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Deformations and types of some Riemann surfaces of infinite genus

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Introduction. In this paper we shall investigate the Riemann surfaces of infinite genus, in particular, the surface of the class $O_{HD}-O_G$, that is hyperbolic but has no non-constant harmonic functions with finite Dirichlet integrals. It is well known that these surfaces have many complicated properties. The compactification theory, it is true, have made them clear to some extent (cf. Constantinescu-Cornea [2], Sario-Nakai [9]). It seems, however, that they are not sufficiently clarified.

First we consider the regular Green lines on a hyperbolic Riemann surface R issuing from a point $z_0 \in R$. Let K_0 be a parametric disk containing the point z_0 and $\lambda(\alpha, \beta)$ be the extremal distance in $R-K_0$ between two regular Green lines with angles α, β. Now we shall define $\delta(z_0, R)$ by the integral mean of $\lambda(\alpha, \beta)$. Intuitively speaking, it will represent the mean diameter of the ideal boundary of the Riemann surface R (for the precise definition, see p. 412). By using the potential theory on the Kuramochi compactification, we get then a theorem; *a Riemann surface R belongs to the class* $O_{HD}-O_G$ *if and only if* $\delta(z_0, R)$ *vanishes for some (any)* z_0 (section 1).

Next we deform a regular hyperbolic Riemann surface R by squeezing along Green lines. More precisely, we cut R along some Green lines to obtain a planar subregion that is mapped conformally onto the unit disk with radial or incised radial slits clustering only to the circumference. By a natural conformal sewing of those slits, we obtain a Riemann surface which is conformally equivalent to R. We deform the slits radially according to a real parameter t. For each t we obtain a Riemann surface $R(t)$ by the conformal sewing of the disk with so deformed slits. We can consider, for each $R(t)$, the extremal distances $\lambda(t; \alpha, \beta)$ and the quantity $\delta(t)=\delta(z_0, R(t))$. Then it will be shown that $\delta(t)$ *moves upper semicontinuously* (section 2).

Finally, we shall give examples of deformations, for which $\delta(t)$ moves continuously or discontinuously (section 3).

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1. Let R be a hyperbolic Riemann surface. Fix a point z_0 in R and denote by $g(z, z_0)$ the Green function of R with pole at z_0. Consider the single-valued function $r(z)=\exp(-g(z, z))$ and the differential $d\theta(z)=-{}^*dg(z, z_0)$. Clearly $0\leqq r(z)<1$ for $z\in R$. Set for $0<\rho<1$

$$G_\rho=\{z\in R;\ r(z)<\rho\},\ C_\rho=\partial G_\rho.$$

Although $\theta(z)$ is not single-valued in $R-\{z_0\}$, it is harmonic locally on $R-\{z_0\}$. An open arc γ is called a Green arc if $d\theta\neq 0$ on γ and a branch of θ is constant on γ. The totality of Green arcs is partially ordered by inclusion. In this sense a maximal Green arc is called a *Green line.* Denote by

$$\boldsymbol{G}=\boldsymbol{G}(R, z_0)$$

the set of Green lines L issueing from z_0; then $z_0\in\bar{L}$.

For a sufficiently small ρ, G_ρ is regular and relatively compact in R and $\bar{G}_\rho$ is mapped conformally onto $\{|w|\leqq 1\}$ by the single-valued function $w=f(z)=\frac{1}{\rho}r(z)\exp(i\theta(z))$. Hereafter we fix such a ρ and use the notation $K_0=\bar{G}_\rho$. Each point z on $\partial K_0=C_\rho$ is represented by the coordinate $\theta\in[0, 2\pi)$ where $z=f^{-1}(\exp(i\theta))$. Using this we designate $L\in\boldsymbol{G}(R, z_0)$ by

$$L=L_\theta$$

where θ is the coordinate of the point $L\cap K_0$. We may write

$$\boldsymbol{G}(R, z_0)=\{L_\theta;\ \theta\in[0, 2\pi)\}.$$

For $L_\alpha, L_\beta\in\boldsymbol{G}(R, z_0)$, we denote

$$\Gamma(L_\alpha, L_\beta)$$

by the set of *locally rectifiable curves*

$$c=\{z(t);\ 0<t<1\}$$

in $R-K_0$ *which join* L_α and L_β; that is, c is relatively compact,

$$z(t)\in R-K_0-L_\alpha-L_\beta \qquad 0<t<1,$$

$$\bigcap_{\varepsilon>0}\overline{\{z(t);\ 0<t<\varepsilon\}}\subset L_\alpha\cap(R-K_0),$$

$$\bigcap_{\varepsilon>0}\overline{\{z(t);\ 1-\varepsilon<t<1\}}\subset L_\beta\cap(R-K_0).$$

Of course, $R-K_0-L_\alpha-L_\beta$ may have two components.

