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Radiation conditions and spectral theory for 2-body Schrödinger operators with "oscillating" long-range potentials I

——the principle of limiting absorption——

By

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Introduction

In this paper we shall consider the exterior boundary-value problem for the Schrödinger equation

(0.1) $$Lu = -\Delta u + V(x)u = \lambda u + f(x) \quad \text{in} \quad \Omega,$$

where Δ is the n-dimensional Laplacian and λ is contained in the continuous spectrum of L. Ω is a domain of $\boldsymbol{R}^n$ exterior to some smooth compact boundary $\partial\Omega$, on which we require the homogeneous Dirichlet or Robin boundary condition. The potential $V(x)$ is a real-valued function and is assumed to be decomposed as

$$V(x) = V_1(x) + V_2(x) + V_3(x),$$

where $V_3(x)$ represents short-range potentials:

(0.2) $$V_3(x) = O(r^{-1-\delta}) \ (r = |x|) \text{ at infinity for some } \delta > 0,$$

$V_2(x)$ represents long-range potentials without oscillation at infinity:

(0.3) $$V_2(x) = O(r^{-\delta}) \quad \text{and} \quad \partial_j V_2(x) = O(r^{-1-\delta}) \qquad (\partial_j = \partial/\partial x_j)$$

and $V_1(x)$ represents "oscillating" long-range potentials which satisfy the following conditions:

(0.4) $$V_1(x) = O(1),\ \partial_r V_1(x) = O(r^{-1}) \quad (\partial_r = \partial/\partial r) \quad \text{and}$$
$$\partial_r^2 V_1(x) + a V_1(x) = O(r^{-1-\delta}) \qquad \text{for some} \quad a \geq 0.$$

We require some more conditions on the angular derivatives of $V_1(x)$ (the precise conditions on $V(x)$ will be given in § 8).

The main purpose of this paper is to derive the unique existence of solutions of

(0.1) by means of the limiting absorption method. Of course we have to seek solutions in a class which contains $L^2(\Omega)$, whereas $f(x)$ should be chosen from a class contained in $L^2(\Omega)$. These classes will be characterized by a *radiation condition* at infinity. In this sense, to find the radiation condition attached to equation (0.1) is a most important problem of this paper.

As a consequence, we can show that there exists some real constant Λ_δ such that (Λ_δ, ∞) is contained in the absolutely continuous spectrum of the selfadjoint operator L in $L^2(\Omega)$. Λ_δ depends on the value $E(\gamma)$, $0<\gamma<\min\{4\delta, 2\}$, defined by

$$E(\gamma)=\limsup_{r\to\infty}\frac{1}{\gamma}\{r\partial_r V_1(x)+\gamma V_1(x)\}\,. \tag{0.5}$$

As we see by an example of von Neumann and Wigner (see Example II-3 of §9), we can not in general expect that $(0, \infty)$ becomes the continuous spectrum of L even if $V_1(x)$ itself behaves like $O(r^{-1})$ at infinity. In our previous work [11], we have proved that if we put

$$E_0=\inf_{0<\gamma<2} E(\gamma), \tag{0.6}$$

then in (E_0, ∞) is not contained the point spectrum of L. In general $\Lambda_\delta \geq E_0$ (see (8.2)).

The principle of limiting absorption can also be applied to eigenfunction expansions (or spectral representations) and scattering theory for the operator L, which will be studied in a forthcoming paper.

We note here that our results can be extended to a more general second order elliptic operators including the term of the Zeeman effect (see § 10).

The first rigorous proof of the principle of limiting absorption is due to Eidus [2] who dealt with equation (0.1) with $V(x)$ behaving like $O(r^{-(n+1+\delta)/2})$ $(\delta>0)$ at infinity. In these few years, Eidus' results have been generalized to short-range potentials by Jäger [5], Saito [12], Agmon [1], Kuroda [7] and Mochizuki [9], and to "non-oscillating" long-range potentials by Ikebe-Saito [4] and Lavine [8].

Here we give some simple examples which satisfy (0.4) and are not covered by any previous result.

$$V(x)=\frac{c}{\log r}\,, \tag{0.7}$$

$$V(x)=\sin(\log r)\,, \tag{0.8}$$

$$V(x)=\frac{c\sin br}{r}\,, \tag{0.9}$$

where b, c are real constants. We have $\Lambda_\delta=E_0=0$ for (0.7) and $\Lambda_\delta=E_0=\sqrt{5}/2$ for (0.8). However, $\Lambda_\delta=E_0+b^2/4$, where $E_0=|bc|/2$, for (0.9). Note that for the first two examples we have $a=0$ in (0.4). On the other hand, for the last example $a=b^2$. Thus, oscillation at infinity of the potential may have bad influence on the continuous spectrum.

The principle of limiting absorption is based on the selfadjointness of the operator L and some uniqueness results for equation (0.1). It is not difficult to show that $L=-\Delta+V(x)$ and the boundary condition determine a selfadjoint operator in $L^2(\Omega)$ if we restrict the near singularity behavior of $V(x)$. On the other hand, the desired uniqueness theorem has been proved as a *growth property* of solutions in our previous papers (Uchiyama [13], Mochizuki [10] and Mochizuki-Uchiyama [11]). The radiation condition will bridge the uniqueness theorem and the existence theorem (cf., Eidus [2]), and will be defined for solutions u of (0.1) as follows:

$$(0.10)\qquad u\in L^2_{\frac{-1-\alpha}{2}}(\Omega)\quad\text{and}\quad \partial_r u+k_\pm(x,\lambda)u\in L^2_{\frac{-1+\beta}{2}}(\Omega)$$

for some $\alpha>0$ and $\beta>0$, where $L^2_\nu(\Omega)$ $(-\infty<\nu<\infty)$ is the weighted L^2-space given in § 1 and $k_\pm(x,\lambda)$ is a complex-valued function which solves the equation

$$(0.11)\qquad V(x)-\lambda+\partial_r k_\pm(x,\lambda)+\frac{n-1}{r}k_\pm(x,\lambda)-k_\pm(x,\lambda)^2=O(r^{-1-\delta})$$

for $r=|x|$ large. Once the radiation condition (0.10) is defined well, we can follow almost the same line of proof given in Mochizuki [9] for short-range potentials.

The remainder of this paper will be organized as follows: The first 7 sections will develop a semi-abstract theory under assuming the existence of the solution $k_\pm(x,\lambda)$ of (0.11). In § 1 we first summarize the required properties of $k_\pm(x,\lambda)$ as assumptions, and then state the main results (Theorems $1\sim5$) without proof. These theorems are proved in § 3 ~ § 7. § 2 is devoted to prove some propositions which will be used to show the theorems. The concrete form of the conditions on $V(x)$ is given in § 8. The existence and required properties of $k_\pm(x,\lambda)$ are proved there. In § 9 we give several examples. As is noted above, our results can be generalized to a more general second order elliptic operators if we give a slight modification of the radiation condition. We discuss them in § 10. Finally, in Appendix, we explain how we get to equation (0.11) to generalize the original Sommerfeld radiation condition, and how we find the 'special' solution $k_\pm(x,\lambda)$.

§ 1. Notation and Results

1. 1. Notation. First we shall list the notation which will be used freely in the sequel.

$\boldsymbol{R}$ is all real numbers; $\boldsymbol{C}$ is all complex numbers. $\mathrm{Re}\,\kappa$ and $\mathrm{Im}\,\kappa$ denote the real part and imaginary part of $\kappa\in\boldsymbol{C}$, respectively; $\sqrt{\kappa}$ denotes the branch of the square root of $\kappa\in\boldsymbol{C}$ with $\mathrm{Im}\sqrt{\kappa}\geq0$; $i=\sqrt{-1}$.

$$f\cdot g=\sum_{j=1}^n f_j g_j\quad\text{for}\quad f=(f_1,\dots,f_n)\quad\text{and}\quad g=(g_1,\dots,g_n).$$

$|f|=\sqrt{f\cdot\bar f}$ for $f=(f_1,\dots,f_n)\in\boldsymbol{C}^n$, where $\bar f_j$ is the complex conjugate of f_j and $\bar f=(\bar f_1,\dots,\bar f_n)$.

$$x=(x_1,\dots,x_n)\in\boldsymbol{R}^n,\quad r=|x|\quad\text{and}\quad \tilde x=x/|x|.$$

$$S(R)=\{x;\ |x|=R\} \quad \text{for} \quad R>0.$$

$$B(R,\ R')=\{x;\ R<|x|<R'\} \quad \text{for} \quad 0<R<R'.$$

$$B(R)=\{x;\ |x|>R\} \quad \text{for} \quad R>0.$$

$$\Omega(R)=\{x\in\Omega;\ |x|<R\} \quad \text{for} \quad R>0.$$

$$\partial_j=\partial/\partial x_j,\ \nabla=(\partial_1,\dots,\partial_n) \quad \text{and} \quad \partial_r=\tilde{x}\cdot\nabla.$$

For $G\subset\Omega$ and $\nu\in\boldsymbol{R}$, $L^2_\nu(G)$ denotes the Hilbert space of all functions $f(x)$ such that $(1+r)^\nu f(x)$ is square integrable in G; the innerproduct and norm of $L^2_\nu(G)$ are denoted by

$$(f,\ g)_{\nu,G}=\int_G(1+r)^{2\nu}f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\|_{\nu,G}=\sqrt{(f,f)_{\nu,G}},$$

respectively; in case $\nu=0$ or $G=\Omega$, we shall omit the subscript 0 or Ω as $L^2(G)$, $\|\ \|_G$, $\|\ \|_\nu$, $\|\ \|$ etc. $H^j(G)$ $(j=1,\ 2)$ denotes the class of L^2-functions in G such that all distribution derivatives up to j belong to $L^2(G)$. H^j_{loc} denotes the class of locally H^j-functions in $\bar{\Omega}=\Omega\cup\partial\Omega$. $C^j(G)$ is all j-times continuously differentiable functions in G; $C_0^\infty(G)$ is all C^∞-functions with support in G. Q_μ denotes the class of functions $V(x)$ satisfying the "Stummel condition", that is, for some $\mu>0$

$$\begin{cases} \sup\limits_{x\in\Omega}\int_{|x-y|<1}|V(y)|^2|x-y|^{-n+4-\mu}dy<\infty & (\text{if } n\geq 4) \\ \sup\limits_{x\in\Omega}\int_{|x-y|<1}|V(y)|^2dy<\infty & (\text{if } n\leq 3)\ . \end{cases}$$

1.2. Radiation condition. We consider the Schrödinger operator $-\Delta+V(x)$ in an infinite domain Ω in $\boldsymbol{R}^n$ with smooth compact boundary $\partial\Omega$ lying inside some sphere $S(R_0)$ of radius R_0. We do not exclude the case that $\partial\Omega$ is empty and $\Omega=\boldsymbol{R}^n$. Throughout this paper, we assume that $V(x)$ is a real-valued function belonging to Q_μ for some $\mu>0$. The differential operator $-\Delta+V(x)$ is regarded as acting on functions in H^2_{loc} satisfying the Dirichlet or Robin boundary condition

$$(1.1) \qquad Bu=\begin{Bmatrix} u \quad \text{or} \\ \nu\cdot\nabla u+d(x)u \end{Bmatrix}=0 \quad \text{on} \quad \partial\Omega$$

in the distribution sense in $\partial\Omega$, where $\nu=(\nu_1,\dots,\nu_n)$ is the outer unit normal to the boundary $\partial\Omega$ and $d(x)$ is a real-valued smooth function on $\partial\Omega$.

We define the operator L acting in the Hilbert space $L^2(\Omega)$ as follows:

$$(1.2) \qquad \begin{cases} \mathscr{D}(L)=\{u\in H^2(\Omega);\ Bu=0 \quad \text{on} \quad \partial\Omega\} \\ Lu=-\Delta u+V(x)u \qquad \text{for} \quad u\in\mathscr{D}(L)\,. \end{cases}$$

Then it is known (cf., e.g., Ikebe-Kato [3] and Mochizuki [9]) that *the operator L is selfadjoint and lower semi-bounded in $L^2(\Omega)$.*

In the following, we consider the boundary-value problem

$$(1.3)\qquad \begin{cases} -\Delta u + V(x)u - \zeta u = f(x) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

for suitably chosen $\zeta \in \boldsymbol{C}$ and $f(x) \in L^2(\Omega)$. The selfadjointness of L shows that if $\zeta \in \boldsymbol{C} \backslash \boldsymbol{R}$, (1.3) has a unique L^2-solution for given any $f(x) \in L^2(\Omega)$. To extend the unique existence theorem to real ζ contained in the continuous spectrum of L, it is necessary to consider the operator $-\Delta + V(x)$ in a wider class of functions in H^2_{loc}, whereas $f(x)$ should be chosen from a more restricted class. The domain of $-\Delta + V(x)$ will be characterized by means of a *radiation condition* at infinity.

