{1,d} = 0$ by (5. 5) Chap. XV [2].

(ii) For any $T \in \mathscr{I}(R)$, $M = \mathrm{Hom}_R(\omega, T) \in \mathscr{P}(R)$ by (4. 3). Clearly $\mathrm{Supp}_R M \subseteq \mathrm{Supp}_R(\omega)$ and $M \in \mathscr{D}(R)$ by (4. 4). By (1. 7) with $r = \dim(R)$ there exists a convergent spectral sequence

$$E_2^{p,q} = E_R^p(M, D_R^{r-q}(R)) \Longrightarrow \mathrm{Tor}_{p+q-r}^R(M, R).$$

While by the duality (3. 5), $E_R^p(M, D_R^{r-q}(R)) \cong \mathrm{Ext}_R^{d-p}(D_R^{r-q}(R), T)$. $E_2^{p,q} = 0$ if $d - p < 0$ or if $r - q < d$ and by the same argument as above (i), $\mathrm{Ext}_R^1(\omega, T) = E_2^{d-1, r-d} = 0$.

We have shown the following theorem which is the purpose of this section.

(4. 6) Theorem. *Assume that* R *is a Noetherian local ring of* depth $(R) = d$ *which is a quotiont ring of a local Gorenstein ring* S *of dimension* s. *Let* $\omega = \mathrm{Ext}_S^{s-d}(R, S)$, *then the subcategory* $\mathscr{D}(R)$ *of* $\mathscr{P}(R)$ *is equivalent to the category* $\mathscr{I}(R)$ *by the functors* $\omega \otimes_R (*) : \mathscr{D}(R) \longrightarrow \mathscr{I}(R)$ *and* $\mathrm{Hom}_R(\omega, *) : \mathscr{I}(R) \longrightarrow \mathscr{D}(R)$.

Moreover, those functors correspond C.-M. *modules to* C.-M. *modules in each categories and respect the supports of the modules.*

§ 5. Applications.

In this section, applications of preceding results are given (mainly in the case when the canonical module K_R of the local ring R exists.) We besides introduce another duality theorem with respect to the functor $\mathrm{E}_R^i(*, *)$, which is extended from one given in the case when R is a Gorenstein local ring in [5].

(5. 1) **Definition.** *Assume that* R *is* C.-M. (*of* $\dim(R)=d$). *A f.-g.* R-*module* K_R *is called the canonical module of* R *if* $K_R\otimes_R\hat{R}=\mathrm{D}_R^d(R)$. *Note that* $K_R\in\mathscr{I}(R)$ *since* $\mathrm{D}_R^d(R)\in\mathscr{I}(\hat{R})$.

(5. 2) **Lemma** ([5] (1. 3. 2)). *If* N *is a* C.-M. R-*module of* dim $(N)=n$, *then* $\mathrm{D}_R^n(N)$ *is also* C.-M. *of dimension* n *over* $\hat{R}$ *and* $\hat{N}=\mathrm{D}_R^n\mathrm{D}_R^n(N)$.

Proof. By (1. 7), $\mathrm{E}_R^i(R, \mathrm{D}_R^n(N))\cong\mathrm{Tor}_{i-n}^{\hat{R}}(\hat{R}, \hat{N})$. By the definition $\mathrm{D}_R^i(\mathrm{D}_R^n(N))=\mathrm{E}_R^i(R, \mathrm{D}_R^n(N))$.

(5. 3) **Lemma.** *Assume that* $N\in\mathscr{F}(R)$ *is* C.-M. *of dimension* n, *then*
(i) $N\in\mathscr{I}(R)$ *if and only if* $\mathrm{D}_R^n(N)\in\mathscr{P}(\hat{R})$ *and*
(ii) $N\in\mathscr{P}(R)$ *if and only if* $\mathrm{D}_R^n(N)\in\mathscr{I}(\hat{R})$.

Proof. We may assume that $R=\hat{R}$. By (1. 7), we have $\mathrm{Tor}_i^R(k, \mathrm{D}_R^n(N))\cong\mathrm{E}_R^{i+n}(k, N)\cong\mathrm{Hom}_R(\mathrm{Ext}_R^{i+n}(k, N), \mathrm{I}_R(k))$. Thus we see that $\beta_i^R(\mathrm{D}_R^n(N))=0$ for $i\gg0$ if and only if $\mu_R^{i+n}(N)=0$ for $i\gg0$. (i) is proved.