Let $\{R_n\}_{n=1}^\infty$ be a regular exhaustion of R with $R_1\supset K_0$. For $L_\alpha\in\boldsymbol{G}$

$$L_\alpha^{(n)}=L_\alpha\cap(\bar{R}_n-K_0)$$

is composed of a finite number of analytic arcs. We denote

$$\Gamma(L_\alpha^{(n)}, L_\beta^{(n)})$$

by the curve family joining $L_\alpha^{(n)}$ and $L_\beta^{(n)}$ in $R-K_0$ in the same way. For a curve family Γ in $R-K_0$, $\lambda(\Gamma)$ means the *extremal length* of the family Γ (cf. Sario-Oikawa [8]).

Lemma 1. (*Continuity lemma, Suita* [10]). *Let Γ, Γ_1, Γ_2,... be curve families in a Riemann surface. If $\Gamma_1 \subset \Gamma_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} \Gamma_n = \Gamma$, then*

$$\lim_{n\to\infty} \lambda(\Gamma_n) = \lambda(\Gamma).$$

Lemma 2. $\lim_{n\to\infty} \lambda(\Gamma(L_\alpha^{(n)}, L_\beta^{(n)})) = \lambda(\Gamma(L_\alpha, L_\beta))$.

Proof. We denote $\Gamma_n(L_\alpha, L_\beta)$ by the set of curves in $\Gamma(L_\alpha, L_\beta)$ contained in $R_n - K_0$. Then,

$$\Gamma_1(L_\alpha, L_\beta) \subset \Gamma_2(L_\alpha, L_\beta) \subset \cdots,$$

and

$$\bigcup_{n=1}^{\infty} \Gamma_n(L_\alpha, L_\beta) = \Gamma(L_\alpha, L_\beta).$$

By Lemma 1,

$$\lim_{n\to\infty} \lambda(\Gamma_n(L_\alpha, L_\beta)) = \lambda(\Gamma(L_\alpha, L_\beta)).$$

Since

$$\Gamma_n(L_\alpha, L_\beta) \subset \Gamma(L_\alpha^{(n)}, L_\beta^{(n)}),$$

we obtain that

$$\lambda(\Gamma_n(L_\alpha, L_\beta)) \geqq \lambda(\Gamma(L_\alpha^{(n)}, L_\beta^{(n)})).$$

While, for any $c \in \Gamma(L_\alpha^{(n)}, L_\beta^{(n)})$, there exists $c' \in \Gamma(L_\alpha, L_\beta)$ with $c \supset c'$. Thus,

$$\lambda(\Gamma(L_\alpha^{(n)}, L_\beta^{(n)})) \geqq \lambda(\Gamma(L_\alpha, L_\beta)),$$

and

$$\lim_{n\to\infty} \lambda(\Gamma(L_\alpha^{(n)}, L_\beta^{(n)})) = \lambda(\Gamma(L_\alpha, L_\beta)).$$

We call the normalized Lebesgue measure

$$dm(\theta) = \frac{1}{2\pi} d\theta$$

on $\partial K_0 = [0, 2\pi)$ a Green measure. For each $L_\theta \in \boldsymbol{G}(R, z_0)$ we write

$$d_\theta = \sup\{r(z);\ z \in L_\theta\}.$$

Clearly $\rho \leqq d_\theta \leqq 1$. If $d_\theta < 1$ then we call L_θ a singular Green line, otherwise regular. We know that $\boldsymbol{E} = \boldsymbol{E}(R, z_0) = \{\theta;\ L_\theta \text{ is singular}\}$ is an F_σ-set in $[0, 2\pi)$ and $m(\boldsymbol{E}) = 0$ (Brelot-Choquet [1]). Hereafter we use an abbrebiation

$$\lambda(\Gamma(L_\alpha, L_\beta))=\lambda(\alpha, \beta).$$

Let R_D^* be the *Royden compactification of* R, *and* Δ_D be the *harmonic boundary* (cf. [2], [9]).

Lemma 3. (*Nakai* [5]). *For every open set* U *in* R_D^*-R

$$\underline{m}\{\theta;\ (\bar{L}_\theta^D-L_\theta-\{z_0\})\subset U\}\geqq\mu(U),$$

where $\underline{m}$ *is the inner measure induced by* m, μ *is the harmonic measure of* R_D^*-R *with respect to* z_0 *and* $\bar{A}^D$ *means the closure of* A *in* R_D^*.

Let dm × dm be the product measure on $[0, 2\pi)\times[0, 2\pi)$ induced by m, and $\boldsymbol{I}=\{\theta\in[0, 2\pi);\ L_\theta$ is regular$\}$.
The set $\boldsymbol{I}$ is a G_δ-set and $m(\boldsymbol{I})=1$.

Theorem 1. *The function* $\lambda(\alpha, \beta)$ *of* (α, β) *is upper-semi-continuous on* $\boldsymbol{I}\times\boldsymbol{I}$.

