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Uniform convergence of the upwind finite element approximation for semilinear parabolic problems

By

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Introduction

In the paper [18] we proposed a finite element approximation corresponding to the upwind finite differencing to diffusion equations with drift terms, and obtained L^∞-stability and L^2 error estimates. This approximation enables us to obtain L^∞-stable schemes under weaker conditions than the conventional finite element method. The purpose of this paper is to present a class of upwind finite element schemes for semilinear parabolic problems and to show the L^∞-convergence. We also derive the rate of convergence, which is optimal in this type of approximation.

The finite element method is usually considered as the Ritz-Galerkin method using piecewise polynomial base functions. Therefore, for parabolic problems as well as elliptic ones, it is natural to derive the error estimates in the L^2-sense (e.g., Douglas-Dupont [5], Wheeler [23], Thomée-Wahlbin [19], Fujita-Mizutani [7]). On the other hand, in the L^∞-sense, Fujii [6] obtained stability conditions for a class of finite element schemes for the diffusion equation. Considering the finite element scheme as a step-by-step approximation of an evolution equation in the lumped mass space, and using the approximation theory for semi-groups in the space of continuous functions, Ushijima [20] proved L^∞-convergence for semilinear parabolic problems without presenting any convergence rate. (cf. Nakagawa-Ushijima [14], Ushijima [21]).

Our standpoint for the finite element scheme is to regard it as a finite difference scheme defined on an irregular mesh, which is obtained by substituting each local base function to the test function of the weak form derived from the original equation. From such a standpoint we use a skillful technique of the finite difference method in dealing with the nonlinear term. To put it concretely, our L^∞-stable schemes include an artificial term which was introduced by Mimura [11] to obtain a class of L^∞-stable finite difference schemes for semilinear parabolic systems. (cf. Mimura-Kametaka-Yamaguti [10]).

Viewing a finite element scheme as a finite difference scheme has not been taken so frequently. The main reason is considered that the derived finite difference scheme does not have "local consistency" except some cases of special triangulation of the domain. (cf. Waltz-Fulton-Cyrus [22], Yamamoto-Tokuda [24]) For parabolic problems, however, since the inconsistency stems only from the highest derivative terms with respect to the spatial variables, we can overcome the difficulty by using the result of uniform convergence of the corresponding linear elliptic problems, which is an extension to the upwind finite element scheme of the result obtained by Ciarlet and Raviart [4] for the conventional finite element scheme.

The contents of this paper is as follows. In § 1, two main theorems are stated with some basic assumptions. In § 2, some properties of the derived finite difference equations are discussed. In § 3 (resp. § 4), by making use of a theorem concerned with the corresponding elliptic equations, the main theorem for the explicit (resp. implicit) scheme is proved. In § 5, the theorem used in § 3 and § 4 is proved. In § 6, some remarks are given.

§ 1. Presentation of results

Let Ω be a bounded domain in $\boldsymbol{R}^n$ $(n \geqq 2)$ with a C^3-class boundary Γ or a polyhedral domain in $\boldsymbol{R}^n$, and T be a fixed positive number. Consider the problem,

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - (\boldsymbol{b}\cdot\boldsymbol{\nabla})u + f(x, t, u) & \text{in} \quad Q = \Omega \times (0, T), \\ u = g(x, t) & \text{on} \quad \Sigma = \Gamma \times (0, T), \\ u = u^0(x) & \text{in} \quad \Omega \quad \text{at} \quad t = 0, \end{cases}$$

where $\boldsymbol{b} = (b_1(x), \dots, b_n(x))$ is a given vector-valued Lipschitz continuous function in $\bar{\Omega}$ and

$$\boldsymbol{b}\cdot\boldsymbol{\nabla} = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\,.$$

Assumption 1. f, g and u^0 satisfy the following conditions:

i) $f(x, t, u) = f_0(x, t) + f_1(x, t, u)u$, where f_0 and f_1 are continuous in $\bar{Q}$ and in $\bar{Q} \times \boldsymbol{R}$ respectively. Furthermore, f_1 is continuously differentiable in $u \in \boldsymbol{R}$ and satisfies

$$(1.2) \qquad f_1(x, t, u) \leqq M \qquad \text{for} \quad (x, t) \in \bar{Q},\ u \in \boldsymbol{R},$$

where M is a constant;

ii) g is Lipschitz continuous in $t \in [0, T]$ uniformly with respect to $x \in \Gamma$;

iii) u^0 is continuous in $\bar{\Omega}$;

iv) g and u^0 satisfy the compatibility condition

$$(1.3) \qquad g(x, 0) = u^0(x) \qquad \text{for} \quad x \in \Gamma.$$

In order to obtain the approximate solutions we discretize (1.1). We triangulate $\bar{\Omega}$ and obtain a set of closed n-simplices $\{T_j\}_{j=1}^{N_E}$ and a set of nodal points $\{P_i\}_{i=1}^{N+N_B}$ satisfying the following four conditions:

i) the interiors of T_i and T_j, $i \neq j$, are disjoint;

ii) any one of the faces of T_i is a face of another n-simplex T_j or a portion of the boundary of the polyhedron $\bigcup_{j=1}^{N_E} T_j$;

iii) all the nodal points lying on the boundary of the polyhedron $\bigcup_{j=1}^{N_E} T_j$ exist on Γ;

iv) P_i, $i=1,\dots, N$, exist in Ω and P_i, $i=N+1,\dots, N+N_B$, exist on Γ.

Define $h(T_j)$, $\rho(T_j)$, h, κ, Ω_h, and Γ_h:

$h(T_j)$ = the diameter of the smallest ball containing T_j;

$\rho(T_j)$ = the diameter of the largest ball contained in T_j;

$h = \max\{h(T_j);\ j=1,\dots, N_E\}$;

κ = the minimum perpendicular length of all the simplices;

Ω_h = the interior of $\bigcup_{j=1}^{N_E} T_j$;

Γ_h = the boundary of Ω_h.

We denote by $\mathscr{T}_h$ $(=\{T_j\}_{j=1}^{N_E})$ a triangulation of Ω satisfying the above conditions. (N_E, N, and N_B may vary depending on a triangulation.)

Remark 1.1

i) Obviously it holds that

$$\kappa < h. \tag{1.4}$$

ii) In the case where Ω is a polyhedral domain, we can take $\Omega_h = \Omega$ and then we do not have to consider any extensions of functions defined in Ω in the subsequent sections.

Assumption 2. Triangulation $\mathscr{T}_h$ is regular and of acute type; i.e.,

i) there exists a constant γ (>1) independent of triangulation such that

$$h(T_k) \leqq \gamma\rho(T_k) \qquad \text{for all} \quad T_k \in \mathscr{T}_h,$$

ii) it holds that

$$\sigma(T_k) \leqq 0 \qquad \text{for all} \quad T_k \in \mathscr{T}_h,$$

where $\sigma(T_k) = \max_{0\leqq i<j\leqq n} \cos(\nabla\lambda_i, \nabla\lambda_j)$, and λ_i, $i=0,\dots, n$, are the barycentric coordinates with respect to the vertices of the n-simplex T_k.

With each nodal point P_i we associate functions ϕ_{ih} and $\bar{\phi}_{ih}$ such that

i) ϕ_{ih} is linear on each n-simplex and $\phi_{ih}(P_j) = \delta_{ij}$ for $i, j=1,\dots, N+N_B$,

ii) $\bar{\phi}_{ih} \in L^2(\Omega_h)$ is the characteristic function of the barycentric domain D_i associated with P_i; i.e.,

$$D_i = \bigcup_k \{D_i^k;\ T_k \in \mathscr{T}_h \text{ such that } P_i \text{ is a vertex of } T_k\},$$

where

$$D_i^k = \bigcap_{j=1}^{n} \{x;\, x \in T_k \text{ and } \lambda_{i_j}(x) \leqq \lambda_i(x)\}$$

and $\lambda_i, \lambda_{i_1}, \dots, \lambda_{i_n}$ are the barycentric coordinates with respect to $P_i, P_{i_1}, \dots, P_{i_n}$, the vertices of T_k.

Define V_h, V_{0h} and a lumping operator $-$:

V_h = the linear span of ϕ_{ih}, $i=1,\dots, N+N_B$;

V_{0h} = the linear span of ϕ_{ih}, $i=1,\dots, N$;

$-$: $C(\bar{\Omega})$ or $C(\bar{\Omega}_h) \longrightarrow L^2(\Omega_h)$

$$v \longmapsto \bar{v} = \sum_{i=1}^{N+N_B} v(P_i)\bar{\phi}_{ih}.$$

We now introduce the upwind finite element. A n-simplex $T_j \in \mathscr{T}_h$ is called an upwind finite element at a nodal point P_i if the following two conditions are satisfied:

i) P_i is a vertex of T_j,

ii) $T_j - \{P_i\}$ meets the oriented half line with end point P_i of direction $\boldsymbol{b}(P_i)$.

We denote by $T_{c(i)}$ the upwind finite element at P_i, by selecting arbitrary one in the case where there exist some upwind finite elements at the point. Define a linear operator B_h from V_h into $L^2(\Omega_h)$ by

$$B_h v_h = \sum_{i=1}^{N} B_{hi} v_h\, \bar{\phi}_{ih} \qquad \text{for} \quad v_h \in V_h,$$

where

$$B_{hi} v_h = \boldsymbol{b}(P_i) \cdot \boldsymbol{\nabla} v_h|_{T_{c(i)}}.$$

We consider the following discretized problems of (1.1).