(ii) follows from (i) by (5. 2).

(5. 4) **Theorem** (c. f. (1. 1. 4) [5]). *Assume that there exists the canonical module* K_R *of a* C.-M. *local ring* $(R, \mathfrak{M}, k)$ *of dimension* d. *Then if* $M\in\mathscr{P}(R)$, *for* $i=0, \ldots, d$,

$$\mathrm{E}_R^i(N, M\otimes_R K_R)=\mathrm{Ext}_{\hat{R}}^{d-i}(\hat{M}, \hat{N})$$

for any $N\in\mathscr{F}(R)$.

Proof. We may assume that $R=\hat{R}$ then $K_R=\mathrm{D}_R^d(R)$. Put $\mathrm{T}^i(M)=\mathrm{E}_R^{d-i}(N, M\otimes_R K_R)$. By (4. 5) (i), $K_R\otimes_R(*)$ is an exact functor on $\mathscr{D}(R)=\mathscr{P}(R)$ to $\mathscr{I}(R)$. Since $\mathrm{Id}_R(M\otimes_R K_R)=d$, $\mathrm{E}_R^i(N, M\otimes_R K_R)=0$ for $i>d$. $\mathrm{T}^0(*)=\mathrm{E}_R^d(N, (*)\otimes_R K_R)$ is therefore a left exact contravariant functor on $\mathscr{P}(R)$. For any $M\in\mathscr{P}(R)$, $\mathrm{T}^0(M)\cong\mathrm{Hom}_R(M, N)$. In fact, if $M=R^{(n)}$ for some $n>0$, $\mathrm{T}^0(M)=\mathrm{E}_R^d(N, R^{(n)}\otimes_R \mathrm{D}_R^d(R))\cong\mathrm{Tor}_0^R(N, R^{(n)})\cong\mathrm{Hom}_R(R^{(n)}, N)$ by (**1. 7**). For a general M, consider the finite presentation of M.

By (1. 7), $T^i(R)=E_R^{d-i}(N, K_R)=E_R^{d-i}(N, D_R^d(R))\cong \mathrm{Tor}_{-i}^R(N, R)=0$ for $i>0$. Q. E. D.

As an application of our duality theorem (3. 5) and the theorem above, we shall give the following theorem which is an extension of (1. 3. 9) [5].

(**5. 5**) **Theorem.** *Assume that R is a* C.-M. *local ring of* $\dim(R)=d$. *Let $M\in\mathscr{P}(R)$ be* C.-M. *of* $\dim(M)=m$ *and $N\in\mathscr{F}(R)$ be also* C.-M. *of* $\dim(N)=n$ *such that* $\mathrm{Tor}_i^R(M, N)=0$ *for $i>0$. Then* (**i**) $M\otimes_R N$ *is a* C.-M. *module of dimension $m+n-d$.*
If further there exists the canonical module K_R,
(**ii**) $D_R^{m+n-d}(M\otimes_R N)\cong D_n^m(M\otimes_R K_R)\otimes_R D_R^n(N)$ *and*
(**iii**) $\mathrm{Tor}_i^R(D_R^m(M\otimes_R K_R), D_R^n(N))=0$ *for $i>0$.*

Proof. We may assume that $R=\hat{R}$. Let E. and F. denote the free resolutions of M and N, respectively. There exist two convergent spectral sequences

$$(\#1)\quad \mathrm{Ext}_R^p(M, \mathrm{Ext}_R^q(N, N_R))$$

and $$(\#2)\quad \mathrm{Ext}_R^p(N, \mathrm{Ext}_R^q(M, K_R))$$

with the same abutement $H^{p+q}(\mathrm{Hom}_R(E.\otimes_R F., K_R))$, which is isomorphic to $\mathrm{Ext}_R^{p+q}(M\otimes_R N, K_R)$ since the complex E. $\otimes$F. provides the free resolution of $M\otimes_R N$.