The Sommerfeld radiation condition is used in Eidus [2] to determine the domain of $-\Delta + V(x)$ with $V(x)$ tending sufficiently rapidly to zero as $|x| \to \infty$. We shall improve this radiation condition so as to be able to apply to a wider class of potentials $V(x)$. For this purpose, the essential point is in obtaining a complex-valued function $k(x, \zeta)$ which solves the following "Riccati type equation" for $r = |x|$ large:

$$(1.4)\qquad V(x) - \zeta + \partial_r k(x, \zeta) + \frac{n-1}{r} k(x, \zeta) - k(x, \zeta)^2 = 0(r^{-1-\delta}),$$

where $\operatorname{Re}\zeta$ is in the continuous spectrum of L and δ is a positive constant. The function $k(x, \zeta)$ will be used to define a *modified* radiation condition. In Appendix, we shall explain how we get to the "Riccati type equation" (1.4).

To make clear the role of the function $k(x, \zeta)$, we assume first the existence and some properties of $k(x, \zeta)$, and construct a semi-abstract theory. After proving all the theorems, we give in § 8 the explicit form of conditions on $V(x)$ under which all the assumptions stated below are satisfied.

Assumption 1. There exist real constants $\delta > 0$, Λ_δ and a real function $\gamma(\lambda)$ of $\lambda > \Lambda_\delta$ such that

$$(1.5)\qquad 0 < \gamma(\lambda) < \min\{4\delta, 2\}$$

and the following *growth property* holds: Let $u \in H^2_{\text{loc}}$ satisfy the equation

$$(1.6)\qquad -\Delta u + V(x)u - \lambda u = 0 \quad \text{in } \Omega$$

with real $\lambda > \Lambda_\delta$. If we have the inequality

$$(1.7)\qquad \int_{B(R_0)} (1+r)^{-1+\beta} |u|^2 dx < \infty \quad \text{for some } \beta > \frac{1}{2}\gamma(\lambda),$$

then u must identically vanish in Ω.

Assumption 2. We put

$$(1.8)\qquad \begin{cases} \Pi_\delta^+ = \{\zeta \in \boldsymbol{C};\ \operatorname{Re}\zeta > \Lambda_\delta \ \text{ and } \ \operatorname{Im}\zeta \geq 0\} \\ \Pi_\delta^- = \{\zeta \in \boldsymbol{C};\ \operatorname{Re}\zeta > \Lambda_\delta \ \text{ and } \ \operatorname{Im}\zeta \leq 0\}, \end{cases}$$

where $\delta > 0$ and Λ_δ real are constants given in Assumption 1, and let $K^{\pm}$ be any compact set in $\Pi_\delta^\pm$. Then there exists an $R_1 = R_1(K^\pm) \geq R_0$ and a complex-valued *continuous* function $k(x, \zeta) = k_\pm(x, \zeta)$ defined for $(x, \zeta) \in B(R_1) \times K^\pm$ which satisfies equation (1.4), namely, if we put

$$(1.9) \qquad q_\pm(x, \zeta) = V(x) - \zeta + \partial_r k_\pm(x, \zeta) + \frac{n-1}{r} k_\pm(x, \zeta) - k_\pm(x, \zeta)^2,$$

then there exists a constant $C_1 = C_1(K^\pm) > 0$ such that

$$(\text{A2–1}) \qquad |q_\pm(x, \zeta)| \leq C_1 r^{-1-\delta} \quad \text{for any} \quad (x, \zeta) \in B(R_1) \times K^\pm.$$

Further, we assume that there exist constants $C_j = C_j(K^\pm) > 0\ (j = 2 \sim 5)$ and $\beta = \beta(K^\pm) > 0$ such that $k_\pm(x, \zeta)$ satisfies the following inequalities for any $(x, \zeta) \in B(R_1) \times K^\pm$:

$$(\text{A2–2}) \qquad |k_\pm(x, \zeta)| \leq C_2,$$

$$(\text{A2–3}) \qquad \mp \operatorname{Im} k_\pm(x, \zeta) \geq C_3,$$

$$(\text{A2–4}) \qquad \operatorname{Re} k_\pm(x, \zeta) - \frac{n-1-\beta}{2r} \geq C_4 r^{-1},$$

$$(\text{A2–5}) \qquad |\nabla k_\pm(x, \zeta) - \tilde{x}\partial_r k_\pm(x, \zeta)| \leq C_5 r^{-1-\delta}.$$

Finally, we assume that β in (A2–4) can be chosen as follows:

$$(\text{A2–6}) \qquad \frac{1}{2}\gamma(\operatorname{Re} \zeta) < \beta < 2\delta \ \text{ for any } \zeta \in K^\pm, \quad \text{and} \quad \beta \leq 1.$$

Now we fix any compact set $K^\pm$ in $\Pi_\delta^\pm$, and consider equation (1.3) with $\zeta \in K^\pm$ and $f(x) \in L^2_{\frac{1+\beta}{2}}(\Omega)$.

Definition 1.1. A solution u of (1.3) with $\zeta \in K^\pm$ is said to satisfy the (outgoing [or incoming]) radiation condition if we have

$$(1.10) \qquad u \in L^2_{\frac{-1-\alpha}{2}}(\Omega) \quad \text{and} \quad \partial_r u + k_\pm(x, \zeta)u \in L^2_{\frac{-1+\beta}{2}}(B(R_1)),$$

where R_1 and β are as in Assumption 2 and α is a positive constant satisfying

$$(1.11) \qquad \alpha + \beta \leq \min\{2\delta, 2\} \quad \text{and} \quad \alpha \leq \beta.$$

Definition 1.2. A solution u of (1.3) with $\zeta \in K^\pm$ is called an outgoing [or incoming] solution if it also satisfies the radiation condition (1.10).

Remark 1.1. The restriction $\beta \leq 1$ is not essential, and we can weaken the conditions on α and β in (1.10) as follows:

$$(1.12) \qquad \alpha + \beta \leq \min\{2\delta, 2\}, \quad 0 < \alpha \leq \beta \quad \text{and}$$

$$\frac{1}{2}\gamma(\operatorname{Re} \zeta) < \beta < \min\{2\delta, 2\} \ \text{ for any } \zeta \in K^\pm.$$

Thus, the apriori inequalities (Theorem 2 stated below) for outgoing [incoming] solutions of (1.3) can be slightly generalized in case $\delta > 1/2$. In this paper, however, we do not enter into this problem since (A2–4) on $k_{\pm}(x, \zeta)$ has to be replaced by a rather complicated condition.

1.3. Theorems. The main results of this paper are summarized in the following theorems.

Theorem 1 (*uniqueness*). *For any $\zeta \in K^+$ $[K^-]$ and $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$, the boundary-value problem* (1.3) *has at most one outgoing [incoming] solution.*

Theorem 2 (*apriori estimates*). *There exists a constant $C = C(K^{\pm}) > 0$ such that for any outgoing [incoming] solution $u = u(\cdot, \zeta)$ of* (1.3) *with $\zeta \in K^{\pm}$ and $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$, we have*

$$\|u(\zeta)\|_{\frac{-1-\alpha}{2}, B(R)} \leq CR^{-\alpha/2}\|f\|_{\frac{1+\beta}{2}} \text{ for any } R \geq R_1, \tag{1.13}$$

$$\|\nabla u(\zeta) + \tilde{x}k_{\pm}(x, \zeta)u(\zeta)\|_{\frac{-1+\beta}{2}, B(R_1)} \leq C\|f\|_{\frac{1+\beta}{2}}, \tag{1.14}$$

$$\|u(\zeta)\|_{\frac{-1-\alpha}{2}} \leq C\|f\|_{\frac{1+\beta}{2}}. \tag{1.15}$$

Theorem 3 (*principle of limiting absorption*). (a) *For any non-real $\zeta \in K^{\pm}$ and $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$, there exists a unique outgoing [incoming] solution of* (1.3), *which coincides with the L^2-solution.*

(b) *Let $\{\zeta_l\}_{l=1,2,\dots}$ be a non-real sequence in $K^{\pm}$ such that $\zeta_l \to \zeta_0 \in K^{\pm}$ as $l \to \infty$, let $\{f_l\}$ be a sequence in $L^2_{\frac{1+\beta}{2}}(\Omega)$ such that $f_l \to f_0$ strongly in $L^2_{\frac{1+\beta}{2}}(\Omega)$ as $l \to \infty$, and let $\{u_l\}$ be the corresponding sequence of outgoing [incoming] solutions of* (1.3) *with $\zeta = \zeta_l$ and $f = f_l$. Then $\{u_l\}$ converges in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$ to a function u_0 as $l \to \infty$, and u_0 becomes an outgoing [incoming] solution of* (1.3) *with $\zeta = \zeta_0$ and $f = f_0$.*

As a corollary of the above theorems, we have the following

Theorem 4 (*existence and property*). *For any $\zeta \in K^{\pm}$ and $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$, there exists a unique outgoing [incoming] solution $u = u(\cdot, \zeta, f)$ of* (1.3). *Moreover, if we define the operator $\mathscr{R}_\zeta$: $K^{\pm} \times L^2_{\frac{1+\beta}{2}}(\Omega) \ni (\zeta, f) \to u(\cdot, \zeta, f) \in L^2_{\frac{-1-\alpha}{2}}(\Omega)$ by*

$$[\mathscr{R}_\zeta f](x) = u(x, \zeta, f), \tag{1.16}$$

then $\mathscr{R}_\zeta f$ depends continuously on $(\zeta, f) \in K^{\pm} \times L^2_{\frac{1+\beta}{2}}(\Omega)$.

Finally, we return to the selfadjoint operator L defined by (1.2). For the spectrum of L, we have the following results.

Theorem 5. (a) *Let $\{\mathscr{E}(\lambda); \lambda \in \boldsymbol{R}\}$ denote the spectral measure of L, and for* *any Borel set* $e \Subset (\Lambda_\delta, \infty)$ *choose* α, β *and* $\mathscr{R}_\zeta$ *depending on* $K^\pm = \{\lambda \pm i\varepsilon;\ \lambda \in \bar{e},\ 0 \le \varepsilon \le 1\}$ *as above. Then for any* $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$ *and* $g \in L^2_{\frac{1+\alpha}{2}}(\Omega)$, *we have*

$$(1.17)\qquad (\mathscr{E}(e)f,\ g) = \frac{1}{2\pi i}\int_e (\mathscr{R}_{\lambda+i0}f - \mathscr{R}_{\lambda-i0}f,\ g)\,d\lambda,$$

where the integrand of the right side means the duality between $L^2_{-\frac{1-\alpha}{2}}(\Omega)$ *and* $L^2_{\frac{1+\alpha}{2}}(\Omega)$:

$$(1.18)\qquad (\mathscr{R}_{\lambda\pm i0}f,\ g) = \int_\Omega [\mathscr{R}_{\lambda\pm i0}f](x)\overline{g(x)}\,dx.$$

(b) *The part of* L *in* $\mathscr{E}((\Lambda_\delta, \infty))L^2(\Omega)$ *is absolutely continuous, i.e.,* $(\mathscr{E}(\lambda)f, f)$ *for* $f \in L^2(\Omega)$ *is absolutely continuous with respect to the Lebesgue measure in* $\lambda \in (\Lambda_\delta, \infty)$.

§2. Some estimating propositions

In this section, we prepare some estimating inequalities related to the outgoing [incoming] solutions of (1.3).

Proposition 2.1. *Let* $u \in L^2_\nu(\Omega)$, $\nu \in \boldsymbol{R}$, *be a solution of* (1.3) *with* $\zeta \in K^\pm$ *and* $f \in L^2_\nu(\Omega)$. *Then we have for some* $C_6 = C_6(K^\pm) > 0$

$$(2.1)\qquad \|\nabla u\|_\nu \le C_6\{\|u\|_\nu + \|f\|_\nu\}.$$

Proof. In the case where u satisfies the Dirichlet boundary condition, (2.1) can be proved by the same argument as in the proof of Lemma 1 of Mochizuki-Uchiyama [11]. For u satisfying the Robin boundary condition, we can also follow the same line of proof if we have the following: for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$(2.2)\qquad \int_{\partial\Omega} |d(x)|\,|u|^2 dS \le \int_{\Omega(R_0+1)} \{\varepsilon|\nabla u|^2 + C(\varepsilon)|u|^2\}dx$$

and

$$(2.3)\qquad \int_{\Omega(R_0)} |V(x)|\,|u|^2 dx \le \int_{\Omega(R_0+1)} \{\varepsilon|\nabla u|^2 + C(\varepsilon)|u|^2\}dx.$$

Inequality (2.2) is well known, and inequality (2.3) can be proved by use of near-singularity behaviors of the elementary solution of Δ and the fact that $V(x) \in Q_\mu$ (cf., e.g., Mochizuki [9]). q.e.d.