Explicit scheme:

$$(1.5)\quad \begin{cases} \text{Find } \{u_h^k\}_{k=0,\dots,N_T} \subset V_h \text{ such that} \\ \left(\dfrac{\bar{u}_h^{k+1} - \bar{u}_h^k}{\tau}, \bar{\phi}_h\right)_h = -a_h(u_h^k, \phi_h) - (B_h u_h^k, \bar{\phi}_h)_h \\ \qquad + (\bar{f}_0(k\tau) + (\bar{f}_1(x, k\tau, \bar{u}_h^k) - M)\bar{u}_h^{k+1} + M\bar{u}_h^k, \bar{\phi}_h)_h \\ \qquad\qquad \text{for all} \quad \phi_h \in V_{0h},\ k=0,\dots, N_T-1, \\ u_h^{k+1}(P_i) = g(P_i, (k+1)\tau) \qquad \text{for} \quad i=N+1,\dots, N+N_B,\ k=0,\dots, N_T-1, \\ u_h^0(P_i) = u^0(P_i) \qquad \text{for} \quad i=1,\dots, N+N_B. \end{cases}$$

Implicit scheme:

$$(1.6)\quad \begin{cases} \text{Find } \{u_h^k\}_{k=0,\dots,N_T} \subset V_h \text{ such that} \\ \left(\dfrac{\bar{u}_h^{k+1} - \bar{u}_h^k}{\tau}, \bar{\phi}_h\right)_h = -a_h(u_h^{k+1}, \phi_h) - (B_h u_h^{k+1}, \bar{\phi}_h)_h \\ \qquad + (\bar{f}_0(k\tau+\tau) + (\bar{f}_1(x, k\tau, \bar{u}_h^k) - M)\bar{u}_h^{k+1} + M\bar{u}_h^k, \bar{\phi}_h)_h \end{cases}$$

$$\left|\begin{array}{ll} & \text{for all} \quad \phi_h \in V_{0h}, \ k=0,\dots, N_T-1, \\ u_h^{k+1}(P_i)=g(P_i, (k+1)\tau) & \\ & \text{for} \quad i=N+1,\dots, N+N_B, \ k=0,\dots, N_T-1, \\ u_h^0(P_i)=u^0(P_i) & \text{for} \quad i=1,\dots, N+N_B, \end{array}\right.$$

where τ is a time mesh, $N_T=[T/\tau]$,

$$(u, v)_h=\int_{\Omega_h} u \cdot v\, dx,$$

and

$$a_h(u, v)=\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_h.$$

Our main results are the following two theorems.

Theorem 1.1. *Under Assumptions* 1, 2, *and the condition*

$$\tau \leqq \frac{\kappa^2}{(n+1)+\|\boldsymbol{b}\|_{0,\infty,\Omega}\,\kappa}, \tag{1.7}$$

(1.5) *is L^∞-stable and if the exact solution u of* (1.1) *belongs to* $C^{1+1,1}(\bar{Q}) \cap C^{2+1,0}(\bar{Q}) \cap C^{0,1+\frac{1}{2}}(\bar{Q})$, *then the solution u_h^k of* (1.5) *satisfies the estimate*

$$\max_{x\in\bar{\Omega}, k=0,\dots,N_T} |u_h^k(x)-u(x, k\tau)| \leqq ch, \tag{1.8}$$

where c is a nonnegative constant independent of h, τ, and κ and

$$\|\boldsymbol{b}\|_{0,\infty,\Omega}=\max_{x\in\bar{\Omega}}\{\sum_{i=1}^{n} b_i(x)^2\}^{1/2}.$$

Theorem 1.2. *Under Assumptions* 1 *and* 2, (1.6) *is unconditionally L^∞-stable and if the exact solution u of* (1.1) *belongs to* $C^{1+1,1}(\bar{Q}) \cap C^{2+1,0}(\bar{Q}) \cap C^{0,1+1}(\bar{Q})$, *then the solution u_h^k of* (1.6) *satisfies the estimate*

$$\max_{x\in\bar{\Omega}, k=0,\dots,N_T} |u_h^k(x)-u(x, k\tau)| \leqq c(\tau+h), \tag{1.9}$$

where c is a nonnegative constant independent of h, τ and κ.

Remark 1.2.

i) In (1.8) and (1.9), u_h^k are supposed to be extended to $\Omega-\Omega_h$ in such a way that *u_h^k are constant along the normal to Γ.*
ii) More precise results about the L^∞-stability are shown in Corollaries 3.1 and 4.1.
iii) We cannot expect ch^2 (resp. $c(\tau+h^2)$) in (1.8) (resp. (1.9)), because the upwind finite element approximation corresponds to the one-sided difference approximation whose trancation error is of order h.
iv) Notations of function spaces used in this paper are listed below.

For $0<\alpha\leqq 1$ and nonnegative integers l, m,

$$|u|_{l+\alpha,\infty,\Omega}=\sup\left\{\frac{|D_x^\beta u(x)-D_x^\beta u(y)|}{|x-y|^\alpha};\ |\beta|=l,\ x,\ y\in\Omega\right\},$$

$$|u|_{(l+\alpha,m),\infty,Q}=\sup\left\{\frac{|D_x^\beta D_t^m u(x,t)-D_x^\beta D_t^m u(y,t)|}{|x-y|^\alpha};\ |\beta|=l,\ x,\ y\in\Omega,\ t\in(0,T)\right\},$$

$$|u|_{(l,m+\alpha),\infty,Q}=\sup\left\{\frac{|D_x^\beta D_t^m u(x,t)-D_x^\beta D_t^m(x,s)|}{|t-s|^\alpha};\ |\beta|=l,\ x\in\Omega,\ t,\ s\in(0,T)\right\},$$

$\|u\|_{l,\infty,\Omega}=\sup\{|D_x^\beta u(x)|;\ |\beta|\leqq l,\ x\in\Omega\}$,

$\|u\|_{(l,m),\infty,Q}=\sup\{|D_x^\beta D_t^j u(x,t)|;\ |\beta|\leqq l,\ j=0,\dots,m,\ x\in\Omega,\ t\in(0,T)\}$,

$\|u\|_{l+\alpha,\infty,\Omega}=\|u\|_{l,\infty,\Omega}+|u|_{l+\alpha,\infty,\Omega}$,

$\|u\|_{(l+\alpha,m),\infty,Q}=\|u\|_{(l,m),\infty,Q}+|u|_{(l+\alpha,m),\infty,Q}$,

$\|u\|_{(l,m+\alpha),\infty,Q}=\|u\|_{(l,m),\infty,Q}+|u|_{(l,m+\alpha),\infty,Q}$,

$C^l(\bar{\Omega})=\{u;\ u$ is continuously differentiable up to order l in $\bar{\Omega}\}$,

$C^{l,m}(\bar{Q})=\{u;\ u$ is continuously differentiable up to order l in x and up to order m in t in $\bar{Q}\}$,

$C^{l+\alpha}(\bar{\Omega})=\{u;\ u\in C^l(\bar{\Omega})$ and $\|u\|_{l+\alpha,\infty,\Omega}<+\infty\}$,

$C^{l+\alpha,m}(\bar{Q})=\{u;\ u\in C^{l,m}(\bar{Q})$ and $\|u\|_{(l+\alpha,m),\infty,Q}<+\infty\}$,

$C^{l,m+\alpha}(\bar{Q})=\{u;\ u\in C^{l,m}(\bar{Q})$ and $\|u\|_{(l,m+\alpha),\infty,Q}<+\infty\}$,

$$|g|_{(0,0+1),\infty,\Sigma}=\sup\left\{\frac{|g(x,t)-g(x,s)|}{|t-s|},\ x\in\Gamma,\ t,\ s\in(0,T)\right\}.$$

For $1\leqq p<+\infty$,

$|u|_{l,p,\Omega}=\{\sum_{|\beta|=l}\|D_x^\beta u\|_{L^p(\Omega)}^p\}^{1/p}$,

$\|u\|_{l,p,\Omega}=\{\sum_{j=0}^{l}|u|_{j,p,\Omega}^p\}^{1/p}$,

$W^{l,p}(\Omega)=\{u;\ u$ is measurable in Ω, $\|u\|_{l,p,\Omega}<+\infty\}$.

Also, the following notations are used throughout this paper:

$c=c(A_1,\dots,A_j)$ means that c is a nonnegative or positive constant depending only on $A_1,\dots,A_j$;

$$<i,j>=\{i,\ i+1,\dots,j\}\qquad\text{for integers}\quad i<j;$$

θ with super- and/or sub-scripts denotes a real number whose absolute value is bounded by 1.

§2. Preliminaries

In this section we introduce a difference operator related to the spatial derivative terms of (1.1). After examining some properties of the difference operator, we rewrite (1.5) and (1.6) by making use of it.

Let $L_h=\{L_{hi}\}_{i=1}^N$ be a difference operator such that

$$L_{hi}v=\frac{1}{m_{ii}}\sum_{j=1}^{N+N_B}(a_{ij}+b_{ij})v_j \qquad \text{for} \quad i\in <1, N>, \tag{2.1}$$

where $v_j=v(P_j)$, $j\in <1, N+N_B>$, and

$$m_{ii}=(\bar{\phi}_{ih}, \bar{\phi}_{ih})_h, \quad a_{ij}=a_h(\phi_{jh}, \phi_{ih}), \quad b_{ij}=(B_h\phi_j, \bar{\phi}_{ih})_h.$$

Lemma 2.1. *Under Assumption* 2, *it holds that for* $i\in <1, N>$ *and* $j\in <1, N+N_B>$,

$$a_{ij}\leqq 0 \quad \text{if} \quad i\neq j \quad \textit{and} \quad 0<\frac{a_{ii}}{m_{ii}}\leqq\frac{n+1}{\kappa_i^2}, \tag{2.2}$$

$$b_{ij}\leqq 0 \quad \text{if} \quad i\neq j \quad \textit{and} \quad 0\leqq\frac{b_{ii}}{m_{ii}}\leqq\frac{|b(P_i)|}{\kappa_i}, \tag{2.3}$$

$$\sum_{j=1}^{N+N_B}a_{ij}=0 \quad \textit{and} \quad \sum_{j=1}^{N+N_B}b_{ij}=0, \tag{2.4}$$

where κ_i *is the minimum perpendicular length of all the simplices which contain* P_i *as a vertex and*

$$|\boldsymbol{b}(P_i)|=\{\sum_{j=1}^{n}b_j(P_i)^2\}^{1/2}.$$

Proof. (2.2) is the result obtained by Fujii [6]. Let us prove (2.3). Let $T_{c(i)}$ have vertices $P_i, P_{i_1},\dots, P_{i_n}$ and $\lambda_0, \lambda_1,\dots, \lambda_n$ be the barycentric coordinates with respect to them. Observing that

$$b_{ij}=m_{ii}\boldsymbol{b}(P_i)\cdot\nabla\phi_{jh}|_{T_{c(i)}},$$

we have

$$b_{ij}=\begin{cases} m_{ii}\boldsymbol{b}(P_i)\cdot\nabla\lambda_l & \text{if} \quad j=i_l\in\{i, i_1,\dots, i_n\} \\ 0 & \text{if} \quad j\notin\{i, i_1,\dots, i_n\}, \end{cases} \tag{2.5}$$

where $i_0=i$. Let $\boldsymbol{p}_k$ be the vector $\overrightarrow{P_iP_{i_k}}$, $k\in <1, n>$. A brief calculation shows

$$\nabla\lambda_l\cdot\boldsymbol{p}_k=\delta_{lk} \qquad \text{for} \quad l, k\in <1, n>, \tag{2.6}$$

$$\nabla\lambda_0\cdot\boldsymbol{p}_k=-1 \qquad \text{for} \quad k\in <1, n>, \tag{2.7}$$

$$(2.8) \qquad |\nabla \lambda_0| \leqq \frac{1}{\kappa_l}.$$

In view of the definition of the upwind finite element, we note that there exist non-negative numbers c_k^l, $k \in <1, n>$, such that

$$(2.9) \qquad \boldsymbol{b}(P_i) = -\sum_{k=1}^{n} c_k^l \boldsymbol{p}_k.$$

Combining (2.5)$\sim$(2.9), we obtain (2.3).