Since both M and N are C.-M., we have by (#1) and (#2), (##1) $\mathrm{Ext}_R^i(M, D_R^n(N))\cong \mathrm{Ext}_R^{i+d-n}(M\otimes_R N, K_R)\cong D_R^{n-i}(M\otimes_R N)$ and (##2) $\mathrm{Ext}_R^i(N, D_R^m(M))\cong D_R^{m-i}(M\otimes_R N)$. Since $M\otimes_R K_R$ is C.-M. of dimension m, $\mathrm{Ext}_R^i(M, D_R^n(N))\cong E_R^{d-i}(D_R^n(N), (M\otimes_R K_R)\cong \mathrm{Tor}_{d-i-m}^R(D_R^n(N), D_R^m(M\otimes_R K_R))$, by (5. 4). By (##1),
$\mathrm{Tor}_{d-i-m}^R(D_R^n(N), D_R^m(M\otimes_R K_R))\cong D_R^{n-i}(M\otimes_R N)$. Putting $i=d-m$, the assertion (ii) is sustained and besides $D_R^{n-i}(M\otimes_R N)=0$ if $i>d-m$.

If we show that $D_R^{m-i}(M\otimes_R N)=0$ for $i<d-n$, the remaining assertions follow. By (3. 5) and (1. 7), $\mathrm{Ext}_R^i(D_R^n(N), M\otimes_R K_R)\cong E_R^{d-i}(M, D_R^n(N))\cong \mathrm{Tor}_{d-i-n}^R(M, N)$, it vanishes for $i<d-n$. On the other hand, by (5. 2) and (4. 5), $(0:N)_R=(0:D_R^n(N))_R$ and any $(M\otimes K_R)$-sequence is M-sequence. It follows that $\mathrm{grade}_M N=d-n$. Since any M-sequence is $D_R^m(M)$-sequence, $\mathrm{geade}_{D_R^m(M)}N=\mathrm{grade}_M N$ and by (##2), $D_R^{m-i}(M\otimes_R N)=0$ for $i<d=n$. Q. E. D.

(**5. 6**) **Corollary.** *Under the same situation as* (5. 5), *assume further that $N\in\mathscr{I}(R)$, then $M\otimes_R N\in\mathscr{I}(R)$.*

Proof. By (5. 3), $D^n_R(N)\in\mathscr{P}(\hat{R})$ and $D^m_R(M\otimes_R K_R)\in\mathscr{P}(\hat{R})$. By (5. 5) (ii) and (iii), $D^{m+n-d}_R(M\otimes_R N)\in\mathscr{P}(\hat{R})$ and again by (5. 3) $M\otimes_R N\in\mathscr{I}(R)$.

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References

[1] Bass, H., On the ubiquity of Gorenstein rings, Math. Z. 82.
[2] Cartan, H. and Eilenberg, S., Homological algebra, Prinston.
[3] Foxby, H.-B., Quasi-perfect modules over Cohen-Macaulay rings. Math. Nachrichten. 66, 1975.
[4] Herzog, J. and Kunz, E., Der Kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Math. 238 Springer 1971
[5] Herzog, J., Komplex, Auflösungen und Dualität in der lokalen Algebra, in preprint.
[6] Grothendieck, A., Local Cohomology, Lecture Notes in Math. 41 Springer 1967.
[7] Hochster M., Topics in the homological theory of modules over comutative rings, Conference Board of the Mathematical Sciences 24 A. M. S.
[8] Peskine, C. and Szpiro L., Dimension projective finie et cohomologie locale, I. H. E. S. Publ. Math. 42, 1973.
[9] Sharp, R. Y, Local cohomology theory in commutative algebra Quart. J. Nath Oxford (2), 21 (1970).
[10] Sharp, R. Y., Some results on the vanishing of local cohomology modules, Proc. London Math. Soc. (3) 30 (1975).
[11] Sharp, R. Y., The construction of a module of finite projective dimension from a finitely generated module of finite injective dimension, Comment. Math. Helvetici 50 (1975).
[12] Sharp, R. Y, Finitely generated modules of finite injectve dimension over certain Cohen-Macaulay rings, Proc. London Math. Soc. (3) 25. (1972).