Proposition 2.2. *Let* u *be an outgoing [incoming] solution of* (1.3) *with* $\zeta \in K^\pm$ *and* $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$. *If* $\operatorname{Im}\zeta \ne 0$, *then we have* $u \in L^2(\Omega)$ *and*

$$(2.4)\qquad |\operatorname{Im}\zeta|\,\|u\| \le \|f\|.$$

Proof. By the Green formula, we have

$$\begin{aligned}\operatorname{Im}\int_{\Omega(R)} f\bar{u}\,dx &= \operatorname{Im}\int_{\Omega(R)} \{-\Delta u + V(x)u - \zeta u\}\bar{u}\,dx \\ &= -\operatorname{Im}\int_{S(R)} (\partial_r u)\bar{u}\,dS - \operatorname{Im}\zeta\int_{\Omega(R)} |u|^2 dx.\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\operatorname{Im}\zeta\int_{\Omega(R)} |u|^2 dx - \int_{S(R)} \operatorname{Im} k_\pm(x, \zeta)|u|^2 dS \\ = -\operatorname{Im}\int_{S(R)} \{\partial_r u + k_\pm(x, \zeta)u\}\bar{u}\,dS - \operatorname{Im}\int_{\Omega(R)} f\bar{u}\,dx.\end{aligned} \tag{2.5}$$

As we see in (A2–3), for $\zeta \in K^+$ $[K^-]$ both $\operatorname{Im}\zeta$ and $-\operatorname{Im} k_+(x, \zeta)$ $[-\operatorname{Im} k_-(x, \zeta)]$ are non-negative [non-positive] for $|x| \geq R_1$. Thus, we have from (2.5)

$$\begin{aligned}|\operatorname{Im}\zeta|\int_{\Omega(R)} |u|^2 dx \\ \leq \int_{S(R)} |\partial_r u + k_\pm(x, \zeta)u|\,|u|\,dS + \int_{\Omega(R)} |f|\,|u|\,dx\end{aligned} \tag{2.6}$$

for $R \geq R_1$. From the radiation condition (1.10) (since we have chosen $\alpha \leq \beta$) it follows that

$$\liminf_{R\to\infty} \int_{S(R)} |\partial_r u + k_\pm(x, \zeta)u|\,|u|\,dS = 0.$$

Thus, letting $R \to \infty$ in (2.6) and then dividing both sides by $\|u\|$, we obtain (2.4). q.e.d.

Proposition 2.3. *Let u be an outgoing [incoming] solution of* (1.3) *with $\zeta \in K^\pm$ and $f \in L^2_{\frac{1+\beta}{2}}(\Omega)$. Then there exists a $C_7 = C_7(K^\pm) > 0$ such that for any $R \geq R_1$*

$$\begin{aligned}\|u\|^2_{\frac{-1-\alpha}{2}, B(R)} \leq C_7(1+R)^{-\alpha}\{\|\partial_r u + k_\pm(x, \zeta)u\|^2_{\frac{-1+\beta}{2}, B(R)} \\ + \|u\|^2_{\frac{-1-\alpha}{2}} + \|f\|^2_{\frac{1+\alpha}{2}}\}.\end{aligned} \tag{2.7}$$

Proof. By (2.5) and (A2–3)

$$\begin{aligned}C_3\int_{S(R)} |u|^2 dS &\leq \int_{S(R)} |\operatorname{Im} k_\pm(x, \zeta)|\,|u|^2 dS \\ &\leq \int_{S(R)} |\partial_r u + k_\pm(x, \zeta)u|\,|u|\,dS + \int_{\Omega(R)} |f|\,|u|\,dx\end{aligned} \tag{2.8}$$

for $R \geq R_1$. Multiplying by $(1+R)^{-1-\alpha}$ on both sides of (2.8) and integrating over (R', ∞) with respect to R, where $R' \geq R_1$, we have

$$C_3\|u\|^2_{\frac{-1-\alpha}{2},B(R')}$$

$$\leq \int_{B(R')}(1+r)^{-1-\alpha}|\partial_r u+k_\pm(x,\zeta)u||u|dx$$

$$+\alpha^{-1}(1+R')^{-\alpha}\|u\|_{\frac{-1-\alpha}{2}}\|f\|_{\frac{1+\alpha}{2}}.$$

Since

$$\int_{B(R')}(1+r)^{-1-\alpha}|\partial_r u+k_\pm(x,\zeta)u||u|dx$$

$$=\int_{B(R')}(1+r)^{\frac{-\alpha-\beta}{2}}(1+r)^{\frac{-2-\alpha+\beta}{2}}|\partial_r u+k_\pm(x,\zeta)u||u|dx$$

$$\leq\frac{1}{2}(1+R')^{\frac{-\alpha-\beta}{2}}\{\|\partial_r u+k_\pm(x,\zeta)u\|^2_{\frac{-1+\beta}{2},B(R')}+\|u\|^2_{\frac{-1-\alpha}{2}}\},$$

noting that $\alpha\leq\beta$ and choosing $C_7=(1+\alpha^{-1})C_3^{-1}$, we obtain (2.7). q.e.d.

Proposition 2.4. *Let u be an outgoing [incoming] solution of (1.3) with $\zeta\in K^\pm$ and $f\in L^2_{\frac{1+\beta}{2}}(\Omega)$. Then there exist a constant $C_8=C_8(K^\pm)>0$ such that*

$$\|\nabla u+\tilde{x}k_\pm(x,\zeta)u\|_{\frac{-1+\beta}{2},B(R_1)}\leq C_8\{\|u\|_{\frac{-1-\alpha}{2}}+\|f\|_{\frac{1+\beta}{2}}\}. \tag{2.9}$$

Remark 2.1. The assumption that $\beta\leq 1$ (cf., (A2–6)) is essentially used only to prove this proposition.

Proof. For the sake of simplicity, we put

$$\theta=\theta(x,\zeta)=\nabla u+\tilde{x}k_\pm(x,\zeta)u. \tag{2.10}$$

Then it follows from the first equation of (1.3) that

$$\begin{aligned}\nabla\cdot\theta&=k_\pm\tilde{x}\cdot\theta+\left\{\Delta u-k_\pm^2u+(\partial_r k_\pm)u+\frac{n-1}{r}k_\pm u\right\}\\&=k_\pm\tilde{x}\cdot\theta+q_\pm(x,\zeta)u-f(x).\end{aligned} \tag{2.11}$$

Multiply by $\tilde{x}\cdot\bar{\theta}$ on both sides of (2.11) and take the real part. Then we have

$$-\mathrm{Re}\{\nabla\cdot\theta(\tilde{x}\cdot\bar{\theta})\}+\mathrm{Re}k_\pm|\tilde{x}\cdot\theta|^2+\mathrm{Re}\{(q_\pm u-f)\tilde{x}\cdot\bar{\theta}\}=0. \tag{2.12}$$

Here

$$-\mathrm{Re}\{\nabla\cdot\theta(\tilde{x}\cdot\bar{\theta})\}=-\mathrm{Re}\nabla\cdot\{\theta(\tilde{x}\cdot\bar{\theta})\}+\mathrm{Re}\{\theta\cdot\nabla(\tilde{x}\cdot\bar{\theta})\}$$

$$=-\mathrm{Re}\,\nabla\cdot\left\{\theta(\tilde{x}\cdot\bar{\theta})-\frac{1}{2}\tilde{x}|\theta|^2\right\}+\left(\mathrm{Re}k_\pm-\frac{n-1}{2r}\right)|\theta|^2$$

$$-\mathrm{Re}\,k_\pm|\tilde{x}\cdot\theta|^2+\frac{1}{r}\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}+\mathrm{Re}\{(\nabla k_\pm-\tilde{x}\partial_r k_\pm)\cdot\bar{\theta}u\},$$

since we have

$$\begin{aligned}\nabla(\tilde{x}\cdot\bar{\theta}) &= \nabla(\tilde{x}\cdot\nabla\bar{u}+\overline{k_{\pm}u})\\ &= \frac{1}{r}\{\nabla\bar{u}-\tilde{x}(\tilde{x}\cdot\nabla\bar{u})\}+(\tilde{x}\cdot\nabla)\nabla\bar{u}+\overline{k_{\pm}\nabla u}+\overline{(\nabla k_{\pm})u}\\ &= \frac{1}{r}\{\bar{\theta}-\tilde{x}(\tilde{x}\cdot\bar{\theta})\}+(\tilde{x}\cdot\nabla)\bar{\theta}-\tilde{x}(\overline{\tilde{x}\cdot\nabla k_{\pm})u}-\tilde{x}\overline{k_{\pm}}(\tilde{x}\cdot\bar{\theta})+\overline{k_{\pm}}\bar{\theta}+\overline{(\nabla k_{\pm})u},\end{aligned}$$

where $\tilde{x}\cdot\nabla=\partial_r$. Thus it follows from (2.12) that

$$\begin{aligned}\text{(2.13)}\qquad \operatorname{Re}\nabla\cdot\left\{\theta(\tilde{x}\cdot\bar{\theta})-\frac{1}{2}\tilde{x}|\theta|^2\right\} &= \left(\operatorname{Re}k_{\pm}-\frac{n-1}{2r}\right)|\theta|^2\\ &+\frac{1}{r}\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}+\operatorname{Re}\{(\nabla k_{\pm}-\tilde{x}\partial_r k_{\pm})\cdot\bar{\theta}u\}\\ &+\operatorname{Re}\{(q_{\pm}u-f)\tilde{x}\cdot\bar{\theta}\}\,.\end{aligned}$$

Let $\phi(r)\geq 0$ be a C^1-, monotone increasing function of $r\in[0,\infty)$ such that $\phi(r)=0$ for $r\leq R_1$ and $\phi(r)=1$ for $r\geq R_1+1$. Multiply by $\phi(r)r^{\beta}$ on both sides of (2.13) and integrate over $B(R_1, R)$, where $R\geq R_1+1$. Then integration by parts gives

$$\begin{aligned}&\int_{S(R)} r^{\beta}\left\{|\tilde{x}\cdot\theta|^2-\frac{1}{2}|\theta|^2\right\}dS\\ &=\int_{B(R_1,R)}\phi(r)r^{\beta}\Big[\left(\operatorname{Re}k_{\pm}-\frac{n-1-\beta}{2r}\right)|\theta|^2+\frac{1-\beta}{r}\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}\\ &+\operatorname{Re}\{(\nabla k_{\pm}-\tilde{x}\partial_r k_{\pm})\cdot\bar{\theta}u\}+\operatorname{Re}\{(q_{\pm}u-f)\tilde{x}\cdot\bar{\theta}\}\Big]dx\\ &+\int_{B(R_1,R_1+1)}\phi'(r)r^{\beta}\left\{|\tilde{x}\cdot\theta|^2-\frac{1}{2}|\theta|^2\right\}dx\,.\end{aligned}$$

Note that $0<\alpha\leq\beta$ and $\alpha+\beta\leq\min\{2\delta, 2\}$ by (1.11), and that $(1-\beta)\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}\geq 0$. Then it follows from (A2–1), (A2–3), (A2–4) and (A2–5) that

$$\begin{aligned}\text{(2.14)}\qquad \int_{S(R)} r^{\beta}|\tilde{x}\cdot\theta|^2 dS &\geq \int_{B(R_1+1,R)} r^{\beta}\{C_4 r^{-1}|\theta|^2\\ &-C_5 r^{-1-\delta}|\theta|\,|u|-C_1 r^{-1-\delta}|\theta|\,|u|-|f|\,|\theta|\}dx\\ &-\frac{1}{2}\int_{B(R_1,R_1+1)}\phi'(r)r^{\beta}|\theta|^2 dx.\end{aligned}$$

By the Schwartz inequality, we have for any $\varepsilon>0$

$$\begin{aligned}&-\{C_5 r^{-1-\delta}|u|+C_1 r^{-1-\delta}|u|+|f|\}|\theta|\\ &\geq -\varepsilon r^{-1}|\theta|^2-C_9(K^{\pm},\varepsilon)(r^{-1-2\delta}|u|^2+r|f|^2)\,.\end{aligned}$$

Thus, noting the radiation condition

$$\tilde{x}\cdot\theta=\partial_r u+k_\pm(x,\zeta)u\in L^2_{\frac{-1+\beta}{2}}(B(R_1)),$$

we can let $R\to\infty$ in (2.14) to obtain

$$(2.15)\qquad (C_4-\varepsilon)\int_{B(R_1+1)} r^{-1+\beta}|\theta|^2dx$$
$$\le C_9\int_{B(R_1+1)}\{r^{-1-2\delta+\beta}|u|^2+r^{1+\beta}|f|^2\}dx$$
$$+\frac{1}{2}C_{10}\int_{B(R_1,R_1+1)}|\theta|^2dx,$$

where ε is chosen so small that $C_4-\varepsilon\ge C_4/2$ and $C_{10}=\max_{x\in B(R_1,R_1+1)}\phi'(r)r^\beta$. Since $|\theta|^2\le 2|\nabla u|^2+2C_2^2|u|^2$ by (2.10) and (A2–2), it follows from Proposition 2.1 that

$$(2.16)\qquad \int_{B(R_1,R_1+1)}|\theta|^2dx\le C_{11}\{\|u\|^2_{\frac{-1-\alpha}{2}}+\|f\|^2_{\frac{-1-\alpha}{2}}\}.$$

Moreover, we have by (1.11)

$$-1-2\delta+\beta\le -1-\alpha.$$

Thus, (2.9) follows from (2.15) and (2.16). The proof is completed. q.e.d.