(2.4) is obtained by noting that

$$\sum_{j=1}^{N+N_B} \phi_{jh} = 1 \qquad \text{in} \quad \Omega_h.$$

This completes the proof.

For later use we present equivalent schemes to (1.5) and (1.6), making use of the difference operator L_h.

Explicit scheme:

$$(2.10) \qquad \begin{cases} \dfrac{u_{hi}^{k+1} - u_{hi}^k}{\tau} = -L_{hi}u_h^k + f_{0i}^k + f_1(P_i, k\tau, u_{hi}^k)u_{hi}^{k+1} - M(u_{hi}^{k+1} - u_{hi}^k) \\ \qquad\qquad\qquad \text{for} \quad i \in <1, N>, \ k \in <0, N_T - 1>, \\ u_{hi}^{k+1} = g(P_i, (k+1)\tau) \quad \text{for} \quad i \in <N+1, N+N_B>, \ k \in <0, N_T - 1>, \\ u_{hi}^0 = u_i^0 \quad \text{for} \quad i \in <1, N+N_B>. \end{cases}$$

Implicit scheme:

$$(2.11) \qquad \begin{cases} \dfrac{u_{hi}^{k+1} - u_{hi}^k}{\tau} = -L_{hi}u_h^{k+1} + f_{0i}^{k+1} + f_1(P, k\tau, u_{hi}^k)u_{hi}^{k+1} - M(u_{hi}^{k+1} - u_{hi}^k) \\ \qquad\qquad\qquad \text{for} \quad i \in <1, N>, \ k \in <0, N_T - 1>, \\ u_{hi}^{k+1} = g(P_i, (k+1)\tau) \quad \text{for} \quad i \in <N+1, N+N_B>, \ k \in <0, N_T - 1>, \\ u_{hi}^0 = u_i^0 \quad \text{for} \quad i \in <1, N+N_B>, \end{cases}$$

where $u_{hi}^k = u_h^k(P_i)$, $f_{0i}^k = f_0(P_i, k\tau)$ and $u_i^0 = u^0(P_i)$.

(2.10) and (2.11) are derived by substituting $\phi_h = \phi_{ih}$ in (1.5) and (1.6) respectively and by dividing both sides by m_{ii}.

Remark 2.1. Except some cases of special triangulation, L_h has no local consistency in the conventional sense: for any sufficiently smooth solution u of (1.1), $-L_{hi}u$ does not approximate $(\Delta u - (\boldsymbol{b} \cdot \nabla)u)(P_i)$; i.e.,

$$-L_{hi}u = (\Delta u - (\boldsymbol{b} \cdot \nabla)u)(P_i) + O(1).$$

(See Waltz, Fulton and Cyrus [22].)

§3. Proof of Theorem 1.1.

In this section we prove Theorem 1.1 by using the result of uniform convergence for the corresponding elliptic problem (Theorem 3.1), which will be proved in §5.

We first state without proof a result about the approximate domain Ω_h (cf. Ciarlet and Wagschal [3]).

Lemma 3.1. *Let $\mathscr{T}_h$ be regular (i.e., i) of Assumption 2). Then, for sufficiently small h, there exists a bijective mapping ν from Γ onto Γ_h along the normal to Γ at that point and we have*

$$\sup_{x\in\Gamma} \operatorname{dist}(x, \nu(x)) \leqq ch^2, \tag{3.1}$$

where $c=c(\Omega, n)$.

Let Ω_1 be a bounded domain containing $\bar{\Omega}$. By Lemma 3.1, without loss of generality, we can assume that

$$\Omega_h \subset \Omega_1 \qquad \text{for all} \quad h. \tag{3.2}$$

Consider the following elliptic problem with $\boldsymbol{b}$ stated in (1.1),

$$\begin{cases} -\Delta v + (\boldsymbol{b}\cdot\nabla)v + \mu v = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \Gamma, \end{cases} \tag{3.3}$$

where $f(x)\in L^p(\Omega)$ and $g(x)\in C(\Gamma)$ are given functions and μ is a nonnegative constant greater than or equal to some nonnegative constant $\mu_0(\gamma, n, \|\boldsymbol{b}\|_{0,\infty,\Omega})$. (See (5.24).) We discretize (3.3) by the finite element scheme,

$$\begin{cases} \text{Find } v_h \in V_h \text{ such that} \\ a_h(v_h, \phi_h) + (B_h v_h, \bar{\Phi}_h)_h + \mu(\bar{v}_h, \bar{\Phi}_h)_h = (\tilde{f}, \phi_h)_h \qquad \text{for all} \quad \phi_h \in V_{0h}, \\ v_h(P_i) = g(P_i) \qquad \text{for} \quad i \in \langle N+1, \dots, N+N_B \rangle, \end{cases} \tag{3.4}$$

where $\tilde{f}\in L^p(\Omega_1)$ is an extension of f to Ω_1 satisfying that

$$\|\tilde{f}\|_{0,p,\Omega_1} \leqq c(p, \Omega)\|f\|_{0,p,\Omega}. \tag{3.5}$$

Define an interpolation operator I_h from $C(\bar{\Omega})$ onto V_h by

$$(I_h v)(P_i) = v(P_i) \qquad \text{for} \quad i \in \langle 1, N+N_B \rangle. \tag{3.6}$$

Theorem 3.1. *Suppose Assumption 2 and that $p>n$. If the exact solution v of (3.3) belongs to $W^{2,p}(\Omega)$, then we have, for the solution v_h of (3.4),*

$$\|v_h - I_h v\|_{0,\infty,\Omega_h} \leqq ch\|v\|_{2,p,\Omega}, \tag{3.7}$$

where $c=c(\gamma, \Omega, n, p, \mu, \|\boldsymbol{b}\|_{0+1,\infty,\Omega})$ and

$$\|\boldsymbol{b}\|_{0+1,\infty,\Omega} = \sup\left\{\frac{|\boldsymbol{b}(x)-\boldsymbol{b}(y)|}{|x-y|};\ x, y\in\Omega\right\} + \|\boldsymbol{b}\|_{0,\infty,\Omega}.$$

Theorem 3.1 is an extension to the upwind finite element scheme of the result obtained by Ciarlet and Raviart [4]. For the proof we are required to develop an approximation theory for the upwind finite element term, so we shall show the complete proof in § 5.

Now, we fix a linear operator Ξ which is decided from Ω. In the case when $\partial\Omega$ is smooth, Ξ extends functions defined in Ω to Ω_1, satisfying

$$(3.8) \qquad \|\Xi v\|_{m,p,\Omega_1} \leqq c(p, \Omega)\,\|v\|_{m,p,\Omega} \quad \text{for} \quad v \in W^{m,p}(\Omega),\ m \in <0, 3>,\ 1 \leqq p < +\infty,$$

$$(3.9) \qquad \|\Xi v\|_{m,\infty,\Omega_1} \leqq c(\Omega)\,\|v\|_{m,\infty,\Omega} \quad \text{for} \quad v \in C^m(\bar{\Omega}),\ m \in <0, 3>,$$

and

$$(3.10) \qquad \|\Xi v\|_{m+\alpha,\infty,\Omega_1} \leqq c(\Omega)\,\|v\|_{m+\alpha,\infty,\Omega} \quad \text{for} \quad v \in C^{m+\alpha}(\bar{\Omega}),\ 0 < \alpha \leqq 1,\ m \in <0, 2>.$$

Since Ω has the C^3-class boundary Γ, we can construct such a Ξ by the method of Nikolskij. (See Nečas [15], Theorem 3.9, p. 75 or Mizohata [12], Theorem 3.10, p. 183). In the case where Ω is a polyhedral domain we define $\Xi = I$.

Define a linear operator E_h from $C(\bar{\Omega}) \cap W^{2,p}(\Omega)$ into V_{0h} by

$$(3.11) \qquad E_h v = \frac{v_h - I_h v}{h} \quad \text{for} \quad v \in C(\bar{\Omega}) \cap W^{2,p}(\Omega),$$

where v_h is the solution of (3.4) with $\mu = \mu_0$ corresponding to

$$\tilde{f} = -\Delta \Xi v + (\boldsymbol{b} \cdot \nabla)\Xi v + \mu_0 \Xi v \quad \text{and} \quad g = v.$$

It has been noted in Remark 2.1 that $L_{hi}(I_h v)$ does not approximate $(-\Delta v + (\boldsymbol{b} \cdot \nabla)v)(P_i)$. However, if we replace $I_h v$ by $I_h v + hE_h v$, which is nothing but v_h, the desired result is obtained.

Lemma 3.2. *Let v belong to $C^{2+1}(\bar{\Omega})$. Then, we have*

$$(3.12) \qquad \|E_h v\|_{0,\infty,\Omega_h} \leqq c\|v\|_{1+1,\infty,\Omega},$$

$$(3.13) \qquad L_{hi}(I_h v + hE_h v) = (-\Delta v + (\boldsymbol{b} \cdot \nabla)v)(P_i) + \theta_i ch \|\nabla v\|_{1+1,\infty,\Omega} \quad \text{for} \quad i \in <1, N>,$$

where $c = c(\gamma, \Omega, n, \|\boldsymbol{b}\|_{0+1,\infty,\Omega})$.

Proof. Fix $p > n$ and set

$$f = -\Delta v + (\nabla \cdot \boldsymbol{b})v + \mu_0 v.$$

Obviously, f belongs to $L^p(\Omega)$ and $\tilde{f}(= -\Delta \Xi v + (\boldsymbol{b} \cdot \nabla)\Xi v + \mu_0 \Xi v)$ is an extension of f satisfying (3.5) by virtue of (3.8). Applying Theorem 3.1 with $\mu = \mu_0$, we have

$$(3.14) \qquad \|E_h v\|_{0,\infty,\Omega_h} \leqq c\|v\|_{2,p,\Omega}.$$

Combining (3.14) and the trivial inequality

$$\|v\|_{2,p,\Omega} \leqq c(p,\, \Omega)\, \|v\|_{1+1,\infty,\Omega},$$

we obtain (3.12).