Proof. For $\alpha\in\boldsymbol{I}$, $L_\alpha(r)=L_\alpha\cap\{z;\ \rho\leqq r(z)\leqq r\}$ $(\rho<r<1)$ is a compact set in R, and $\bigcup_{r>\rho} L_\alpha(r)=L_\alpha-\operatorname{int}K_0$. Fix such an r. There exist planar neighbourhoods U_α, V_α of $L_\alpha(r)$ with $U_\alpha\subset\bar{U}_\alpha\subset V_\alpha$ and a sufficiently small $\eta_\alpha>0$ such that $L_\theta(r)\subset U_\alpha$ for $|\theta-\alpha|<\eta_\alpha$, $\theta\in\boldsymbol{I}$. Similarly for $\beta\in\boldsymbol{I}$ $(\alpha\neq\beta)$ there exist U_β, V_β, η_β and $L_{\theta'}(r)\subset U_\beta$ for $|\theta'-\beta|<\eta_\beta$, $\theta'\in\boldsymbol{I}$. We may assume that $\bar{V}_\alpha\cap\bar{V}_\beta=\phi$. It is not difficult to construct a quasi-conformal mapping $\Phi_{\theta,\theta'}(z)$ of $R-K_0$ onto itself such that $\Phi_{\theta,\theta'}|_{R-K_0-V_\alpha-V_\beta}$ is the identity mapping. $\Phi_{\theta,\theta'}$ maps $L_\alpha(r)$, $L_\beta(r)$ onto $L_\theta(r)$, $L_{\theta'}(r)$ respectively, and the maximal dilatation of $\Phi_{\theta,\theta'}$ converges to 1 for $(\theta, \theta')\to(\alpha, \beta)$, and moreover $\Phi_{\theta,\theta'}(\Gamma(L_\alpha(r), L_\beta(r)))=\Gamma(L_\theta(r), L_{\theta'}(r))$. Thus

$$\lim_{\substack{(\theta,\theta')\to(\alpha,\beta)\\ (\theta,\theta')\in\boldsymbol{I}\times\boldsymbol{I}}}\lambda(\Gamma(L_\theta(r), L_{\theta'}(r)))=\lambda(\Gamma(L_\alpha(r), L_\beta(r)))\qquad(\alpha, \beta)\in\boldsymbol{I}\times\boldsymbol{I}.$$

As in Lemma 2, for $(\theta, \theta')\in\boldsymbol{I}\times\boldsymbol{I}$

$$\lambda(\Gamma(L_\theta(r), L_{\theta'}(r)))\searrow\lambda(\Gamma(L_\theta, L_{\theta'}))\qquad(r\nearrow 1).$$

Theorem 2. *The function* $\lambda(\alpha, \beta)$ *is non-negative and bounded on* $[0, 2\pi)\times[0, 2\pi)$.

Proof. Since we have set $K_0=\{r(z)\leqq\rho\}$, for an appropriate $\rho'>\rho$ a planar annulus $\{\rho<r(z)<\rho'\}$ is mapped conformally onto $D=\{1<|w|<\rho'/\rho\}$ by the function $w=f(z)=\frac{1}{\rho}r(z)\exp(i\theta(z))$. Then $L_\alpha\cap\{\rho<r(z)<\rho'\}$ is represented in D as a radial cross cut, and

$$\lambda(\alpha, \beta)\leqq 2\pi\Big/\log\frac{\rho'}{\rho}.$$

Thus $\lambda(\alpha, \beta)$ is an integrable function on $[0, 2\pi)\times[0, 2\pi)$, and

$$\iint_{[0,2\pi)^2}\lambda(\alpha, \beta)dm(\alpha)dm(\beta)=\delta(R, z_0, K_0)=\delta(R, z_0)$$

exists.

Theorem 3. *A hyperbolic Riemann surface R belongs to the class* O_{HD} *if and only if* $\delta(R, z_0, K_0)=0$.

Proof (*Sufficiency*). If $R\notin O_{HD}$, then Δ_D has at least two points. Since μ is supported on Δ_D, there are mutually disjoint open sets U_1, U_2 of R_D^*-R with $\mu(U_i)>0$ $(i=1, 2)$. By Lemma 3, there exist mutually disjoint measurable sets $F_1, F_2\subset[0, 2\pi)$ such that $m(F_i)>0$ $(i=1, 2)$ and $\bar{L}_\alpha^D\cap\bar{L}_\beta^D-K_0=\phi$ for $(\alpha, \beta)\in F_1\times F_2$.
Hence there is a C^1-class Dirichlet function f on $R-K_0$ such that $f\leqq 0$ on $\bar{L}_\alpha^D\cap(R_D^*-K_0)$ and $f\geqq 1$ on $\bar{L}_\beta^D\cap(R_D^*-K_0)$ (cf. [8]). Consider the linear density $\rho|dz|=|\mathrm{grad} f|\,|dz|$ on $R-K_0$, then for any $c\in\Gamma(L_\alpha, L_\beta)$ $\int_c\rho|dz|\geqq 1$ and $\iint_{R-K_0}\rho^2dx\,dy=D_{R-K_0}(f)<\infty$. Thus $\lambda(\alpha, \beta)>0$ for $(\alpha, \beta)\in F_1\times F_2$ with $m^2(F_1\times F_2)>0$. This is a contradiction.