Corollary 2.1. *Suppose that $\delta>1/2$ in Assumptions* 1 *and* 2, *and let $f(x)\in L_1^2(\Omega)$. Then for any outgoing [incoming] solution of* (1.3) *with $\zeta\in K^\pm$, we have $\nabla u+\tilde{x}k_\pm(x,\zeta)u\in L^2(B(R_1))$ and*

$$(2.17)\qquad \|\nabla u+\tilde{x}k_\pm(x,\zeta)u\|_{B(R_1)}\le C_8\{\|u\|_{\frac{-1-\alpha}{2}}+\|f\|_1\},$$

where $\alpha\le\min\{2\delta,2\}-1$.

Proof. If $\delta>1/2$, we can choose $\beta=1$ in (2.9). q.e.d.

§3. Proof of Theorem 1

In the case where $\mathrm{Im}\,\zeta\ne 0$, the uniqueness theorem easily follows from Proposition 2.2. Thus, to complete the proof of Theorem 1, we have only to show the uniqueness for real $\zeta=\lambda\pm i0\in K^\pm$. For this aim we use the growth property of solutions of the homogeneous equation (1.6) (Assumption 1).

Let u be an outgoing [incoming] solution of (1.6) with real $\zeta=\lambda\pm i0\in K^\pm$ which also satisfies the boundary condition (1.1). Applying the Green formula, we have for $s\ge R_1$

$$0=-\mathrm{Im}\int_{\Omega(s)}\{-\Delta u+V(x)u-\lambda u\}\bar{u}dx$$
$$=\mathrm{Im}\int_{S(s)}(\partial_r u)\bar{u}dS=\mathrm{Im}\int_{S(s)}\{\partial_r u+\mathrm{Re}\,k_\pm(x,\lambda)u\}\bar{u}dS$$

$$= \int_{S(s)} \frac{1}{\operatorname{Im} k_{\pm}(x, \lambda)} \operatorname{Re} [\{\partial_r u + \operatorname{Re} k_{\pm}(x, \lambda) u\} \overline{i \operatorname{Im} k_{\pm}(x, \lambda) u}] dS$$

$$= \frac{1}{2} \int_{S(s)} \frac{1}{\operatorname{Im} k_{\pm}(x, \lambda)} \{ |\partial_r u + k_{\pm}(x, \lambda) u|^2 - |\partial_r u + \operatorname{Re} k_{\pm}(x, \lambda) u|^2 - |\operatorname{Im} k_{\pm}(x, \lambda)|^2 |u|^2 \} dS.$$

Thus, it follows from (A2–3) and the radiation condition (1.10) that

$$\int_{B(R)} (1+r)^{-1+\beta} |u|^2 dx \le C_3^{-2} \int_{B(R)} (1+r)^{-1+\beta} |\partial_r u + k_{\pm}(x, \lambda) u|^2 dx < \infty$$

for $R \ge R_1$. Since β satisfies (A2–6), this shows that u satisfies condition (1.7) of Assumption 1. Hence $u=0$ in Ω, and the proof is complete.

§4. Proof of Theorem 2

We prepare two lemmas which follow from Propositions 2.1, 2.3 and 2.4.

Lemma 4.1. *Let $\{\zeta_l\}$ be a bounded sequence in $K^{\pm}$, let $\{f_l\}$ be a bounded sequence in $L^2_{\frac{1+\beta}{2}}(\Omega)$ and let $\{u_l\}$ be the corresponding sequence of outgoing [incoming] solutions of* (1.3) *with $\zeta=\zeta_l$ and $f=f_l$. Then $\{u_l\}$ is precompact in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$ if it is bounded in the same space.*

Proof. Suppose that $\{u_l\}$ is bounded in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$. Then by Proposition 2.1 with $\nu = \frac{-1-\alpha}{2}$ and the Rellich compactness criterion, we see that for any $R>0$

$$\{u_l\} \text{ is precompact in } L^2(\Omega(R)). \tag{4.1}$$

On the other hand, Propositions 2.3 and 2.4 assert that for any $\varepsilon>0$, if we choose $R \ge R_1$ sufficiently large, then

$$\sup_l \|u_l\|_{\frac{-1-\alpha}{2}, B(R)} < \varepsilon. \tag{4.2}$$

The precompactness of $\{u_l\}$ in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$ then follows from (4.1) and (4.2). q.e.d.

Lemma 4.2. *Let $\{u_l\}$ be as in Lemma 4.1. Suppose that $\zeta_l \to \zeta_0$ and $u_l \to u_0$ in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$ as $l \to \infty$. Then u_0 satisfies the radiation condition* (1.10) *with $\zeta=\zeta_0$.*

Proof. Since $\{u_l\}$ is bounded in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$, we see from Proposition 2.4 that $\{\partial_r u_l + k_{\pm}(x, \zeta_l) u_l\}$ is also bounded in $L^2_{\frac{-1+\beta}{2}}(B(R))$ for some $R=R(K^{\pm}) \ge R_1$, and hence there exists a convergent subsequence, which we also write as $\{\partial_r u_l + k_{\pm}(x, \zeta_l) u_l\}$, in the weak topology of $L^2_{\frac{-1+\beta}{2}}(B(R))$. We denote by w the limit function.

Then for any $\phi \in C_0^\infty(B(R))$

$$(4.3) \qquad \int_{B(R)} w\bar{\phi}dx = \lim_{l\to\infty}\int_{B(R)} \{\partial_r u_l + k_\pm(x, \zeta_l)u_l\}\bar{\phi}dx$$

$$= -\int_{B(R)} u_0\{\partial_r\bar{\phi} + \frac{n-1}{r}\bar{\phi} - k_\pm(x, \zeta_0)\bar{\phi}\}dx,$$

where to obtain the last equality, we have used the condition that $\{u_l\}$ converges to u_0 in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$. (4.3) implies that

$$w = \partial_r u_0 + k_\pm(x, \zeta_0)u_0.$$

Since $w \in L^2_{\frac{-1+\beta}{2}}(B(R))$, we conclude that u_0 satisfies the radiation condition (1.10).

q.e.d.

Proof of Theorem 2. Note that (1.14) follows from Proposition 2.4 and (1.15), and (1.13) follows from Proposition 2.3, (1.14) and (1.15). Thus, we have only to prove inequality (1.15).

We shall prove (1.15) by contradiction. If we assume contrary, we can choose a sequence $\{\zeta_l\}$ in $K^\pm$ and sequences $\{f_l\}$ in $L^2_{\frac{1+\beta}{2}}(\Omega)$ and $\{u_l\}$ of outgoing [incoming] solutions of (1.3) with $\zeta=\zeta_l$ and $f=f_l$ as follows:

$$(4.4) \qquad \|u_l\|_{\frac{-1-\alpha}{2}} = 1 \quad \text{and} \quad \|f_l\|_{\frac{1+\beta}{2}} \le \frac{1}{l}.$$

Since $\{f_l\}$ is bounded in $L^2_{\frac{1+\beta}{2}}(\Omega)$, it follows from Lemma 4.1 that $\{u_l\}$ is precompact in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$. Let $\{u_{l'}\}$ be a convergent subsequence and denote the limit function by u_0. Then

$$(4.5) \qquad \|u_0\|_{\frac{-1-\alpha}{2}} = 1$$

and u_0 satisfies the equation

$$(4.6) \qquad -\Delta u_0 + V(x)u_0 - \zeta_0 u_0 = 0 \quad \text{in} \quad \Omega$$

in the distribution sense. The ellipticity of $-\Delta + V(x)$ implies that $u_0 \in H^2_{\mathrm{loc}}$. Moreover, u_0 satisfies the boundary condition (1.1) since it is satisfied by each $u_{l'}$. Lemma 4.2 asserts that u_0 satisfies the radiation condition. Hence, by the uniqueness theorem (Theorem 1), u_0 must identically vanish in Ω. This contradicts to (4.5) and the proof is complete.

§5. Proof of Theorem 3

(a) By (A2–2) and Proposition 2.1 with $\nu = \frac{-1+\beta}{2} \le 0$, we see that for any $\zeta \in K^\pm$ and $f \in L^2_{\frac{1+\beta}{2}}(\Omega) \subset L^2(\Omega)$, the L^2-solution of (1.3) satisfies the radiation condition (1.10). Since we have the uniqueness theorem, this proves assertion (a).

(b) It follows from Lemma 4.1 and (1.15) that $\{u_l\}$ is precompact in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$. Let $\{u_{l'}\}$ be a convergent subsequence and let u_0 be the limit function. Then by the same reasoning as in the proof of Theorem 2, u_0 satisfies the boundary-value problem (1.3) and the radiation condition (1.10) with $\zeta=\zeta_0$ and $f=f_0$. The uniqueness theorem shows that this u_0 is a unique accumulation point of $\{u_l\}$. Hence, $\{u_l\}$ itself converges to the outgoing [incoming] solution u_0.

The proof of Theorem 3 is complete.

§6. Proof of Theorem 4

The first assertion of Theorem 4 is obvious from Theorems 1 and 3. Let $\{\zeta_l\}$ and $\{f_l\}$ be convergent sequences in $K^{\pm}$ and $L^2_{\frac{1+\beta}{2}}(\Omega)$, respectively. Assume that $\zeta_l \to \zeta_0$ and $f_l \to f_0$ as $l \to \infty$. Then as is proved in Theorem 3, $\{u(\cdot, \zeta_l, f_l)\}$ becomes a Cauchy sequence in $L^2_{\frac{-1-\alpha}{2}}(\Omega)$ which converges to $u(\cdot, \zeta_0, f_0)$ as $l \to \infty$. This proves the continuity of $\mathscr{R}_\zeta f$, and hence Theorem 4 is proved.

§7. Proof of Theorem 5

(a) We have only to show (1.17) in the case where e is an interval: $e=(\lambda_1, \lambda_2)$, where $\Lambda_\delta<\lambda_1<\lambda_2<\infty$. Note that if $\varepsilon>0$, $\mathscr{R}_{\lambda\pm i\varepsilon}$ coincides with the resolvent $(L-\lambda\mp i\varepsilon)^{-1}$ of the selfadjoint operator L. Since L has no eigenvalues in (Λ_δ, ∞), by the Stieltjes inversion formula we have for any $f, g \in L^2(\Omega)$

$$(7.1)\qquad (\mathscr{E}(e)f, g)=\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1}^{\lambda_2}(\mathscr{R}_{\lambda+i\varepsilon}f-\mathscr{R}_{\lambda-i\varepsilon}f, g)\,d\lambda.$$

Let $f\in L^2_{\frac{1+\beta}{2}}(\Omega)$ and $g\in L^2_{\frac{1+\alpha}{2}}(\Omega)$. Then Theorem 4 shows that $(\mathscr{R}_{\lambda\pm i\varepsilon}f, g)$ is continuous in $\lambda\in e$ and converges to $(\mathscr{R}_{\lambda\pm i0}f, g)$ (cf., (1.18)) uniformly in $\lambda\in e$ as $\varepsilon\downarrow 0$. Hence we have (1.17).

(b) Since $L^2_{\frac{1+\beta}{2}}(\Omega)\subset L^2_{\frac{1+\alpha}{2}}(\Omega)$ by (1.11), and is dense in $L^2(\Omega)$, assertion (b) easily follows from assertion (a).

The proof of Theorem 5 is complete.

§8. Applications: Sufficient conditions on $V(x)$

In this section we shall show that Assumptions 1 and 2 hold for potentials $V(x)$ which satisfy the following conditions.

(V1) $V(x)$ is a real-valued function which belongs to the "Stummel class" Q_μ for some $\mu>0$.