We now prove(3.13). Since $I_h v + hE_h v = v_h$, we have

$$(3.15)\qquad L_{hi}(I_h v + hE_h v) = \frac{1}{m_{ii}}\{a_h(v_h,\, \phi_{ih}) + (B_h v_h,\, \bar{\Phi}_{ih})_h\}$$

$$= \frac{1}{m_{ii}}\{(-\Delta \Xi v + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\Xi v + \mu_0 \Xi v,\, \phi_{ih})_h - \mu_0(\bar{v}_h,\, \bar{\Phi}_{ih})_h\}\,.$$

In the support of ϕ_{ih}, it holds that

$$(3.16)\qquad -\Delta \Xi v + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\Xi v + \mu_0 \Xi v = (-\Delta v + (\boldsymbol{b}\cdot\boldsymbol{\nabla})v + \mu_0 v)(P_i) + \theta_i(1+\mu_0)h\|\boldsymbol{\nabla} v\|_{1+1,\infty,\Omega}.$$

Combining (3.15), (3.16), (3.12) and that

$$(1,\, \phi_{ih})_h = (1,\, \bar{\Phi}_{ih})_h = m_{ii},$$

we obtain (3.13). This completes the proof.

Lemma 3.3. *Let* $v_h^k \in V_h$, $k \in <0,\, N_T>$, *satisfy*

$$(3.17)\qquad v_{hi}^{k+1} = \frac{v_{hi}^k - \tau L_{hi} v_h^k + \tau r_i^k v_{hi}^k + \tau s_i^k}{1+\tau q_i^k} \qquad \textit{for}\quad i \in <1,\, N>,\ k \in <0,\, N_T - 1>,$$

and

$$(3.18)\qquad |v_{hi}^{k+1} - v_{hi}^k| \leqq \tau S \qquad \textit{for}\quad i \in <N+1,\, N+N_B>,\ k \in <0,\, N_T - 1>,$$

where q_i^k, r_i^k *and* s_i^k *are given functions defined on the nodal points* P_i, $i \in <1,\, N>$, *for* $k \in <0,\, N_T>$, *satisfying*

$$q_i^k \geqq 0,\ |r_i^k| \leqq R \quad \textit{and} \quad |s_i^k| \leqq S \qquad \textit{for}\quad i \in <1,\, N>,\ k \in <0,\, N_T>,$$

where R and S are nonnegative constants.

Then, under the condition (1.7), *we have*

$$(3.19)\qquad \max_{i\in<1,N+N_B>} |v_{hi}^k| \leqq e^{RT} \max_{i\in<1,N+N_B>} |v_{hi}^0| + \frac{e^{RT}-1}{R} S \qquad \textit{for}\quad k \in <0,\, N_T>.$$

Proof. From (2.1) we have

$$(3.20)\qquad v_{hi}^{k+1} = \frac{\left(1 - \tau \dfrac{a_{ii}+b_{ii}}{m_{ii}}\right) v_{hi}^k - \dfrac{\tau}{m_{ii}} \sum_{j\neq i} (a_{ij}+b_{ij}) v_{hj}^k + \tau r_i^k v_{hi}^k + \tau s_i^k}{1+\tau q_i^k}.$$

By (2.2), (2.3) and (1.7), it holds that

$$1-\tau\frac{a_{ii}+b_{ii}}{m_{ii}}\geqq 0, \qquad -\frac{\tau}{m_{ii}}(a_{ij}+b_{ij})\geqq 0$$

for $i\in <1, N>, j\in <1, N+N_B>, i\neq j$. Hence, we have

$$|v_{hi}^{k+1}|\leqq\left(1-\tau\frac{a_{ii}+b_{ii}}{m_{ii}}\right)|v_{hi}^k|-\frac{\tau}{m_{ii}}\sum_{j\neq i}(a_{ij}+b_{ij})|v_{hj}^k|$$

$$+\tau|r_i^k|\,|v_{hi}^k|+\tau|s_i^k| \qquad \text{for} \quad i\in <1, N>.$$

By (2.4) we obtain

$$\max_{i\in<1,N>}|v_{hi}^{k+1}|\leqq(1+\tau R)\max_{i\in<1,N+N_B>}|v_{hi}^k|+\tau S. \tag{3.21}$$

From (3.18) and (3.21), we have

$$\max_{i\in<1,N+N_B>}|v_{hi}^{k+1}|\leqq(1+\tau R)\max_{i\in<1,N+N_B>}|v_{hi}^k|+\tau S \qquad \text{for} \quad k\in <0, N_T-1>,$$

which implies (3.19). This completes the proof.

Corollary 3.1. *Under Assumptions* 1, 2 *and the condition* (1.7), *the solution* u_h^k *of* (1.5) *satisfies the estimate*

$$\max_{i\in<1,N+N_B>}|u_{hi}^k|\leqq e^{|M|T}\|u^0\|_{0,\infty,\Omega}+\frac{e^{|M|T}-1}{|M|}$$

$$\times\max(\|f_0\|_{0,\infty,Q}, |g|_{(0,0+1),\infty,\Sigma}) \qquad \textit{for} \quad k\in <0, N_T>. \tag{3.22}$$

Proof. From (2.10) we observe u_h^k satisfies (3.17) and (3.18) by taking

$$q_i^k=M-f_1(P_i, k\tau, u_{hi}^k), \quad r_i^k=M, \quad \text{and} \quad s_i^k=f_{0i}^k$$

$$\text{for} \quad i\in <1, N>, \ k\in <0, N_T>$$

and

$$R=|M|, \quad S=\max(\|f_0\|_{0,\infty,Q}, |g|_{(0,0+1),\infty,\Sigma}).$$

Applying Lemma 3.3, we obtain (3.22). This completes the proof.

Proof of Theorem 1.1. Set $v_h^k\in V_{0h}$, $k\in <0, N_T>$, as follows:

$$v_h^k=u_h^k-I_hu(k\tau)-hE_hu(k\tau).$$

We look for a difference equation which v_h^k satisfies. Setting $u_i^k=u(P_i, k\tau)$ and $c=c(\gamma, \Omega, n, \|\boldsymbol{b}\|_{0+1,\infty,\Omega})$, we have

$$\frac{v_{hi}^{k+1}-v_{hi}^k}{\tau}=\frac{u_{hi}^{k+1}-u_{hi}^k}{\tau}-\frac{u_i^{k+1}-u_i^k}{\tau}-h\left(E_h\frac{u(k\tau+\tau)-u(k\tau)}{\tau}\right)(P_i) \tag{3.23}$$

$$=\frac{u_{hi}^{k+1}-u_{hi}^k}{\tau}-\frac{\partial u}{\partial t}(P_i, k\tau)+\theta_{1i}^k\tau^{1/2}|u_t|_{(0,0+\frac{1}{2}),\infty,Q}$$

$$+\theta_{2i}^k ch\|u_t\|_{(1+1,0),\infty,Q},$$

(3.24) $$L_{hi}v_h^k = L_{hi}u_h^k - L_{hi}(I_h u(k\tau) + hE_h(u(k\tau)))$$
$$= L_{hi}u_h^k - (-\Delta u + (\boldsymbol{b}\cdot\nabla)u)(P_i, k\tau) + \theta_{3i}^k ch\|\nabla u\|_{(1+1,0),\infty,Q},$$

(3.25) $$Mv_{hi}^k = Mu_{hi}^k - Mu_i^k + \theta_{4i}^k cMh\|u\|_{(1+1,0),\infty,Q},$$

and

(3.26) $$\{f_1(P_i, k\tau, u_{hi}^k) - M\}v_{hi}^{k+1}$$
$$= \{f_1(P_i, k\tau, u_{hi}^k) - M\}u_{hi}^{k+1} - \Big\{f_1(P_i, k\tau, u_i^k) + (u_{hi}^k - u_i^k)$$
$$\times \frac{\partial f_1}{\partial u}(P_i, k\tau, \xi_i^k) - M\Big\}\{u_i^k + \theta_{5i}^k \tau^{1/2}|u|_{(0,0+\frac{1}{2}),\infty,Q}\}$$
$$+ \theta_{6i}^k\{f_1(P_i, k\tau, u_{hi}^k) - M\}ch\|u\|_{(1+1,0),\infty,Q}$$
$$= \{f_1(P_i, k\tau, u_{hi}^k) - M\}u_{hi}^{k+1} - f_1(P_i, k\tau, u_i^k)u_i^k + Mu_i^k$$
$$- v_{hi}\frac{\partial f_1}{\partial u}(P_i, k\tau, \xi_i^k)u_i^k + \theta_{7i}^k\Big\{\frac{\partial f_1}{\partial u}(P_i, k\tau, \xi_i^k)u_i^k\Big\}ch\|u\|_{(1+1,0),\infty,Q}$$
$$- \theta_{5i}^k\{f_1(P_i, k\tau, u_{hi}^k) - M\}\tau^{1/2}|u|_{(0,0+\frac{1}{2}),\infty,Q}$$
$$+ \theta_{6i}^k\{f_1(P_i, k\tau, u_{hi}^k) - M\}ch\|u\|_{(1+1,0),\infty,Q},$$

where ξ_i^k is an intermediate value between u_{hi}^k and u_i^k. Since u_{hi}^k are bounded by Corollary 3.1, so are ξ_i^k, i.e., it holds that

$$|\xi_i^k| \leqq U \qquad \text{for} \quad i \in \langle 1, N\rangle,\ k \in \langle 0, N_T\rangle,$$

where

$$U = \max\{\|u\|_{0,\infty,Q}, \text{ the right of (3.22)}\}.$$

Combining (3.23)~(3.26), we obtain

(3.27) $$\frac{v_{hi}^{k+1} - v_{hi}^k}{\tau} = -L_{hi}v_h^k + \{f_1(P_i, k\tau, u_{hi}^k) - M\}v_{hi}^{k+1}$$
$$+ \Big\{M + \frac{\partial f_1}{\partial u}(P_i, k\tau, \xi_i^k)u_i^k\Big\}v_{hi}^k + s_i^k$$
$$\text{for} \quad i \in \langle 1, N\rangle,\ k \in \langle 0, N_T - 1\rangle,$$

where $s_i^k = \theta_i^k \hat{c}(\tau^{1/2} + h)$ and

$$\hat{c} = \{\|u\|_{(1+1,1),\infty,Q} + \|u\|_{(2+1,0),\infty,Q} + \|u\|_{(0,1+\frac{1}{2}),\infty,Q}\}$$
$$\times c\Big(U\cdot \sup_{Q\times(-U,U)}\Big|\frac{\partial f_1}{\partial u}(x, t, u)\Big|, \sup_{Q\times(-U,U)}|f_1(x, t, u)|, M,$$
$$\|\boldsymbol{b}\|_{0+1,\infty,\Omega}, \gamma, \Omega, n\Big).$$