To prove necessity we need some preparations. We use the following notations and notions according to Constantinescu-Cornea [2]; the Kuramochi compactification R_K^* of R, the minimal points set Δ_1 of R_K^*-R, the Kuramochi capacity $\tilde{C}(\cdot)$, thin (dünn) sets, the Kuramochi kernel $\tilde{g}_a$ with pole at a and potentials $\tilde{p}$. The set Δ_0 means $R_K^*-R-\Delta_1$.

If $R\in O_{HD}-O_G$, then there is only one point $\{a\}$ in R_K^*-R with positive harmonic measure and $R_K^*-R-\{a\}$ has zero harmonic measure. We know that $\tilde{C}(\{a\})>0$ follows. Moreover almost all (with respect to the Green measure m) $L_\theta\in\boldsymbol{G}(R, z_0)$ terminates at $\{a\}$ in R_K^*, i.e.

$$m(\{\theta;\ \bar{L}_\theta^K-L_\theta-\{z_0\}=\{a\}\})=1,$$

where $\bar{A}^K$ means the closure of A in R_K^* ([4]).
Following conditions for a set $A\subset R_K^*$ are mutually equivalent (cf. [2]);

(1) A is thin at $\{a\}$

(2) If $A\ni a$ then $\tilde{C}(\{a\})=0$. If $\overline{A-(\Delta_0\cup\{a\})}^K\ni a$ then there is a potential $\tilde{p}$ such that $\tilde{p}(a)<\infty$, $\lim_{\substack{b\to a\\ b\in A-(\Delta_0\cup\{a\})}}\tilde{p}(b)=\infty$.

Lemma 4. *If* $R\in O_{HD}-O_G$, *then almost all* $L_\theta\in\boldsymbol{G}$ *is not thin at* $\{a\}$.

Proof. Assume that $\bar{L}_\theta^K-L_\theta-\{z_0\}=\{a\}$ and L_θ is not thin at $\{a\}$. Clearly $\overline{L_\theta-(\Delta_0\cup\{a\})}^K=\bar{L}_\theta^K\ni a$. Thus there is a potential $\tilde{p}$ such that $\tilde{p}(a)<\infty$ and $\lim_{\substack{b\to a\\ b\in L_\theta-(\Delta_0\cup\{a\})}}\tilde{p}(b)=\infty$. Hence $\lim_{\substack{b\to a\\ b\in\bar{L}_\theta^K-(\Delta_0\cup\{a\})}}\tilde{p}(b)=\infty$ and $\overline{\bar{L}_\theta^K-(\Delta_0\cup\{a\})}^K\supset\bar{L}_\theta^K\ni a$.
By (1), (2), $\bar{L}_\theta^K$ is thin at $\{a\}$, but $\tilde{C}(\{a\})=0$. This is a contradiction.

Lemma 5. *Let* $R\in O_{HD}-O_G$. *A Green line* L_α *is not thin at* $\{a\}$ *if and only if* $\bar{L}_\alpha^D\supset\Delta_D$.

Proof. If $R\in O_{HD}-O_G$, then Δ_D consists of only one point $\{a_D\}$ (cf. [9]).

Let ω be the harmonic measure of the ideal boundary of R with respect to $R-K_0$. Set

$\mathscr{F}=\{u;$ Dirichlet function on $R-K_0$, $u=\omega$ on $\partial K_0 \cup (L_\alpha - K_0)\}$, then $\mathscr{F}\ni\omega$. We know that L_α is not thin at $\{a\}$ if and only if $D_{R-K_0}(\omega)=\min\{D_{R-K_0}(u);\ u\in\mathscr{F}\}$ (cf. [2], [11]).

Assume that L_α is thin at $\{a\}$, then there is an HD-function ω_{L_α} on $R-K_0-L_\alpha$ with minimum Dirichlet integral among $\mathscr{F}$ such that $\omega_{L_\alpha}\neq\omega$. The function $\omega-\omega_{L_\alpha}$ is a non-constant HD-function on $D=R-K_0-L_\alpha$ with zero boundary value on ∂D. Hence, $D\notin SO_{HD}$. And $\bar{L}_\alpha^D \cap \{a\}=\phi$ (cf. [9]).