(V2) There exists an $R_0>0$ such that $B(R_0)\subset\Omega$ and $V(x)$ is decomposed in $B(R_0)$ as

$$V(x)=V_1(x)+V_2(x)+V_3(x),$$

where $V_3(x)$ is a short-range potential:

(V2–1) $$V_3(x)=O(r^{-1-\delta}) \qquad \text{for some} \quad \delta>0$$

and $V_2(x)$ is a *non-oscillating* long-range potential:

(V2–2) $$V_2(x)=O(r^{-\delta}) \quad \text{and} \quad \nabla V_2(x)=O(r^{-1-\delta}).$$

$V_1(x)$ is a *oscillating* long-range potential, and we require the following (V2–3) and (V2–4):

(V2–3) $$V_1(x)=O(1),\ \partial_r V_1(x)=O(r^{-1}) \quad \text{and}$$
$$\partial_r^2 V_1(x)+aV_1(x)=O(r^{-1-\delta}) \qquad \text{for some} \quad a\geq 0,$$

(V2–4) $$\nabla V_1(x)-\tilde{x}\partial_r V_1(x)=O(r^{-1-\delta}) \quad \text{and}$$
$$\nabla \partial_r V_1(x)-\tilde{x}\partial_r^2 V_1(x)=O(r^{-1-\delta}).$$

Hereafter we assume that $0<\delta\leq 1$ which does not restrict the generality.

(V3) For the operator $-\Delta+V(x)$, the unique continuation property holds.

Remark 8.1. If $a<0$ in the condition $\partial_r^2 V_1+aV_1=O(r^{-1-\delta})$, then by a straightforward calculation (cf., Lemma 8.1), we have $V_1=O(r^{-1-\delta})$. Namely, $V_1(x)$ becomes a short-range potential.

Remark 8.2. (V3) can be verified for $V(x)$ satisfying a Hölder condition except at a finite number of singularities.

To show that Assumptions 1 and 2 hold for the above potential, we put

(8.1) $$E(\gamma) = \limsup_{r\to\infty} \frac{1}{\gamma} \{r\partial_r V_1(x)+\gamma V_1(x)\}$$

for $\gamma>0$, and define Λ_δ as follows:

(8.2) $$\Lambda_\delta=\inf_{\gamma\in\Gamma_\delta} E(\gamma)+\frac{a}{4}\ ; \quad \Gamma_\delta=(0,\ \min\{4\delta,\ 2\})\ .$$

By condition (V2–3), we see $E(\gamma)<\infty$ for any $\gamma>0$. Moreover, as is proved in Lemma 2 of Mochizuki-Uchiyama [11], we have

(8.3) $$E_0=\inf_{0<\gamma<2} E(\gamma)>-\infty.$$

Proposition 8.1. *Let δ and Λ_δ be as given above. Then there exists a non-increasing function $\gamma(\lambda)$ of $\lambda>\Lambda_\delta$ which satisfies the following inequalities*:

(8.4) $$0<\gamma(\lambda)<\min\{4\delta,\ 2\},$$

(8.5) $$E(\gamma(\lambda))+\frac{a}{4}<\Lambda_\delta+\frac{1}{3}(\lambda-\Lambda_\delta)<\lambda.$$

Moreover, Assumption 1 *with this* $\gamma(\lambda)$ *holds.*

Proof. The existence of $\gamma(\lambda)$ satisfying (8.4) and (8.5) follows from (8.2). Since $a\geq 0$ in (8.5), we have $0<\gamma(\lambda)<2$ and $E_0\leq E(\gamma(\lambda))<\lambda$. Then as is proved in Mochizuki [10] (cf., also Uchiyama [13] and Mochizuki-Uchiyama [11]), u satisfying (1.6) and (1.7) must identically vanish in Ω. Thus, Assumption 1 holds and the proposition is proved. q.e.d.

To proceed into verifying Assumption 2, we need some lemmas.

Lemma 8.1. *If* $a>0$ *in* (V2–3), *then we have*

$$V_1(x)=O(r^{-1}) \qquad \textit{at infinity}. \tag{8.6}$$

Proof. If $a=b^2>0$ $(b>0)$, noting the equality

$$\begin{aligned}\partial_r(e^{-br}\partial_r V_1)-b\partial_r(e^{-br}V_1)&=e^{-br}(\partial_r^2 V_1-2b\partial_r V_1+b^2V_1)\\&=e^{-br}\{O(r^{-1-\delta})+O(r^{-1})\}\,,\end{aligned}$$

we have

$$-Ce^{-br}r^{-1}\leq\partial_r(e^{-br}\partial_r V_1)-b\partial_r(e^{-br}V_1)\leq Ce^{-br}r^{-1}$$

for any $r\geq R$, where the constants $C>0$ and $R\geq R_0$ have been chosen sufficiently large. Integrating this with respect to r from r to ∞ and noting the inequality

$$\int_r^\infty e^{-br}r^{-1}dr\leq r^{-1}\int_r^\infty e^{-br}dr=\frac{e^{-br}}{br}\,,$$

we obtain

$$\partial_r V_1(x)-\frac{C}{br}\leq bV_1(x)\leq\partial_r V_1(x)+\frac{C}{br}\,.$$

This shows (8.6) and the proof is completed. q.e.d.

Lemma 8.2. *The following inequalities hold*:

$$\Lambda_\delta-a/4\geq E_0\geq\limsup_{r\to\infty}V_1(x)\,. \tag{8.7}$$

Proof. We have only to show that $E_0\geq\limsup_{r\to\infty}V_1(x)$. Assume the contrary. Then there exists a constant C such that

$$E(\gamma)<C<\limsup_{r\to\infty}V_1(x) \qquad \text{for some} \quad \gamma\in(0,\,2)\,. \tag{$*$}$$

It follows from (8.1) and the first inequality of $(*)$ that

$$\frac{1}{\gamma}r^{1-\gamma}\partial_r(r^\gamma V_1(r\tilde{x}))<C$$

for r large. Multiplying both sides by $\gamma r^{-1+\gamma}$ and integrating over $(r_0,\,s)$, where r_0 is chosen sufficiently large, we have

$$s^{\gamma}V_1(s\tilde{x})-r_0^{\gamma}V_1(r_0\tilde{x})<C(s^{\gamma}-r_0^{\gamma}).$$

Hence,

$$V_1(s\tilde{x})<C+s^{-\gamma}\{r_0^{\gamma}V_1(r_0\tilde{x})-Cr_0^{\gamma}\}$$

for any $s>r_0$. This contradicts to the second inequality of (∗), and the lemma is proved. q.e.d.

Lemma 8.3. *$E(\gamma)$ is a non-increasing, continuous function of $\gamma\in(0, 2]$.*

Proof. For any pair γ, γ' in (0, 2], we have from (8.1)

$$E(\gamma)-E(\gamma')\leq\frac{|\gamma-\gamma'|}{\gamma}\{|E(\gamma')|+\sup_{x\in B(R_0)}|V_1(x)|\}.$$

This implies the continuity of $E(\gamma)$. Moreover, if $0<\gamma'<\gamma\leq 2$, we have

$$E(\gamma)-E(\gamma')\leq-\frac{\gamma-\gamma'}{\gamma}\{E(\gamma')-\limsup_{r\to\infty}V_1(x)\}\leq 0$$

since $E(\gamma')\geq\limsup_{r\to\infty}V_1(x)$ by Lemma 8.2. Thus, $E(\gamma)$ is non-increasing in $\gamma\in(0, 2]$. q.e.d.

Now we define the domain $\Pi_\delta^{\pm}$ by (1.8) and (8.2). Let $K^{\pm}$ be any compact set of $\Pi_\delta^{\pm}$, and let

(8.8) $$d=d(K^{\pm})=\inf_{\zeta\in K^{\pm}}(\operatorname{Re}\zeta-\Lambda_\delta).$$

To determine the constant $\beta=\beta(K^{\pm})$ satisfying (A2–6), we put

(8.9) $$\gamma(K^{\pm})=\sup_{\zeta\in K^{\pm}}\gamma(\operatorname{Re}\zeta)=\gamma(\Lambda_\delta+d),$$

where $\gamma(\lambda)$ is the non-increasing function given in Proposition 8.1. Then by (8.4) and (8.5) we have

(8.10) $$0<\gamma(K^{\pm})<\min\{4\delta, 2\},$$

(8.11) $$E(\gamma(K^{\pm}))+\frac{a}{4}<\Lambda_\delta+\frac{d}{3}.$$

Taking account of these inequalities, we can choose positive constants $\beta=\beta(K^{\pm})$ and $\alpha=\alpha(K^{\pm})$ as follows:

(8.12) $$\frac{1}{2}\gamma(K^{\pm})<\beta<2\delta \quad\text{and}\quad \beta\leq 1;$$

(8.13) $$0<\alpha\leq 2\delta-\beta \quad\text{and}\quad \alpha\leq\beta.$$

Remark 8.3. If $V_1(x)$ satisfies (V2–3) with $\delta>1/2$, as is seen in (8.12), we can choose $\beta=1$ for any $K^{\pm}$.

Remark 8.4. If $\partial_r V_1(x)=o(r^{-1})$ in (V2–3) (see §9; Examples **I**–1 and **I**–2), we have from (8.1) $E(\gamma)=\limsup_{r\to\infty} V_1(x)$ for any $\gamma>0$, and hence $E(\gamma)$ is independent of γ. In this case, β can be any constant satisfying

$$0<\beta<2\delta \quad \text{and} \quad \beta\le 1. \tag{8.14}$$

In fact, if we put $\gamma(\lambda)=\gamma$ (constant), where $0<\gamma<2\beta$, then (8.4), (8.5) and (8.12) are satisfied with this $\gamma(\lambda)=\gamma$.

Next we define $k_\pm(x, \zeta)$ as follows (cf., Appendix):

$$k_\pm(x, \zeta)=-i\sqrt{\zeta-\eta V_1(x)-V_2(x)}+\frac{n-1}{2r} + \frac{-\eta\partial_r V_1(x)}{4\{\zeta-\eta V_1(x)-V_2(x)\}} \quad \text{for} \quad \zeta\in\Pi_\delta^\pm, \tag{8.15}$$

where

$$\eta=\frac{4\zeta}{4\zeta-a}. \tag{8.16}$$

It follows from (8.16) that

$$\mathrm{Im}(\zeta-\eta V_1-V_2)=\mathrm{Im}\,\zeta\left\{1+\frac{4aV_1}{|4\zeta-a|^2}\right\}$$

and

$$\mathrm{Re}\,(\zeta-\eta V_1-V_2)=\mathrm{Re}\,\zeta-V_1-\mathrm{Re}\,\frac{aV_1}{4\zeta-a}-V_2.$$

For any $\zeta\in\Pi_\delta^\pm$ we have from Lemma 8.2

$$\mathrm{Re}\,\zeta>\limsup_{r\to\infty} V_1(x) \qquad \text{if} \quad a=0,$$

and from Lemmas 8.1 and 8.2

$$V_1(x)=O(r^{-1}) \quad \text{and} \quad \mathrm{Re}\,\zeta>a/4 \qquad \text{if} \quad a>0.$$

Moreover, we have $V_2(x)=O(r^{-\delta})$ by (A2–2). Thus, it follows that there exist constants $R_{10}=R_{10}(K^\pm)\ge R_0$ and $C=C(K^\pm)\ge 1$ such that

$$0\le \pm\mathrm{Im}\,\{\zeta-\eta V_1(x)-V_2(x)\}\le C; \tag{8.17}$$

$$C^{-1}\le \mathrm{Re}\,\{\zeta-\eta V_1(x)-V_2(x)\}\le C \tag{8.18}$$

for any $(x, \zeta)\in B(R_{10})\times K^\pm$.

Proposition 8.2. *For β and $k_\pm(x, \zeta)$ given as above, Assumption 2 holds.*

Proof. The continuity of $k_\pm(x, \zeta)$ easily follows from (8.15), (8.17) and (8.18). Further, we have (A2–2) for $|x|\ge R_{10}$ sufficiently large. (A2–6) is obvious from (8.12). Note that

$$\operatorname{Im} k_{\pm}(x, \zeta) = -\operatorname{Re}\sqrt{\zeta-\eta V_1-V_2}+\operatorname{Im}\left[\frac{-\eta\partial_r V_1}{4(\zeta-\eta V_1-V_2)}\right].$$

Here

$$\operatorname{Re}\sqrt{\zeta-\eta V_1-V_2}\geq\sqrt{\operatorname{Re}(\zeta-\eta V_1-V_2)}>C^{-1/2}[\leq-\sqrt{\operatorname{Re}(\zeta-\eta V_1-V_2)}<-C^{-1/2}]$$

if $\zeta\in K^+$ $[K^-]$ and $|x|$ is sufficiently large. Thus, (A2–3) follows if we note that $\partial_r V_1(x)=O(r^{-1})$ at infinity. Note that

$$\nabla k_{\pm}-\tilde{x}\partial_r k_{\pm}=\frac{i\eta(\nabla V_1-\tilde{x}\partial_r V_1)+i(\nabla V_2-\tilde{x}\partial_r V_2)}{2\sqrt{\zeta-\eta V_1-V_2}}$$
$$+\frac{-\eta(\nabla\partial_r V_1-\tilde{x}\partial_r^2 V_1)}{4(\zeta-\eta V_1-V_2)}-\frac{\eta\partial_r V_1\{\eta(\nabla V_1-\tilde{x}\partial_r V_1)+(\nabla V_2-\tilde{x}\partial_r V_2)\}}{4(\zeta-\eta V_1-V_2)^2}.$$

Then (A2–5) follows from (V2–2) and (V2–4).