From (3.27) we observe v_h^k satisfies (3.17) and (3.18) by taking

$$q_i^k = M - f_1(P_i, k\tau, u_{hi}^k)$$

$$r_i^k = M + \frac{\partial f_1}{\partial u}(P_i, k\tau, \xi_i^k)\, u_i^k,$$

$$R = |M| + U\cdot \sup_{Q\times(-U,U)} \left|\frac{\partial f_1}{\partial u}(x, t, u)\right|,$$

and

$$S = \hat{c}(\tau^{1/2} + h).$$

Applying Lemma 3.3, we have

$$\max_{k\in<0,N_T>} \|v_h^k\|_{0,\infty,\Omega_h} \leqq e^{RT}\|v_h^0\|_{0,\infty,\Omega_h} + \frac{e^{RT}-1}{R}\hat{c}(\tau^{1/2}+h).$$

Hence, it holds

$$\text{(3.28)}\qquad \max_{k\in<0,N_T>} \|u_h^k - \Xi u(k\tau)\|_{0,\infty,\Omega_h} \leqq \max_{k\in<0,N_T>} \|I_h u(k\tau) - \Xi u(k\tau)\|_{0,\infty,\Omega_h}$$
$$+ h \max_{k\in<0,N_T>} \|E_h u(k\tau)\|_{0,\infty,\Omega_h} + he^{RT}\|E_h u(0)\|_{0,\infty,\Omega_h}$$
$$+ \frac{e^{RT}-1}{R}\hat{c}(\tau^{1/2}+h).$$

By the interpolation theory, the first term of the right of (3.28) is bounded by $c(\gamma, \Omega, n)h^2\|u\|_{(2,0),\infty,Q}$. Applying Lemma 3.2, we obtain

$$\text{(3.29)}\qquad \max_{k\in<0,N_T>} \|u_h^k - \Xi u(k\tau)\|_{0,\infty,\Omega_h} \leqq c(\tau^{1/2}+h),$$

where $c = c(\hat{c}, \gamma, \Omega, n, \|u\|_{(2,0),\infty,Q}, R, T)$.

For $x\in\Omega-\Omega_h$, remembering i) of Remark 1.2, we have

$$u_h^k(x) - u(x, k\tau) = \{u_h^k(y) - u(y, k\tau)\} + \{u(y, k\tau) - u(x, k\tau)\},$$

where $y = v(z)\in\Gamma_h$ such that the normal $\overline{yz}$ at z to Γ contains x. By Lemma 3.1 and (3.29), we obtain

$$\text{(3.30)}\qquad \max_{k\in<0,N_T>} \|u_h^k - u(k\tau)\|_{0,\infty,\Omega-\Omega_h} \leqq c(\tau^{1/2}+h),$$

where $c = (\hat{c}, \gamma, \Omega, n, \|u\|_{(2,0),\infty,Q}, R, T)$. Combining (1.4), (1.7), (3.29) and (3.30), we get (1.8). This completes the proof.

§4. Proof of Theorem 1.2

Theorem 1.2 can be proved in a similar way to the proof of Theorem 1.1. So, in this section, we only state the corresponding results to Lemma 3.3 and Corollary 3.1 without proofs.

Lemma 4.1. *Let $v_h^k \in V_h$, $k \in <0, N_T>$ satisfy*

$$v_{hi}^{k+1} + \tau L_{hi} v_h^{k+1} + \tau q_i^{k+1} v_{hi}^{k+1} = v_{hi}^k + \tau r_i^k v_{hi}^k + \tau s_i^k \quad \text{for} \quad i \in <1, N>, \ k \in <0, N_T - 1>, \tag{4.1}$$

and

$$|v_{hi}^{k+1} - v_{hi}^k| \leqq \tau S \quad \text{for} \quad i \in <N+1, N+N_B>, \ k \in <0, N_T - 1>, \tag{4.2}$$

where q_i^k, r_i^k and s_i^k are given functions defined on the nodal points satisfying

$$q_i^k \geqq 0, \quad |r_i^k| \leqq R \quad |s_i^k| \leqq S \quad \text{for} \quad i \in <1, N>, \ k \in <0, N_T>,$$

where R and S are nonnegative constants.

Then, we have

$$\max_{i \in <1, N+N_B>} |v_{hi}^k| \leqq e^{RT} \max_{i \in <1, N+N_B>} |v_{hi}^0| + \frac{e^{RT} - 1}{R} S \quad \text{for} \quad k \in <0, N_T>. \tag{4.3}$$

Corollary 4.1. *Under Assumptions* 1 *and* 2, *the solution u_h^k of* (1.6) *satisfies the estimate*

$$\max_{i \in <1, N+N_B>} |u_{hi}^k| \leqq e^{|M|T} \|u^0\|_{0,\infty,\Omega} + \frac{e^{|M|T} - 1}{|M|} \times \max(\|f_0\|_{0,\infty,Q}, |g|_{(0,0+1),\infty,\Sigma}) \quad \text{for} \quad k \in <0, N_T>. \tag{4.4}$$

§ 5. Uniform convergence for elliptic problems

In this section we prove Theorem 3.1. The upwind finite element approximation has such a feature that the domain where the approximate function is used does not coincide with the domain where the data for the approximation are taken; i.e., the former is the barycentric domain D_i and the latter is the upwind finite element $T_{c(i)}$. Therefore, we first develop a theory for the upwind finite element approximation. We next prove Theorem 3.1 according to Ciarlet and Raviart [4].

For later reference we begin by stating the following two lemmas without proofs.

Lemma 5.1. (*cf. Ciarlet and Raviart* [2], *Theorem* 5) *Suppose $\mathscr{T}_h$ is regular. For $1 \leqq p < +\infty$, there exists a positive constant $c(\gamma, p, n)$ such that*

$$\|I_h v - v\|_{1,p,\Omega_h} \leqq ch|v|_{2,p,\Omega_h} \quad \text{for} \quad v \in W^{2,p}(\Omega_1), \ p > n/2, \tag{5.1}$$

and

$$\|\overline{I_h} v - v\|_{0,p,\Omega_h} \leqq ch|v|_{1,p,\Omega_h} \quad \text{for} \quad v \in W^{1,p}(\Omega_1), \ p > n. \tag{5.2}$$

In particular, it holds that

$$\|\bar{\psi}_h-\psi_h\|_{0,p,\Omega_h}\leqq ch|\psi_h|_{1,p,\Omega_h} \qquad \text{for} \quad \psi_h\in V_h. \tag{5.3}$$

Lemma 5.2. (*cf.* *Fujii* [6]) *Suppose* $\mathcal{T}_h$ *is regular. Then, there exist positive constants* $c_1(\gamma, p, n)$ *and* $c_2(\gamma, p, n)$ *such that*

$$c_1\|\psi_h\|_{0,p,\Omega_h}\leqq\|\bar{\psi}_h\|_{0,p,\Omega_h}\leqq c_2\|\psi_h\|_{0,p,\Omega_h} \qquad \text{for} \quad \psi_h\in V_h. \tag{5.4}$$

Now let B_1 be the open ball in $\boldsymbol{R}^n$ with center at origin O of radius 1. Let $\mathcal{G}$ be the set of (open) polyhedral domains containing O such that $G\in\mathcal{G}$ if and only if the closure $\bar{G}$ is divided into a union of closed n-simplices T_i^G satisfying the following conditions:

i) O is a vertex of T_i^G for all i;
ii) $T_i^G\subset\bar{B}_1$ for all i;
iii) the interiors of T_i^G are pairwise disjoint;
iv) any one of the face of T_i^G is either a face of another T_i^G or else is a portion of the boundary of G;
v) there exists a constant γ (>1) independent of G such that

$$h(T_i^G)\leqq\gamma\rho(T_i^G) \qquad \text{for all} \quad i;$$

vi) there exists i_0 such that $T_{i_0}^G$ has at least one vertex on the boundary of B_1.

Lemma 5.3. *Let* $\mathcal{G}$ *and* B_1 *be those defined above. Then, for* $1<p<+\infty$ *and* $0\leqq m<+\infty$, *there exists a uniformly bounded extension operator from* $W^{m,p}(G)$ *into* $W^{m,p}(B_1)$; *i.e., there exists a positive constant* $c(\gamma, p, m, n)$ *such that, for any* $G\in\mathcal{G}$ *and* $v\in W^{m,p}(G)$, *there exists an extension* $\tilde{v}$ *of* v *to* B_1 *satisfying*

$$\|\tilde{v}\|_{m,p,B_1}\leqq c\|v\|_{m,p,G}. \tag{5.5}$$

Proof. We first prove that there exists a positive constant $\delta(\gamma, n)$ such that

$$B_\delta\subset G \qquad \text{for all} \quad G\in\mathcal{G}, \tag{5.6}$$

where $B_\delta=\{x;\ |x|<\delta\}$. From the condition v) of $\mathcal{G}$, each $\bar{G}$ consists of a finite number of T_i^G. The number is bounded by a constant depending only on γ and n. By vi) and v), the minimum length of $\overline{OP_j^G}$, where P_j^G is any vertex of G, is greater than some positive constant $c(\gamma, n)$, which implies (5.6).

For the proof of (5.5), we use the variant of Calderon's extension theorem (cf. Morrey [13], p. 74). Each $G\in\mathcal{G}$ is a strongly Lipschitz domain and its Lipschitz constant is bounded by some positive constant $c'(\gamma)$. Making use of (5.6), we can obtain a partition of unity $\{\zeta_j^G\}$ such that the support of ζ_j^G does not become so small for any j and G. This completes the proof.

By the following lemma we observe that the result obtained by Bramble and Hilbert [1] holds uniformly for G belonging to $\mathcal{G}$.

Lemma 5.4. *Let* $\mathcal{G}$ *be as above. For* $1<p<+\infty$ *and* $0\leqq m<+\infty$, *there exists a positive constant* $c(\gamma, p, n, m)$ *such that*

(5.7) $\inf_{q\in\mathscr{P}_m}\|v+q\|_{m+1,p,G}\leqq c|v|_{m+1,p,G}$ *for all* $G\in\mathscr{G}$ *and* $v\in W^{m+1,p}(G)$,

where $\mathscr{P}_m$ *is the space of polynomials of degree* $\leqq m$ *defined in* $\boldsymbol{R}^n$.