Conversely, assume $\bar{L}_\alpha \cap \{a_D\}=\phi$, then $D\notin SO_{HD}$ i.e. there exists a non-constant HD-function u on D with null boundary value on ∂D. This function u is uniquely determined under $u(a_D)=1$. Then, $\omega-cu\in\mathscr{F}(c$: real), and

$$(\omega-cu,\ \omega-cu)=(\omega,\ \omega)+c^2(u,\ u)-2c(\omega,\ u).$$

By the Royden decomposition of u on $R-K_0$, it is follows that

$$u=\omega+(u-\omega).$$

Hence, $(\omega,\ u-\omega)=0$ and $(\omega,\ u)=(\omega,\ \omega)>0$.
Thus for an appropriate $c>0$,

$$(\omega-cu,\ \omega-cu)<(\omega,\ \omega).$$

That is, L_α is thin at $\{a\}$.
Here, (,) means the inner product by Dirichlet integrals on $R-K_0$.

(*Proof of the necessity*) Let $R\in O_{HD}-O_G$ and $\delta(R,\ z_0,\ K_0)>0$. By Lemmata 4, 5, there are $\alpha,\ \beta\in\boldsymbol{I}\ (\alpha\neq\beta)$ such that $\bar{L}_\alpha^D\cap\bar{L}_\beta^D\ni a_D$ and $\lambda(\alpha,\ \beta)>0$. We can prove the existence of a bounded continuous Dirichlet function u on R such that $u=1$ on $L_\alpha-K_0$ and $u=0$ on $L_\beta-K_0$. While, u is extendable continuously on R_D^*. This is a contradiction. (*Existence of u.*) Let u_n be an HBD-function on $R-K_0-L_\alpha^{(n)}-L_\beta^{(n)}$ such that $u_n=1$ on $L_\alpha^{(n)}$, $u_n=0$ on $L_\beta^{(n)}$, $\partial u_n/\partial v=0$ on ∂K_0(normal derivatives) and u_n has $\mathscr{L}_0$ – behaviour near the ideal boundary, where $\mathscr{L}_0$ is a principal operator (cf. [7]). Then,

$$\lambda(\Gamma(L_\alpha^{(n)},\ L_\beta^{(n)}))=D_{R-K_0}(u_n)^{-1}\geqq\lambda(\alpha,\ \beta)>0.$$

For $m>n$,

$$D_{R-K_0}(u_m,\ u_n)=\int_{L_\alpha^{(m)}+L_\beta^{(m)}} u_m{}^*du_n=\int_{L_\alpha^{(m)}}{}^*du_n=\int_{L_\alpha^{(n)}}{}^*du_n=D_{R-K_0}(u_n),$$

and

$$D_{R-K_0}(u_m-u_n)=D_{R-K_0}(u_m)-D_{R-K_0}(u_n).$$

While,

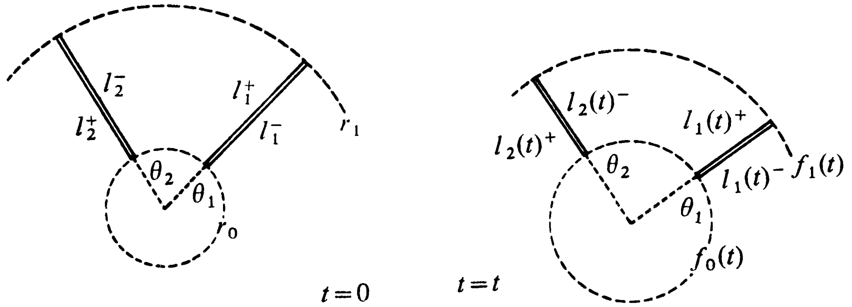
$$D_{R-K_0}(u_n)\leqq D_{R-K_0}(u_m)\leqq\lambda(\alpha,\ \beta)^{-1}<\infty.$$

Thus, $D_{R-K_0}(u_m-u_n)\to 0$ $(m, n\to\infty)$.
Hence, $\{u_n\}$ converges to an *HBD*-function u compact-uniformly on $R-K_0$ and $u=1$ on $L_\alpha\cap(R-K_0)$, $u=0$ on $L_\beta\cap(R-K_0)$.

2. Hereafter, we assume that *R is a regular Riemann surface.* The critical points of $g(z, z_0)$ are at most countable and isolated in R. We consider a deformation of R such that the critical points move continuously along Green lines.

First, cut R according to the cutting process of Sario (Sario [7]) along Green lines issueing from some critical points to obtain a planar subregion G, then $f(z)=r(z)\exp(i\theta(z))$ is single-valued on G and maps G onto the unit disk without radial and incised radial slits clustering nowhere in the unit disk, say D. The function $f(z)$ is a homeomorphism between $G\cup\partial G$ and $D\cup\partial D$ in the sense of the prime ends. The prime ends of G are identified in R as points of ∂G. We obtain a surface $S(D)$ conformally equivalent to R by sewing the sides of the slits of D according to the correspondence between the prime ends of G and D by $f(z)$. In this sewing, the sides identified have same r-coordinates in the sense of the polar coordinate system in the unit disk, and the minus sides correspond to the plus sides, where we say plus or minus with respect to the argument. Clearly the end points of the slits correspond to the critical points of g. The converse is not true and in each slit there are a finite number of points corresponding to the critical points.