Next we prove (A2–4).

$$\operatorname{Re} k_{\pm}(x, \zeta)-\frac{n-1}{2r}=\operatorname{Im}\sqrt{\zeta-\eta V_1-V_2}+\frac{1}{4}\operatorname{Re}\left[\frac{-\eta\partial_r V_1}{\zeta-\eta V_1-V_2}\right].$$

For $\gamma=\gamma(K^{\pm})$ given by (8.9), we put

$$I_\delta=\operatorname{Re}\left[\frac{-\eta\partial_r V_1}{\zeta-\eta V_1-V_2}\right]+\frac{\gamma}{r}=\frac{\gamma}{r}\operatorname{Re}\left[\frac{\zeta-\frac{\eta}{\gamma}(r\partial_r V_1+\gamma V_1)-V_2}{\zeta-\eta V_1-V_2}\right].$$

Since $\zeta/\eta=\zeta-a/4$, it follows that

$$I_\delta=\frac{\gamma\left\{\operatorname{Re}\zeta-\frac{a}{4}-V_1-\operatorname{Re}\frac{1}{\eta}V_2\right\}\left\{\operatorname{Re}\zeta-\frac{a}{4}-\frac{1}{\gamma}(r\partial_r V_1+\gamma V_1)-\operatorname{Re}\frac{1}{\eta}V_2\right\}}{r\left|\zeta-\frac{a}{4}-V_1-\frac{1}{\eta}V_2\right|^2}$$
$$+\frac{\gamma\left(\operatorname{Im}\zeta-\operatorname{Im}\frac{1}{\eta}V_2\right)^2}{r\left|\zeta-\frac{a}{4}-V_1-\frac{1}{\eta}V_2\right|^2}.$$

Here we choose $R\geq R_{10}$ so large that $\Lambda_\delta-\frac{a}{4}-V_1(x)>-\frac{d}{3}$ (cf., (8.7)), $-\operatorname{Re}\frac{1}{\eta}V_2(x)>-\frac{d}{3}$ (cf., (V2–2)) and $E(\gamma)-\frac{1}{\gamma}(r\partial_r V_1+\gamma V_1)>-\frac{d}{3}$ (cf., (8.1)) for $|x|\geq R$, where $d>0$ is as given in (8.8). Then

$$\operatorname{Re}\zeta-\frac{a}{4}-V_1-\operatorname{Re}\frac{1}{\eta}V_2=\operatorname{Re}\zeta-\Lambda_\delta+\Lambda_\delta-\frac{a}{4}-V_1-\operatorname{Re}\frac{1}{\eta}V_2>\frac{d}{3}>0$$

and

$$\operatorname{Re}\zeta-\frac{a}{4}-\frac{1}{\gamma}(r\partial_r V_1+\gamma V_1)-\operatorname{Re}\frac{1}{\eta}V_2=\operatorname{Re}\zeta-\Lambda_\delta+\Lambda_\delta-\left(E(\gamma)+\frac{a}{4}\right)$$
$$+E(\gamma)-\frac{1}{\gamma}(r\partial_r V_1+\gamma V_1)-\operatorname{Re}\frac{1}{\eta}V_2>0$$

for $|x|\geq R$, and hence we have $I_\delta>0$. If we note that

$$\mathrm{Im}\sqrt{\zeta-\eta V_1-V_2}\geq 0 \qquad \text{(see Notation given in § 1)},$$

it then follows that

$$\mathrm{Re}k_\pm-\frac{n-1-\beta}{2r}>\frac{\beta}{2r}-\frac{\gamma}{4r}=\frac{2\beta-\gamma}{4r}.$$

This proves (A2–4) since we have $2\beta>\gamma$ by (8.12).

Finally, we prove (A2–1). It follows from (1.9) and (8.15) that

$$q_\pm(x,\zeta)=(1-\eta)V_1+\frac{-\eta\partial_r^2 V_1}{4(\zeta-\eta V_1-V_2)}+\frac{(n-1)(n-3)}{4r^2}$$
$$-\frac{5\eta^2(\partial_r V_1)^2+4\eta\partial_r V_1\partial_r V_2}{16(\zeta-\eta V_1-V_2)^2}+\frac{i\partial_r V_2}{2\sqrt{\zeta-\eta V_1-V_2}}+V_3.$$

If $a=0$ in (V2–3), then $\eta=1$ and (A2–1) follows from (V2). If $a>0$ in (V2–3), then we have

$$(1-\eta)V_1+\frac{-\eta\partial_r^2 V_1}{4(\zeta-\eta V_1-V_2)}=\left(1-\eta+\frac{a\eta}{4\zeta}\right)V_1-\frac{\eta(\partial_r^2 V_1+aV_1)}{4(\zeta-\eta V_1-V_2)}$$
$$+\frac{a\eta V_1(\eta V_1+V_2)}{4\zeta(\zeta-\eta V_1-V_2)}=0(r^{-1-\delta}),$$

since $1-\eta+\frac{a\eta}{4\zeta}=0$ by (8.16) and $V_1=O(r^{-1})$ by Lemma 8.1. Thus, (A2–1) also holds in this case. Proposition 8.2 is proved. q.e.d.

By means of the above two propositions, we see that for the potential $V(x)$ satisfying (V1), (V2) and (V3) all the results (Theorems $1\sim5$) stated in § 1 hold.

Remark 8.5. We can replace (V2–3) by the following more general conditions (cf., Lemma 8.1):

(V2 – 3)′ $\quad V_1(x)=O(r^{-1+\nu})$, $\partial_r V_1(x)=O(r^{-1})$ and

$$\partial_r^2 V_1(x)+a(r)V_1(x)=O(r^{-1-\delta}),$$

where ν is a constant such that

(8.19) $$0\leq\nu\leq 1$$

and $a(r)$ is a real function of $r\geq R_0$ such that

(8.20) $$a(r)=O(r^{-2\nu}),\ a'(r)=O(r^{-\nu-\delta})\ \text{and}\ a''(r)=O(r^{-\nu-\delta}).$$

In fact, if we replace Λ_δ by

(8.21) $$\tilde{\Lambda}_\delta=\max\{\inf_{\gamma\in\Gamma_\delta}E(\gamma),\ \inf_{\gamma\in\Gamma_\delta}E(\gamma)+\limsup_{r\to\infty}a(r)/4\}$$

and $k_\pm(x,\zeta)$ by

(8.22)
$$\tilde{k}_\pm(x, \zeta) = -i\sqrt{\zeta-\eta(r)V_1(x)-V_2(x)} + \frac{n-1}{2r} + \frac{-\partial_r[\eta(r)V_1(x)]}{4\{\zeta-\eta(r)V_1(x)-V_2(x)\}};$$

(8.23)
$$\eta(r)=\frac{4\zeta}{4\zeta-a(r)},$$

then Propositions 8.1 and 8.2 can be proved for these $\tilde{\Lambda}_\delta$ and $\tilde{k}_\pm(x, \zeta)$ by the same argument as given above.

§ 9. Examples

I. First we consider potentials $V_1(x)$ of the form

(9.1)
$$V_1(x)=V_1(r)=\phi(\log r) \qquad \text{at infinity,}$$

where $\phi(t)$ is a real function of $t>0$ satisfying the following conditions:

(9.2)
$$\phi(t),\ \phi'(t) \quad \text{and} \quad \phi''(t) \text{ are bounded at infinity.}$$

This type of potentials satisfies (V2–3) with $a=0$ and (V2–4) for any $0<\delta\le 1$, since we have

$$V_1(r)=O(1),\ V_1'(r)=O(r^{-1}) \quad \text{and} \quad V_1''(r)=O(r^{-2}).$$

Let $\delta>0$ be as given in conditions (V2–1) on $V_3(x)$ and (V2–2) on $V_2(x)$. Then, by (8.2)

(9.3)
$$\Lambda_\delta=\inf_{\gamma\in\Gamma_\delta} E(\gamma)=\inf_{\gamma\in\Gamma_\delta}[\limsup_{r\to\infty}\frac{1}{\gamma}\{\phi'(\log r)+\gamma\phi(\log r)\}];$$
$$\Gamma_\delta=(0,\ \min\{4\delta,\ 2\}),$$

and it follows from Theorem 5 that the spectrum of $L=-\Delta+V(x)$ ($V(x)=V_1(r)+V_2(x)+V_3(x)$) contained in (Λ_δ, ∞) is absolutely continuous.

I–1. Let $\phi(t)=0$. Then $V_1(x)=0$ and

(9.4)
$$V(x)=V_2(x)+V_3(x).$$

In this case, we have from (9.3)

$$\Lambda_\delta=E(\gamma)=0 \qquad \text{for any} \quad \delta>0 \quad \text{and} \quad \gamma>0,$$

and hence the continuous spectrum $(0, \infty)$ of L is absolutely continuous (this result is already obtained by Ikebe-Saito [4] and Lavine [8]).

I–2. Let

(9.5)
$$\phi(t)=o(1) \quad \text{and} \quad \phi'(t)=o(1) \quad \text{at infinity.}$$

Then we also have from (9.3)

$$\Lambda_\delta = E(\gamma) = 0 \quad \text{for any} \quad \delta > 0 \quad \text{and} \quad \gamma > 0.$$

The potentials

(9.6) $$V_1(r) = \frac{c}{\log r} \qquad (c\text{: real constant}),$$

(9.7) $$V_1(r) = \frac{c}{\log(\log r)},$$

(9.8) $$V_1(r) = \frac{\sin(\log r)}{\log r}$$

are typical examples which satisfy not only (9.2) but also (9.5).

I–3. Let $\phi(t) = \sin t$, i.e.,

(9.9) $$V_1(r) = \sin(\log r).$$

Then, as we see in Mochizuki-Uchiyama [11], L has no eigenvalues in $(\sqrt{5}/2, \infty)$. From (9.3) it follows that

$$\Lambda_\delta = \inf_{\gamma \in \Gamma_\delta} \left[\limsup_{r \to \infty} \frac{1}{\gamma} \{\cos(\log r) + \gamma \sin(\log r)\}\right]$$
$$= \inf_{\gamma \in \Gamma_\delta} \sqrt{1 + \gamma^{-2}} = \sqrt{1 + \max\left\{\frac{1}{16\delta^2}, \frac{1}{4}\right\}}.$$

Thus, in general $\Lambda_\delta \geq \sqrt{5}/2$. However, if $\delta \geq 1/2$ in conditions on $V_2(x)$ and $V_3(x)$, we have $\Lambda_\delta = \sqrt{5}/2$.

II. Next we give examples which satisfy (V2–3) with $a > 0$. Let $V_1(x)$ be an oscillating long-range potential of the form

(9.10) $$V_1(x) = \psi(x) \frac{\sin br}{r} \qquad (b\text{; real constant}),$$

where $\psi(x)$ is a real function satisfying the following conditions at infinity:

(9.11) $$\psi(x) = O(1), \quad \nabla\psi(x) = O(r^{-\delta}) \quad \text{and} \quad \nabla^2\psi(x) = O(r^{-\delta}).$$

Then $V_1(x)$ satisfies (V2–3) with $a = b^2$ and (V2–4). By (8.2)

(9.12) $$\Lambda_\delta = \inf_{\gamma \in \Gamma_\delta} E(\gamma) + b^2/4$$
$$= |b| \max\{1/4\delta, 1/2\} \limsup_{r \to \infty} \{\psi(x) \cos br\} + b^2/4,$$

and (Λ_δ, ∞) becomes the absolutely continuous spectrum of L. Note that, in this case, the essential spectrum of L consists of $[0, \infty)$ since $V(x)$ tends to zero at infinity. Moreover, as we have proved in [11], there exist no eigenvalues in (E_0, ∞), where

(9.13) $$E_0 = \inf_{0 < \gamma < 2} E(\gamma) = \frac{|b|}{2} \limsup_{r \to \infty} \{\psi(x) \cos br\},$$

i.e., (E_0, ∞) is the continuous spectrum of L. In this sense, our above result is incomplete, and it remains as an open problem to study a more precise structure of the spectrum of L in $(0, \Lambda_\delta]$.