Proof. Assume that there is no such constant. Then, it follows that there exists a sequence $\{G_j, v_j\}$ such that $G_j\in\mathscr{G}$, $v_j\in W^{m+1,p}(G_j)$ and that

$$\inf_{q\in p_m}\|v_j+q\|_{m+1,p,G_j}=1 \quad\text{and}\quad |v_j|_{m+1,p,G_j}\longrightarrow 0. \tag{5.8}$$

By selecting a subsequence we may assume furthermore that there exists a polyhedral domain $G\in\mathscr{G}$ such that G_j converges to G, i.e.,

$$\chi_j\longrightarrow\chi \text{ pointwise in } B_1, \tag{5.9}$$

where χ_j (resp. χ) is the characteristic function of G_j (resp. G). Take a sequence $\{q_j\}\subset\mathscr{P}_m$ satisfying

$$1\leqq\|v_j+q_j\|_{m+1,p,G_j}\leqq 2.$$

Applying Lemma 5.3, we can construct $\tilde{w}_j\in W^{m+1,p}(B_1)$, a uniformly bounded extension of $w_j=v_j+q_j$. Since $\{\tilde{w}_j\}$ is bounded in $W^{m+1,p}(B_1)$, we can select a subsequence, which is also denoted by $\tilde{w}_j$, such that

$$\tilde{w}_j\longrightarrow w \text{ weakly in } W^{m+1,p}(B_1) \text{ and strongly in } W^{m,p}(B_1), \tag{5.10}$$

where w is a function belonging to $W^{m+1,p}(B_1)$.

We first claim that

$$|w|_{m+1,p,G}=0. \tag{5.11}$$

For any $\phi\in C_0^\infty(G)$, it holds that $\operatorname{supp}[\phi]\subset G_j$ for sufficiently large j. Therefore, using (5.10) and (5.8), we have, for $|\alpha|=m+1$,

$$\begin{aligned}\int_G D^\alpha w\,\phi\,dx&=(-1)^{|\alpha|}\lim_{j\to+\infty}\int_G\tilde{w}_jD^\alpha\phi\,dx\\&=(-1)^{|\alpha|}\lim_{j\to+\infty}\int_{G_j}\tilde{w}_jD^\alpha\phi\,dx\\&=\lim_{j\to+\infty}\int_{G_j}D^\alpha v_j\,\phi\,dx\\&=0,\end{aligned}$$

which yields (5.11).

We next claim that for all $q\in\mathscr{P}_m$

$$\|w+q\|_{m+1,p,G}\geqq 1. \tag{5.12}$$

From (5.8) it holds, for any $q\in\mathscr{P}_m$,

$$1\leqq\sum_{|\alpha|\leqq m}\|D^\alpha(v_j+q_j)+D^\alpha q\|_{0,p,G_j}^p+|v_j|_{m+1,p,G_j}^p. \tag{5.13}$$

By making use of (5.10) and (5.9), we have, for $|\alpha|\leqq m$,

$$\begin{aligned}&\|\chi_j(D^\alpha w_j+D^\alpha q)-\chi(D^\alpha w+D^\alpha q)\|_{0,p,B_1}\\&\leqq\|\chi_jD^\alpha w_j-\chi_jD^\alpha w\|_{0,p,B_1}+\|\chi_jD^\alpha w-\chi D^\alpha w\|_{0,p,B_1}+\|\chi_jD^\alpha q-\chi D^\alpha q\|_{0,p,B_1}\\&\leqq\|D^\alpha\tilde{w}_j-D^\alpha w\|_{0,p,B_1}+\|(\chi_j-\chi)D^\alpha w\|_{0,p,B_1}+\|(\chi_j-\chi)D^\alpha q\|_{0,p,B_1}\\&\longrightarrow 0\quad\text{as}\quad j\longrightarrow+\infty,\end{aligned}$$

which implies that

$$(5.14)\qquad \|D^\alpha(v_j+q_j)+D^\alpha q\|_{0,p,G_j}\longrightarrow\|D^\alpha w+D^\alpha q\|_{0,p,G}\quad\text{as}\quad j\longrightarrow+\infty.$$

Combining (5.8), (5.13), and (5.14), we get (5.12).

But, (5.11) and (5.12) contradict the result for the fixed domain G (cf. Bramble and Hilbert [1], Theorem 1),

$$\inf_{q\in\mathscr{P}_m}\|w+q\|_{m+1,p,G}\leqq c|w|_{m+1,p,G}\qquad\text{for all}\quad w\in W^{m+1,p}(G),$$

where c is a positive constant independent of w. Hence, there is a constant $c(\gamma, p, n, m)$ such that (5.7) holds. This completes the proof.

For $G\in\mathscr{G}$, we can find the barycentric domain D^G associated with O as $\bar{G}$ is the union of T_i^G. Define a linear operator Π_j^G from $W^{2,p}(G)$ into $\mathscr{P}_1$ such that, for $v\in W^{2,p}(G)$, $\Pi_j^G v$ coincides with v at $n+1$ nodal points of T_j^G, where $p>n/2$ and T_j^G is one of n-simplices constituting $\bar{G}$. We regard Π_j^G as an operator from $W^{2,p}(G)$ into $W^{1,p}(D^G)$. Then, we have

Lemma 5.5. *Let $\mathscr{G}$ be as above. Then, for $p>n/2$, there exists a constant $c_3(\gamma, p, n)$ such that*

$$(5.15)\qquad \|v-\Pi_j^G v\|_{1,p,D^G}\leqq c_3|v|_{2,p,G}\qquad\text{for all } j,\quad G\in\mathscr{G},\quad\text{and}\quad v\in W^{2,p}(G).$$

Proof. By the definition of $\mathscr{G}$ and $p>n/2$, we observe Π_j^G is continuous from $W^{2,p}(G)$ into $W^{1,p}(D^G)$ and that Π_j^G is uniformly bounded for j and $G\in\mathscr{G}$. Noting that $\Pi_j^G q=q$ on D^G for any $q\in\mathscr{P}_1$, and using Lemma 5.4 with $m=1$, we obtain (5.15) in a similar way to that of Ciarlet and Raviart [2, Lemma 6]. This completes the proof.

We now have the following estimate for the upwind finite element approximation.

Lemma 5.6. *Suppose a triangulation $\mathscr{T}_h$ is regular and that V_h is the finite element space defined in § 1. If $p>n/2$, then there exists a constant $c(\gamma, p, n)$ such that, for any $v\in W^{2,p}(\Omega_1)$ and $\psi_h\in V_{0h}$,*

$$(5.16)\qquad |(B_hI_hv,\bar{\psi}_h)_h-((\boldsymbol{b}\cdot\boldsymbol{\nabla})v,\psi_h)_h|\leqq ch\|\boldsymbol{b}\|_{0+1,\infty,\Omega}\|v\|_{2,p,\Omega_1}\|\psi_h\|_{1,p',\Omega_h},$$

where $p'=\dfrac{p}{p-1}$.

Proof. Set

$$Q_h v = \sum_{i=1}^{N+N_B} \boldsymbol{b}(P_i)\cdot\boldsymbol{\nabla} v(x)\bar{\Phi}_{ih}.$$

By Hölder's inequality, (5.3), and (5.1), we have

$$\text{(5.17)}\qquad \text{the left of (5.16)} \leqq |(B_h I_h v - Q_h v,\ \bar{\psi}_h)_h| + |(Q_h v,\ \bar{\psi}_h - \psi_h)_h| + |(Q_h v - (\boldsymbol{b}\cdot\boldsymbol{\nabla})v,\ \psi_h)_h|$$

$$\leqq \sum_{i=1}^{N} \|B_{hi}(I_h v) - \boldsymbol{b}(P_i)\cdot\boldsymbol{\nabla} v(x)\|_{0,p,D_i}\|\bar{\psi}_h\|_{0,p',\Omega_h}$$

$$+ c(\gamma, p, n)h\|\boldsymbol{b}\|_{0,\infty,\Omega}|v|_{1,p,\Omega_1}|\psi_h|_{1,p',\Omega_h}$$

$$+ h|\boldsymbol{b}|_{0+1,\infty,\Omega}|v|_{1,p,\Omega_1}\|\psi_h\|_{0,p',\Omega_h}.$$

Let us estimate the first term. Given any interior nodal point P_i, we let h_i' be the maximum length of all the sides containing P_i, and S_i be the union of all the n-simplices containing P_i. Define an affine mapping F_i from $\boldsymbol{R}_\xi^n$ onto $\boldsymbol{R}_x^n$ such that $x = h_i'\xi + x^{(i)}$, where $x^{(i)}$ is the coordinate of P_i. It is easy to see that $F_i^{-1}(S_i) \in \mathscr{G}$. Using F_i^{-1}, (5.15), and F_i, we obtain

$$\|B_{hi}(I_h v) - \boldsymbol{b}(P_i)\cdot\boldsymbol{\nabla} v\|_{0,p,D_i} \leqq c_3 h_i'\|\boldsymbol{b}\|_{0,\infty,\Omega}|v|_{2,p,S_i},$$

which implies that

$$\text{(5.18)}\qquad \sum_{i=1}^{N} \|B_{hi}(I_h v) - \boldsymbol{b}(P_i)\cdot\boldsymbol{\nabla} v\|_{0,p,D_i} \leqq c_3(n+1)h\|\boldsymbol{b}\|_{0,\infty,\Omega}|v|_{2,p,\Omega_1}.$$

Combining (5.17), (5.18), and (5.4), we get (5.16). This completes the proof.