We move radially the points of the slits corresponding to the critical points instead of deforming the holes representing genus of $R=S(D)$ along Green lines. In this case the classes of slits identified deform by the same parameters. For instance, consider two slits $l_i=\{r_0\leqq r\leqq r_1, \theta=\theta_i\}$ $(i=1, 2)$ and sew l_1^+ side on l_2^- and l_2^+ side on l_1^- side, and consider a pair of continuous functions $f_0(t), f_1(t)$ $(t\geqq 0)$ such that $f_0(0)=r_0, f_1(0)=r_1, f_0(t)<f_1(t)$. At time t, l_i is deformed to $l_i(t)=\{f_0(t)\leqq r\leqq f_1(t), \theta=\theta_i\}$ and $l_1(t)^+$, $l_1(t)^-$ are sewn on $l_2(t)^-$, $l_2(t)^+$ respectively. Let $D(t)$ be the slit region at time t. Then we get a Riemann surface $R(t)$ by the conformal sewing of the slit region $D(t)$. At $t=0$, $R(0)=R=S(D)$. For the simplicity of the description, we assume that *the critical points of the Green function correspond to the end points of the slits.*



Here we treat the case that the *slits are contracted*, that is, in the example above, $f_0(t)$ increases and $f_1(t)$ decreases monotonously as time passes.

Thus, for each class of slits identified, we give a pair of functions $\{f_{0,n}(t), f_{1,n}(t)\}_n$ such that;

(1) $f_{0,n}(t)$, $f_{1,n}(t)$ are continuous on $t \geqq 0$,

(2) $f_{0,n}(t) < f_{1,n}(t)$,

(3) $f_{0,n}(t)$ monotonously increases, $f_{1,n}(t)$ monotonously decreases,

(4) $\{f_{0,n}(0), f_{1,n}(0)\}_n$ represents the r-coordinates of the end points of the slits in $D(0)$ or $R(0)$,

(5) $f_{1,n}(t) \equiv 1$ for incised slits.

Since the Riemann surface $R(t)$ is obtained from $D(t)$ the unit disk without radial slits and incised radial slits by the conformal sewing, the function $-\log r$ is the Green function of $R(t)$ with pole at 0. Thus a radious which doesn't cross the slits is a regular Green line in $R(t)$ and vice versa. Hence, we may assume that *regular Green line* L_α *which is represented by* $\{0 \leq r < 1, \theta = \alpha\}$ *is common to each* $R(t)$. Moreover we may assume that the neighbourhood K_0 of $z_0 = 0$ is common to each $R(t)$. Even when R is not regular, we can obtain $S(D)$ and also deform R. We, however, need additional conditions for these deformations (cf. [3]).

Let $\Gamma(t; L_\alpha, L_\beta)$ be the family of curves joining L_α and L_β in $R(t) - K_0$ then $\Gamma(0; L_\alpha, L_\beta) = \Gamma(L_\alpha, L_\beta)$. Define $\lambda(t; \alpha, \beta) = \lambda(\Gamma(t; L_\alpha, L_\beta))$, then $\lambda(0; \alpha, \beta) = \lambda(\alpha, \beta)$. Let $\{r_n\}$ be a sequence such that $r_1 < r_2 < \cdots r_n \nearrow 1 (n \to \infty)$ and set $R_n = \{z; r(z) < r_n\}$. We assume that $R_1 \supset K_0$. Note that regular exhaustion $\{R_n\}$ is common to each $R(t)$. We denote $R_n \cap R(t)$. We can consider the curve families $\Gamma(t; L_\alpha^{(n)}, L_\beta^{(n)})$, $\Gamma_n(t; L_\alpha, L_\beta)$ in each $R(t)$ as in sec. 1. Thus for fixed t,

$$\lim_{n\to\infty} \lambda(\Gamma(t; L_\alpha^{(n)}, L_\beta^{(n)})) = \lim_{n\to\infty} \lambda(\Gamma_n(t; L_\alpha, L_\beta)) = \lambda(t; \alpha, \beta) \quad \alpha, \beta \in \boldsymbol{I}.$$

The sequence $\lambda(\Gamma_n(t; L_\alpha, L_\beta))$ is monotonously decreasing.

Theorem 5. *For* $\alpha, \beta \in \boldsymbol{I}$, $\lambda(t; \alpha, \beta)$ *is an upper-semi-continuous function of* t.

Proof. We see the continuity of $\lambda(\Gamma_n(t; L_\alpha, L_\beta))$ with respect to t.