II–1. Let $\psi(x)=c$ (real constant), i.e.,

$$V_1(x)=\frac{c\sin br}{r}, \tag{9.14}$$

and assume that $\delta\geq 1/2$ in conditions on $V_2(x)$ and $V_3(x)$. Then it follows from (9.12) that $\Lambda_\delta=\Lambda_\delta(b)=|bc|/2+b^2/4$, whereas $E_0=|bc|/2$. Note that $\Lambda_\delta\to 0$ as $b\to 0$. This is reasonable since $V_1(x)$ tends as $b\to 0$ to the Coulomb potential c/r (in some sense) for which the absolutely continuous spectrum consists of $(0, \infty)$.

II–2. Let $\psi(x)=cx_1/r$, i.e.,

$$V_1(x)=\frac{cx_1\sin br}{r^2}. \tag{9.15}$$

Since $\psi(x)$ satisfies (9.11) with $\delta=1$, in this case, we also have $\Lambda_\delta=|bc|/2+b^2/4$ if $\delta\geq 1/2$ in conditions on $V_2(x)$ and $V_3(x)$.

II–3. The potential

$$V(x)=\frac{-32\sin r[g(r)^3\cos r-3g(r)^2\sin^3 r+g(r)\cos r+\sin^3 r]}{[1+g(r)^2]^2} \tag{9.16}$$

in $\boldsymbol{R}^3$, where $g(r)=2r-\sin 2r$, is given by von Neumann and Wigner as an example which has the eigenvalue $+1$ with eigenfunction

$$u(x)=\frac{\sin r}{r[1+g(r)^2]}.$$

$V(x)$ can be decomposed as

$$V(x)=-\frac{8\sin 2r}{r}+V_3(x),$$

where $V_3(x)=O(r^{-2})$ as $r\to\infty$. Thus, by **II–1**, we see that $(8, \infty)$ is the continuous spectrum of L and $(9, \infty)$ is the absolutely continuous spectrum.

II–4. The potential

$$V(x)=\frac{-32k^2\alpha^2\sin kr[(kr+1/2\alpha)\cos kr-\sin kr]}{[1+\alpha h(r)]^2} \tag{9.17}$$

in $\boldsymbol{R}^3$, where k, α are non-zero real constants and $h(r)=2kr-\sin 2kr$, is given by Moses and Tuan as an example which has the eigenvalue $+k^2$ with eigenfunction

$$u(x)=\frac{\sin kr}{r[1+\alpha h(r)]}.$$

The above $V(x)$ has the following property:

$$V(x)=\frac{-4k\sin 2kr}{r}+V_3(x)\,,$$

where $V_3(x)=O(r^{-2})$ as $r\to\infty$. Thus we see that $(4k^2, \infty)$ is the continuous spectrum of L and $(5k^2, \infty)$ is the absolutely continuous spectrum.

II–5. Generalizing the above two examples, Kato [6] gave the potential

$$V(x)=-\frac{f''}{f}+2\frac{f'^2}{f^2}-2k\cot kr\frac{f'}{f}-\frac{(n-1)(n-3)}{4r^2} \tag{9.18}$$

in $\mathbf{R}^n$, where

$$f=f(r)=r^\alpha-\alpha\int_0^r r^{\alpha-1}\cos 2krdr \qquad (\alpha>0).$$

Here we assume $\alpha>1/2$ and consider the operator $L=-\Delta+V(x)$ defined for functions in $\Omega=\{x;\ |x|>\pi/k\}$ satisfying the Dirichlet boundary condition. Then $+k^2$ is the eigenvalue with eigenfunction

$$u(x)=\frac{\sin kr}{f(r)}r^{-(n-1)/2}.$$

In fact, $f(r)=r^\alpha\{1+O(r^{-1})\}$ at infinity. Since we have

$$V(x)=-\frac{4k\alpha\sin 2kr}{r}+O(r^{-2}) \quad \text{as} \quad r\longrightarrow\infty,$$

it follows that $(4\alpha k^2, \infty)$ is the continuous spectrum of L and $(4\alpha k^2+k^2, \infty)$ is the absolutely continuous spectrum.

III. Finally, we give an example for which condition (V2–3)′ of Remark 8.5 is applicable. Let

$$V_1(x)=V_1(r)=\frac{\sin r^\mu}{r^\mu} \qquad (0<\mu\le 1)\,. \tag{9.19}$$

This potential behaves like

$$\begin{cases} V_1(r)=O(r^{-\mu}); \\ V_1'(r)=\dfrac{\mu\cos r^\mu}{r}+O(r^{-1-\mu})=O(r^{-1}); \\ V_1''(r)=\dfrac{-\mu^2\sin r^\mu}{r^{2-\mu}}+O(r^{-2})=O(r^{-1-(1-\mu)})\,. \end{cases} \tag{9.20}$$

Namely, if we put $a(r)=\mu^2r^{-2(1-\mu)}$, $V_1(r)$ satisfies (V2–3)′ with $\nu=1-\mu$ and $\delta=1$. We assume that $\delta>1/2$ in conditions on $V_2(x)$ and $V_3(x)$. Then by (8.21) we have

$$\tilde{\Lambda}_\delta=\tilde{\Lambda}_{1/2}=\frac{\mu}{2}+\frac{\mu^2}{4}\lim_{r\to\infty}r^{-2(1-\mu)},$$

and $(\tilde{\Lambda}_\delta, \infty)$ becomes the absolutely continuous spectrum of L. If $0<\mu<1$, we have $\tilde{\Lambda}_\delta=\mu/2$, and if $\mu=1$, we have $\tilde{\Lambda}_\delta=3/4$, which coincides with the result obtained in **II**–1.

As we see in (9.20), the potential (9.19) satisfies also (V2–3) with $a=0$ and $\delta=1-\mu$ if $0<\mu<1$. However, in this situation, the result becomes bad especially for $\mu>1/2$. In fact, we have from (8.2)

$$\Lambda_\delta=\Lambda_{1-\mu}=\inf_{\gamma\in\Gamma_{1-\mu}} E(\gamma)=\max\left\{\frac{\mu}{4(1-\mu)},\ \frac{\mu}{2}\right\}.$$

Hence, if $\mu>1/2$, we have $\Lambda_\delta=\frac{\mu}{4(1-\mu)}$, which tends to ∞ as $\mu\to 1$.

§10. Extension of the results to more general second order elliptic equations

In this section, we shall show that all the previous results (Theorems 1~5) can be extended to the exterior boundary-value problem

$$(10.1)\qquad P(x, D)u=-\sum_{j,k=1}^{n}(\partial_j+ib_j(x))a_{jk}(x)(\partial_k+ib_k(x))u+V(x)u$$
$$=\zeta u+f(x)\quad \text{in}\quad \Omega;$$

$$(10.2)\qquad Bu=\left\{\begin{array}{l} u \quad \text{or} \\ \sum_{j,k=1}^{n}\nu_j a_{jk}(x)(\partial_k+ib_k(x))u+d(x)u \end{array}\right\}=0\quad \text{on}\quad \partial\Omega,$$

where $V(x)$ satisfies conditions (V1) and (V2) in §8, and the unique continuation property (V3) is assumed to the operator $P(x, D)$. $a_{jk}(x)$ and $b_j(x)$ are required further to satisfy the following conditions.

(AB1) $a_{jk}(x)$ and $b_j(x)$ are real-valued smooth functions in $\bar{\Omega}=\Omega\cup\partial\Omega$; $a_{jk}(x)=a_{kj}(x)$ and there exists a constant $C_0\geq 1$ such that

$$C_0^{-1}|\xi|^2\leq\sum_{j,k=1}^{n}a_{jk}(x)\xi_j\xi_k\leq C_0|\xi|^2$$

for any $\xi\in R^n$ and $x\in\bar{\Omega}$.

(AB2) As $r\longrightarrow\infty$

(AB2–1) $a_{jk}(x)-\delta_{jk}=O(r^{-\delta})$ and $\nabla a_{jk}(x)=O(r^{-1-\delta})$,

(AB2–2) $\partial_j b_k(x)-\partial_k b_j(x)=O(r^{-1-\delta})$,

where δ_{jk} is the Kronecker delta and δ is a positive constant as given in (V2).

For the sake of simplicity, we put $D_j=\partial_j+ib_j(x)$, $D=(D_1,\dots, D_n)$ and $A=A(x)=(a_{jk}(x))$; that is $P(x, D)=-D\cdot AD+V(x)$.

We define the operator L in $L^2(\Omega)$ by

$$(10.3)\qquad \begin{cases} \mathscr{D}(L)=\{u;\ u\in L^2(\Omega)\cap H^2_{loc},\ D\cdot ADu\in L^2(\Omega),\ Bu|_{\partial\Omega}=0\} \\ Lu=P(x, D)u \qquad \text{for}\quad u\in\mathscr{D}(L). \end{cases}$$

Then L is selfadjoint and lower semi-bounded in $L^2(\Omega)$ (see, e.g., Mochizuki [9]). Moreover, the *growth property* (Assumption 1) holds for solutions of the equation

$$(10.4)\qquad P(x, D)u-\lambda u=0 \quad \text{in}\quad \Omega$$

with real $\lambda>\Lambda_\delta$ (Mochizuki [10]).

Let $\Pi_\delta^\pm$, α, β and $k_\pm(x, \zeta)$ be as in §8. For solutions of (10.1), (10.2) with ζ in a compact set $K^\pm$ of $\Pi_\delta^\pm$ and $f\in L^2_{\frac{1+\beta}{2}}(\Omega)$, the radiation condition has to be modified as follows:

$$(10.5)\qquad u\in L^2_{\frac{-1-\alpha}{2}}(\Omega) \quad\text{and}\quad \tilde{x}\cdot A\{Du+\tilde{x}\hat{k}_\pm(x, \zeta)u\}\in L^2_{\frac{-1+\beta}{2}}(B(R_1))$$

for some $R_1=R_1(K^\pm)\geq R_0$, where

$$(10.6)\qquad \hat{k}_\pm(x, \zeta)=k_\pm(x, \zeta)(\tilde{x}\cdot A\tilde{x})^{-1/2}.$$

Outgoing [incoming] solutions are defined by means of (10.5) as in Definition 1.2.

We know that Theorems $1\sim5$ can easily be proved if one can establish Propositions $2.1\sim2.4$, where in all assertions $\nabla u+\tilde{x}k_\pm(x, \zeta)u$ should be replaced by $A\{Du+\tilde{x}\hat{k}_\pm(x, \zeta)u\}$. Propositions $2.1\sim2.3$ can be proved without any essential change of arguments. Thus, Proposition 2.4 remains as the only one assertion which has to be carefully checked. It should be noted that in the proof of Propositions $2.1\sim2.3$ we never make use of (AB2), which will be used to prove Proposition 2.4.

We put

$$(10.7)\qquad \theta=Du+\tilde{x}\hat{k}_\pm(x, \zeta)u.$$

Then it follows from equation (10.1) that

$$(10.8)\qquad -D\cdot A\theta+\hat{k}_\pm(x, \zeta)\tilde{x}\cdot A\theta+\hat{q}_\pm(x, \zeta)u-f(x)=0;$$

$$(10.9)\qquad \hat{q}_\pm(x, \zeta)=q_\pm(x, \zeta)+\left\{\nabla\cdot(A\tilde{x}(\tilde{x}\cdot A\tilde{x})^{-1/2})-\frac{n-1}{r}\right\}k_\pm(x, \zeta)$$
$$+\{(\tilde{x}\cdot A\tilde{x})^{-1/2}\tilde{x}\cdot A\nabla-\tilde{x}\cdot\nabla\}k_\pm(x, \zeta).$$

By (AB2–1)

$$\nabla\cdot(A\tilde{x}(\tilde{x}\cdot A\tilde{x})^{-1/2})-\frac{n-1}{r}=(\tilde{x}\cdot A\nabla)(\tilde{x}\cdot A\tilde{x})^{-1/2}$$
$$+(\tilde{x}\cdot A\tilde{x})^{-1/2}(\nabla\cdot A\tilde{x})-\frac{n-1}{r}=O(r^{-1-\delta})$$

and

$$(\tilde{x}\cdot A\tilde{x})^{-1/2}\tilde{x}\cdot A\nabla-\tilde{x}\cdot\nabla=O(r^{-\delta})\cdot\nabla.$$

Hence, we see that there exist constants $R_1=R_1(K^{\pm})\geq R_0$ and $C_1=C_1(K^{\pm})>0$ such that

$$(10.10)\qquad |\hat{q}_{\pm}(x,\zeta)|\leq C_1r^{-1-\delta}\quad\text{for any}\quad (x,\zeta)\in B(R_1)\times K^{\pm}.$$

This corresponds to (A2–1) in Assumption 2. Using again (AB2–1), we have the inequalities corresponding to (A2–2)$\sim$(A2–5):

$$(10.11)\qquad |\hat{k}_{\pm}(x,\zeta)|\leq C_2,$$

$$(10.12)\qquad \mp\operatorname{Im}\hat{k}_{\pm}(x,\zeta)\geq C_3,$$

$$(10.13)\qquad \operatorname{Re}\hat{k}_{\pm}(x,\zeta)-\frac{n-1-\beta}{2r}\geq C_4r^{-1},$$

$$(10.14)\qquad |(\tilde{x}\cdot A\tilde{x})\nabla\hat{k}_{\pm}(x,\zeta)-\tilde{x}(\tilde{x}\cdot A\nabla)\hat{k}_{\pm}(x,\zeta)|\leq C_5r^{-1-\delta}$$

for any $(x,\zeta)\in B(R_1)\times K^{\pm}$.