Lemma 5.7. *Let there be given a triangulation $\mathscr{T}_h$ satisfying Assumption 2. Given any function $w_h \in V_h$, we let $w_{h,\alpha}$ denote the function of V_h which satisfies for $i \in <1,\ N+N_B>$*

$$\text{(5.19)}\qquad w_{h,\alpha}(P_i) = \begin{cases} w_h(P_i) - \alpha & \text{if}\quad w_h(P_i) > \alpha \\ 0 & \text{if}\quad w_h(P_i) \leqq \alpha, \end{cases}$$

where α is a given nonnegative number satisfying

$$\text{(5.20)}\qquad \alpha \geqq w_h(P_i) \qquad \text{for}\quad i \in <N+1,\ N+N_B>.$$

Then, there exists a positive constant $c(\Omega)$ and a nonnegative constant $\mu_0(\gamma, n, \|\boldsymbol{b}\|_{0,\infty,\Omega})$ such that, for all $\mu \geqq \mu_0$,

$$\text{(5.21)}\qquad a_h(w_h,\ w_{h,\alpha}) + (B_h w_h,\ \bar{w}_{h,\alpha})_h + \mu(\bar{w}_h,\ \bar{w}_{h,\alpha})_h \geqq c\|w_{h,\alpha}\|_{1,2,\Omega_h}^2.$$

Proof. We first prove that

$$\text{(5.22)}\qquad \text{the left of (5.21)} \geqq a_h(w_{h,\alpha},\ w_{h,\alpha}) + (B_h w_{h,\alpha},\ \bar{w}_{h,\alpha})_h + \mu(\bar{w}_{h,\alpha},\ \bar{w}_{h,\alpha})_h,$$

which is an extension of the result obtained by Ciarlet and Raviart [4, Lemma 2]. According to it, we divide the indices of all nodal points into two disjoint sets,

$$I_1=\{i;\ i\in <1,\ N+N_B>,\ w_h(P_i)>\alpha\},$$

$$I_0=\{i;\ i\in <1,\ N+N_B>,\ w_h(P_i)\leqq\alpha\}.$$

Then, we can write

$$w_{h,\alpha}=\sum_{i\in I_1}(w_{hi}-\alpha)\phi_{ih},$$

$$w_h-w_{h,\alpha}=\sum_{j\in I_0}w_{hj}\phi_{jh}+\sum_{j\in I_1}\alpha\phi_{jh},$$

where $w_{hk}=w_h(P_k)$, $k=i, j$. Using the notations in (2.1), we have

$$\begin{aligned}
&a_h(w_h-w_{h,\alpha},\ w_{h,\alpha})+(B_h(w_h-w_{h,\alpha}),\ \bar{w}_{h,\alpha})_h+\mu(\bar{w}_h-\bar{w}_{h,\alpha},\ \bar{w}_{h,\alpha})_h\\
&\quad=\sum_{j\in I_0}\sum_{i\in I_1}(a_{ij}+b_{ij})w_{hj}(w_{hi}-\alpha)\ +\sum_{j\in I_1}\sum_{i\in I_1}(a_{ij}+b_{ij}+\mu m_{ii}\delta_{ij})\alpha(w_{hi}-\alpha)\\
&\quad=\sum_{j\in I_0}\sum_{i\in I_1}(a_{ij}+b_{ij})(w_{hj}-\alpha)(w_{hi}-\alpha)\\
&\qquad+\sum_{j\in I_1\cup I_0}\sum_{i\in I_1}(a_{ij}+b_{ij}+\mu m_{ii}\delta_{ij})\alpha(w_{hi}-\alpha).
\end{aligned}$$

By (2.2) and (2.3), we get $a_{ij}+b_{ij}\leqq 0$ for $i\neq j$, which implies that the first term is nonnegative. Noting that $I_1\subset <1,\ N>$ by (5.20), and applying (2.4), we have

$$\text{the second term}=\sum_{i\in I_1}\alpha(w_{hi}-\alpha)\mu m_{ii}\geqq 0.$$

Therefore, we obtain (5.22).

We next prove that

$$|(B_h w_{h,\alpha},\ \bar{w}_{h,\alpha})_h|\leqq c(\gamma,\ n)\,\|\boldsymbol{b}\|_{0,\infty,\Omega}|w_{h,\alpha}|_{1,2,\Omega_h}\|\bar{w}_{h,\alpha}\|_{0,2,\Omega_h}. \tag{5.23}$$

By i) of Assumption 2, the ratio of the volumes of any two n-simplices sharing with a nodal point is bounded by a constant $c(\gamma, n)$, which gives (5.23). Using the constant $c(\gamma, n)$ in (5.23), we fix μ_0 as follows,

$$\mu_0=\frac{1}{2}c(\gamma,\ n)^2\|\boldsymbol{b}\|^2_{0,\infty,\Omega}. \tag{5.24}$$

Combining (5.22)~(5.24), we have

$$\begin{aligned}
&a_h(w_h,\ w_{h,\alpha})+(B_h w_h,\ \bar{w}_{h,\alpha})_h+\mu(\bar{w}_h,\ \bar{w}_{h,\alpha})_h\\
&\geqq|w_{h,\alpha}|^2_{1,2,\Omega_h}-c(\gamma,\ n)\|\boldsymbol{b}\|_{0,\infty,\Omega}|w_{h,\alpha}|_{1,2,\Omega_h}\|\bar{w}_{h,\alpha}\|_{0,2,\Omega_h}\\
&\quad+\mu\|\bar{w}_{h,\alpha}\|^2_{0,2,\Omega_h}\\
&\geqq\frac{1}{2}|w_{h,\alpha}|^2_{1,2,\Omega_h}+(\mu-\mu_0)\|\bar{w}_{h,\alpha}\|^2_{0,2,\Omega_h}
\end{aligned}$$

$$\geqq \frac{1}{2}|w_{h,\alpha}|_{1,2,\Omega_h}^2.$$

Noting that $w_{h,\alpha} \in V_{0h}$ and using Poincaré's inequality and (3.1), we obtain (5.21). This completes the proof.

Lemma 5.8. *Suppose* $1 \leqq q < +\infty$ *and that* Ω_h *and* Ω_1 *satisfy* (3.1) *and* (3.2). *Then, there exists a positive constant* $c(\Omega, q, n)$ *such that*

$$(5.25) \qquad \|g\|_{0,q,\Omega_h-\Omega} \leqq ch^{2/q}\|g\|_{1,q,\Omega_1} \qquad \text{for} \quad g \in W^{1,q}(\Omega_1).$$

Proof. We begin by proving that

$$(5.26) \qquad \left\{\int_0^d dx_n \int_{R^{n-1}} |g(x', x_n)|^q dx'\right\}^{1/q} \leqq c(q) d^{1/q}\|g\|_{1,q,R^n}$$

$$\text{for} \quad g \in C_0^\infty(R^n),$$

where $d(<1)$ is a positive constant and $x' = (x_1, \dots, x_{n-1})$. Using that

$$g(x', x_n) = g(x', 0) + \int_0^{x_n} \frac{\partial g}{\partial x_n}(x', t)dt,$$

we have for $0 < x_n < d$

$$|g(x', x_n)|^q \leqq c(q)|g(x', 0)|^q + x_n^{q/q'} \int_0^d \left|\frac{\partial g}{\partial x_n}(x', t)\right|^q dt,$$

where q' is defined by $\frac{1}{q} + \frac{1}{q'} = 1$. By the boundedness of the trace operator, we obtain

$$\int_0^d dx_n \int_{R^{n-1}} |g(x', x_n)|^q dx' \leqq c(q) d \int_{R^{n-1}} |g(x', 0)|^q dx'$$

$$+ \int_0^d x_n^{q/q'} dx_n \int_{R^{n-1}} dx' \int_0^d \left|\frac{\partial g}{\partial x_n}(x', t)\right|^q dt$$

$$\leqq c(q)(d + d^q)\|g\|_{1,q,R^n}^q,$$

which implies (5.26).

By making use of a partition of unity, of C^1-mappings (a portion of Γ is transferred into $\{(x', x_n);\ x_n = 0\}$ by a map), and of (5.26) and (3.1), we obtain (5.25) for all $g \in C_0^\infty(R^n)$. Density arguments yield (5.25) for all $g \in W^{1,q}(\Omega_1)$. This completes the proof.

Proof of Theorem 3.1. Let w_h be $v_h - I_h v \in V_{0h}$ and $w_{h,\alpha}$ be as (5.19) for $\alpha \geqq 0$. We first prove that

$$(5.27) \qquad \|w_{h,\alpha}\|_{1,2,\Omega_h}^2 \leqq ch\|v\|_{2,p,\Omega}\|w_{h,\alpha}\|_{1,p',\Omega_h},$$

where $c = c(\gamma, \Omega, n, p, \mu, \|b\|_{0+1,\infty,\Omega})$ and $p' = \frac{p}{p-1}$. Let $\tilde{v} = \Xi v$, where Ξ is the extension operator defined in § 3. Set $f_1 \in L^p(\Omega_1)$ as follows,

$$f_1 = -\Delta\tilde{v} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\tilde{v} + \mu\tilde{v}.$$

Since $w_{h,\alpha} \in V_{0h}$, we have

$$a_h(\tilde{v}, w_{h,\alpha}) + ((\boldsymbol{b}\cdot\boldsymbol{\nabla})\tilde{v}, w_{h,\alpha})_h + \mu(\tilde{v}, w_{h,\alpha})_h = (f_1, w_{h,\alpha})_h. \tag{5.28}$$

Combining (3.4) and (5.28), we obtain

$$\begin{aligned} &a_h(w_h, w_{h,\alpha}) + (B_h w_h, \bar{w}_{h,\alpha})_h + \mu(\bar{w}_h, \bar{w}_{h,\alpha})_h \\ &= (\tilde{f}, w_{h,\alpha})_h - a_h(I_h v, w_{h,\alpha}) - (B_h I_h v, \bar{w}_{h,\alpha})_h - \mu(\overline{I_h v}, \bar{w}_{h,\alpha})_h \\ &= (\tilde{f} - f_1, w_{h,\alpha})_h - a_h(I_h v - \tilde{v}, w_{h,\alpha}) - \{(B_h I_h v, \bar{w}_{h,\alpha})_h - ((\boldsymbol{b}\cdot\boldsymbol{\nabla})\tilde{v}, w_{h,\alpha})_h\} \\ &\quad - \mu\{(\overline{I_h v}, \bar{w}_{h,\alpha})_h - (\tilde{v}, w_{h,\alpha})_h\}. \end{aligned} \tag{5.29}$$

Let us estimate each term of the right of (5.29). Defining $w_{h,\alpha} = 0$ on $\Omega_1 - \Omega_h$ and applying Lemma 5.8, (3.1), and (3.5), we have

$$\begin{aligned} |\text{the first term}| &\leqq \left\{\int_{\Omega_h - \Omega} |\tilde{f} - f_1|^p dx\right\}^{1/p} \left\{\int_{\Omega_h - \Omega} |w_{h,\alpha}|^{p'} dx\right\}^{1/p'} \\ &\leqq c(\Omega, p, n) h^{2/p'} \|v\|_{2,p,\Omega} \|w_{h,\alpha}\|_{1,p',\Omega_h}. \end{aligned} \tag{5.30}$$

By (5.1)~(5.4) and Lemma 5.6, we have

$$|\text{the second term}| \leqq c(\gamma, p, n) h \|v\|_{2,p,\Omega} |w_{h,\alpha}|_{1,p',\Omega_h}, \tag{5.31}$$

$$|\text{the third term}| \leqq c(\gamma, p, n) h \|\boldsymbol{b}\|_{0+1,\infty,\Omega} \|v\|_{2,p,\Omega} \|w_{h,\alpha}\|_{1,p',\Omega_h}, \tag{5.32}$$

and

$$\begin{aligned} |\text{the fourth term}| &\leqq \mu\{|(\overline{I_h v} - \tilde{v}, \bar{w}_{h,\alpha})_h| + |(\tilde{v}, \bar{w}_{h,\alpha} - w_{h,\alpha})_h|\} \\ &\leqq \mu c(\gamma, p, n) h \|v\|_{1,p,\Omega} \|w_{h,\alpha}\|_{1,p',\Omega_h}. \end{aligned} \tag{5.33}$$

Combining (5.29)~(5.33), applying Lemma 5.7, and noting $1 < p' < 2$, we obtain (5.27).