If the end points of the slits don't cross ∂R_n at $t = t_0$, then the finite Riemann surfaces $R_n \cap R(t)$ ($|t - t_0| < \delta$) are quasi-conformally equivalent for a suitable δ. As in theorem 1, consider the neighbourhoods U_i ($i = 1, 2, \ldots, k(n)$) of slits in $D(t)$ and construct quasi-conformal mappings Φ_t: $R_n \cap R(t) \to R_n \cap R(t_0)$ such that $\Phi_t|_{R_n \cap R(t) - \cup U_i} =$ identity and $\Phi_t(\Gamma_n(t; L_\alpha, L_\beta)) = \Gamma_n(t_0; L_\alpha, L_\beta)$ and the maximal dilatation of Φ_t converges to 1 as $t \to t_0$. Then the continuity of $\lambda(\Gamma_n(t; L_\alpha, L_\beta))$ at $t = t_0$ is easily proved.

When the end points of the slits cross ∂R_n at $t = t_0$, we consider, representively, the slit which is in R_n for $t_0 < t < t_0 + \varepsilon$ and crosses R_n for $t \leqq t_0$. It is clear that $\lim_{t \to t_0 - 0} \lambda(\Gamma_n(t; L_\alpha, L_\beta)) = \lambda(\Gamma_n(t_0; L_\alpha, L_\beta))$.

For simplicity we assume that only two slits are sewn and these slits cross ∂R_n. Let $l_i = \{f_0(t_0) \leqq r \leqq f_1(t_0), \theta = \theta_i\}$ ($i = 1, 2$) be slits at $t = t_0$ and be deformed to $l_i(t) = \{f_0(t) \leqq r \leqq f_1(t), \theta = \theta_i\}$. Then $f_0(t_0) \leqq f_0(t), f_1(t_0) = r_n \geqq f_1(t)$. We may assume that $r_n > f_1(t)$. Denote slits $l'_n(t) = \{f_1(t) \leqq r \leqq r_n, \theta = \theta_i\}$. Divide $\Gamma_n(t) = \Gamma_n(t; L_\alpha, L_\beta)$ into two classes

$$\Gamma'_n(t) = \{c \in \Gamma_n(t); c \text{ does not cross } l'_i(t) \quad i = 1, 2\}$$

$$\Gamma''_n(t)=\{c\in\Gamma_n(t);\ c \text{ crosses } l'_1(t) \text{ or } l'_2(t)\}$$

Then

$$\lambda(\Gamma'_n(t))^{-1}+\lambda(\Gamma''_n(t))^{-1}\geqq\lambda(\Gamma_n(t))^{-1}\geqq\lambda(\Gamma'_n(t))^{-1}.$$

It is not difficult to show that

$$\lambda(\Gamma'_n(t))\to\lambda(\Gamma_n(t_0)) \qquad \text{for} \quad t\to t_0+0,$$

and

$$\lambda(\Gamma''_n(t))\to\infty \qquad (t\to t_0+0).$$

Thus

$$\lambda(\Gamma_n(t;\ L_\alpha,\ L_\beta))\to\lambda(\Gamma_n(t_0;\ L_\alpha,\ L_\beta)) \qquad t\to t_0.$$

Similarly we can prove other cases.

Clearly $\lambda(t;\ \alpha,\ \beta)$ being uniformly bounded with respect to t, α, β, and upper-semi-continuous with respect to $(\alpha,\beta)\in \boldsymbol{I}\times\boldsymbol{I}$ for fixed t, there exists a quantity

$$\iint_{[0,2\pi)^2}\lambda(t;\ \alpha,\ \beta)dm(\alpha)dm(\beta)=\delta(t).$$

Theorem 6. *The function* $\delta(t)$ *is an upper-semi-continuous function of* t $(t\geqq 0)$.

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence converging to t_0, then Lebesgue's theorem gives,

$$\overline{\lim_{n\to\infty}}\,\delta(t_n)=\overline{\lim_{n\to\infty}}\iint_{[0,2\pi)^2}\lambda(t_n;\ \alpha,\ \beta)dm(\alpha)dm(\beta)$$

$$\leqq\iint_{I\times I}\overline{\lim_{h\to\infty}}\,\lambda(t_n;\ \alpha,\ \beta)dm(\alpha)dm(\beta)\leqq\iint_{[0,2\pi)^2}\lambda(t_0;\ \alpha,\ \beta)dm(\alpha)dm(\beta)=\delta(t_0).$$

Corollary. If $\delta(t_0)=0$ (or, equivalently $R(t_0)\in O_{HD}-O_G$), then $\delta(t)$ is continuous at $t=t_0$.

3. In this section we concern the Riemann surface $R=R(0)$ obtained by the conformal sewing of a radial slit region whose slits are distributed in a sequence of annuli. Let $\{r_n\}$ be a sequence such that;

$$0<r_0<r_1<\cdots<r_{n-1}<r_n<\cdots$$

and $r_n\nearrow 1$ $(n\to\infty)$.