Now, we multiply by $\tilde{x}\cdot A\bar{\theta}$ on both sides of (10.8) and take the real parts. Then we have

$$(10.15)\qquad \operatorname{Re}\nabla\cdot\left\{A\theta(\tilde{x}\cdot A\bar{\theta})-\frac{1}{2}A\tilde{x}(\theta\cdot A\bar{\theta})\right\}$$
$$=\left\{\operatorname{Re}\hat{k}_{\pm}-\frac{1}{2}(\nabla\cdot A\tilde{x})\right\}(\theta\cdot A\bar{\theta})+\frac{1}{r}\{|A\theta|^2-|\tilde{x}\cdot A\theta|^2\}$$
$$+\operatorname{Re}[\{(\tilde{x}\cdot A\tilde{x})\nabla\hat{k}_{\pm}-\tilde{x}(\tilde{x}\cdot A\nabla)\hat{k}_{\pm}\}\cdot A\bar{\theta}u]$$
$$+\operatorname{Re}[\sum_{j,k,l,p}a_{jp}(\partial_ja_{kl})\tilde{x}_l\theta_p\bar{\theta}_k]$$
$$-\operatorname{Im}[\sum_{j,k}(A\tilde{x})_k(\partial_jb_k-\partial_kb_j)(A\theta)_j\bar{u}]$$
$$+\operatorname{Re}[(\hat{q}_{\pm}u-f)\tilde{x}\cdot\bar{\theta}].$$

This corresponds to equation (2.13). Making use of this equation and applying (10.10)$\sim$(10.14), (AB1) and (AB2), we can follow the same line of proof of Proposition 2.4 to get inequality (2.9) with $\nabla u+\tilde{x}k_{\pm}(x,\zeta)u$ replaced by $Du+\tilde{x}\hat{k}_{\pm}(x,\zeta)u$. Hence, Proposition 2.4 holds true for outgoing [incoming] solutions of (10.1), (10.2).

Remark 10.1. If we assume $a_{jk}(x)-\delta_{jk}=O(r^{-1-\delta})$ in (AB2–1), then as the radiation condition we can use the following

$$(10.5)'\qquad u\in L^2_{\frac{-1-\alpha}{2}}(\Omega)\quad\text{and}\quad \tilde{x}\cdot ADu+k_{\pm}(x,\zeta)u\in L^2_{\frac{-1+\beta}{2}}(B(R_1)).$$

For if we put $\theta=Du+\tilde{x}k_{\pm}(x,\zeta)u$, then θ satisfies

$$(10.8)'\qquad -D\cdot A\theta+k_{\pm}(x,\zeta)\tilde{x}\cdot A\theta+\check{q}_{\pm}(x,\zeta)u-f=0,$$

where

$$\check{q}_\pm(x,\zeta)=q_\pm(x,\zeta)+\left\{\nabla\cdot A\tilde{x}-\frac{n-1}{r}\right\}k_\pm(x,\zeta) \tag{10.9)'$$

$$+\{\tilde{x}\cdot A\nabla-\tilde{x}\cdot\nabla\}k_\pm(x,\zeta)-k_\pm(x,\zeta)^2\{(\tilde{x}\cdot A\tilde{x})-1\},$$

and by assumptions we have $\check{q}_\pm(x,\zeta)=O(r^{-1-\delta})$.

Appendix

Let us explain how we found the radiation condition (8.15).

For solutions u of the Helmholtz equation

$$-\Delta u-\zeta u=f(x) \quad \text{in} \quad \Omega, \tag{1}$$

the Sommerfeld radiation condition is given in the form

$$u=O(r^{-(n-1)/2}) \quad \text{and} \quad \partial_r u-i\sqrt{\zeta}\,u=O(r^{-(n+1)/2}) \tag{2}$$

at infinity. This condition is used in Eidus' classical paper [2] to prove the principle of limiting absorption for the boundary-value problem (1.3) with $V(x)$ behaving like $O(r^{-(n+1+\delta)/2})$ at infinity. Note that (2) can be written as

$$u\in L^2_{\frac{-1-\varepsilon}{2}}(\Omega) \quad \text{and} \tag{3}$$

$$\partial_r u-i\sqrt{\zeta}\,u=e^{i\sqrt{\zeta}r}\partial_r(e^{-i\sqrt{\zeta}r}u)\in L^2_{\frac{1-\varepsilon}{2}}(\Omega) \qquad \text{for any} \quad \varepsilon>0.$$

In order to generalize Eidus' results to more general potentials, (3) should be modified in the form

$$u\in L^2_{\frac{-1-\alpha}{2}}(\Omega) \quad \text{and} \quad e^{-\rho(x,\zeta)}\partial_r(e^{\rho(x,\zeta)}u)\in L^2_{\frac{-1+\beta}{2}}(\Omega), \tag{4}$$

where α, β are positive constants satisfying $\alpha+\beta\le\min\{2\delta,2\}$. Suppose that $\rho(x,\zeta)$ depends only on r and ζ. Then $e^\rho u$ satisfies the equation

$$\{-\Delta+2\rho'\partial_r+q(x,\zeta)\}(e^\rho u)=e^\rho f(x); \tag{5}$$

$$q(x,\zeta)=V(x)-\zeta+\rho''(r,\zeta)+\frac{n-1}{r}\rho'(r,\zeta)-\rho'^2(r,\zeta). \tag{6}$$

In this equation we hope to make $q(x,\zeta)=O(r^{-1-\delta})$ at infinity. This implies the "Riccati" equation (1.4). Namely, if we put

$$\rho'(r,\zeta)=k(r,\zeta), \tag{7}$$

then we have

$$V(x)-\zeta+k'+\frac{n-1}{r}k-k^2=O(r^{-1-\delta}). \tag{8}$$

Let $V(x)$ be a short-range potential: $V(x)=O(r^{-1-\delta})$. Then (8) has a special solution

$$k(r, \zeta)=-i\sqrt{\zeta}+\frac{n-1}{2r}. \tag{9}$$

Making use of this $k(r, \zeta)$, we can represent (4) as follows:

$$u\in L^2_{\frac{-1-\alpha}{2}}(\Omega) \quad\text{and}\quad \partial_r u+k(r, \zeta)u\in L^2_{\frac{-1+\beta}{2}}(\Omega). \tag{10}$$

This is the radiation condition used in Mochizuki [9] (cf., also Ikebe-Saito [4], where is used the same radiation condition for some long-range potentials).

In the present case, however, the function (9) does not satisfy equation (8) since the behavior at infinity of the potential is not so simple. So we have to make a modification of (9). For this purpose, we first assume that $V_1(x)$ depends only on r and $V_2(x)=0$. Put

$$k(r, \zeta)=-i\sqrt{\zeta-\eta V_1(r)}+h(r, \zeta), \tag{11}$$

where η is a complex number, $h(r, \zeta)$ is a complex function and $\operatorname{Re}\zeta$ and r should be chosen sufficiently large. Then it follows from (6) and (7) that

$$\begin{aligned} q&=V_1+V_3-\zeta+k'+\frac{n-1}{r}k-k^2 \\ &=(1-\eta)V_1+V_3+h'+\frac{n-1}{r}h-h^2 \\ &\quad+2i\sqrt{\zeta-\eta V_1}\left\{h-\frac{n-1}{2r}-\frac{-\eta V_1'}{4(\zeta-\eta V_1)}\right\}. \end{aligned} \tag{12}$$

Our aim is to find some η and $h(r, \zeta)$ which make $q=O(r^{-1-\delta})$. We put

$$h(r, \zeta)=\frac{n-1}{2r}+\frac{-\eta V_1'}{4(\zeta-\eta V_1)}, \tag{13}$$

and substitute this in (12). Then

$$\begin{aligned} q&=(1-\eta)V_1+\frac{-\eta V_1''}{4(\zeta-\eta V_1)}-\frac{5\eta^2 V_1'^2}{16(\zeta-\eta V_1)^2}+\frac{(n-1)(n-3)}{4r^2}+V_3 \\ &=(1-\eta)V_1+\frac{-\eta V_1''}{4(\zeta-\eta V_1)}+O(r^{-1-\delta}), \end{aligned}$$

since we have assumed that $V_1'=O(r^{-1})$ and $V_3=O(r^{-1-\delta})$. Let $a=0$ in (V2–3). Then since $V_1''=O(r^{-1-\delta})$, choosing

$$\eta=1, \tag{14}$$

we have $q=O(r^{-1-\delta})$. Next, let $a>0$ in (V2–3). Then, as is proved in Lemma 8.3, $V_1=O(r^{-1})$ and hence

$$(1-\eta)V_1+\frac{-\eta V_1''}{4(\zeta-\eta V_1)}=-\frac{\eta\left\{V_1''-\frac{4\zeta(1-\eta)}{\eta}V_1\right\}}{4(\zeta-\eta V_1)}+O(r^{-2})\,.$$

Thus, choosing

$$\eta=\frac{4\zeta}{4\zeta-a}, \tag{15}$$

we have $q=O(r^{-1-\delta})$. It follows from (11), (13), (14) and (15)

$$k(r,\zeta)=-i\sqrt{\zeta-\eta V_1(r)}+\frac{n-1}{2r}+\frac{-\eta V_1'(r)}{4\{\zeta-\eta V_1(r)\}}\;;\eta=\frac{4\zeta}{4\zeta-a}\,. \tag{16}$$

As is shown above this $k(r, \zeta)$ solves equation (8).

The function $k_\pm(x, \zeta)$ defined by (8.15) comes from (16) if we replace $\eta V_1(r)$ by $\eta V_1(x)+V_2(x)$ and $V_1'(r)$ by $\partial_r V_1(x)$ (note that we do not assume that $V_2(x)$ has second order derivatives). Conditions (V2–4) on $V_1(x)$ and (V2–2) on $V_2(x)$ are required in order to guarantee that this replacement causes no serious difficulties (cf., Proposition 8.2).

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References

[1] Agmon, S., Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Norm. Sup. Pisa (4) **2** (1975), 151–218.
[2] Eidus, D. M., The principle of limit absorption, Mat. Sb. **57** (1962), 13–44.=Amer. Math. Soc. Transl. (2) **47** (1965), 157–191.
[3] Ikebe, T. and T. Kato, Uniqueness of the selfadjoint extension of singular elliptic operators, Arch. Rational Mech. Ann. **9** (1962), 77–92.
[4] Ikebe, T. and Y. Saitō, Limiting absorption method and absolute continuity for the Schrödinger operator, J. Math. Kyoto Univ. **7** (1972), 513–542.
[5] Jäger, W., Ein gewöhnlicher Differentialoperator zweiter Ordnung für Functionen mit Werten in einem Hilbertraum, Math. Z. **113** (1970), 68–98.
[6] Kato, T., Growth properties of solutions of the reduced wave equation with a variable coefficient, Comm. Pure Appl. Math. **12** (1959), 403–425.
[7] Kuroda, S. T., Scattering theory for differential operators, I, operator theory, J. Math. Soc. Japan **25** (1973), 75–104.
[8] Lavine, R., Absolute continuity of positive spectrum for Schrödinger operators with long-range potentials, J. Functional Analysis **12** (1973), 30–54.
[9] Mochizuki, K., Spectral and scattering theory for second order elliptic differential operators in an exterior domain, *Lecture Note Univ. Utah*, Winter and Spring 1972.
[10] Mochizuki, K., Growth properties of solutions of second order elliptic differential equations, J. Math. Kyoto Univ. **16** (1976), 351–373.

[11] Mochizuki, K. and J. Uchiyama, On eigenvalues in the continuum of 2-body or many-body Schrödinger operators, Nagoya Math. J. **70** (1978).
[12] Saitō, Y., The principle of limiting absorption for second order differential equations with operator-valued coefficients, Publ. RIMS, Kyoto Univ. **7** (1971/72), 581–619.
[13] Uchiyama, J., Lower bounds of growth order of solutions of Schrödinger equations with homogeneous potentials, Publ. RIMS, Kyoto Univ. **10** (1975), 425–444.