According to Ciarlet and Raviart [4], we define

$$E(\alpha) = \{x;\ x \in \Omega_h,\ w_{h,\alpha}(x) > 0\},$$

and $\zeta(\alpha) = \operatorname{mes} E(\alpha)$. By Hölder's inequality and Sobolev's lemma, we have

$$\|w_{h,\alpha}\|_{1,p',\Omega_h} \leqq c(p') \zeta(\alpha)^{\frac{1}{p'} - \frac{1}{2}} \|w_{h,\alpha}\|_{1,2,\Omega_h}, \tag{5.34}$$

and

$$\|w_{h,\alpha}\|_{0,q,\Omega_h} \leqq c(q, n, \Omega) \|w_{h,\alpha}\|_{1,2,\Omega_h}, \tag{5.35}$$

where $q = \dfrac{2n}{n-2}$ in $n \neq 2$ and we take $q > \dfrac{2p}{p-2}$ in $n = 2$. Combining (5.27), (5.34), and (5.35), we obtain

$$\|w_{h,\alpha}\|_{0,q,\Omega_h} \leqq ch\|v\|_{2,p,\Omega}\,\zeta(\alpha)^{\frac{1}{p'}-\frac{1}{2}}, \tag{5.36}$$

where $c=c(\gamma, \Omega, n, p, \mu, \|\boldsymbol{b}\|_{0+1,\infty,\Omega}, q)$.

Now take $\beta>\alpha$. Then, we can write

$$\bar{E}(\beta)=\bigcup_j \{T_j;\ \text{there exists a vertex } P_k^j \text{ of } T_j \text{ such that } w_h(P_k^j)>\beta\}.$$

Therefore, we have

$$w_{h,\alpha}(x)\geqq(\beta-\alpha)\lambda_k^j\geqq 0 \qquad \text{on} \quad T_j\in\bar{E}(\beta),$$

where λ_k^j is the barycentric coordinate associated with P_k^j in T_j. Noting $E(\beta)\subset E(\alpha)$ and that

$$\int_{T_j}|\lambda_k^j|^q dx=(q+n+1)B(n+1,\ q+1)\,\mathrm{mes}\,T_j,$$

where B is the beta function, we get

$$\|w_{h,\alpha}\|_{0,q,\Omega_h}\geqq(\beta-\alpha)\{(q+n+1)B(n+1,\ q+1)\}^{1/q}\zeta(\beta)^{1/q}. \tag{5.37}$$

Combining (5.36) and (5.37), we obtain

$$\zeta(\beta)\leqq\left(\frac{ch\|v\|_{2,p,\Omega}}{\beta-\alpha}\right)^q\zeta(\alpha)^{q\left(\frac{1}{p'}-\frac{1}{2}\right)} \qquad \text{for} \quad \beta>\alpha\geqq 0, \tag{5.38}$$

where $c=c(\gamma, \Omega, n, p, \mu, \|\boldsymbol{b}\|_{0+1,\infty,\Omega}, q)$.

Since $q\left(\frac{1}{p'}-\frac{1}{2}\right)>1$, by a result of Stampacchia [17, Lemma 4.1] we obtain

$$\zeta(c'ch\|v\|_{2,p,\Omega})=0,$$

where $c'=c'(\zeta(0), q, p')$ is monotone increasing with respect to $\zeta(0)$. Therefore, we have

$$v_h(x)-(I_hv)(x)\leqq ch\|v\|_{2,p,\Omega} \qquad \text{for all} \quad x\in\Omega_h, \tag{5.39}$$

where $c=c(\gamma, \Omega, n, p, \mu, \|\boldsymbol{b}\|_{0+1,\infty,\Omega})$. Replacing v by $-v$, we have

$$-v_h(x)+(I_hv)(x)\leqq ch\|v\|_{2,p,\Omega} \qquad \text{for all} \quad x\in\Omega_h. \tag{5.40}$$

Combining (5.39) and (5.40), we obtain (3.7). This completes the proof.

§6. Concluding remarks

We have discussed hitherto uniform convergence of the upwind finite element approximation for semilinear parabolic problems. In dealing with the nonlinear term, the assumption (1.2) has been essential. Now we can replace it by the weaker condition

$$f_1(x, t, u)\leqq M \qquad \text{for} \quad (x, t)\in\bar{Q},\ u\in[U_1, U_2], \tag{6.1}$$

where U_1 (resp. U_2) is the minimum (resp. maximum) value of the exact solution u in $\bar{Q}$. Actually, consider a continuous function $\hat{f}_1(x, t, u)$ such that $\hat{f}_1$ is continuously differentiable in u and that

$$\hat{f}_1 \begin{cases} =f_1 & \text{in} \quad \bar{Q}\times[U_1, U_2] \\ \leqq M+1 & \text{otherwise.} \end{cases}$$

Then, replacing f_1 by $\hat{f}_1$ does not give rise to any change in the solution in $\bar{Q}$. Since $\hat{f}_1$ satisfies i) of Assumption 1, we can apply Theorems 1.1 and 1.2 to the schemes (1.5) and (1.6) with $\hat{f}_1$ and $M+1$ in place of f_1 and M respectively. By the same argument, for the problems of blow-up type such that the solution tends to infinity at a finite time T^*, we can also show the uniform convergence in $\bar{\Omega}\times[0, T^*-\varepsilon]$ for any $\varepsilon>0$. (cf. Nakagawa-Ushijima [14])

Our method can be extended straightforward to the problems with Neumann condition,

$$\begin{cases} \dfrac{\partial u}{\partial t}=\Delta u-(\boldsymbol{b}\cdot\nabla)u+f(x, t, u) & \text{in} \quad Q, \\ \dfrac{du}{d\boldsymbol{\nu}}=g(x, t) & \text{on} \quad \Sigma, \\ u=u^0 & \text{in } \Omega \text{ at } t=0, \end{cases}$$

if $\boldsymbol{b}$ satisfies

$$\boldsymbol{b}\cdot\boldsymbol{\nu}\geqq 0 \qquad \text{for all} \quad x\in\Gamma,$$

where $\boldsymbol{\nu}$ is the outer normal to Γ.

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References

[1] Bramble, J. H. and Hilbert, S. R., Bounds for a class of linear functionals with applications to Hermite interpolation, *Numer. Math.*, **16** (1971), 362–369.

[2] Ciarlet, P. G. and Raviart, P. A., General Lagrange and Hermite interpolation in $\mathbf{R}^n$ with applications to finite element methods, *Arch. Rational Mech. Anal.*, **46** (1971), 177–199.

[3] Ciarlet, P. G. and Wagschal, C., Multipoint Taylor formulas and applications to the finite element method, *Numer. Math.*, **17** (1971), 84–100.

[4] Ciarlet, P. G. and Raviart, P. A., Maximum principle and uniform convergence for the

finite element method, *Computer Methods in Applied Mechanics and Engineering*, **2** (1973), 17–31.

[5] Douglas, J., Jr. and Dupont, T., Galerkin methods for parabolic equations, *SIAM J. Numer. Anal.*, **7** (1970), 575–626.

[6] Fujii, H., Some remarks on finite element analysis of time-dependent field problems, *Theory and practice in finite element structural analysis*, ed. by Yamada, Y. and Gallagher, R. H., 91–106, Univ. of Tokyo Press, Tokyo, 1973.

[7] Fujita, H. and Mizutani, A., On the finite element method for parabolic equations, I; approximation of holomorphic semi-groups, *J. Math. Soc. Japan*, **28** (1976), 749–771.

[8] Kikuchi, F., The discrete maximum principle and artificial viscosity in finite element approximations to convective diffusion equations, ISAS Report, **550** (1977).

[9] Lesaint, P. and Raviart, P. A., On a finite element method for solving the neutron transport equation, Mathematical Aspects of Finite Elements in Partial Differential Equations, ed. by de Boor, C., 89–123, Academic Press, New York, 1974.

[10] Mimura, M., Kametaka, Y. and Yamaguti, M., On a certain difference scheme for some semilinear diffusion system, *Proc. Japan Acad.*, **47** (1971), 385–387.

[11] Mimura, M., Finite difference method for a class of semilinear parabolic systems related to physical and biological problems, Doct. Thesis, Kyoto Univ. 1973.

[12] Mizohata, S., *The theory of partial differential equations*, Cambridge Univ. Press, London, 1973. (translation of *Henbibun-Hoteishiki Ron*, Iwanami, Tokyo, 1965)

[13] Morrey, C. B., *Multiple integrals in the calculus of variations*, Springer, Berlin, 1966.

[14] Nakagawa, T. and Ushijima, T., Finite element analysis of the semilinear heat equation of blow-up type, to appear in *Topics in Numerical Analysis III*, Academic Press.

[15] Nečas, J., *Les méthodes directes en théorie des équations elliptiques*, Prague: Acad. Tchécoslovaque des Science 1967.

[16] Richtmyer, R. D. and Morton, K. W., *Difference methods for initial-value problems*, 2nd edn., Wiley, New York, 1967.

[17] Stampacchia, G., Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier* (*Grenoble*), **15** (1965), 189–258.

[18] Tabata, M., A finite element approximation corresponding to the upwind finite differencing, *Memoirs of Numerical Mathematics*, **4** (1977), 47–63.

[19] Thomée, V. and Wahlbin, L., On Galerkin methods in semilinear parabolic problems, *SIAM J. Numer. Anal.*, **12** (1975), 378–389.

[20] Ushijima, T., On the lumped mass approximation of semilinear parabolic equations, 1975-annual report of the trial research in large scale computation supported by Japan Ministry of Education, 90–131.

[21] Ushijima, T., On the uniform convergence for the lumped mass approximation of the heat equation, J. Fac. Sci. Univ. Tokyo, **24** (1977), 477–490.

[22] Waltz, J. E., Fulton, R. E. and Cyrus, N. J., Accuracy and convergence of finite element approximation, *Proc. 2nd Conf. Matrix Meth. Struct. Mech.*, 995–1027, Wright-Patterson Air Force Bace, Ohio, 1968.

[23] Wheeler, M., A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations, *SIAM J. Numer. Anal.*, **10** (1973), 723–759.

[24] Yamamoto, Y. and Tokuda, N., A note on convergence of finite element solutions, *Int. J. for Numerical Methods in Engineering*, **3** (1971), 485–493.