We assume that the slit region D in the former section has slits on

$$r_{2n}\leqq r\leqq r_{2n+1}$$

$$\theta=\theta_1,\ \theta_2,\dots,\ \theta_{k(n)} \quad n=0,\ 1,\dots,\ (k(n)<\infty).$$

We classify the slits such that the members in each class are on the same annulus, and identify the sides of slits as in the former section. Then the natural conformal structure makes a Riemann surface (cf. [12]). Moreover we assume that the slits being on the annuli $r_{2n} \leqq r \leqq r_{2n+1}$ are deformed according to the same parameters $\{f_{0,n}(t), f_{1,n}(t)\ (t \geqq 0)\}$ $n=0, 1, \ldots$. For simplicity, we fix $f_{0,n}(t)$ as the identity functions and decrease $f_{1,n}(t)$ so as to contruct the slits. We reparameterize this deformation as follows. Let $l_n = r_{2n+1} - r_{2n}$ be the length of slits at $t=0$ ($n=0, 1, \ldots$, and $b_n = r_{2n+2} - r_{2n+1}$ be the width of planar annuli $r_{2n+1} < |z| < r_{2n+2}$ at $t=0$. At time t, these quantities reduce to $\varphi_n(t) \cdot l_n = r_{2n+1}(t) - r_{2n}$ and $\psi_n(t) \cdot b_n = r_{2n+2} - r_{2n+1}(t)$ respectively, where $r_{2n+1}(t)$ is the r-coordinate of the point where r_n has gone at time t. The function $\varphi_n(t)$ (resp. $\psi_n(t)$) is defined on $t \geqq 0$,

$$1 \geqq \varphi_n(t) > 0 \qquad (\text{resp. } \psi_n(t) \geqq 1)$$

and monotonously decreasing (increasing) and

$$\varphi_n(t) \cdot l_n + \psi_n(t) \cdot b_n = r_{2n+2} - r_{2n}$$

are independent of t. Clearly,

Lemma 6. *Let $0 \leqq t_1 < t_2$. If there is an $M \geqq 1$ with*

$$M \geqq \varphi_n(t_1)/\varphi_n(t_2),\ \psi_n(t_1)/\psi_n(t_2) \geqq M^{-1},$$

then $R(t_1)$ and $R(t_2)$ are mutually quasi-conformally equivalent.

An estimation of the maximal dilatation of the quasi-conformal mapping gives easily;

Lemma 7. *If $\varphi_n(t)/\varphi_n(t_0) \to 1$ and $\psi_n(t)/\psi_n(t_0) \to 1$ uniformly with respect to n for $t \to t_0$, then $\delta(t) \to \delta(t_0)$.*

Now we concern the case that the types of Riemann surfaces change at $t=t_0$ and the behaviour of $\delta(t)$ at the point. We assume that,

$$R(t) \in O_{HD}(-O_G) \quad \text{and} \quad \delta(t)=0 \qquad t<t_0,$$

$$R(t) \notin O_{HD} \quad \text{and} \quad \delta(t)>0 \qquad t>t_0.$$

There are two cases;

(1) $R(t_0) \in O_{HD}$ i.e. $\delta(t_0)=0$,

(2) $R(t_0) \notin O_{HD}$ i.e. $\delta(t_0)>0$.

In case (1), $\delta(t)$ is continuous at $t=t_0$, in case (2) $\delta(t)$ is not continuous at $t=t_0$. If we assume that $r_{2n+2} - r_{2n+1} = r_{2n+1} - r_{2n}$ ($n=0, 1, \ldots$) at $t=0$, then the conditions for ψ_n in lemmata 6, 7 can be omitted. Even under this additional condition, according as Tôki [12], a sufficient number of slits and the ingenious sewings of the slit region give $R(0) \in O_{HB} - O_G \subset O_{HD} - O_G$. While, if we contruct the slits sufficiently, then $R(2) \notin O_{HD}$ (cf. [3]). We assume, in the case, that the length l_n of slits in $R(0)$ reduce to m_n in $R(2)$ at $t=2$, and the type changes at $t=1$.

Example for (1).

$$\varphi_n(t)=\begin{cases} 1 & 0\leq t\leq 1 \\ (m_n/l_n-1)\cdot n\cdot(t-1)+1 & 1\leq t\leq 1+1/n \\ m_n/l_n & 1+1/n\leq t \end{cases}$$

The function $\delta(t)$ is continuous on $t\geq 0$.

Example for (2).

$$\varphi_n(t)=\begin{cases} (m_n/l_n-1)\cdot t+1 & 0\leqq t\leqq 1 \\ m_n/l_n & 1\leqq t. \end{cases}$$

The function $\delta(t)$ is not continuous only at $t=1$.

In this case, $R(0)$ is quasi-conformally equivalent to $R(t)$ for $t<1$, but not to $R(1)$. We don't know yet whether $R(1)$ is a boundary point of the Teichmüller space containing $R(0)$ as an interior point.

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